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Asymptotic $\mathcal{W}$-symmetries in three-dimensional higher-spin gauge theories

A. Campoleoni, S. Fredenhagen and S. Pfenninger

Max-Planck-Institut für Gravitationsphysik
Albert-Einstein-Institut
Am Mühlenberg 1
14476 Golm, GERMANY

andrea.campoleoni@aei.mpg.de, stefan.fredenhagen@aei.mpg.de, stefan.pfenninger@aei.mpg.de

Abstract

We discuss how to systematically compute the asymptotic symmetry algebras of generic three-dimensional bosonic higher-spin gauge theories in backgrounds that are asymptotically AdS. We apply these techniques to a one-parameter family of higher-spin gauge theories that can be considered as large N limits of $SL(N) \times SL(N)$ Chern-Simons theories, and we provide a closed formula for the structure constants of the resulting infinite-dimensional non-linear $\mathcal{W}$-algebras. Along the way we provide a closed formula for the structure constants of all classical $\mathcal{W}_N$ algebras. In both examples the higher-spin generators of the $\mathcal{W}$-algebras are Virasoro primaries. We eventually discuss how to relate our basis to a non-primary quadratic basis that was previously discussed in literature.

Contents

1 Introduction

In a three-dimensional space-time the little group of massless particles does not admit representations with arbitrary helicity. Nevertheless, one can still consider the analogues of the field equations that for $D > 3$ describe the free propagation of spin-s massless particles, although in $D = 2+1$ they do not propagate any local degree of freedom for $s > 1$. Any non-linear completion of these field equations defines a three-dimensional higher-spin (HS) gauge theory, at least as a classical theory. The first member of this class is just Einstein gravity, which does not admit wave solutions in $D = 2+1$. Despite the lack of propagating degrees of freedom, these toy models already display many features of their higher-dimensional counterparts. They thus often provide a manageable testing ground for various ideas on field theories involving higher spins.

A marked simplification arises when one considers pure HS gauge theories without matter: in $D = 2+1$ the coupling of massless higher-spin fields to gravity can be described by a Chern-Simons (CS) action for any value of the cosmological constant [1]. On the other hand, in higher space-time dimensions the non-linear field equations of Vasiliev [2] require a non-vanishing cosmological constant (see [3, 4] for a review), while on flat backgrounds a classification of cubic vertices is now available [5], but a complete interacting theory is still lacking (see [6] for an account of the state of the art).

Even if it is not needed to handle a full interacting theory, a negative cosmological constant plays an important role in $D = 2+1$. In the gravitational sector it allows for a richer space of solutions, still without gravitational waves but with black holes [7]. It also allows to build three-dimensional counterparts of Vasiliev's models [8], describing the coupling of two or four scalars to a HS gauge sector. The latter toy models provide a natural arena to test – with suitable extrapolations – various proposals that relate holographically Vasiliev's theory to conformal field theories. This interplay was first investigated by Sezgin and Sundell [9] (see also [10] for earlier suggestions), while recently various groups independently elaborated upon a conjecture by Klebanov and Polyakov, that links Vasiliev's theory on AdS_4 to the large N limit of the three-dimensional critical $O(N)$ vector model [11, 12].

A first step toward the characterisation of possible two-dimensional CFT duals was performed in [13, 14], with an analysis of the asymptotic symmetries of some three-dimensional classical HS gauge theories. The outcome generalises an earlier result by Brown and Henneaux [15], that defined the class of asymptotically Anti de Sitter solutions of three-dimensional Einstein's equations such that it contains all physically relevant solutions and all its elements have finite boundary charges. In $D = 2+1$ these conditions allow for an enhancement of asymptotic symmetries from the AdS algebra to two copies of a centrally extended Virasoro algebra, with a central charge that grows with the AdS radius. Adding massless higher-spin fields maintains the conformal symmetry and actually extends it: each Virasoro algebra is replaced by a centrally extended non-linear $\mathcal{W}$-algebra, with the same central charge as in pure gravity (see [16] for an introduction to

$\mathcal{W}$-algebras). The classical asymptotic $\mathcal{W}$-symmetries of [13, 14] were then shown to survive even at the quantum level in [17], and these observations led Gaberdiel and Gopakumar to conjecture in [18] a duality between suitable large N limits of minimal model CFT's with $\mathcal{W}_N \times \mathcal{W}_N$ symmetry and the Vasiliev-like models of [8] (see also [19, 20, 21] for some recent additional checks of this conjecture and [22, 23] for its extension to other classes of minimal models).

In [13, 14, 19] the identification of asymptotic symmetries rests heavily upon the CS formulation of the HS dynamics, somehow following the derivation of [24] of the original Brown-Henneaux result. On the one hand, the CS action enables one to define quite straightforwardly the boundary charges [25]. On the other hand, CS theories defined on a manifold with boundary admit a non-trivial boundary dynamics, generically described by a Wess-Zumino-Witten (WZW) action (see, for instance, [26] and references therein). As a result, the Poisson structure on the phase space of boundary excitations is generically an affine algebra. However, even if the action of three-dimensional HS gauge theories can be cast in a CS form, not all solutions of the CS theory are admissible classical higher-spin configurations. Selecting the class of asymptotically AdS solutions imposes a constraint on the phase space of the boundary theory and $\mathcal{W}$-algebras emerge as Dirac-bracket algebras on the constrained phase space.

In the mathematical literature this way of constructing classical $\mathcal{W}$-algebras out of affine algebras is known as Drinfeld-Sokolov (DS) reduction (see [27] for a review), and it was first applied to WZW theories at the end of the eighties [28]. It associates a classical centrally extended $\mathcal{W}$-algebra to any semisimple Lie algebra, independent of whether it is the gauge algebra of a sensible toy model for higher-spin interactions or not. In this paper we propose a procedure to compute the structure constants of any $\mathcal{W}$-algebra that can be obtained by a DS reduction, and we apply it to a class of algebras that are relevant to the study of higher spins.

In Section 2 we begin by recalling the interplay between the DS reduction and the asymptotic symmetries of three-dimensional HS gauge theories. Then we present our procedure to "reduce" a generic affine algebra. We eventually use this tool to shed light on some general properties of the resulting $\mathcal{W}$-algebras and to analyse some examples that lay outside of the class of algebras considered in the following sections. Let us already mention that the DS construction rests upon a gauge choice: different choices lead to different bases for the $\mathcal{W}$-algebra. Our analysis is focused on the so called "highest-weight gauge", thus providing an alternative to similar results that were previously obtained in the so called "U-gauge" (see chapter 9 of [27]). The highest-weight choice gives a $\mathcal{W}$-algebra where all generators are primaries with respect to the lowest-spin ones. In the absence of spin-1 generators all of them are thus Virasoro primaries. In the general case one can easily recover a basis with this property by shifting the Virasoro current with the Sugawara energy-momentum tensor built from spin-1 currents.

In Section 3 we return to three-dimensional HS gauge theories. Besides them there is a one-parameter family that plays a distinguished role, and we focus on it. It was first discussed in [29, 30, 31, 32, 33] in the late eighties, when the subject was in its infancy. All its members are bosonic theories describing, for generic values of a parameter λ, the coupling to gravity of a set of massless fields where each integer spin from 3 onwards appears once. The interest in this family is twofold: on the one hand, for integer $\lambda = N$ the CS action reduces to that of a $SL(N) \times SL(N)$ theory, while all other values of λ provide a sort of large N limit of this rather natural class of toy models for HS interactions, out of which the case $N = 2$ coincides with Einstein gravity [34, 35]. On the other hand, whenever no truncations arise, the field content is the same as in the gauge sector of Vasiliev's models, that are actually built upon the same gauge algebras [8]. The corresponding $\mathcal{W}$-algebras are thus expected to emerge also in the large N limit of $\mathcal{W}_N \times \mathcal{W}_N$ minimal models, as discussed in [19, 20, 21].

Even if for different values of λ the field contents coincide, the gauge algebras are inequivalent [29, 33]. Therefore, as already pointed out in [19], different λ lead to inequivalent asymptotic symmetries. These are given by two copies of an infinite-dimensional non-linear $\mathcal{W}$-algebra, that we denote by $\mathcal{W}_\infty[\lambda]$ as in [19]. This family of $\mathcal{W}$-algebras was introduced independently in [36] and [37], in a non-primary basis with at most quadratic non-linearities appearing in the Poisson brackets. In Section 3 we use the machinery developed in Section 2 to provide a closed formula for all structure constants of $\mathcal{W}_\infty[\lambda]$ in a Virasoro-primary basis. Our formula reproduces the results for the first few spins that were computed in [19]. Setting $\lambda = N$ also gives a closed formula for the structure constants of $\mathcal{W}_N$ in a Virasoro-primary basis. In Section 4 we eventually discuss how to relate systematically our presentation of $\mathcal{W}_\infty[\lambda]$ to the quadratic basis of [36]. The paper closes with a summary of our results and some appendices. Appendix A summarises the structure constants of the one-parameter family of higher spin algebras that we consider. In Appendix B we collected the proofs of some formulae appearing in the main text, in particular the formula for the structure constants of $\mathcal{W}_\infty[\lambda]$. Finally, in Appendix C we display the Poisson brackets of the examples of $\mathcal{W}$-algebras that we discuss in the main body of the text.

Let us finally stress that the characterisation of asymptotically AdS solutions that triggers our analysis goes beyond the identification of proper fall-off conditions at spatial infinity. Rather, it selects exact solutions of the field equations. The study of exact solutions is another interesting arena where one may take advantage of the simplicity of three-dimensional toy models to extract information on their higher-dimensional counterparts. Exact solutions of Vasiliev's models in $D > 3$ were first obtained in [38] (see [4] for a review), while recently solutions displaying various similarities with gravity black holes were presented in [39]. On the three-dimensional side various issues on exact solutions were discussed in [8, 40, 14, 41, 42, 43]. In Section 3.1.1 we discuss how one could extend an earlier proposal of [14] in order to express our exact solutions in an alternative form involving only Lorentz-invariant metric-like fields (see [44] for a review of the metric-like formulation). Aside from making more transparent the identification between CS theories and HS gauge theories, we hope that this interplay between alternative approaches provides useful tools to better understand the exact solutions already discussed in literature, such as the intriguing HS generalisations of gravity black holes [39, 40, 42, 43].

2 Asymptotic symmetries from Drinfeld-Sokolov reduction

A three-dimensional pure higher-spin (HS) gauge theory coupled to gravity in backgrounds that are asymptotically AdS can be described by a Chern-Simons (CS) theory supplemented by suitable boundary conditions [13, 14]. These translate into the Drinfeld-Sokolov (DS) constraint on the centrally extended loop algebra that appears on the boundary of a CS theory. Therefore, asymptotic symmetries are described by the $\mathcal{W}$-algebras that arise from the DS reduction. In this section we first review the DS reduction in the context of HS gauge theories. We then provide an algorithm to perform it in the highest-weight gauge, from which one obtains $\mathcal{W}$-algebras in a basis where all fields are primaries with respect to the lowest spin ones.

2.1 Higher-spin gauge theories in $D = 2+1$

In $D = 2+1$ Einstein gravity with a negative cosmological constant is equivalent to a $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ Chern-Simons theory [34, 35]. In fact, up to boundary terms, one can rewrite the Einstein-Hilbert action as

$$S \,=\, S_{CS}[A] \,-\, S_{CS}[\widetilde{A}]\,, \tag{2.1}$$

with

$$S_{CS}[A] \,=\, \frac{k}{4\pi}\int \mathrm{tr}\left(\, A\wedge dA \,+\, \frac{2}{3}\, A\wedge A\wedge A\,\right). \tag{2.2}$$

The fields A and $\widetilde{A}$ are $sl(2,\mathbb{R})$-valued differential forms so that, for instance, $A = A_\mu{}^i\, J_i\, dx^\mu$, where the J_i generate the $sl(2,\mathbb{R})$ algebra. We normalise the invariant form entering the CS action such that

$$\mathrm{tr}\,(J_i J_j) \,=\, \frac{1}{2}\,\eta_{ij} \quad \Rightarrow \quad k \,=\, \frac{l}{4G}\,, \tag{2.3}$$

where l denotes the AdS radius and G is Newton's constant. The standard first-order formulation of gravity is recovered by considering the combinations

$$e \,=\, \frac{l}{2}\left(A - \widetilde{A}\right)\,, \qquad\qquad \omega \,=\, \frac{1}{2}\left(A + \widetilde{A}\right)\,, \tag{2.4}$$

that identify the dreibein and the spin connection.

In a similar fashion, the first-order formulation of the free dynamics of massless bosonic symmetric fields $\varphi_{\mu_1\ldots\,\mu_s}$ with $s \geq 2$ involves a vielbein-like 1-form and an auxiliary 1-form which generalises the spin connection [45, 46]. In $D = 2+1$ these two differential forms have the same structure (i.e. the same fibre indices), and one can consider linear combinations of them as in (2.4). This eventually allows one to build bosonic massless higher-spin gauge theories out of $G \times G$ Chern-Simons theories [1]. The action has the same form as (2.1), but now the gauge fields A and $\widetilde{A}$ take values in a (possibly infinite-dimensional) Lie algebra $\mathfrak{g}$ admitting a non-degenerate bilinear invariant form. Vielbeine and spin connections are identified through (2.4), while the invariance of the action under

$$\delta A \,=\, dA \,+\, [\,A\,,\,\lambda\,]\,, \qquad\qquad \delta\widetilde{A} \,=\, d\widetilde{A} \,+\, [\,\widetilde{A}\,,\,\tilde{\lambda}\,] \tag{2.5}$$

leads to two different kinds of gauge transformations generated by the parameters

$$\xi \,=\, \frac{l}{2}\left(\lambda - \tilde{\lambda}\right)\,, \qquad\qquad \Lambda \,=\, \frac{1}{2}\left(\lambda + \tilde{\lambda}\right)\,. \tag{2.6}$$

Those generated by ξ correspond to local translations in pure gravity (that in $D = 2+1$ are equivalent to diffeomorphisms [35]), while those generated by Λ extend the usual local Lorentz transformations:

$$\delta e \,=\, d\xi \,+\, [\,\omega\,,\,\xi\,] \,+\, [\,e\,,\,\Lambda\,]\,, \tag{2.7a}$$

$$\delta\omega \,=\, d\Lambda \,+\, [\,\omega\,,\,\Lambda\,] \,+\, \frac{1}{l^2}\,[\,e\,,\,\xi\,]\,. \tag{2.7b}$$

Even if no local fluctuations are present, one can define the "spectrum" of the theory by looking at the transformation properties of the fields under Lorentz transformations. It is thus fixed by the choice of a $sl(2,\mathbb{R})$ subalgebra in $\mathfrak{g}$ that – together with the corresponding one coming from the second copy of $\mathfrak{g}$ – identifies the gravitational sector. Once this selection is made one can consider the branching of $\mathfrak{g}$ under the adjoint action of the "gravitational" $sl(2,\mathbb{R})$, so that

$$\mathfrak{g} \,=\, sl(2,\mathbb{R}) \,\oplus\, \left(\bigoplus_{\ell\,,\,a} \mathfrak{g}^{(\ell,a)}\right)\,. \tag{2.8}$$

Each $\mathfrak{g}^{(\ell,a)}$ has dimension $2\ell+1$ with $2\ell \in \mathbb{N}$, while the index a accounts for possible multiplicities. For infinite-dimensional algebras we thus discard by hypothesis $sl(2)$ embeddings that would bring on infinite-dimensional irreducible representations in (2.8). The branching of $\mathfrak{g}$ induces the decomposition

$$A(x) \,=\, A_\mu{}^i(x) J_i\, dx^\mu \,+\, \sum_{\ell\,,\,a}\,\sum_{m\,=\,-\,\ell}^{\ell} A_\mu^{[a]\,\ell,m}(x)\,(W^\ell_m)_{[a]}\, dx^\mu\,, \tag{2.9}$$

and a similar one for $\widetilde{A}$. Here the $(W^\ell_m)_{[a]}$ generate $\mathfrak{g}^{(\ell,a)}$, while the J_i generate $sl(2,\mathbb{R})$ as in pure gravity. Let us now focus for a while on $sl(2)$ embeddings that do not involve any half-integer ℓ. In this case the dimension of each $\mathfrak{g}^{(\ell,a)}$ equals the number of independent off-shell fiber components of the vielbein or of the spin connection associated to a fully symmetric tensor $\varphi^{[a]}{}_{\mu_1\ldots\mu_{\ell+1}}$ [45, 46]. As a result, for any integer ℓ the 1-forms

$$e_\mu^{[a]\,\ell,m} \,=\, \frac{l}{2}\left(A_\mu^{[a]\,\ell,m} - \widetilde{A}_\mu^{[a]\,\ell,m}\right)\,, \qquad\qquad \omega_\mu^{[a]\,\ell,m} \,=\, \frac{1}{2}\left(A_\mu^{[a]\,\ell,m} + \widetilde{A}_\mu^{[a]\,\ell,m}\right) \tag{2.10}$$

can be identified with the vielbein and the spin connection of a spin-$(\ell+1)$ field. The spectrum is thus specified by the "spins" and the multiplicities appearing in (2.8). Additional comments on this identification will be presented in Section 3.1.1.

The choice of the principal $sl(2,\mathbb{R})$ embedding in a finite-dimensional non-compact simple Lie algebra $\mathfrak{g}$ always fits into this scheme. In this case each ℓ in (2.8) corresponds to one of the exponents of $\mathfrak{g}$, so that all values of ℓ are integers and greater or equal to 1. The simplest examples in this class are $SL(N) \times SL(N)$ CS theories with a principally embedded gravitational sector. Besides the graviton, they involve fields with spin $(\ell+1) = 3, \ldots, N$. An infinite-dimensional counterpart of the latter HS gauge theories, with a similar but unbounded spectrum, will be studied in Section 3. An ampler discussion of $SL(N) \times SL(N)$ theories can be found in [14], while here we would like to briefly discuss the subtleties that the choice of alternative embeddings entails.

First of all, in general (2.8) also contains half-integer values of ℓ. The corresponding 1-forms in (2.10) cannot be associated to any tensorial field. On the other hand, for each half-integer ℓ their number equals the number of off-shell components of a *spinorial* field ${\psi^\alpha}_{\mu_1 \ldots \mu_{\ell+1/2}}$. Since no local degrees of freedom are involved, the spinorial nature of the resulting fields does not prevent one from applying the previous construction even starting from a purely bosonic gauge algebra. A slight generalisation is also possible, since spinorial fields do not require a spin-connection in the first-order formulation [45, 47, 48]. The doubling of $\mathfrak{g}$ would then lead to a doubling of the number of these fields in the spectrum. The same is true for spin-1 fields, corresponding to $sl(2,\mathbb{R})$ singlets in (2.8). As a result, in general one could even consider $G \times \widetilde{G}$ Chern-Simons theories, provided that $\widetilde{\mathfrak{g}}$ is the subalgebra of $\mathfrak{g}$ needed to reconstruct vielbeine and spin connections for the fields of integer spin $s \geq 2$.[1] The basic building blocks of a toy model for HS interactions would still be available: one has a set of fields that do not propagate any local degree of freedom and that transform under Lorentz transformations as those used to describe the free propagation of "higher spins", in the sense specified at the beginning of the Introduction.

However, we also know the action of these theories, and we can check if it also displays the features that one would like to associate to a sensible toy model for HS interactions. The simplest feature to analyse is the structure of kinetic terms, that is dictated by the structure of the invariant form appearing in (2.2). To study it, we have to fix our notation. We denote the generators of $sl(2,\mathbb{R})$ by $J_\pm, J_0$ and we choose the convention

$$[\, J_+ \,,\, J_- \,] \,=\, 2J_0 \quad , \quad [\, J_\pm \,,\, J_0 \,] \,=\, \pm\, J_\pm \,. \tag{2.11}$$

We also choose a basis of $\mathfrak{g}^{(\ell,a)}$ such that

$$[\, J_i \,,\, (W^\ell_m)_{[a]} \,] \,=\, (i\ell - m)\, (W^\ell_{i+m})_{[a]} \,, \tag{2.12}$$

[1]An analogue construction is standard in the context of pure supergravity theories in $D = 2+1$ where, besides the graviton, only fields with spin $s < 2$ appear. This allows one to consider super-CS theories built upon generic $G \times \widetilde{G}$ supergroups or simply upon $SL(2,\mathbb{R}) \times G$ [34].

where $i = 0, \pm 1$, and we identified $J_{\pm 1} \equiv J_{\pm}$. Eq. (2.12) forces the Killing form to satisfy

$$\mathrm{tr}\left((W^k_m)_{[a]}\, (W^\ell_n)_{[b]} \right) \,=\, (-1)^{\ell-m}\, \frac{(\ell+m)!(\ell-m)!}{(2\ell)!}\, \delta^{k,\ell}\, \delta_{m+n,0}\, (N_\ell)_{ab}\,, \tag{2.13}$$

with

$$(N_\ell)_{ab} \,=\, \mathrm{tr}\left((W^\ell_\ell)_{[a]} (W^\ell_{-\ell})_{[b]} \right)\,. \tag{2.14}$$

Eq. (2.13) implies $(N_\ell)_{ab} = (-1)^{2\ell}(N_\ell)_{ba}$, so that the matrix N_ℓ is symmetric for integer values of ℓ and skew symmetric for half-integer values of ℓ. This leads respectively to symmetric or skew symmetric kinetic terms. While unfamiliar, the latter are precisely as pertains to the bosonic nature of these spinorial fields, and are instrumental in order to attain a non-trivial kinetic term for them, unless further prescriptions are introduced, like a grading of the gauge algebra. At any rate, the models that we are considering involve more than one field, so that an issue should be checked: the relative sign between different kinetic terms. Even if no local fluctuations are available in three dimensions, one could still require that no sign differences are present, as it is crucial in higher space-time dimensions. This is not the case for a generic choice of $\mathfrak{g}$ and of a $sl(2)$ embedding in it[2]. The relative signs between kinetic terms are also affected by the choice of a real form for $\mathfrak{g}$. For instance, as we shall discuss in Section 3.1, these considerations select the real form $sl(N,\mathbb{R})$ in the case of $SL(N) \times SL(N)$ CS theories that we mentioned before.

2.2 Asymptotic symmetries

We are now going to discuss the asymptotic symmetries of asymptotically-AdS configurations. Therefore, our CS theories have to be defined on manifolds $\mathcal{M}$ with a cylindrical boundary $\partial\mathcal{M}$ parameterised by a time-coordinate t and an angular coordinate θ. In order to fix our notation, in this section we first briefly recall the main features of CS theories on manifolds of this type following the reviews [26, 49]. Then, following [14], we discuss how the conditions selecting asymptotically-AdS configurations translate into the Drinfeld-Sokolov constraint.

Let us begin by focusing on a single chiral sector, say the one involving A. As reviewed in [26, 14], it is always possible to choose the gauge

$$A_\rho \,=\, b^{-1}(\rho)\, \partial_\rho\, b(\rho)\,, \tag{2.15}$$

where ρ is a radial coordinate and $b(\rho)$ is an arbitrary function taking values in the gauge group G. The gauge (2.15) is preserved by residual gauge transformations with parameters

$$\Lambda \,=\, b^{-1}(\rho)\, \lambda(t,\theta)\, b(\rho)\,, \tag{2.16}$$

[2] While this paper was in preparation, HS gauge theories based on a non-principally embedded gravitational sector were discussed in [43] (see also [14] for previous comments). Its authors proposed to consider all possible embeddings in a given gauge algebra as different phases of a common theory, related by a breaking of the Lorentz-like symmetries of (2.7). The opportunities opened by this observation could well overcome our reservations, but still any attempt to extrapolate possible results to higher dimensions should face the subtleties that we remarked here.

and on shell it implies

$$A_\theta \;=\; b^{-1}(\rho)\, a(t,\theta)\, b(\rho)\,. \tag{2.17}$$

Here, $\lambda(t,\theta)$ and $a(t,\theta)$ are arbitrary $\mathfrak{g}$-valued functions. One can then impose the boundary condition

$$\left(\frac{A_t}{l} - A_\theta\right)\bigg|_{\partial\mathcal{M}} \;=\; 0 \tag{2.18}$$

that cancels the boundary term appearing in the variation of the action. In pure AdS gravity, (2.18) is satisfied by all BTZ backgrounds, so that we can safely use it to select the space of asymptotically-AdS solutions. Requiring (2.18) on the boundary forces $A_t = l\, A_\theta$ everywhere in the bulk and removes the gauge invariance, because both a and λ must depend on $(\frac{t}{l} - \theta)$ so that there is no more an arbitrary time dependence.

We are left with the $\mathfrak{g}$-valued function $a(\theta)$ on which the gauge transformations generated by (2.16) act as

$$\delta_\lambda a(\theta) \;=\; \lambda'(\theta) \,+\, [\, a(\theta)\,,\, \lambda(\theta)\,]\,, \tag{2.19}$$

where a prime denotes a derivative in θ. These are not proper gauge transformations [26, 49], but rather global symmetries generated by the boundary charges

$$Q(\lambda) = -\,\frac{k}{2\pi}\int d\theta\, \mathrm{tr}\left(\lambda(\theta)a(\theta)\right)\,, \tag{2.20}$$

where k times the trace denotes the invariant bilinear form that is used to define the CS action. The latter observation suffices to fix the canonical structure of the boundary theory since

$$\delta_\lambda a(\theta) \;=\; \{Q(\lambda), a(\theta)\} \tag{2.21}$$

implies

$$\{Q(\lambda), Q(\eta)\} = -\,\frac{k}{2\pi}\int d\theta\, \mathrm{tr}\left(\eta(\theta)\delta_\lambda a(\theta)\right)\,. \tag{2.22}$$

This Poisson algebra is the centrally extended loop algebra of $\mathfrak{g}$ (see, for instance, [26, 49]), and it induces an analogue Poisson structure on the space of on-shell configurations $a(\theta)$, that accounts for the boundary degrees of freedom.

The other chiral sector, involving $\widetilde{A}$, can be treated in a similar fashion, but with some small variations needed to ensure the invertibility of the dreibein. This is guaranteed if one reaches the following on-shell parameterisation,

$$l^{-1}\widetilde{A}_t \;=\; -\,\widetilde{A}_\theta \;=\; b(\rho)\,\tilde{a}(t,\theta)\, b^{-1}(\rho)\,, \qquad \widetilde{A}_\rho \;=\; b(\rho)\,\partial_\rho\, b^{-1}(\rho)\,, \tag{2.23}$$

and restricts the $b(\rho)$ appearing both in (2.15) and (2.23) to take values in the "gravitational" subgroup of G. Even if the dreibein is always invertible, in [14] (see also [13, 42]) we argued that (2.23) and the corresponding condition for A do not provide a satisfactory on-shell parameterisation of the space of asymptotically-AdS configurations. We thus proposed to also require a finite difference between them and the AdS solution at the boundary,

$$(A - A_{AdS})\bigg|_{\partial\mathcal{M}} \;=\; \mathcal{O}(1)\,, \tag{2.24}$$

and similarly for $\widetilde{A}$. Eq. (2.24) translates into the Drinfeld-Sokolov condition on $a(\theta)$ and thus recovers the conformal symmetry discovered by Brown and Henneaux in the metric formulation of gravity [15]. A similar characterisation of the space of asymptotically-AdS solutions was discussed in the context of (super)gravity theories in [24, 50, 51].

To display the consequences of (2.24) on $a(\theta)$ it is convenient to denote the adjoint action of the $sl(2,\mathbb{R})$ generators J_i on the elements of $\mathfrak{g}$ by L_i,

$$L_i\, x := [\, J_i\, ,\, x\,] \quad \text{for } x \in \mathfrak{g}\ . \tag{2.25}$$

The L_i satisfy the same commutation relations (2.11) as the J_i. Since by hypothesis we consider only $sl(2)$ embeddings that branch $\mathfrak{g}$ into a (possibly infinite) sum of finite-dimensional $sl(2)$-irreducible representations, the eigenvalues of L_0 are half integers. We can thus decompose the gauge algebra into eigenvectors of L_0 of negative, zero or positive eigenvalues,

$$\mathfrak{g} \,=\, \mathfrak{g}_< \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_>\ . \tag{2.26}$$

The Drinfeld-Sokolov condition amounts to the constraint that $a(\theta) - J_+$ has no components corresponding to the negative spectrum of L_0,

$$a(\theta) - J_+ \in\, \mathfrak{g}_0 \oplus \mathfrak{g}_>\ , \tag{2.27}$$

or equivalently,

$$P_{\mathfrak{g}_<}\,(a(\theta) - J_+) \,=\, 0\ , \tag{2.28}$$

where $P_{\mathfrak{g}_<}$ is the orthogonal projector onto $\mathfrak{g}_<$. In HS gauge theories one has to impose an analogous condition also on $\tilde{a}(\theta)$, but the analysis proceeds along the same lines as the one for $a(\theta)$. Therefore, in the following we continue to focus on $a(\theta)$.

The constraints (2.28) are in general of first class[3] and they generate gauge transformations. To get to the reduced phase space, one has to fix the gauge. A natural gauge choice is the so-called highest-weight gauge

$$a(\theta) \,=\, J_+ \,+\, a_-(\theta)\ , \tag{2.29}$$

where a_- satisfies[4]

$$L_-\, a_-(\theta) \,=\, 0\ . \tag{2.30}$$

This choice fixes the gauge completely. In the restricted class of solutions satisfying (2.24) the boundary degrees of freedom are thus described by $a_-(\theta)$ [28, 52]. We shall now analyse the symmetries of the constrained boundary theory, which correspond to the asymptotic symmetries of the HS models that we introduced in Section 2.1.

[3]When L_0 admits half-integers eigenvalues some constraints are second class. However, this does not affect the possibility to reach the gauge (2.29) [52].

[4]Note that L_- increases the L_0 eigenvalue, so that $a_-(\theta)$ can be expanded in a set of highest weight eigenvectors for L_0. This rather unintuitive association follows from the convention (2.11) that we chose for the $sl(2,\mathbb{R})$ algebra.

2.3 Drinfeld-Sokolov reduction in highest-weight gauge

To find the symmetries of the constrained theory, we look at the set of symmetry transformations (2.19) that leave the form (2.29) of $a(\theta)$ invariant,

$$L_-(\delta_\lambda a) \;=\; 0 \;. \tag{2.31}$$

This condition translates into

$$L_- \,(\, \partial_\theta + [\, a_-(\theta), \,\cdot\,]\,)\, \lambda(\theta) \;+\; L_- L_+ \lambda(\theta) \;=\; 0 \;. \tag{2.32}$$

The operator $L_- L_+$ can be rewritten as

$$L_- L_+ \;=\; -\,\Delta \;+\; L_0 (L_0 - 1) \;, \tag{2.33}$$

where we introduced the quadratic Casimir

$$\Delta \;=\; L_0^2 \;-\; \frac{1}{2}\,(\, L_+ L_- + L_- L_+ \,) \;. \tag{2.34}$$

In the basis of $\mathfrak{g}$ introduced in (2.12) the operator $(\Delta - L_0(L_0 - 1))$ acts as

$$(\, \Delta - L_0(L_0 - 1)\,)\,(W^\ell_m)_{[a]} \;=\; (\ell - m)(\ell + m + 1)\,(W^\ell_m)_{[a]} \;, \tag{2.35}$$

i.e. by multiplication with a number that is non-zero for $m \neq \ell$. We denote by $\mathfrak{g}_-$ ($\mathfrak{g}_+$) the space of highest (lowest) weight states,

$$x \in \mathfrak{g}_- \;\Leftrightarrow\; L_- x = 0\,, \qquad x \in \mathfrak{g}_+ \;\Leftrightarrow\; L_+ x = 0\,. \tag{2.36}$$

In general, $\mathfrak{g}_-$ and $\mathfrak{g}_+$ can have a non-trivial intersection which contains the $sl(2)$ singlets. We also introduce the projection operators $P_\pm$ onto $\mathfrak{g}_\pm$, respectively. The operator $(\Delta - L_0(L_0 - 1))$ is invertible on the orthogonal complement of $\mathfrak{g}_+$, and we define

$$R := \;-\,\frac{1}{\Delta - L_0(L_0 - 1)}\;(\, 1 - P_+\,) \;. \tag{2.37}$$

In particular we have

$$R\, L_- L_+ \;=\; L_- L_+\, R \;=\; 1 - P_+ \;. \tag{2.38}$$

Furthermore we introduce the covariant derivative

$$D_\theta := \;\partial_\theta \;+\; [\, a_-(\theta), \,\cdot\,] \;, \tag{2.39}$$

which commutes with L_-, because $L_- a_- = 0$.

Applying R to (2.32) and taking into account (2.38), we eventually obtain

$$\lambda(\theta) \;=\; \lambda_+(\theta) \;-\; R L_- D_\theta \lambda(\theta) \;. \tag{2.40}$$

Here, $\lambda_+(\theta) = P_+\lambda(\theta)$ is the lowest-weight part of λ. Eq. (2.40) is solved by

$$\lambda(\theta) \;=\; \frac{1}{1+RL_-D_\theta}\,\lambda_+(\theta)\;, \tag{2.41}$$

which expresses the gauge parameter λ in terms of its lowest-weight part.[5] Inserting the solution (2.41) into the expression (2.19) for $\delta_\lambda a$, we find

$$\delta_\lambda a(\theta) \;=\; P_-\,\frac{1}{1+D_\theta RL_-}\,D_\theta\lambda_+(\theta) \;=\; P_-\sum_{n=0}^{\infty}\Big(-D_\theta RL_-\Big)^n D_\theta\lambda_+(\theta)\;. \tag{2.42}$$

This finally expresses $\delta_\lambda a$ in terms of λ_+. One might be worried about the infinite series appearing in (2.42). For a gauge parameter with definite $sl(2)$ quantum numbers as $\lambda_+(\theta) = \epsilon(\theta)W^\ell_\ell$, however, the series expansion in (2.42) stops at the term with $n = 2\ell$. This is because each term $D_\theta RL_-$ involves the application of L_-, and $(L_-)^{2\ell+1}W^\ell_\ell = 0$, while L_- commutes with D_θ and R does not change the $sl(2)$ quantum numbers. The indices $[a]$ of (2.9) do not play any role in this argument and thus we omitted them for simplicity.

In order to identify the Poisson structure on the reduced phase space one can then substitute (2.42) in (2.22). It is also possible to display the Poisson brackets between fields of defined conformal spin. To this end one can expand $a_-(\theta)$ and the independent part of the gauge parameter, encoded in $\lambda_+(\theta)$, in the basis (2.12):

$$a_-(\theta) \;=\; \frac{2\pi}{k}\left(\mathcal{L}(\theta)J_- + \sum_{\ell\,,\,a}\mathcal{W}^{[a]}_\ell(\theta)\,(W^\ell_{-\ell})_{[a]}\right)\;, \tag{2.43a}$$

$$\lambda_+(\theta) \;=\; \epsilon(\theta)J_+ + \sum_{\ell\,,\,a}\epsilon^{[a]}_\ell(\theta)\,(W^\ell_\ell)_{[a]}\;. \tag{2.43b}$$

Here $[a]$ is a colour index, while ℓ is a $sl(2)$ quantum number. The charges (2.20) which generate the transformations (2.42) then read

$$Q(\lambda_+) \;=\; \int d\theta\,\epsilon(\theta)\,\mathcal{L}(\theta) \;-\; \sum_{\ell,\,a,\,b}(N_\ell)_{ab}\int d\theta\,\epsilon^{[a]}_\ell(\theta)\,\mathcal{W}^{[b]}_\ell(\theta)\,, \tag{2.44}$$

with the matrices $(N_\ell)_{ab}$ defined in (2.14). By substituting (2.44) in (2.21) one can eventually read off the Poisson brackets $\{\mathcal{W}^{[a]}_i(\theta),\mathcal{W}^{[b]}_j(\theta')\}$. If all values of ℓ are integers, one can diagonalise $(N_\ell)_{ab}$ and thus determine all Poisson brackets involving $\mathcal{W}^{[a]}_\ell$ by looking at the gauge transformations generated by $\epsilon^{[a]}_\ell$. If some half-integer values of ℓ appear in (2.8) one can at most make $(N_\ell)_{ab}$ block-diagonal, with a sequence of 2×2 blocks. This means that the Poisson brackets of a given field can be extracted from the gauge transformations generated by the gauge parameter with the "partner" colour charge.

[5] A similar formula appears in [53].

The determination of the Poisson brackets can also formulated covariantly in colour indices. To this end it is convenient to denote the inverse of the matrix $(N_\ell)_{ab}$ by $(N_\ell)^{ab}$, and to consider

$$\delta_{i\,[a]}\,\mathcal{W}_j^{[b]}(\theta) = \sum_c (N_j)^{bc}\,\mathrm{tr}\left((W_j^j)_{[c]}\,\frac{1}{1+D_\theta R L_-}\,D_\theta\,\epsilon(\theta)(W_i^i)_{[a]}\right)\,, \tag{2.45}$$

where $\sum_c (N_j)^{bc}\,\mathrm{tr}(W^j_{j[c]}\,\cdot\,)$ selects the coefficient in front of the generator $(W^j_{-j})_{[b]}$, and the variation is taken with respect to $\lambda_+(\theta) = \epsilon(\theta)(W_i^i)_{[a]}$. From (2.21) we have

$$\delta_{i\,[a]}\,\mathcal{W}_j^{[b]}(\theta') \,=\, -\sum_c (N_i)_{ac}\int d\theta''\epsilon(\theta'')\left\{\mathcal{W}_i^{[c]}(\theta'')\,,\,\mathcal{W}_j^{[b]}(\theta')\right\}, \tag{2.46}$$

so that we can obtain the Poisson brackets from the variation by

$$\left\{\mathcal{W}_i^{[a]}(\theta)\,,\,\mathcal{W}_j^{[b]}(\theta')\right\} \,=\, -\sum_c (N_i)^{ac}\delta_{i\,[c]}\,\mathcal{W}_j^{[b]}(\theta')\Big|_{\epsilon(\theta')\,=\,\delta(\theta-\theta')}\,. \tag{2.47}$$

2.4 General properties

The Poisson algebra that we obtained in the last section by the DS reduction of a centrally extended loop algebra of course depends very much on the detailed structure of the algebra $\mathfrak{g}$ that we started with and on the choice of a $sl(2,\mathbb{R})$ embedding. On the other hand, there are a few general properties that we want to discuss here.

2.4.1 Primary basis

The DS reduction in highest-weight gauge always leads to a presentation of the resulting $\mathcal{W}$-algebra in a basis where all generators are primaries with respect to the lowest spin ones. If no spin-1 fields are present – as in the case of principal $sl(2,\mathbb{R})$ embeddings – all generators are thus automatically Virasoro primaries.

If spin-1 fields are present, corresponding to $sl(2)$ singlets in (2.8), we can compute their Poisson brackets with the other fields by evaluating the gauge transformations generated by $\lambda_+ = \epsilon(\theta)(W_0^0)_{[a]}$. From the previous discussion we know that we only have to evaluate the term at zeroth order in the series (2.42),

$$D_\theta\lambda_+ \,=\, \epsilon'\,(W_0^0)_{[a]} \,-\, \epsilon\,\mathcal{Q}_{[a]}\,a_-\,. \tag{2.48}$$

Here, in analogy with (2.25), we denoted by $\mathcal{Q}_{[a]}$ the adjoint action of $(W_0^0)_{[a]}$ on the elements of the algebra,

$$\mathcal{Q}_{[a]}\,x := \left[\,(W_0^0)_{[a]},\,x\,\right] \quad \text{for } x\in\mathfrak{g}\,. \tag{2.49}$$

These operators cannot modify the $sl(2)$ quantum numbers, so that we can describe their action by

$$\mathcal{Q}_{[a]}(W_m^\ell)_{[b]} \,=\, \sum_c (f_\ell)^c{}_{ab}\,(W_m^\ell)_{[c]}\,. \tag{2.50}$$

Therefore, all terms in (2.48) belong to $\mathfrak{g}_-$ and the projector P_- does not induce any modification. Expanding $a_-(\theta)$ as in (2.43a) we eventually get

$$\delta_{[a]}\mathcal{W}_\ell^{[b]} \;=\; -\,\epsilon\sum_c (f_\ell)^b{}_{ac}\,\mathcal{W}_\ell^{[c]} \;+\; \frac{k}{2\pi}\,\epsilon'\,\delta_a{}^b\,\delta_{0,\ell}\,. \tag{2.51}$$

In the singlet sector one can always diagonalise the matrix $(N_\ell)_{ab}$ appearing in (2.44), but we can also substitute (2.51) in (2.47) to get

$$\Big\{\mathcal{W}_0^{[a]}(\theta),\mathcal{W}_0^{[b]}(\theta')\Big\} \;=\; \delta(\theta-\theta')\sum_c (f_0)^{ba}{}_c\,\mathcal{W}_0^{[c]}(\theta') \;+\; \frac{k}{2\pi}\,(N_0)^{ab}\,\partial_\theta\delta(\theta-\theta')\,, \tag{2.52a}$$

$$\Big\{\mathcal{W}_0^{[a]}(\theta),\mathcal{W}_\ell^{[b]}(\theta')\Big\} \;=\; \delta(\theta-\theta')\sum_c (f_\ell)^{ba}{}_c\,\mathcal{W}_\ell^{[c]}(\theta') \quad \text{for } \ell\geq 1\,. \tag{2.52b}$$

In both cases one index of $(f_\ell)^b{}_{ac}$ is raised using $(N_0)^{ab}$, that is the inverse of the Killing metric of the subalgebra of $\mathfrak{g}$ spanned by the $sl(2)$ singlets. The spin-1 fields $\mathcal{W}_0^{[a]}$ thus generate the centrally extended Kac-Moody subalgebra (2.52a). They always appear when non-principal $sl(2)$ embeddings are considered, so that the presence of a Kac-Moody subalgebra is a neat signature of this class of DS reduction (see also [52] for more comments on this setup).

Let us now evaluate the Poisson bracket of an arbitrary field with the Virasoro current. In this case it is sufficient to consider the gauge transformations generated by $\lambda_+(\theta) = \epsilon(\theta)J_+$. In the computation the singlets behave differently with respect to all other fields and it is convenient to split $a_-(\theta)$ as

$$a_-(\theta) \;=\; \hat{a}_-(\theta) \;+\; \frac{2\pi}{k}\sum_a \mathcal{W}_0^{[a]}(\theta)(W_0^0)_{[a]}\,. \tag{2.53}$$

To proceed we only have to compute the series in (2.42) up to $n=2$:

$$D_\theta\lambda_+ = \; L_+\Big(\epsilon' J_0 - \epsilon\,\hat{a}_-\Big)\,, \tag{2.54a}$$

$$\Big(-D_\theta R L_-\Big)D_\theta\lambda_+ = \;-\frac{\epsilon''}{2}\,L_+J_- \,+\, \epsilon'(L_0+1)\hat{a}_- \,+\, \epsilon\,\hat{a}_-' \,+\, \frac{2\pi}{k}\,\epsilon\sum_a \mathcal{W}_0^{[a]}\mathcal{Q}_{[a]}\hat{a}_-\,, \tag{2.54b}$$

$$\Big(-D_\theta R L_-\Big)^2 D_\theta\lambda_+ = \;\frac{\epsilon'''}{2}\,J_-\,. \tag{2.54c}$$

Summing all contributions that remain after the projection by P_- we eventually find

$$\delta_\lambda a \;=\; \frac{\epsilon'''}{2}J_- \,+\, \epsilon'(L_0+1)a_- \,+\, \epsilon\,a_-' \,+\, \frac{2\pi}{k}\,\epsilon\sum_a \mathcal{W}_0^{[a]}\,\mathcal{Q}_{[a]}\,\hat{a}_- \qquad \text{for } \lambda_+ = \epsilon J_+\,. \tag{2.55}$$

Expanding $a_-(\theta)$ as in (2.43a), then leads to

$$\delta\mathcal{L} \;=\; \epsilon\,\mathcal{L}' \,+\, 2\,\epsilon'\mathcal{L} \,+\, \frac{k}{4\pi}\,\epsilon'''\,, \tag{2.56a}$$

$$\delta\,\mathcal{W}_\ell^{(a)} \;=\; \epsilon\,\mathcal{W}_\ell^{(a)\,\prime} \,+\, (\ell+1)\,\epsilon'\,\mathcal{W}_\ell^{[a]} \,+\, \epsilon\,\frac{2\pi}{k}\sum_{b\,,\,c}(f_\ell)^a{}_{bc}\,\mathcal{W}_0^{[b]}\mathcal{W}_\ell^{[c]} \quad \text{for } \ell\geq 1\,, \tag{2.56b}$$

while the fields with $\ell = 0$ are left invariant. If no spin-1 fields are present, the terms containing $\mathcal{W}_0^{[a]}$ disappear, and the field $\mathcal{W}_\ell^{[a]}$ transforms as a primary field of conformal weight $(\ell+1)$. Even in the general case one can easily express the result in a Virasoro primary basis by redefining the Virasoro current as

$$\widehat{\mathcal{L}} := \mathcal{L} - \frac{\pi}{k} \sum_{a\,,\,b} (N_0)_{ab}\, \mathcal{W}_0^{[a]}\, \mathcal{W}_0^{[b]}\,. \tag{2.57}$$

With respect to the improved Sugawara Virasoro current $\widehat{\mathcal{L}}$ the Poisson brackets read

$$\left\{\widehat{\mathcal{L}}(\theta)\,,\, \widehat{\mathcal{L}}(\theta')\right\} = \delta(\theta-\theta')\,\widehat{\mathcal{L}}'(\theta') - 2\,\partial_\theta\delta(\theta-\theta')\,\widehat{\mathcal{L}}(\theta') - \frac{k}{4\pi}\,\partial_\theta^3\delta(\theta-\theta')\,, \tag{2.58a}$$

$$\left\{\widehat{\mathcal{L}}(\theta)\,,\, \mathcal{W}_\ell^{[a]}(\theta')\right\} = \delta(\theta-\theta')\,\mathcal{W}^{(a)\prime}(\theta') - (\ell+1)\,\partial_\theta\delta(\theta-\theta')\,\mathcal{W}_\ell^{(a)}(\theta')\,. \tag{2.58b}$$

2.4.2 Central terms

Eqs. (2.52a) and (2.58a) display a central term, but central terms do not arise only in the Kac-Moody or Virasoro subalgebras. On the other hand, their structure is rather rigid and does not depend on the particular loop algebra to which one applies the DS procedure. They can be computed by substituting the covariant derivative D_θ with an ordinary derivative in (2.42). When we consider a transformation by a gauge parameter of a given conformal spin, say $\lambda_+(\theta) = \epsilon(\theta)(W_\ell^\ell)_{[a]}$, we find

$$\delta_\lambda a\,\big|_{D_\theta = \partial_\theta} = \frac{(-1)^{2\ell}}{(2\ell)!}\,\epsilon^{(2\ell+1)}\,(W_{-\ell}^\ell)_{[a]}\,, \tag{2.59}$$

where the exponent between parentheses denotes the action of the corresponding number of derivatives on ϵ. To reach this result one just has to combine $L_+ W_{\ell-(n+1)}^\ell = (n+1) W_{\ell-n}^\ell$ (following from (2.12)) with eq. (2.38). In conclusion, central terms only appear in the Poisson brackets between fields of the same conformal weight,

$$\left\{\mathcal{W}_\ell^{[a]}(\theta)\,,\, \mathcal{W}_\ell^{[b]}(\theta')\right\} = \frac{k}{2\pi(2\ell)!}\,(N_\ell)^{ab}\,\partial_\theta^{2\ell+1}\delta(\theta-\theta') + \dots\,, \tag{2.60}$$

where $(N_\ell)^{ab}$ is the inverse of the matrix introduced in (2.14).

2.4.3 Polynomials in the Virasoro generators

There is another property of the $\mathcal{W}$-algebra that does not depend on the particular Lie algebra $\mathfrak{g}$ chosen as a starting point for the DS reduction. It is the presence of non-linearities in the resulting Poisson algebra. This follows from the repeated application of the covariant derivative in (2.42), but here we would like to display explicitly a class of polynomial terms that does not depend on the structure of $\mathfrak{g}$ but only on the branching (2.8). These are the polynomials that involve just the Virasoro generator $\mathcal{L}$, whose structure depends only on the commutators (2.12).

One can compute the polynomials containing only $\mathcal{L}$ by considering only the Virasoro part in the covariant derivative,

$$D_\theta \;\to\; \partial_\theta \,+\, \frac{2\pi}{k}\,\mathcal{L}(\theta)\,L_- \,. \tag{2.61}$$

Furthermore, as in (2.59), it is convenient to consider separately the contributions to $\delta_\lambda a$ coming from gauge parameters of different spin. Let us for simplicity elide colour indices and consider $\lambda_+(\theta) = \epsilon(\theta) W^\ell_\ell$. This leads to

$$\begin{aligned} \delta_\lambda a\,\Big|_{D_\theta = \partial + \frac{2\pi}{k}\mathcal{L}L_-} \;=\; & \frac{(-1)^{2\ell}}{(2\ell)!} \sum_{r=0}^{\lceil \ell \rceil} \left(\frac{2\pi}{k}\right)^r \sum_{p_1=0}^{2(\ell-r)+1} \; \sum_{p_2=0}^{2(\ell-r)+1-p_1} \cdots \sum_{p_r=0}^{2(\ell-r)+1-\sum_1^{r-1}p_t} \\ & \times\, C[\,\ell\,]_{p_1\dots p_r}\, \mathcal{L}^{(p_1)} \dots \mathcal{L}^{(p_r)}\, \epsilon^{(2(\ell-r)+1-\sum_1^r p_t)}\; W^\ell_{-\ell}\,, \end{aligned} \tag{2.62}$$

where the extremum of the sum over r is ℓ or $\ell+1/2$, depending on whether ℓ is integer or half integer. As a result, fields with half-integer spin also admit pure-Virasoro terms that do not contain derivatives, while if the spin is integer there is at least one derivative in all terms of (2.62). The "structure constants" appearing in (2.62) read

$$\begin{aligned} C[\,\ell\,]_{p_1\dots p_r} \;=\; & \sum_{i_1=p_1}^{2(\ell-r)+1} \; \sum_{i_2=\langle p_1+p_2-i_1\rangle_+}^{2(\ell-r)+1-i_1} \cdots \sum_{i_r=\langle \sum_1^r p_a - \sum_1^{r-1} i_t\rangle_+}^{2(\ell-r)+1-\sum_1^{r-1} i_t} \; \prod_{s=1}^{r} \binom{\sum_1^s p_t - \sum_1^{s-1} i_t}{p_s} \\ & \times \left(2\,(r-s)+1+\sum_{t=1}^{r-s+1} i_t\right)\left(2\,(l-r+s)-\sum_{t=1}^{r-s+1} i_t\right), \end{aligned} \tag{2.63}$$

where $\langle i\rangle_+ = \max(0,i)$. The details of their computation are presented in Appendix B.

Note that the tensors $C[\,\ell\,]_{p_1\dots p_r}$ are not symmetric for interchanges of the indices p_s. As a result a symmetrisation is needed in order to extract the coefficient appearing in front of a non-ordered combination of derivatives of $\mathcal{L}$. Let us also stress that the terms displayed here can only appear in the brackets $\{\mathcal{W}^{[a]}_\ell(\theta), \mathcal{W}^{[b]}_\ell(\theta')\}$, precisely as in the discussion of the central terms (that actually come from the term of order zero in the sum over r). This is due to the proportionality of (2.62) to $W^\ell_{-\ell}$ and to the fact that the adjoint action of $sl(2)$ generators cannot modify the quantum number ℓ. This is no longer true if one considers the full covariant derivative in (2.42), so that the computation of the remaining terms in the Poisson algebra requires the knowledge of the detailed structure of $\mathfrak{g}$. In the following we shall exploit it in a couple of examples, before presenting the main results of this paper in Section 3, where we apply this procedure to compute the asymptotic symmetries of $SL(N,\mathbb{R}) \times SL(N,\mathbb{R})$ HS gauge theories and their $N\to\infty$ limits.

2.5 Non-principal embeddings: two examples

In this section we present the main features of DS reductions based on non-principal $sl(2,\mathbb{R})$ embeddings by studying in detail two examples. The interpretation of the $\mathcal{W}$-algebras we are going to discuss as asymptotic symmetries of classical HS gauge theories could be not completely straightforward due to the comments we presented at the end of Section 2. On the other hand, we feel it could be instructive to also examine the features of non-principal DS reductions, also in view of possible applications to other classes of minimal models aside from those considered in [18, 22, 23]. See also [43] for a discussion of their possible role in HS gauge theories.

In particular, we perform the DS reductions associated to two non-principal embeddings of $sl(2,\mathbb{R})$ in $sl(N,\mathbb{R})$. The algebra $sl(N,\mathbb{R})$ admits a number of inequivalent $sl(2)$ embeddings equal to the number of partitions of N [54]. One of them is just the trivial embedding that cannot be used to build a HS gauge theory, and whose DS reduction gives the affine extension of $sl(N,\mathbb{R})$ rather than a $\mathcal{W}$-algebra [52]. In the $sl(3,\mathbb{R})$ case there is thus only one non-trivial non-principal sl(2) embedding and we present its DS reduction, together with the one associated to the corresponding embedding of $sl(2,\mathbb{R})$ in $sl(4,\mathbb{R})$. In fact, inequivalent embeddings can be obtained by embedding different $n \times n$ representations of $sl(2,\mathbb{R})$ in the fundamental of $sl(N,\mathbb{R})$. In both cases we consider the "next-to-principal" $sl(2,\mathbb{R})$ embeddings in $sl(N,\mathbb{R})$, where one singles out a $(N-1)\times(N-1)$ representation of $sl(2,\mathbb{R})$ in the fundamental of $sl(N,\mathbb{R})$.

2.5.1 The Polyakov-Bershadsky $\mathcal{W}_3^{(2)}$ algebra

We consider here the $sl(2,\mathbb{R})$ embedding in $sl(3,\mathbb{R})$ that branches the fundamental representation as

$$\underline{\mathbf{8}} = \underline{\mathbf{3}} + 2\cdot\underline{\mathbf{2}} + \underline{\mathbf{1}}\,. \tag{2.64}$$

The three-dimensional representation in (2.64) corresponds to the $sl(2,\mathbb{R})$ subalgebra used to implement the DS reduction. Accordingly to this decomposition, the $sl(3,\mathbb{R})$ algebra can be realised in terms of the three $sl(2,\mathbb{R})$ generators W_i^1 $(i=-1,0,1)$, two sets of generators $\psi_m^{[a]}$ $(m=-\frac{1}{2},\frac{1}{2}\,;\ a=1,-1)$ of spin $3/2$ and one generator W_0^0 of spin 1:

$$\left[\,W_i^1\,,\,W_j^1\,\right] \,=\, (i-j)\,W_{i+j}^1\ , \tag{2.65a}$$

$$\left[\,W_i^1\,,\,W_0^0\,\right] \,=\, 0\ , \tag{2.65b}$$

$$\left[\,W_i^1\,,\,\psi_m^{[a]}\,\right] \,=\, \left(\frac{i}{2}-m\right)\psi_{i+m}^{[a]}\ , \tag{2.65c}$$

$$\left[\,W_0^0\,,\,\psi_m^{[a]}\,\right] \,=\, a\,\psi_m^{[a]}\ , \tag{2.65d}$$

$$\left[\,\psi_m^{[a]}\,,\,\psi_n^{[b]}\,\right] \,=\, \frac{a-b}{2}\left(\,W_{m+n}^1\,+\,\frac{3}{2}\,(a-b)\,m(m-n)^2\,W_0^0\,\right)\,. \tag{2.65e}$$

The presence of two sets of generators with half-integer ℓ allows to consider linear combinations of them without spoiling (2.65c), and we defined the $\psi_m^{[a]}$ so that they are eigenvectors of the adjoint action of W_0^0. On the other hand, in agreement with the discussion in Section 2.1, this freedom does not suffice to separate their contributions to the Killing metric.

The DS reduction based on the embedding (2.64) was first performed independently by Polyakov and Bershadsky [55] and the resulting $\mathcal{W}$-algebra is usually denoted by $\mathcal{W}_3^{(2)}$.

In order to perform it with our techniques, it is convenient to introduce the notation

$$a_- \,=\, \frac{2\pi}{k}\left(\mathcal{W}_0(\theta)\,W_0^0 \,+\, \mathcal{W}_{\frac{1}{2}}^{[1]}(\theta)\,\psi_{-\frac{1}{2}}^{[1]} \,+\, \mathcal{W}_{\frac{1}{2}}^{[-1]}(\theta)\,\psi_{-\frac{1}{2}}^{[-1]} \,+\, \mathcal{L}(\theta)\,W_{-1}^1\right), \tag{2.66}$$

and to denote the gauge variation with respect to $\lambda_+(\theta) = \epsilon(\theta)(W_\ell^\ell)_{[a]}$ as $\delta_{\ell\,[a]}$. The general procedure (2.42) leads eventually to the following transformations preserving the highest-weight gauge:

$$\delta_0\mathcal{W}_0 \,=\, \frac{k}{2\pi}\,\epsilon'\,, \qquad \delta_0\mathcal{W}_{\frac{1}{2}}^{[a]} \,=\, -\,a\,\epsilon\,\mathcal{W}_{\frac{1}{2}}^{[a]}\,, \qquad \delta_0\mathcal{L} \,=\, 0\,, \tag{2.67a}$$

$$\delta_1\mathcal{W}_0 \,=\, 0\,, \qquad \delta_1\mathcal{L} \,=\, 2\,\epsilon'\,\mathcal{L} \,+\, \epsilon\,\mathcal{L}' \,+\, \frac{k}{4\pi}\,\epsilon'''\,, \tag{2.67b}$$

$$\delta_1\mathcal{W}_{\frac{1}{2}}^{[a]} \,=\, \frac{3}{2}\,\epsilon'\,\mathcal{W}_{\frac{1}{2}}^{[a]} \,+\, \epsilon\,\mathcal{W}_{\frac{1}{2}}^{[a]\,\prime} \,+\, a\,\frac{2\pi}{k}\,\epsilon\,\mathcal{W}_{\frac{1}{2}}^{[a]}\,\mathcal{W}_0\,, \tag{2.67c}$$

$$\delta_{\frac{1}{2}\,[a]}\mathcal{W}_{\frac{1}{2}}^{[b]} \,=\, \delta_a{}^b\left(-\,\epsilon\,\mathcal{L} \,-\, 2\,a\,\epsilon'\,\mathcal{W}_0 \,-\, a\,\epsilon\,\mathcal{W}_0' \,-\, \frac{2\pi}{k}\,\epsilon\,\mathcal{W}_0\,\mathcal{W}_0 \,-\, \frac{k}{2\pi}\,\epsilon''\right). \tag{2.67d}$$

The resulting $\mathcal{W}$-algebra is obtained by substituting (2.67) in (2.47), and it coincides with the one in [55]. We present it in Appendix C. To compute the Poisson brackets one has to know the structure of the Killing metric of $sl(3,\mathbb{R})$. In our basis it is block-diagonal with one block for each type of field, except for a mixing in the two sets of spin-$\frac{3}{2}$ generators $\psi_m^{[a]}$. In particular, the various matrices $(N_\ell)_{ab}$ of (2.14) are [6]

$$(N_{\frac{1}{2}})_{ab} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\,, \qquad N_0 \,=\, \frac{2}{3}\,. \tag{2.68}$$

Since one block in the Killing form involves generators with different colour indices, the structure of the gauge transformation (2.67d) and the corresponding Poisson bracket (C.1f) differ by more than just a numerical factor, namely also by the colours of the fields that occur.

The fields $\mathcal{W}_{\frac{1}{2}}^{[a]}$ are not Virasoro primaries (see (2.67c)). As described in general in (2.57), and for this example already mentioned in [55], a shift

$$\mathcal{L} \,\to\, \mathcal{L} \,-\, \frac{2\pi}{3k}\,(\mathcal{W}_0)^2 \,\equiv\, \widehat{\mathcal{L}} \tag{2.69}$$

leads to a basis where all fields are primaries with respect to $\widehat{\mathcal{L}}$.

2.5.2 A non-principal $sl(2,\mathbb{R})$ embedding in $sl(4,\mathbb{R})$

We consider here the $sl(2,\mathbb{R})$ embedding in $sl(4,\mathbb{R})$ that branches the fundamental representation as

$$\underline{\mathbf{15}} \,=\, \underline{\mathbf{3}} \,+\, \underline{\mathbf{5}} \,+\, 2\cdot\underline{\mathbf{3}} \,+\, \underline{\mathbf{1}}\,. \tag{2.70}$$

[6] In (2.68) we ordered the generators such that $\mathrm{tr}\left(\psi_{\frac{1}{2}}^{[1]}\psi_{-\frac{1}{2}}^{[-1]}\right) = 1$, while $\mathrm{tr}\left(\psi_{\frac{1}{2}}^{[-1]}\psi_{-\frac{1}{2}}^{[1]}\right) = -1$.

The first three-dimensional representation in (2.70) corresponds to the $sl(2,\mathbb{R})$ subalgebra used to implement the DS reduction. In a HS perspective it would thus be associated to the graviton, while the other two would lead to two coloured massless spin-2 fields. Accordingly to this decomposition, the $sl(4,\mathbb{R})$ algebra can be realised in terms of the three $sl(2,\mathbb{R})$ generators W^1_i $(i=-1,0,1)$, two extra sets of spin-2 generators $\phi^{[a]}_i$ $(i=-1,0,1\,;\ a=1,-1)$, one set of spin-3 generators W^2_m $(m=-2,\dots,2)$ and the $sl(2)$ singlet W^0_0 as

$$\Big[\,W^1_i\,,\,W^1_j\,\Big] \;=\; (i-j)\,W^1_{i+j}\ , \tag{2.71a}$$
$$\Big[\,W^1_i\,,\,\phi^{[a]}_j\,\Big] \;=\; (i-j)\,\phi^{[a]}_{i+j}\ , \tag{2.71b}$$
$$\Big[\,W^1_i\,,\,W^2_m\,\Big] \;=(2i-m)\,W^2_{i+m}\ , \tag{2.71c}$$
$$\Big[\,W^1_i\,,\,W^0_0\,\Big] \;=\; 0\ , \tag{2.71d}$$

$$\Big[\,W^0_0\,,\,\phi^{[a]}_i\,\Big] \;=\; a\,\phi^{[a]}_i\ , \tag{2.71e}$$
$$\Big[\,W^0_0\,,\,W^2_i\,\Big] \;=\; 0\ , \tag{2.71f}$$

$$\Big[\,W^2_m\,,\,W^2_n\,\Big] \;=\; -\,\frac{1}{12}\,(m-n)\Big(2m^2\,+\,2n^2\,-\,mn\,-\,8\Big)\,W^1_{m+n}\ , \tag{2.71g}$$
$$\Big[\,W^2_m\,,\,\phi^{[a]}_i\,\Big] \;=\; \frac{a}{6}\Big(m^2\,+\,6i^2\,-\,3mi\,-\,4\Big)\,\phi^{[a]}_{m+i}\ , \tag{2.71h}$$
$$\Big[\,\phi^{[a]}_i\,,\,\phi^{[b]}_j\,\Big] \;=\; \frac{a-b}{2}\left(-\,a\,(i-j)W^1_{i+j}\,+\,2\,W^2_{i+j}\,-\,\frac{4}{3}\,(2-3|i-j|)\,\delta_{i+j,0}\,W^0_0\right). \tag{2.71i}$$

As in the previous example, we used the freedom in the definition of the $\phi^{[a]}_m$ to let them be eigenvectors of the adjoint action of W^0_0.

With the convention

$$a_- \;=\; \frac{2\pi}{k}\,\Big(\,\mathcal{W}_0(\theta)W^0_0 + \,\mathcal{L}(\theta)W^1_{-1} + \,\mathcal{W}^{[-1]}_1(\theta)\phi^{[-1]}_{-1} + \,\mathcal{W}^{[1]}_1(\theta)\phi^{[1]}_{-1} + \,\mathcal{W}_2(\theta)W^2_{-2}\,\Big)\,, \tag{2.72}$$

the transformations preserving the highest-weight parameterisation of a_- are

$$\delta_0\mathcal{W}_0 \;=\; \frac{k}{2\pi}\,\epsilon'\,, \qquad \delta_0\mathcal{W}^{[a]}_1 \;=\; -\,a\,\epsilon\,\mathcal{W}^{[a]}_1\,, \qquad \delta_0\mathcal{L} \;=\; \delta_0\mathcal{W}_2 \;=\; 0\,, \tag{2.73a}$$

$$\delta_1\mathcal{L} \;=\; 2\,\epsilon'\,\mathcal{L} \,+\, \epsilon\,\mathcal{L}' \,+\, \frac{k}{4\pi}\,\epsilon'''\,, \tag{2.73b}$$
$$\delta_1\mathcal{W}^{[a]}_1 \;=\; 2\,\epsilon'\,\mathcal{W}^{[a]}_1 \,+\, \epsilon\,\Big(\mathcal{W}^{[a]}_1\Big)' \,+\, \frac{2\pi}{k}\,a\,\epsilon\,\mathcal{W}_0\,\mathcal{W}^{[a]}_1\,, \tag{2.73c}$$
$$\delta_1\mathcal{W}_2 \;=\; 3\,\epsilon'\,\mathcal{W}_2 \,+\, \epsilon\,(\mathcal{W}_2)'\ , \tag{2.73d}$$

$$
\begin{aligned}
\delta_{1\,[a]}\mathcal{W}_1^{[b]} \,=\, \delta_a{}^b\bigg(& \frac{3a}{2}\,\epsilon''\,\mathcal{W}_0 \,+\, \frac{3a}{2}\,\epsilon'\,\mathcal{W}_0' \,+\, \frac{a}{2}\,\epsilon\,\mathcal{W}_0'' \\
&+\, 2\,\epsilon'\,\mathcal{L} \,+\, \epsilon\,\mathcal{L}' \,+\, \frac{k}{4\pi}\,\epsilon''' \,+\, 2\,a\,\epsilon\,\mathcal{W}_2 \\
&+\, \frac{2\pi}{k}\,2\,a\,\epsilon\,\mathcal{W}_0\,\mathcal{L} \,+\, \frac{2\pi}{k}\,\frac{3}{2}\,\epsilon'\,\mathcal{W}_0\,\mathcal{W}_0 \,+\, \frac{2\pi}{k}\,\frac{3}{2}\,\epsilon\,\mathcal{W}_0\,\mathcal{W}_0' \\
&+\, \left(\frac{2\pi}{k}\right)^2\frac{a}{2}\,\epsilon\,\mathcal{W}_0\,\mathcal{W}_0\,\mathcal{W}_0\bigg)\,,
\end{aligned} \tag{2.73e}
$$

$$
\begin{aligned}
\delta_{1\,[a]}\mathcal{W}_2 \,=\, &-\frac{5a}{3}\,\epsilon''\,\mathcal{W}_1^{[-a]} \,-\, \frac{5a}{6}\,\epsilon'\,\left(\mathcal{W}_1^{[-a]}\right)' \,-\, \frac{a}{6}\,\epsilon\,\left(\mathcal{W}_1^{[-a]}\right)'' \\
&-\, \frac{2\pi}{k}\,a\,\frac{8}{3}\,\epsilon\,\mathcal{L}\,\mathcal{W}_1^{[-a]} \,-\, \frac{2\pi}{k}\,\frac{5}{2}\,\epsilon'\,\mathcal{W}_0\,\mathcal{W}_1^{[-a]} \,-\, \frac{2\pi}{k}\,\frac{3}{2}\,\epsilon\,\mathcal{W}_0'\,\mathcal{W}_1^{[-a]} \\
&-\, \frac{2\pi}{k}\,\frac{a}{2}\,\epsilon\,\mathcal{W}_0\,\left(\mathcal{W}_1^{[-a]}\right)' - \left(\frac{2\pi}{k}\right)^2 a\,\epsilon\,\mathcal{W}_0\,\mathcal{W}_0\,\mathcal{W}_1^{[-a]}\,,
\end{aligned} \tag{2.73f}
$$

$$
\begin{aligned}
\delta_2\mathcal{W}_2 \,=\, &\frac{5}{6}\,\epsilon'''\,\mathcal{L} \,+\, \frac{5}{4}\,\epsilon''\,\mathcal{L}' \,+\, \frac{3}{4}\,\epsilon'\,\mathcal{L}'' \,+\, \frac{1}{6}\,\epsilon\,\mathcal{L}''' \,+\, \frac{k}{48\pi}\,\epsilon^{(5)} \\
&+\, \frac{2\pi}{k}\,\frac{8}{3}\,\epsilon'\,\mathcal{L}\,\mathcal{L} \,+\, \frac{2\pi}{k}\,\frac{8}{3}\,\epsilon\,\mathcal{L}\,\mathcal{L}' \,+\, \frac{2\pi}{k}\,8\,\epsilon'\,\mathcal{W}_1^{[-1]}\,\mathcal{W}_1^{[1]} \\
&+\, \frac{2\pi}{k}\,4\,\epsilon\,\left(\mathcal{W}_1^{[-1]}\right)'\,\mathcal{W}_1^{[1]} \,+\, \frac{2\pi}{k}\,4\,\epsilon\,\mathcal{W}_1^{[-1]}\,\left(\mathcal{W}_1^{[1]}\right)'\,.
\end{aligned} \tag{2.73g}
$$

In the basis (2.71) the Killing metric splits into blocks for the different field types except for a mixing in the two extra sets of spin-2 generators. More concretely, the matrices $(N_\ell)_{ab}$ of (2.14) become

$$
(N_1)_{ab} \,=\, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,, \qquad N_0 \,=\, \frac{3}{16}\,, \qquad N_2 \,=\, 1\,. \tag{2.74}
$$

In this case the matrix $(N_1)_{ab}$ can be clearly diagonalised, but this would spoil our choice of working with W_0^0 eigenvectors. Note, however, that its eigenvalues are ± 1, so that the kinetic terms of the two coloured spin-2 fields would have opposite sign in the action (2.1).

To obtain a $\mathcal{W}$-algebra in a Virasoro-primary basis, one can again shift $\mathcal{L}$ as

$$
\mathcal{L} \,\to\, \mathcal{L} - \frac{3\pi}{16k}\,\mathcal{W}_0\,\mathcal{W}_0 \,\equiv\, \widehat{\mathcal{L}}\,. \tag{2.75}
$$

The corresponding Poisson algebra is presented in Appendix C.

3 The structure of $\mathcal{W}_\infty[\lambda]$ in a primary basis

In this section we use the techniques developed in Section 2 to study an interesting one-parameter family of higher-spin gauge theories. Their gauge algebras are the direct sum of two copies of the infinite-dimensional Lie algebras called $hs[\lambda]$. After a brief introduction to these algebras we shall stress their link with higher spins by discussing the relation between a suitable basis of their invariant tensors and Fronsdal's metric-like fields. For any λ we shall then compute the structure constants of the $\mathcal{W}$-algebra of asymptotic symmetries in a Virasoro-primary basis.

3.1 The higher-spin algebras $hs[\lambda] \oplus hs[\lambda]$

In Section 2.1 we have seen that in any three-dimensional HS gauge theory a crucial role is played by the gauge subalgebra that describes the gravitational sector of the model. Instead of choosing it among all possible embeddings in a given algebra, one can actually proceed in a different direction and build HS algebras out of products of generators of the "gravitational" $sl(2,\mathbb{R}) \oplus sl(2,\mathbb{R})$ gauge algebra.

For instance, following [29, 30, 31, 32, 33, 19], we start with the universal enveloping algebra of $sl(2,\mathbb{R})$ generated by $J_\pm$ and J_0. We then do the identification

$$C_2 \,:=\, J_0^2 \,-\, \frac{1}{2}\,(\, J_+ J_- + J_- J_+ \,) \,\equiv\, \mu\, \mathbb{1}\,, \tag{3.1}$$

which sets the quadratic Casimir C_2 to a particular value μ that we often parameterise as

$$\mu \,=\, \frac{1}{4}\left(\lambda^2 - 1\right)\,. \tag{3.2}$$

The algebra obtained in that way is spanned by the identity $\mathbb{1}$ and the elements

$$W^\ell_m := (-1)^{\ell-m}\,\frac{(\ell+m)!}{(2\ell)!}\,L_-^{\,\ell-m} J_+^{\,\ell}\,, \qquad \ell \geq 1\,, \quad -\,\ell \leq m \leq \ell\,, \tag{3.3}$$

where L_i denotes the adjoint action of $sl(2,\mathbb{R})$ as in (2.25). From their definition it is manifest that they satisfy the commutators

$$[\, W^1_m \,,\, W^\ell_n\,] \,=\, (\ell\, m - n)\, W^\ell_{m+n}\,, \tag{3.4}$$

and we can identify the generators with $\ell = 1$ with the $sl(2,\mathbb{R})$ ones. The whole set of W^ℓ_m generates a Lie algebra $hs[\lambda]$ whose remaining commutators are fixed by the $sl(2)$ commutators of eq. (2.11). It branches as

$$hs[\lambda] \,=\, \bigoplus_{\ell\,=\,1}^{\infty}\, \mathfrak{g}^{(\ell)} \tag{3.5}$$

under the adjoint action of the defining $sl(2)$ subalgebra. Different values of the parameter μ (related to λ by (3.2)) give algebras that are not isomorphic [29, 33].

This construction shows that one can identify $hs[\lambda]$ with the subspace orthogonal to the identity in the quotient of the universal enveloping algebra of $sl(2,\mathbb{R})$ by the ideal generated by $(C_2 - \mu\mathbb{1})$ [29, 30, 31, 32, 33, 19],

$$\frac{\mathcal{U}(sl(2,\mathbb{R}))}{\langle\, C_2 - \mu \mathbb{1}\,\rangle} \,=\, hs[\lambda] \,\oplus\, \mathbb{C}\,. \tag{3.6}$$

The whole quotient is an associative algebra. The product of the W^ℓ_m was given in [30] (see also [31]) as

$$W^k_m \star W^\ell_n = \frac{1}{2} \sum_{i\,=\,|k-\ell|}^{k+\ell} f_\lambda \left(\begin{array}{cc|c} k & \ell & i \\ m & n & m+n \end{array} \right) W^i_{m+n} \,, \tag{3.7}$$

where the identity is denoted by W^0_0 and the λ-dependent structure constants are defined in Appendix A. The product (3.7) allows one to realise the Lie product on $hs[\lambda]$ as a $\star$-commutator:

$$[\, W^k_m \,, W^\ell_n \,] = \sum_{\substack{i\,=\,|k-\ell|+1 \\ i+k+\ell \text{ odd}}}^{k+\ell-1} f_\lambda \left(\begin{array}{cc|c} k & \ell & i \\ m & n & m+n \end{array} \right) W^i_{m+n} \,. \tag{3.8}$$

The associative product (3.7) was also used in [31] to define an invariant bilinear form on $hs[\lambda]$ as

$$\text{tr} \left(W^k_m W^\ell_n \right) := \frac{6}{(\lambda^2-1)}\, W^k_m \star W^\ell_n \Big|_{W^i_p = 0 \text{ for } i>0} \,, \tag{3.9}$$

i.e. by extracting the term proportional to the identity from the product. The invariant form (3.9) allows to define a CS action based on the algebra $hs[\lambda] \oplus hs[\lambda]$ as in (2.1). Eq. (3.5) then shows that – for a generic value of λ – the spectrum of the corresponding HS gauge theory contains all integer spins from 2 to ∞, and each of them appears only once. However, when λ is integer, the invariant form (3.9) degenerates as one can see from its explicit expression:

$$\text{tr} \left(W^k_m W^\ell_n \right) = (-1)^{\ell-m}\, N_\ell(\lambda)\, \frac{(\ell+m)!(\ell-m)!}{(2\ell)!}\, \delta^{k,\ell}\, \delta_{m+n,0} \,, \tag{3.10a}$$

$$N_\ell(\lambda) = -\,\frac{6\,(\ell!)^2}{(2\ell+1)!} \prod_{i\,=\,2}^{\ell} (i-\lambda)(i+\lambda) \,. \tag{3.10b}$$

The normalisation factors $N_\ell(\lambda)$ follow from the definitions in Appendix A. For integer $\lambda = N$ the CS action (2.1) thus actually corresponds to that of a $sl(N,\mathbb{R}) \oplus sl(N,\mathbb{R})$ theory.[7] Another interesting value of the parameter is $\lambda = 1/2$ that gives the three-dimensional Fradkin-Vasiliev algebra [56].

Before concluding this review, let us notice that the W^ℓ_m with odd ℓ form a subalgebra of $hs[\lambda]$ that we denote by $ho[\lambda]$. As a result, one could well build a HS gauge theory on top of $ho[\lambda] \oplus ho[\lambda]$. For $\lambda \notin \mathbb{N}$ its spectrum contains all even integer spins greater than zero.[8] For $\lambda \in \mathbb{N}$ the truncation to even spins of the $A_N \oplus A_N$ CS theories leads to

[7] The normalisation factor that we introduced in (3.9) thus plays a double role: on the one hand it gives $\text{tr}\left(W^1_1 W^1_{-1} \right) = -1$. On the other hand it guarantees that $\lambda = 1$ still provides a gauge theory involving all spins from 2 to ∞, as in [18].

[8] A similar truncation is available also in higher space-time dimensions where it leads to the so called minimal Vasiliev models (see e.g. [3]).

$B_N \oplus B_N$ gauge algebras for odd N and to $C_N \oplus C_N$ gauge algebras for even N [32], while the D series of simple Lie algebras cannot be recovered in this fashion.

Moreover, commutators of generators with even ℓ (corresponding to fields of odd spin!) can be always expanded in a sum of generators with odd ℓ. On the other hand, mixed commutators (one odd and one even ℓ) always give only terms with even ℓ. As a result, it is possible to rescale by i all generators with even ℓ to get a different real form of $hs[\lambda]$. As already mentioned, for integer $\lambda = N$ the previous construction gives the $sl(N,\mathbb{R})$ real algebra, i.e. the maximal non-compact real form of $sl(N)$. The rescaling by i of the generators with even ℓ leads to the next-to-maximal non-compact real form of $sl(N)$, i.e. $su(\frac{N-1}{2},\frac{N+1}{2})$ for odd N or $su(\frac{N}{2},\frac{N}{2})$ for even N. Additional comments on the possible real forms for $\lambda \notin \mathbb{N}$ can be found in §3.1 of [30]. Note, however, that the rescaling by i reverses the sign of the normalisation factors N_{2j} in (3.10) and, in turn, of the kinetic terms of the fields of odd spin. The maximal non-compact real form seems thus to be preferred to build a HS gauge theory. As noticed in [14], additional subtleties emerge if one consider CS actions built upon two different real forms of the same gauge algebra. In the following we shall avoid all of them by focusing on CS theories based on gauge algebras $hs[\lambda] \oplus hs[\lambda]$, with $hs[\lambda]$ fixed by (3.3) and (3.8).

3.1.1 Metric-like fields and invariant tensors of $hs[\lambda]$

The previous discussion about $hs[\lambda]$ suffices to fully characterise a classical HS gauge theory for any admissible value of λ, in a form which generalises the frame formulation of Einstein gravity. However, even sticking to this algebraic framework one can extract some information on the "metric-like" formulation of these theories, involving Lorentz-invariant symmetric fields $\varphi_{\mu_1 \dots \mu_s}$ (see e.g. [44] for a review). To this end, it is crucial to realise that the two classes of gauge transformations discussed in Section 2.1 play a very different role. Those generated by ξ in (2.7) generalise local translations, while those generated by Λ generalise local Lorentz transformations. As stressed in [14], all metric-like fields should be invariant under local Lorentz-like transformations. Moreover, one can write them in terms of the vielbeine, since the spin connections are just auxiliary fields. In the $sl(3) \oplus sl(3)$ CS theory discussed in detail in [14] these conditions suffice to fix the structure of all fields in the spectrum: the metric $g_{\mu\nu}$ and the spin-3 field $\varphi_{\mu\nu\rho}$. Collecting all vielbeine in the vector $e^A \equiv e_\mu{}^A dx^\mu$ and all generators of the algebra in the vector T_A, up to a normalisation constant they read

$$g \,\sim\, \mathrm{tr}\,(\, e \cdot e\,) \,=\, e^A e^B\, \mathrm{tr}\,(\, T_A T_B\,)\,, \tag{3.11}$$

$$\varphi_3 \,\sim\, \mathrm{tr}\,(\, e \cdot e \cdot e\,) \,=\, \frac{1}{3!}\, e^A e^B e^C\, \mathrm{tr}\,(\, T_{(A} T_B T_{C)}\,)\,, \tag{3.12}$$

where the parentheses denote the symmetrisation of the indices they enclose, with unit normalisation. The cyclicity of the trace guarantees the extended Lorentz invariance. The same result holds for the trace of an arbitrary power of the vielbeine,

$$\delta_\Lambda e \,=\, [\, e\,,\, \Lambda\,] \quad \Rightarrow \quad \delta_\Lambda \mathrm{tr}\,(e^n) \,=\, n\, \mathrm{tr}\,(e^{n-1}[\, e\,,\, \Lambda\,]) \,=\, 0\,. \tag{3.13}$$

As a result, for $s > 3$ the invariance under Lorentz-like transformations does not suffice to fix the structure of $\varphi_{\mu_1 \ldots \mu_s}$. For instance, for $s = 4$ one can consider both $\mathrm{tr}(e^4)$ and $\mathrm{tr}(e^2)\mathrm{tr}(e^2)$ and one has to single out the linear combination that defines $\varphi_{\mu\nu\rho\sigma}$. This freedom corresponds to the existence of two Lorentz-like invariant combinations of rank-4: $\varphi_{\mu\nu\rho\sigma}$ and $g_{(\mu\nu}\, g_{\rho\sigma)}$.

The realisation of the Lie algebra $hs[\lambda]$ as a $\star$-commutator algebra proposed in [30, 31] provides a powerful tool to analyse this problem at least for the first values of the spin. In fact,

$$k_{A_1 \ldots A_s} \,\equiv\, \frac{1}{s!}\, \mathrm{tr}\,(\, T_{(A_1} \ldots T_{A_s)}\,) \,:=\, \frac{6}{(\lambda^2 - 1)\, s!}\; T_{(A_1} \star \, \ldots \, \star \, T_{A_s)} \Big|_{T_A = 0} \tag{3.14}$$

is a symmetric invariant tensor of $hs[\lambda]$ (which coincides with the Killing metric (3.9) for $s = 2$). Its contraction with the vielbeine e^A gives a Lorentz-like invariant tensor of rank s.[9] Metric-like fields should then result from the contraction of the vielbeine with the elements of a particular *basis* of the polynomial ring of invariant tensors of $hs[\lambda]$. This is clear for all $A_N \oplus A_N$, $B_N \oplus B_N$ and $C_N \oplus C_N$ CS theories that can be extracted from the $hs[\lambda] \oplus hs[\lambda]$ one. In fact, their spectra are given by the exponents of the gauge algebras, and are thus in one to one correspondence with the ranks of their independent Casimir operators. Each Casimir operator is, in turn, uniquely associated to a symmetric invariant tensor (see, for instance, [58] and references therein). It is natural to suppose that the same is true even for non-integer λ.

Since the relative coefficients between different invariant tensors of the same rank cannot be fixed by the extended Lorentz invariance, they should be fixed by the additional requirements that $\varphi_{\mu_1 \ldots \mu_s}$ must satisfy:

1. it has to be doubly traceless as its linearised counterpart (see e.g. [44]);
2. in the linearised regime its rewriting in terms of the vielbeine has to reproduce the definition in a free theory.

To impose the first condition one should invert the general definition of the metric (3.11). For this reason we refrain from discussing it here, deferring to future work a full discussion of the problem. On the other hand, the second condition is more tractable and already suffices to fix the structure of spin-4 and spin-5 fields for any λ.

The linearised definition of $\varphi_{\mu_1 \ldots \mu_s}$ can be most conveniently recalled by describing the vielbein $e^{\,\ell,m}$ of (2.10) as a symmetric traceless tensor $e^{a_1 \cdots a_\ell}$ (that has $2\ell + 1$ independent components as $e^{\,\ell,m}$). Denoting the background vielbein by $\bar{e}_\mu{}^a$ and the linearised fluctuations by $h_\mu{}^{a_1 \cdots a_{s-1}}$, for $s > 2$ in a linear regime one has

$$\varphi_{\mu_1 \ldots \mu_s} \sim \, \bar{e}_{(\mu_1}{}^{a_1} \ldots \, \bar{e}_{\mu_{s-1}}{}^{a_{s-1}} h_{\mu_s)\, a_1 \ldots a_{s-1}} \,. \tag{3.15}$$

[9] See [57] for a similar construction for $D > 3$.

This means that, differently from $g_{\mu\nu}$, it is usually assumed that higher-spin fields do not receive any background contribution, while the tracelessness of $h^{a_1\cdots a_{s-1}}$ guarantees the doubly tracelessness of $\varphi_{\mu_1\ldots\mu_s}$.

Imposing the matching of the most general Lorentz-like invariant combination with (3.15) one obtains

$$\varphi_4 \sim \operatorname{tr} e^4 - \frac{1}{10}(3\lambda^2-7)\left(\operatorname{tr} e^2\right)^2, \tag{3.16}$$

$$\varphi_5 \sim \operatorname{tr} e^5 - \frac{5}{21}(3\lambda^2-13)\operatorname{tr} e^2 \operatorname{tr} e^3. \tag{3.17}$$

However, starting from $s=6$ this comparison does not suffice to fix all free coefficients. In fact, the most general Lorentz-like invariant combination reads

$$\varphi_6 \sim \operatorname{tr} e^6 + \alpha(\lambda)\operatorname{tr} e^2 \operatorname{tr} e^4 + \beta(\lambda)\left(\operatorname{tr} e^2\right)^3 + \gamma(\lambda)\left(\operatorname{tr} e^3\right)^2. \tag{3.18}$$

The condition (3.15) gives

$$\alpha(\lambda) = -\frac{5}{6}(\lambda^2-7), \qquad \beta(\lambda) = \frac{1}{42}(6\lambda^4-71\lambda^2+125), \tag{3.19}$$

but the term $\operatorname{tr}(e^3)$ does not admit any background contribution. As a result, $(\operatorname{tr} e^3)^2$ does not contribute at first order as well, and the matching with (3.15) cannot put any constraint on $\gamma(\lambda)$. Some extra information follows from the observation that all φ_s we were able to identify vanish in the $sl(N)\oplus sl(N)$ theories with $N<s$. This is expected because in these cases there are no fields of spin s in the spectrum.[10] One can thus impose the same condition on φ_6 and this forces $\gamma(\lambda)=(1/3)(\lambda^2-1)$ for $N=3,4,5$. It is however not clear to us if this condition suffices to also force the double trace constraint for arbitrary values of λ, while the uniqueness of (3.16) and (3.17) should ensure the double tracelessness of φ_4 and φ_5. An explicit check would anyway provide a non-trivial consistency check of the whole construction, and we hope to report on it soon.

3.2 Gauge transformations preserving the highest-weight gauge

We now take the algebras of the last subsection as starting point for a DS reduction in the highest-weight gauge, and we determine the structure of the corresponding family of infinite-dimensional $\mathcal{W}$-algebras. Since no $sl(2)$-singlets appear in (3.5), all generators will be Virasoro primaries. The asymptotic symmetries of the $hs[\lambda]\oplus hs[\lambda]$ CS theories that we just discussed are given by two copies of the resulting $\mathcal{W}$-algebras, that we denote by $\mathcal{W}_\infty[\lambda]$ as in [19]. The cases with $\lambda\in\mathbb{N}$ and $\lambda=1/2$ were already discussed in [14] and

[10] A general expression for a basis of the centraliser of $sl(N)$ whose elements vanish when their rank exceeds N was considered in [58]. All tensors in this basis are orthogonal to each other and then, in particular, they are all traceless. On the other hand, the invariant tensors that give the metric-like fields do not satisfy this property. Therefore, they coincide with those in [58] only for $\lambda=N<s$.

[13], respectively. A discussion of the general case was also anticipated in [19]. In these works, however, the computation of structure constants was completed only for fields of spin $s \leq 3$ (see also [36, 37] for an earlier treatment of $\mathcal{W}_\infty[\lambda]$ algebras and [59] for an abstract proof that they are actually related to the DS reduction of $hs[\lambda]$).

Before displaying the structure constants, let us notice that one can easily evaluate the maximum order of non-linearity appearing in the Poisson brackets. Consider a gauge parameter of definite spin, $\lambda_+ = \epsilon(\theta) W^\ell_\ell$. When we act on it with the covariant derivative entering (2.42), the term in $W^{\ell_1}_{-\ell_1}$ gives a result with $\ell_{tot} \leq \ell + \ell_1 - 1$ and L_0-eigenvalue $\ell - \ell_1$. If we apply the covariant derivative r-times, we arrive at a L_0-eigenvalue $\sum \ell_i - \ell$ in a representation with $\ell_{tot} \leq \sum \ell_i + \ell - r$. In addition, the action of D_θ is accompanied by at least $r-1$ applications of L_-, that means the L_0-eigenvalue is $\sum \ell_i - \ell + r - 1$. Clearly, if the L_0-eigenvalue exceeds ℓ_{tot}, the expression vanishes, and this happens if $-\ell + r - 1 > \ell - r$, i.e. $r \geq \ell + 1$. In conclusion, in the Poisson brackets of a field with $sl(2)$ label ℓ there can be at most a non-linearity of order ℓ, as in the Virasoro polynomials discussed in Section 2.4.3. Actually, as we shall see, the pure-Virasoro terms are the only ones that saturate this bound. This limitation also accounts, for instance, for the linearity of the Virasoro algebra, that only contains spin-2 fields, and for the quadratic order of non-linearity of the $\mathcal{W}_3$ algebra.

As in subsections 2.4.2 and 2.4.3 the structure constants can be computed by considering the gauge variation induced by a parameter of given spin, say $\lambda_+ = \epsilon(\theta) W^i_i$. The details of the evaluation of the series (2.42) are presented in Appendix B, while here we directly present our result. In this case the general decomposition (2.43a) can be cast in the form

$$a_-(\theta) \,=\, \frac{2\pi}{k} \sum_{j\,=\,1}^{\infty} \mathcal{W}_j(\theta)\, W^j_{-j}\,, \tag{3.20}$$

where we identified $\mathcal{L}$ with $\mathcal{W}_1$ since no ambiguities can arise due to the absence of colour indices. The gauge variation of each $\mathcal{W}_j(\theta)$ with respect to $\lambda_+ = \epsilon(\theta) W^i_i$ reads

$$\begin{aligned} \delta_i \mathcal{W}_j \,=&\, \frac{k}{2\pi(2i)!}\, \epsilon^{(2i+1)}\, \delta_{i,j} \\ &+ \sum_{r\,=\,1}^{i} \sum_{\substack{L\,=\,|i-j|+r \\ i+j+L+r \text{ even}}}^{i+j-r} \sum_{\{a_t\}} \sum_{\{p_t\}} C[i,j]_{a_1\ldots a_r;\, p_1\ldots p_r}\, \mathcal{W}^{(p_1)}_{a_1} \ldots \mathcal{W}^{(p_r)}_{a_r}\, \epsilon^{(\hat{n} - \sum_1^r p_t)}\,. \end{aligned} \tag{3.21}$$

Here, as in (2.62), an exponent between parentheses denotes the action of the corresponding number of derivatives on the field, while $\hat{n}$ denotes the total number of derivatives which is

$$\hat{n} := \, i + j - L - r + 1\,. \tag{3.22}$$

For each r the sums over a's and p's distribute over these indices the "total spin" L and the total number of derivatives $\hat{n}$. They thus read

$$\sum_{\{a_t\}} := \sum_{a_1=1}^{L} \sum_{a_2=1}^{L-a_1} \cdots \sum_{a_{r-1}=1}^{L-\sum_1^{r-2} a_t} \delta_{a_r,\, L-\sum_1^{r-1} a_t} \,, \tag{3.23a}$$

$$\sum_{\{p_t\}} := \sum_{p_1=0}^{\hat{n}} \sum_{p_2=0}^{\hat{n}-p_1} \cdots \sum_{p_r=1}^{\hat{n}-\sum_1^{r-1} p_t} \,. \tag{3.23b}$$

Note that for $i = j$, the terms that saturate the bound on the order of non-linearity are the pure Virasoro terms of Section 2.4.3. On the other hand, for $j < i$ the upper extremum of the sum over r cannot be reached due to the collapsing of the sum over L. As a result, all Poisson brackets involving fields of labels less or equal to ℓ contain polynomials of order strictly lower than ℓ. The only exception is $\{\mathcal{W}_\ell(\theta), \mathcal{W}_\ell(\theta')\}$ that contains a polynomial of order ℓ involving only $\mathcal{L}$ and its first derivative. Eq. (3.21) also shows that the Poisson brackets (obtained from (3.30) below) of $\mathcal{W}_\infty[\lambda]$ are invariant under the map

$$\mathcal{W}_i \to (-1)^{i+1} \mathcal{W}_i \,, \tag{3.24}$$

which is thus an automorphism of $\mathcal{W}_\infty[\lambda]$.

In addition to these structural results, we can even provide a closed formula for the structure constants:

$$\begin{aligned}
C[i,j]_{a_1\ldots a_r;\, p_1\ldots p_r} = {} & \frac{(-1)^{\hat{n}+r-1}}{(2j)!} \left(\frac{2\pi}{k}\right)^{r-1} \sum_{q_1=p_1}^{\hat{n}} \;\sum_{q_2=\langle(p_1+p_2)-q_1\rangle_+}^{\hat{n}-q_1} \cdots \sum_{q_r=\langle\sum_1^r p_t-\sum_1^{r-1} q_t\rangle_+}^{\hat{n}-\sum_1^{r-1} q_t} \\
\times & \sum_{\substack{b_1=\max(|a_r-b_0|+1\,,\,M(r-1,j))\\ a_r+b_0+b_1\ \text{even}}}^{\min\left(a_r+b_0-1\,,\,\sum_1^{r-1} a_t+j-r+1\right)} \cdots \sum_{\substack{b_{r-1}=\max(|a_2-b_{r-2}|+1\,,\,M(1,j))\\ a_2+b_{r-1}+b_r\ \text{even}}}^{\min(a_2+b_{r-2}-1\,,\,a_1+j-1)} \\
\times & \prod_{s=1}^{r} \binom{\sum_1^s q_t - \sum_1^{s-1} p_t}{p_s} [\, j + b_s - \sum_1^{r-s} a_t - \sum_1^{r-s+1} q_t - r + s \,]_{a_{r-s+1}-b_{s-1}+b_s} \\
\times & \; f_\lambda\!\left(\begin{matrix} a_{r-s+1} & b_{s-1} \\ -\,a_{r-s+1} & -j+\sum_1^{r-s+1} a_t + \sum_1^{r-s+1} q_t + r - s \end{matrix} \,\middle|\, \begin{matrix} b_s \\ \ldots \end{matrix} \right) .
\end{aligned} \tag{3.25}$$

Here $[a]_n$ denotes the descending Pochhammer symbol defined in (A.3b), while $\langle a\rangle_+ = \max(0,a)$ and

$$M(s,\ell) := 2\max\left(\{a_t\}_{t=1}^{s}\,,\, \ell\,\right) - \sum_{t=1}^{s} a_t - \ell + s \,. \tag{3.26}$$

By substituting $\lambda = N$ with N integer in (3.25) one obtains a closed formula for the structure constants of the classical $\mathcal{W}_N$ algebra. Note that eq. (3.25) expresses the structure constants of $\mathcal{W}_\infty[\lambda]$ in terms of those of $hs[\lambda]$. The techniques of Appendix B actually allow to express the structure constants of any classical $\mathcal{W}$-algebra (that can be obtained by a DS reduction) in terms of those of the corresponding Lie algebra.

3.2.1 Gauge transformations up to spin 4

In order to better elucidate the structure of our result, we now use the closed formula (3.25) to present all gauge transformations $\delta_i \mathcal{W}_j$ with $i, j < 4$. Letting $\lambda = 4$ the following results determine the structure constants of $\mathcal{W}_4$. We shall focus on variations with $i \leq j$, since those with $j > i$ do not lead to new independent Poisson brackets. Moreover, in each $\delta_i \mathcal{W}_j$ the gauge parameter will be always denoted by ϵ, since no ambiguities can arise. As in (3.21), $\delta_i \mathcal{W}_j$ denotes indeed the component along W^j_{-j} of the gauge variation of $a_-(\theta)$ induced by the gauge parameter $\lambda_+ = \epsilon(\theta) W^i_i$.

The first gauge transformations just display that all fields are primaries with respect to the Virasoro field $\mathcal{L} \equiv \mathcal{W}_1$,

$$\delta_1 \mathcal{L} \,=\, \epsilon\, \mathcal{L}' \,+\, 2\, \epsilon' \mathcal{L} \,+\, \frac{k}{4\pi}\, \epsilon''' \ , \tag{3.27a}$$

$$\delta_1 \mathcal{W}_\ell \,=\, \epsilon\, \mathcal{W}_\ell' \,+\, (\ell+1)\, \epsilon'\, \mathcal{W}_\ell \ , \qquad \ell = 2, 3, \dots \ . \tag{3.27b}$$

Augmenting the conformal weight of the gauge parameter by one we find

$$\begin{aligned} \delta_2 \mathcal{W}_2 \,=\, & -\frac{3}{7}\, (\lambda^2 - 9)\, [\, \mathcal{W}_3' \, \epsilon + 2\, \mathcal{W}_3\, \epsilon' \,] \,+\, \frac{1}{12}\, [\, 2\, \mathcal{L}''' \epsilon + 9\, \mathcal{L}'' \epsilon' + 15\, \mathcal{L}' \epsilon'' + 10\, \mathcal{L}\, \epsilon''' \,] \\ & +\, \frac{16\pi}{3k} \left[\mathcal{L}\mathcal{L}' \epsilon + \mathcal{L}^2 \epsilon' \right] \,+\, \frac{k}{48\pi}\, \epsilon^{(5)} \, . \end{aligned} \tag{3.27c}$$

If one substitutes $\lambda = 3$ in (3.27c), the transformations that we already displayed suffice to build the $\mathcal{W}_3$ algebra (see e.g. [14]) and, as already anticipated, they contain at most quadratic polynomials in the generators. For generic λ one can also look at the component along W^3_{-3} of the gauge variation induced by $\epsilon(\theta) W^2_2$,

$$\begin{aligned} \delta_2 \mathcal{W}_3 \,=\, & -\frac{2}{9}\, (\lambda^2 - 16)\, [\, 2\, \mathcal{W}_4' \, \epsilon + 5\, \mathcal{W}_4\, \epsilon' \,] \\ & +\, \frac{1}{15}\, [\, \mathcal{W}_2''' \, \epsilon + 6\, \mathcal{W}_2'' \, \epsilon' + 14\, \mathcal{W}_2' \, \epsilon'' + 14\, \mathcal{W}_2\, \epsilon''' \,] \\ & +\, \frac{4\pi}{15k}\, [\, 25\, \mathcal{L}' \, \mathcal{W}_2\, \epsilon + 18\, \mathcal{L}\, \mathcal{W}_2' \, \epsilon + 52\, \mathcal{L}\, \mathcal{W}_2\, \epsilon' \,] \ . \end{aligned} \tag{3.27d}$$

The next higher transformation is

$$
\begin{aligned}
\delta_3\mathcal{W}_3 \;=\;& \frac{5}{33}\,(\lambda^2-16)(\lambda^2-25)\,[\,\mathcal{W}_5'\,\epsilon + 2\;\mathcal{W}_5\,\epsilon'\,] \\
&-\;\frac{1}{30}\,(\lambda^2-19)\,[\,\mathcal{W}_3'''\,\epsilon + 5\,\mathcal{W}_3''\,\epsilon' + 9\,\mathcal{W}_3\,\epsilon'' + 6\,\mathcal{W}_3\,\epsilon'''\,] \\
&+\;\frac{1}{360}\left[\,3\,\mathcal{L}^{(5)}\epsilon + 20\,\mathcal{L}^{(4)}\epsilon' + 56\,\mathcal{L}^{(3)}\epsilon'' + 84\,\mathcal{L}''\epsilon^{(3)} + 70\,\mathcal{L}'\epsilon^{(4)} + 28\,\mathcal{L}\,\epsilon^{(5)}\,\right] \\
&-\;\frac{2\pi}{15k}\,(29\lambda^2-284)\left[\,\mathcal{W}_2\mathcal{W}_2'\,\epsilon + \mathcal{W}_2^2\,\epsilon'\,\right] \\
&-\;\frac{28\pi}{15k}(\lambda^2-19)\,[\,\mathcal{L}'\,\mathcal{W}_3\,\epsilon + \mathcal{L}\,\mathcal{W}_3'\,\epsilon + 2\,\mathcal{L}\,\mathcal{W}_3\,\epsilon'\,] \\
&+\;\frac{\pi}{90k}\left[\,177\,\mathcal{L}'\mathcal{L}''\epsilon + 78\,\mathcal{L}\mathcal{L}'''\epsilon + 295\,\mathcal{L}'^2\epsilon' + \;352\,\mathcal{L}\mathcal{L}''\epsilon' + 588\,\mathcal{L}\mathcal{L}'\epsilon'' + 196\mathcal{L}^2\epsilon'''\,\right] \\
&+\;\frac{32\pi^2}{5k^2}\left[\,3\,\mathcal{L}^2\mathcal{L}'\epsilon + 2\,\mathcal{L}^3\epsilon'\,\right] \;+\; \frac{k}{1440\pi}\,\epsilon^{(7)}\,.
\end{aligned}
\tag{3.27e}
$$

Here a cubic polynomial involving only the Virasoro generators appears, while all other polynomials are at most quadratic, in agreement with the discussion following (3.23). Setting $\lambda = 4$ in this and the previous transformations, one obtains the whole $\mathcal{W}_4$ algebra.

The gauge variations (3.27a–3.27e) agree with those presented in eqs. (3.25–3.29) of [19], where their fields L_s are identified with ours by

$$
L_s(\theta) = N_{s-1}(\lambda)\mathcal{W}_{s-1}(\theta) \tag{3.28}
$$

involving the factors N_i defined in (3.10b). The normalisation chosen in [19] leads to Poisson brackets that do not contain poles in λ, so that it is possible to discuss the truncation to $\mathcal{W}_N$ directly at that level. The price to pay is the presence of poles in λ in the definition of $a_-(\theta)$. Therefore, strictly speaking, some intermediate steps of the DS reduction are not well defined for integer λ.

3.3 Poisson bracket algebra

The Poisson brackets of $\mathcal{W}_\infty[\lambda]$ can be obtained from the gauge transformations (3.21) following the general procedure outlined at the end of Section 2.3. In this case there are no colour indices so that (2.44) reads

$$
Q(\lambda_+) \;=\; -\sum_{\ell=1}^{\infty} N_\ell(\lambda)\int d\theta\,\epsilon_\ell(\theta)\,\mathcal{W}_\ell(\theta)\,, \tag{3.29}
$$

where $N_\ell(\lambda)$ denotes the normalisation factors that we introduced in (3.10b). Eq. (2.47) thus simplifies to

$$
\left\{\mathcal{W}_i(\theta)\,,\,\mathcal{W}_j(\theta')\right\} \;=\; -\,\frac{1}{N_i(\lambda)}\,\delta_i\,\mathcal{W}_j(\theta')\Big|_{\epsilon(\theta')\,=\,\delta(\theta-\theta')}\,. \tag{3.30}
$$

Note that for integer λ a diverging factor can appear in this expression since $N_i(\lambda)$ vanishes for $i \geq \lambda$. However, in these cases (3.30) is clearly meaningless since the charges involving $\mathcal{W}_i$ vanish as well. For $\lambda \in \mathbb{N}$ the Poisson algebra is thus generated by the $\mathcal{W}_i$ with $i < \lambda$. The Poisson brackets that one obtains with this procedure have the same structure as the gauge transformations (3.21), barring the substitution

$$\epsilon^{(n)}(\theta') \;\rightarrow\; (-1)^n\, \partial_\theta^{(n)} \delta(\theta - \theta')\,, \tag{3.31}$$

and the normalisation factors $N_i(\lambda)$. The Poisson brackets associated to the explicit gauge transformations (3.27) are collected in Appendix C.

4 A quadratic basis for $\mathcal{W}_\infty[\lambda]$

In the last section we have discussed the algebras $\mathcal{W}_\infty[\lambda]$ that arise from the DS-reduction of $hs[\lambda]$ in the highest-weight gauge. The corresponding Poisson brackets $\{\mathcal{W}_i, \mathcal{W}_j\}$ are non-linear expressions in the fields where the degree of non-linearity is bounded by the minimum of i and j, so that if one considers fields of higher and higher spins, arbitrarily high degrees of non-linearity can appear. On the other hand, the algebra $\mathcal{W}_\infty[\lambda]$ has a basis such that the Poisson brackets involve at most quadratic polynomials in the fields. In this section, we want to derive the explicit basis transformation that relates theses two bases.

We start by a general discussion of gauge freedom in the DS-reduction. We then present in Section 4.1 a recursive algorithm that can be used to determine the transformation to the highest-weight basis for the DS-reduction of any Lie algebra. In Section 4.2 a quadratic basis for $\mathcal{W}_N$ is reviewed, and in Section 4.3 the basis transformation from the highest-weight basis to the quadratic basis is determined. These results are used in Section 4.4 to derive the corresponding basis transformation for $\mathcal{W}_\infty[\lambda]$.

We have seen in Section 2 that the asymptotic $\mathcal{W}$-symmetries arise from the symmetry transformations (2.19) with the constraint (2.28) that implements the asymptotic AdS condition. The gauge transformations generated by these constraints can be used to choose a certain gauge, e.g. the highest-weight gauge (2.29) for $a(\theta)$ that we used in the last sections. This gauge choice is very natural, but in some situations other choices can be more convenient.

To discuss this gauge freedom we introduce the differential operators

$$l_a = \partial_\theta + a(\theta)\ , \tag{4.1}$$

where we have chosen a representation of the Lie algebra $\mathfrak{g}$. On such a differential operator we can act with (θ-dependent) group elements $g(\theta)$ by conjugation,

$$l_a \mapsto g^{-1}(\theta) l_a g(\theta)\ , \tag{4.2}$$

which implies the transformation

$$a(\theta) \mapsto g^{-1}(\theta)\partial_\theta g(\theta) + g^{-1}(\theta)a(\theta)g(\theta) \tag{4.3}$$

on $a(\theta)$. For infinitesimal transformations $g(\theta) = 1 + \lambda(\theta)$ this reduces to (2.19). The AdS or Drinfeld-Sokolov condition (2.28) is respected by transformations $g(\theta)$ that take values in the subgroup $\mathcal{N} \subset G$ that is generated by $\mathfrak{g}_>$ (defined in (2.26)). On each of the gauge orbits $g^{-1}l_a g$ with $g \in \mathcal{N}$ we can pick a representative corresponding to a certain gauge choice.

4.1 Recursive basis transformation

Given any such representative $\partial_\theta + J_+ + b(\theta)$, we now describe a procedure how to recursively find the representative $\partial_\theta + J_+ + a_-(\theta)$ in the highest-weight gauge on the same orbit, and the gauge transformation $g(\theta) \in \mathcal{N}$ that connects the two representatives,

$$g(\theta)^{-1}\left(\partial_\theta + J_+ + b(\theta)\right)g(\theta) = \partial_\theta + J_+ + a_-(\theta) \ . \tag{4.4}$$

We write the representation matrices $g(\theta)$ as $g(\theta) = 1 + h(\theta)$, where $h(\theta)$ has an expansion

$$h(\theta) = h^1(\theta) + h^2(\theta) + \cdots \ , \tag{4.5}$$

with $L_0 h^n = n h^n$, $n > 0$. From (4.4) we obtain

$$-L_+ h(\theta) + a_-(\theta) = \partial h(\theta) + b(\theta) + b(\theta)h(\theta) - h(\theta)a_-(\theta) \ . \tag{4.6}$$

When we act on this equation by RL_-, where R is the operator that was defined in (2.37), we arrive at

$$h(\theta) = -RL_-\left(\partial h(\theta) + b(\theta) + b(\theta)h(\theta) - h(\theta)a_-(\theta)\right) \ . \tag{4.7}$$

On the other hand, when we act on (4.6) by the projector P_-, we find

$$a_-(\theta) = P_-\left(\partial h(\theta) + b(\theta) + b(\theta)h(\theta) - h(\theta)a_-(\theta)\right) \ . \tag{4.8}$$

These two equations, (4.7) and (4.8), can be used to construct the basis transformation recursively. Let

$$b(\theta) = b^0(\theta) + b^1(\theta) + \cdots \quad , \quad a_-(\theta) = a_-^0(\theta) + a_-^1(\theta) + \cdots \ , \tag{4.9}$$

where $L_0 b_n = n b_n$, and similarly for a_-^n. Projecting (4.7) to L_0 eigenvalue n we find

$$h^n = -RL_-\left(\partial h^{n-1} + b^{n-1} + \sum_{1\leq m\leq n-1} b^{n-m-1}h^m - \sum_{1\leq m\leq n-1} h^m a_-^{n-m-1}\right) \ . \tag{4.10}$$

Analogously, we can project (4.8) to L_0 eigenvalue n to obtain

$$a_-^n = P_-\left(\partial h^n + b^n + \sum_{1\leq m\leq n} b^{n-m}h^m - \sum_{1\leq m\leq n} h^m a_-^{n-m}\right) \ . \tag{4.11}$$

Suppose that $b(\theta)$ is given, and that we have determined $h^m(\theta)$ and $a_-^m(\theta)$ for $m < n$. We can use (4.10) to obtain $h^n(\theta)$, and then determine $a_-^n(\theta)$ by (4.11).

4.2 A quadratic basis for $\mathcal{W}_N$

We have seen how to construct in general the basis transformation to the highest-weight basis recursively in the eigenvalues of L_0. We now want to become more specific for the Lie algebras $sl(N)$ and the corresponding $\mathcal{W}$-algebras $\mathcal{W}_N$, for which another natural basis exists. We use the defining representation by square matrices of size N, in which J_+ is represented as

$$J_+ = \begin{pmatrix} 0 & & & & \\ -1 & 0 & & & \\ 0 & -1 & 0 & & \\ & \ddots & \ddots & \ddots & \\ & & 0 & -1 & 0 \end{pmatrix} . \tag{4.12}$$

The Drinfeld-Sokolov condition states that $a(\theta)$ is of the form

$$a(\theta) = J_+ + u(\theta) \ , \tag{4.13}$$

where $u(\theta)$ is upper triangular. The subgroup $\mathcal{N}$ consists of all matrices of the form $1+h$ where h is strictly upper triangular. A natural gauge choice is to demand that $u(\theta)$ only has one non-zero row (the first one),

$$u = \begin{pmatrix} 0 & u_1 & \cdots & u_{N-1} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} . \tag{4.14}$$

With this choice one obtains a basis for the $\mathcal{W}_N$ algebras that has at most quadratic non-linearities in the Poisson brackets. The nice feature of this basis is that there is a simple way to express the gauge transformations and thus the Poisson brackets with the help of pseudo-differential operators, which we shall briefly review in the following (for details see e.g. [27]).

The infinitesimal transformations $g(\theta) = 1 + \lambda(\theta)$ that respect the gauge choice are uniquely determined in terms of the first column of λ. We introduce the differential operators

$$\lambda_i = \sum_{j=0}^{N-1} \lambda_{ij}\, \partial^{N-j-1} \tag{4.15}$$

of order at most $N-1$ corresponding to the rows of λ, the pseudo-differential operator

$$\lambda^0 = \sum_{i=0}^{n-1} \partial^{-N+i} \lambda_{0j} \tag{4.16}$$

corresponding to the first column, and the differential operator

$$L = \partial^N + u_1 \partial^{N-2} + \cdots + u_{N-1} \tag{4.17}$$

associated to the gauge fields. The matrix λ_{ij} is then given by the equations

$$\lambda_i = \partial^{N-i-1}(\lambda^0 L)_+ - (\partial^{N-i-1}\lambda^0)_+ L \tag{4.18}$$

in terms of its first column, where $(\ )_+$ denotes the projection of a pseudo-differential operator to its regular part (the non-negative powers of the differential). This is the analogue of (2.41) in the highest-weight gauge. The gauge transformation can also be expressed in such a way,

$$\delta_\lambda L = L(\lambda^0 L)_+ - (L\lambda^0)_+ L \ . \tag{4.19}$$

From here one can easily obtain the Poisson brackets. As L only appears quadratically on the right-hand side of (4.19), the Poisson brackets are at most quadratic in the fields u_i. This is quite different from the highest-weight basis, where the degree of non-linearity is $N-1$.

4.3 Basis transformation for $\mathcal{W}_N$

The obvious question arises how the gauge choice of the last subsection and the highest-weight gauge are related. As the corresponding Poisson brackets have different orders of non-linearities, the basis transformation has to be non-linear. We could use our recursive construction from Section 4.1 to relate these two bases, but there is another more elegant way of relating them. The operator $l_a = \partial_\theta + a(\theta)$ can be used to define a system of N first order differential equations on some functions $f_0, \ldots, f_{N-1}$,

$$l_a(f_0, \ldots, f_{N-1})^T = 0 \ . \tag{4.20}$$

Due to the specific form (4.13) of $a(\theta)$ this system of differential equations is equivalent to a single N^{th} order differential equation for f_{N-1}. The other functions $f_0, \ldots, f_{N-2}$ are then given in terms of f_{N-1} by

$$f_j = \det\left((\partial + J_+ + u)_{r,s}\right)_{j+1 \le r,s \le N-1} f_{N-1} \ , \tag{4.21}$$

where the determinant is evaluated with the convention that the entries of the first row appear to the left of entries of the second row, and so on. The scalar differential operator L that determines the differential equation for f_{N-1},

$$L f_{N-1} = 0 \ , \tag{4.22}$$

can also be expressed as a determinant [27],

$$L = \det(\partial + J_+ + u) \ . \tag{4.23}$$

Two differential operators l_a and $l_{a'}$ that are related by a gauge transformation in $\mathcal{N}$ give rise to the same scalar differential operator L. If u is of the form (4.14) (all rows vanish except for the first), we have

$$L = \partial^N + u_1 \partial^{N-2} + .. + u_{N-2}\partial + u_{N-1} \ . \tag{4.24}$$

By evaluating the operator L we can thus relate any choice of representatives to the u-basis.

We now want to work out the basis transformation explicitly for the highest-weight basis of $sl(N)$. The matrix representing J_- reads

$$J_- = \begin{pmatrix} 0 & 1\cdot(N-1) & 0 & & & \\ & 0 & 2\cdot(N-2) & 0 & & \\ & & \ddots & \ddots & & \\ & & & 0 & (N-1)\cdot 1 \\ & & & & 0 \end{pmatrix} . \tag{4.25}$$

The highest-weight vectors in $sl(N)$ are $W^m_{-m} = J^m_-$ with $m = 1, \dots, N-1$, so that a_- can be expanded as

$$a_-(\theta) = \frac{2\pi}{k}\left(\mathcal{W}_1(\theta)J_- + \dots + \mathcal{W}_{N-1}(\theta)J_-^{N-1}\right) , \tag{4.26}$$

or, as matrix,

$$a_- = \frac{2\pi}{k}\begin{pmatrix} 0 & \mathcal{W}_1P_1(1) & \mathcal{W}_2P_2(1) & \mathcal{W}_3P_3(1) & \cdots & \mathcal{W}_{N-1}P_{N-1}(1) \\ 0 & 0 & \mathcal{W}_1P_1(2) & \mathcal{W}_2P_2(2) & \cdots & \mathcal{W}_{N-2}P_{N-2}(2) \\ 0 & 0 & 0 & \mathcal{W}_1P_1(3) & \cdots & \mathcal{W}_{N-3}P_{N-3}(3) \\ & & & \ddots & & \\ & & & & 0 & \mathcal{W}_1P_1(N-1) \\ & & & & & 0 \end{pmatrix} . \tag{4.27}$$

Here, the P_s are defined in terms of ascending and descending Pochhammer symbols (see (A.3a) and (A.3b)),

$$P_s(i) = (i)_s[N-i]_s . \tag{4.28}$$

The determinant (4.23) is evaluated by summing over all permutations of the N columns while respecting the row ordering ($l = \partial + J_+ + a_-$),

$$L = \det l = \sum_{\sigma \in S_N} l_{1\sigma(1)}l_{2\sigma(2)} \cdots l_{N\sigma(N)} . \tag{4.29}$$

We can order the terms by their degree of non-linearity in the $\mathcal{W}_m$. The term without any $\mathcal{W}_m$ is simply ∂^N. A linear term comes from the part of the determinant where exactly one entry of the upper triangle contributes:

$$\begin{pmatrix} \ddots & & & & & \\ & \partial & \cdots & \cdots & \mathcal{W}_sP_s(i+1) & \\ & -1 & \ddots & & \vdots & \\ & & \ddots & \ddots & \vdots & \\ & & & -1 & \partial & \\ & & & & & \ddots \end{pmatrix} . \tag{4.30}$$

There is only one permutation such that besides $\mathcal{W}_s P_s(i+1)$ only ∂'s and (-1)'s appear leading to the contribution

$$\frac{2\pi}{k}\partial_\theta^i \mathcal{W}_s P_s(i+1)\partial_\theta^{N-s-i-1} \ .$$

The total contribution linear in $\mathcal{W}_s$ is then

$$L_{W_s} = \frac{2\pi}{k}\sum_{i=0}^{N-s-1}\partial_\theta^i\, \mathcal{W}_s P_s(i+1)\partial_\theta^{N-s-i-1} \tag{4.31}$$

$$= \frac{2\pi}{k}\sum_{i=0}^{N-s-1}\sum_{p=0}^{i}\binom{i}{p}P_s(i+1)\mathcal{W}_s^{(p)}\partial_\theta^{N-s-p-1} \tag{4.32}$$

$$= \frac{2\pi}{k}\sum_{p=0}^{N-s-1}\sum_{i=p}^{N-s-1}\binom{i}{p}P_s(i+1)\mathcal{W}_s^{(p)}\partial_\theta^{N-s-p-1} \tag{4.33}$$

$$= \sum_{p=0}^{N-s-1} C(s,p,N)\mathcal{W}_s^{(p)}\partial_\theta^{N-s-p-1} \ . \tag{4.34}$$

The coefficients $C(s,p,N)$ can be written as

$$C(s,p,N) = \frac{2\pi}{k}\sum_{i=p}^{N-s-1}\binom{i}{p}P_{s-1}(i+1) \tag{4.35}$$

$$= \frac{2\pi}{k}\sum_{j=0}^{N-s-p-1}\frac{(j+1)_p}{p!}(j+p+1)_s[N-j-p-1]_s \tag{4.36}$$

$$= \frac{2\pi}{k}\sum_{j=0}^{N-s-p-1}\frac{(j+1)_{p+s}}{p!}(N-s-p-j)_s \tag{4.37}$$

$$= \frac{2\pi}{k}s!\binom{p+s}{p}(N-s-p)_s\sum_{j=0}^{N-s-p-1}\frac{(s+p+1-N)_j(p+s+1)_j}{(1+p-N)_j j!} \tag{4.38}$$

$$= \frac{2\pi}{k}(s!)^2\binom{p+s}{p}\binom{N-p-1}{s}{}_2F_1(s+p+1-N, s+p+1; 1+p-N; 1) \ . \tag{4.39}$$

The hypergeometric function ${}_2F_1$ at argument 1 is given by

$${}_2F_1(s+p+1-N, s+p+1; 1+p-N; 1) = \frac{(N-p)_{s+p+1}}{(s+1)_{s+p+1}} \ , \tag{4.40}$$

so that after some manipulations we find

$$C(s,p,N) = \frac{2\pi}{k}(s!)^2\binom{p+s}{p}\binom{N+s}{2s+p+1} \ . \tag{4.41}$$

The linear contribution to the determinant is thus

$$L_{\text{linear}} = \frac{2\pi}{k}\sum_s (s!)^2\sum_p\binom{p+s}{p}\binom{N+s}{2s+p+1}\mathcal{W}_s^{(p)}\partial_\theta^{N-s-p-1} \ . \tag{4.42}$$

The higher order terms appear as products of such linear blocks, and the total determinant is given by

$$L = \sum_{r=0}^{\lfloor \frac{N}{2} \rfloor} \left(\frac{2\pi}{k}\right)^r \sum_{s_1,\dots,s_r} \sum_{i_1,\dots,i_r} \partial_\theta^{i_1} \mathcal{W}_{s_1} p_{s_1}(i_1+1)$$
$$\times \partial_\theta^{i_2} \mathcal{W}_{s_2} P_{s_2}(i_1+i_2+s_1+2) \cdots \mathcal{W}_{a_r} P_{a_r}(i_1+\cdots+i_r+s_1+\cdots+s_{r-1}+r) \partial_\theta^{i_{r+1}} \ . \tag{4.43}$$

Here, $i_{r+1} = N - r - \sum_1^r i_j - \sum_1^r s_j$, and all $i_j \geq 0$, $s_j \geq 1$. As in the linear example above, we commute all differentials to the right, which produces derivatives of the fields $\mathcal{W}_{s_j}$, e.g. for the first field $\mathcal{W}_{s_1}$ we have

$$\partial_\theta^{i_1} \mathcal{W}_{s_1}(\theta) = \sum_{m_1=0}^{i_1} \binom{i_1}{m_1} \mathcal{W}_{s_1}^{(m_1)}(\theta) \partial_\theta^{i_1-m_1} \ . \tag{4.44}$$

For the second field $\mathcal{W}_{s_2}$ we find

$$\partial_\theta^{i_1+i_2-m_1} \mathcal{W}_{s_2}(\theta) = \sum_{m_2=0}^{i_1+i_2-m_1} \binom{i_1+i_2-m_1}{m_2} \mathcal{W}_{s_2}^{(m_2)}(\theta) \partial_\theta^{i_1+i_2-m_1-m_2} \ , \tag{4.45}$$

and we introduce similar summation variables m_j counting the derivatives of the other fields $\mathcal{W}_{s_j}$. We also introduce the variables

$$k_l = \sum_{j=1}^{l} i_j \quad , \quad p_j = \sum_{j=1}^{l} m_j \ . \tag{4.46}$$

The determinant L then becomes

$$L = \sum_r \left(\frac{2\pi}{k}\right)^r \sum_{s_1,\dots,s_r} \sum_{0 \leq k_1 \leq \cdots \leq k_r \leq N-r-S} \sum_{0 \leq p_1 \leq \cdots \leq p_r, p_j \leq k_j}$$
$$\mathcal{W}_{s_1}^{(p_1)} \mathcal{W}_{s_2}^{(p_2-p_1)} \cdots \mathcal{W}_{s_r}^{(p_r-p_{r-1})} \partial_\theta^{N-r-p_r-S}$$
$$\times \binom{k_1}{p_1} \binom{k_2-p_2}{p_2-p_1} \cdots \binom{k_r-p_{r-1}}{p_r-p_{r-1}}$$
$$\times P_{s_1}(k_1+1) P_{s_2}(k_2+s_1+2) \cdots P_{s_r}(k_r + \sum_{j=1}^{r-1} a_j + r) \ , \tag{4.47}$$

where $S = \sum_{j=1}^r s_j$. This means that a given term $\mathcal{W}_{s_1}^{(p_1)} \cdots \mathcal{W}_{s_r}^{(p_r-p_{r-1})} \partial_\theta^{N-r-p_r-S}$ appears with a coefficient

$$C(\{s_j\}, \{p_j\}, N) = \left(\frac{2\pi}{k}\right)^r \sum_{k_r=p_r}^{N-S-r} \sum_{k_{r-1}=p_{r-1}}^{k_r} \cdots \sum_{k_1=p_1}^{k_2} \binom{k_1}{p_1} \cdots \binom{k_r-p_{r-1}}{p_r-p_{r-1}}$$
$$\times (k_1+1)_{s_1} \cdots (k_r + \sum_{j=1}^{r-1} s_j + r)_{s_r}$$
$$\times (N-k_1-s_1)_{s_1} \cdots (N-k_r-\sum_{j=1}^{r} s_j - r + 1)_{s_r} \ . \tag{4.48}$$

By comparison with (4.24), these coefficients determine the basis transformation,

$$u_q = \sum_{r=1}^{\lfloor \frac{q+1}{2} \rfloor} \sum_{\substack{\{s\} \\ S+r \leq q+1}} \sum_{\substack{\{p\} \\ p_r = q+1-r-S}} C(\{s_j\},\{p_j\},N) \mathcal{W}_{s_1}^{(p_1)} \cdots \mathcal{W}_{s_r}^{(p_r - p_{r-1})} \,. \tag{4.49}$$

The coefficients (4.48) depend in a simple way on the number of derivatives, p_j, and we can combine the coefficients belonging to a certain set of fields of spins $s_1, \ldots, s_r$ into a generating function with auxiliary variables α_j,

$$\tilde{C}(\{s_j\},\{\alpha_j\},N) = \left(\frac{2\pi}{k}\right)^r \sum_{p_r=0}^{N-r-S} \sum_{p_{r-1}=0}^{p_r} \cdots \sum_{p_1=0}^{p_2} C(\{s_j\},\{p_j\},N) \alpha_1^{p_1} \alpha_2^{p_2-p_1} \cdots \alpha_r^{p_r - p_{r-1}} \tag{4.50}$$

$$\begin{aligned} &= \sum_{k_r=0}^{N-r-S} \cdots \sum_{k_1=0}^{k_2} (k_1+1)_{s_1} \cdots (k_r + \sum_{j=1}^{r-1} s_j + r)_{s_r} \\ &\quad \times (N-k_1-s_1)_{s_1} \cdots (N - k_r - \sum_{j=1}^{r} s_j - r + 1)_{s_r} \\ &\quad \times (1+\alpha_r)^{k_r - k_{r-1}} (1+\alpha_r+\alpha_{r-1})^{k_{r-1}-k_{r-2}} \cdots (1+\alpha_r+\cdots+\alpha_1)^{k_1} \,. \end{aligned} \tag{4.51}$$

By going back to the variables $i_j = k_j - k_{j-1}$, we can write it as a generalised hypergeometric function,

$$\begin{aligned} \tilde{C}(\{s_j\},\{\alpha_j\},N) &= \left(\frac{2\pi}{k}\right)^r (1)_{s_1}(N-s_1)_{s_1} \cdots (r + \sum_{j=1}^{r-1} s_j)_{s_r} (N-r+1-S)_{s_r} \\ &\times \sum_{i_1,\ldots,i_r \geq 0} \frac{(1+s_1)_{i_1}(1-N+s_1)_{i_1}}{(1)_{i_1}(1-N)_{i_1}} \cdots \frac{(r+\sum_{j=1}^{r} s_j)_{i_1+\cdots+i_r} (r-N+\sum_{j=1}^{r} s_j)_{i_1+\cdots+i_r}}{(r+\sum_{j=1}^{r-1} s_j)_{i_1+\cdots+i_r} (r-N+\sum_{j=1}^{r-1} s_j)_{i_1+\cdots+i_r}} \\ &\times (1+\alpha_r)^{i_r} \cdots (1+\alpha_r+\cdots+\alpha_1)^{i_1} \,. \end{aligned} \tag{4.52}$$

The sum is bounded by $i_1 + \cdots + i_r \leq N - r - S$ because the last Pochhammer symbol in the numerator would vanish otherwise.

4.4 Basis transformation for $\mathcal{W}_\infty[\lambda]$

In the last subsection we have determined the basis transformation that relates the highest-weight basis and a quadratic basis for $\mathcal{W}_N$. A quadratic basis also exists for $\mathcal{W}_\infty[\lambda]$. On the one hand, [36] generalised the construction of $\mathcal{W}_N$ via pseudo-differential operators to a one-parameter family of infinite dimensional $\mathcal{W}$-algebras with quadratic non-linearities. On the other hand, [59] showed that these algebras can be understood as a Drinfeld-Sokolov reduction of $hs[\lambda]$. In the work of [59], they used a realisation of $hs[\lambda]$ in terms of infinite matrices, in which J_+ and J_- are given by the infinite analogues of (4.12) and (4.25) with N replaced by λ.

From this construction it is obvious that also the basis transformation of the last subsection can be generalised to $\mathcal{W}_\infty[\lambda]$: we have to work out the transformation for large matrices and then replace the explicit N-dependence in the coefficients by λ. Of course, the size of the matrices is also determined by N, which means that we first have to take N large, until a given coefficient $C(\{s_j\},\{p_j\},N)$ of the basis transformation stabilises to a certain polynomial in N, and then replace N by λ.

The main difficulty is now that in the expression (4.48) the number N also appears in the range of the sums, and thus we cannot directly replace N by λ. Instead we would have to perform the sum to get an expression which manifestly is a polynomial in N such that we can do the replacement. In the example of the linear contribution this is easily possible: the sum can be performed and the final expression (4.41) is written manifestly as a polynomial in N of degree $2s+p+1$.

For the non-linear terms we shall instead encode the coefficients in terms of generating functions, in which the dependence of the summation range on N can be avoided. We already introduced the functions $\tilde{C}(\{s_j\},\{\alpha_j\},N)$ with the auxiliary variables α_j, which are generalised hypergeometric functions. In the expression (4.52) for $\tilde{C}$ we could leave out the N-dependent restriction in the summation range, because this restriction is automatically implemented by the vanishing of one of the Pochhammer symbols, so that we could replace N by λ. This, however, does not lead to the correct answer, because the function obtained in such a way does not depend polynomially on λ. Let us illustrate this in the example of the linear terms. For them, the generating function is

$$\tilde{C}(s,\alpha,N) = \frac{2\pi}{k}(1)_s(N-s)_s \sum_{i\geq 0} \frac{(1+s)_i(1-N+s)_i}{(1)_i(1-N)_i}(1+\alpha)^i \tag{4.53}$$

$$= \frac{2\pi}{k}(1)_s(N-s)_s {}_2F_1(1+s,1-N+s;1-N;1+\alpha)\ . \tag{4.54}$$

The function that one obtains by replacing N by λ is not continuous at $\lambda = N$, but instead one has

$$\begin{aligned}
&\lim_{\lambda\to N}(\lambda-s)_s\, {}_2F_1(1+s,1-\lambda+s;1-\lambda;1+\alpha) \\
&\quad = (N-s)_s\, {}_2F_1(1+s,1-N+s;1-N;1+\alpha) \\
&\qquad + (-1)^s(1+N)_s(1+\alpha)^N {}_2F_1(1+s,1+N+s;1+N;1+\alpha)\ .
\end{aligned} \tag{4.55}$$

This suggests to extrapolate the function $\tilde{C}(s,\alpha,N)$ to

$$\begin{aligned}
\tilde{C}(s,\alpha,\lambda) &= \frac{2\pi}{k}(1)_s(\lambda-s)_s\, {}_2F_1(1+s,1-\lambda+s;1-\lambda;1+\alpha) \\
&\quad - \frac{2\pi}{k}(1)_s(-1)^s(1+\lambda)_s(1+\alpha)^\lambda {}_2F_1(1+s,1+\lambda+s;1+\lambda;1+\alpha)
\end{aligned} \tag{4.56}$$

$$\begin{aligned}
&= \frac{2\pi}{k}(s!)(\lambda-s)_{2s+1}\Bigg(\frac{\Gamma(\lambda)}{\Gamma(1+s+\lambda)}{}_2F_1(1+s,1+s-\lambda;1-\lambda;1+\alpha) \\
&\quad + \frac{\Gamma(-\lambda)}{\Gamma(1+s-\lambda)}(1+\alpha)^\lambda {}_2F_1(1+s,1+s+\lambda;1+\lambda;1+\alpha)\Bigg)\ .
\end{aligned} \tag{4.57}$$

An elementary transformation of the hypergeometric function leads to

$$\tilde{C}(s,\alpha,\lambda) = \frac{2\pi}{k}(s!)^2 \binom{\lambda+s}{2s+1} {}_2F_1(1+s, 1+s-\lambda; 2+2s; -\alpha) \ . \tag{4.58}$$

The coefficient of each power α^p is a polynomial in λ, which coincides with $C(s,p,N)$ in (4.41) when we replace N by λ.

In principle one can apply this procedure also to the general non-linear terms. One replaces N by λ in the generalised hypergeometric function, and determines the discontinuity at $\lambda = N$ to find the correct extrapolation to $\lambda \neq N$. The function one obtains in such a way, however, is not of practical use, unless one finds a transformation to a (generalisation of a) hypergeometric function that has an expansion in α_i. Such a step might be difficult to perform in general, therefore we shall pursue another strategy here. Again, this is illustrated in the case of the linear terms, where we already know the correct answer.

Let us go back to the expression (4.51) for $\tilde{C}$. In the linear case it reads

$$\tilde{C}(s,\alpha,N) = \frac{2\pi}{k} \sum_{i=0}^{N-s-1} (i+1)_s (N-i-s)_s (1+\alpha)^i \ . \tag{4.59}$$

Now, $(N-i-s)_s$ is the coefficient of γ^s in $(s!)(1+\gamma)^{N-i-1}$, and $(i+1)_s$ is the coefficient of β^s in $(-1)^s(s!)(1+\beta)^{-i-1}$. With the help of the auxiliary variables β and γ we can then write $\tilde{C}$ as

$$\tilde{C}(s,\alpha,N) = \frac{2\pi}{k}(s!)^2(-1)^s \sum_{i=0}^{N-s-1} (1+\gamma)^{N-i-1}(1+\beta)^{-i-1}(1+\alpha)^i \Big|_{\gamma^s\beta^s} \ , \tag{4.60}$$

where it is indicated that in an expansion in γ and β we only keep the term involving $\gamma^s\beta^s$. We perform the geometric sum to obtain

$$\tilde{C}(s,\alpha,N) = \frac{2\pi}{k}(s!)^2(-1)^s \frac{(1+\gamma)^{N-1}}{(1+\beta)} \frac{\left(\frac{1+\alpha}{(1+\beta)(1+\gamma)}\right)^{N-s} - 1}{\frac{1+\alpha}{(1+\beta)(1+\gamma)} - 1} \Bigg|_{\gamma^s\beta^s} \ . \tag{4.61}$$

We rewrite the denominator as

$$\left(\frac{1+\alpha}{(1+\beta)(1+\gamma)} - 1\right)^{-1} = \frac{(1+\beta)(1+\gamma)}{\alpha}\left[1 - \frac{1}{\alpha}\left((1+\beta)(1+\gamma)-1\right)\right]^{-1} \ . \tag{4.62}$$

The term in the square bracket is expanded for small β and γ, while α is kept fixed. We arrive at

$$\begin{aligned}\tilde{C}(s,\alpha,N) = {} & \frac{2\pi}{k}(s!)^2(-1)^s \left((1+\gamma)^s(1+\beta)^{s-N}(1+\alpha)^{N-s} - (1+\gamma)^N\right) \\ & \times \sum_{m\geq 0} \alpha^{-m-1}(\beta+\gamma+\beta\gamma)^m \Big|_{\gamma^s\beta^s} \ .\end{aligned} \tag{4.63}$$

When we further expand this expression, the coefficients of $\alpha^p\beta^s\gamma^s$ are polynomials in N, and we can replace N by λ. We obtain

$$\tilde{C}(s,\alpha,\lambda) = \frac{2\pi}{k}(s!)^2(-1)^s \sum_{p\geq 0}\sum_{u,v=0}^{s} \binom{u+v}{v}\binom{s+u}{u+v}\binom{s-\lambda}{s-u}\binom{\lambda-s}{1+p+u+v}\alpha^p \ . \quad (4.64)$$

It can be verified that this result agrees with the previously derived expression (4.58).

The strategy just described can easily be applied to the general non-linear terms. We start from (4.51), and introduce auxiliary variables $\beta_1,\ldots,\beta_r$ and $\gamma_1,\ldots,\gamma_r$. We can then write

$$\begin{aligned}\tilde{C}(\{s_j\},\{\alpha_j\},N) = \left(\frac{2\pi}{k}\right)^r \prod_{j=1}^{r}(s_j!)^2(-1)^{s_j} \sum_{k_r=0}^{N-r-S}\cdots\sum_{k_1=0}^{k_2} \\ \times (1+\gamma_r)^{k_{r-1}-k_r-1}\cdots(1+\gamma_r+\cdots+\gamma_2)^{k_1-k_2-1}(1+\gamma_r+\cdots+\gamma_1)^{N-k_1-1} \\ \times (1+\beta_r)^{k_{r-1}-k_r-1}\cdots(1+\beta_r+\cdots+\beta_1)^{-k_1-1} \\ \times (1+\alpha_r)^{k_r-k_{r-1}}\cdots(1+\alpha_r+\cdots+\alpha_1)^{k_1}\Big|_{\substack{\gamma_1^{s_1}\cdots\gamma_r^{s_r}\\ \beta_1^{s_1}\cdots\beta_r^{s_r}}} \ . \end{aligned} \quad (4.65)$$

When we introduce the notation

$$A_j = \alpha_j+\cdots+\alpha_r \quad , \quad B_j = \beta_j+\cdots+\beta_r \quad , \quad C_j = \gamma_j+\cdots+\gamma_r \ , \quad (4.66)$$

the expression simplifies to

$$\begin{aligned}\tilde{C}(\{s_j\},\{\alpha_j\},N) = \left(\frac{2\pi}{k}\right)^r \prod_{j=1}^{r}(s_j!)^2(-1)^{s_j} \frac{(1+C_1)^N}{\prod_{j=1}^r(1+C_j)(1+B_j)} \sum_{k_r=0}^{N-r-S}\cdots\sum_{k_1=0}^{k_2} \\ \left(\frac{(1+A_1)}{(1+B_1)(1+C_1)}\right)^{k_1}\cdots\left(\frac{(1+A_r)}{(1+B_r)(1+C_r)}\right)^{k_r-k_{r-1}}\Bigg|_{\substack{\gamma_1^{s_1}\cdots\gamma_r^{s_r}\\ \beta_1^{s_1}\cdots\beta_r^{s_r}}} \ . \end{aligned} \quad (4.67)$$

We can now successively evaluate the geometric sums, starting with the sum over k_1 giving

$$\sum_{k_1=0}^{k_2}\left(\frac{(1+A_1)(1+B_2)(1+C_2)}{(1+B_1)(1+C_1)(1+A_2)}\right)^{k_1} = \frac{\left(\frac{(1+A_1)(1+B_2)(1+C_2)}{(1+B_1)(1+C_1)(1+A_2)}\right)^{k_2+1}-1}{\left(\frac{(1+A_1)(1+B_2)(1+C_2)}{(1+B_1)(1+C_1)(1+A_2)}\right)-1} \ . \quad (4.68)$$

We expand the denominator in the auxiliary variables except for α_1 (the denominator vanishes if all auxiliary variables are set to zero; since α_1 appears in all denominators that arise from the geometric sums, it is enough to keep this parameter finite while expanding in all others). Thus we have

$$\begin{aligned}\left(\frac{(1+A_1)(1+B_2)(1+C_2)}{(1+B_1)(1+C_1)(1+A_2)}-1\right)^{-1} = \\ \frac{(1+B_1)(1+C_1)(1+A_2)}{\alpha_1(1+B_2)(1+C_2)}\left\{1+\frac{1}{\alpha_1}\left((1+A_2)-\frac{(1+B_1)(1+C_1)(1+A_2)}{(1+B_2)(1+C_2)}\right)\right\}^{-1} \ . \end{aligned} \quad (4.69)$$

From the expansion of the expression in the curly brackets we only get negative powers of α_1. Therefore, the term coming from the -1 in the numerator of (4.68) does not contribute any non-negative power of α_1, and we can neglect it if we later project on non-negative powers of α_1. Performing all geometric sums in that way we arrive at

$$\begin{aligned}\tilde{C}(\{s_j\},\{\alpha_j\},N) &= \left(\frac{2\pi}{k}\right)^r \prod_{j=1}^{r} (s_j!)^2(-1)^{s_j} \frac{(1+A_1)^{N-S}(1+B_1)^{-N+r+S}(1+C_1)^{r+S}}{\prod_{j=1}^{r}(1+C_j)(1+B_j)}\alpha_1^{-r} \\ &\times \prod_{j=1}^{r}\left\{1+\frac{1}{\alpha_1}\left((1+A_2)-\frac{(1+B_1)(1+C_1)(1+A_{j+1})}{(1+B_{j+1})(1+C_{j+1})}\right)\right\}^{-1}\Bigg|_{\substack{\gamma_1^{s_1}\cdots\gamma_r^{s_r}\\ \beta_1^{s_1}\cdots\beta_r^{s_r}\\ \text{non-neg. powers of }\alpha_1}} . \end{aligned} \tag{4.70}$$

Here, we have set $S = s_1 + \cdots + s_r$, and $A_{r+1} = B_{r+1} = C_{r+1} = 0$. When we expand in the auxiliary variables (with the expansion in α_1 done only after all other expansions have been performed), the coefficients are polynomials in N. Therefore we can replace N by λ in the expression (4.70) and obtain a generating function for the coefficients of the basis transformation for an arbitrary value of the parameter λ. When we do the explicit expansion (for details see Appendix B), we obtain our final result,

$$\begin{aligned} C(\{s_j\},\{p_j\},\lambda) \;=\; &\left(\frac{2\pi}{k}\right)^r \prod_{j=1}^{r}(s_j!)^2(-1)^{s_j} \sum_{\substack{r_j^{(1)},\ldots,r_j^{(6)}\geq 0\\ j=1,\ldots,r-1}} \sum_{a,b,c\geq 0} \sum_{\substack{b_1,\ldots,b_{r-1}\geq 0\\ c_1,\ldots,c_{r-1}\geq 0}} (-1)^{p_r-p_1-a} \\ &\times \binom{2S+p_r-p_1-a-b-c-\sum_{j=1}^{r-1}\left(b_j+c_j+r_j^{(1)}+\cdots+r_j^{(6)}\right)}{S+p_r-p_1-a-b-\sum_{j=1}^{r-1}\left(b_j+r_j^{(2)}+r_j^{(3)}+r_j^{(5)}+r_j^{(6)}\right)} \\ &\times \binom{S+p_r-p_1-a-b-\sum_{j=1}^{r-1}\left(b_j+r_j^{(2)}+r_j^{(3)}+r_j^{(5)}+r_j^{(6)}\right)}{S-b-\sum_{j=1}^{r-1}\left(b_j+r_j^{(2)}+r_j^{(5)}\right)} \\ &\times \binom{-\lambda+r+S-1+\sum_{j=1}^{r-1}r_j^{(3)}}{b}\binom{r+2S-1-b-\sum_{j=1}^{r-1}\left(b_j-r_j^{(3)}+r_j^{(5)}\right)}{c} \\ &\times \binom{\lambda-S}{r+2S+p_r-b-c-\sum_{j=1}^{r-1}\left(b_j+c_j\right)}\binom{r+2S+p_r-b-c-\sum_{j=1}^{r-1}\left(b_j+c_j\right)}{a} \\ &\times \prod_{j=1}^{r-1}\Bigg[(-1)^{r_j^{(3)}+r_j^{(4)}+r_j^{(5)}}\binom{r_j^{(1)}+\cdots+r_j^{(6)}}{r_j^{(1)}}\binom{r_j^{(2)}+\cdots+r_j^{(6)}}{r_j^{(2)}}\cdots\binom{r_j^{(5)}+r_j^{(6)}}{r_j^{(5)}} \\ &\qquad \times \binom{-1-r_j^{(1)}-\cdots-r_j^{(5)}}{b_j}\binom{-1-r_j^{(1)}-\cdots-r_j^{(4)}}{c_j}\binom{p_r-p_j-\sum_{i=j+1}^{r-1}r_i^{(3)}}{p_{j+1}-p_j} \\ &\qquad \times \binom{s_r+\sum_{i=j}^{r-1}\left(s_i-b_i-r_i^{(5)}\right)}{s_j}\binom{s_r+\sum_{i=j}^{r-1}\left(s_i-c_i-r_i^{(4)}\right)}{s_j}\Bigg] . \end{aligned} \tag{4.71}$$

The sum is finite, because the product of the first two binomial coefficients vanishes unless

$$a + \sum_{j=1}^{r-1} \left(r_j^{(3)} + r_j^{(6)} \right) \leq S + p_r - p_1 \tag{4.72}$$

$$b + \sum_{j=1}^{r-1} \left(b_j + r_j^{(2)} + r_j^{(5)} \right) \leq S \tag{4.73}$$

$$c + \sum_{j=1}^{r-1} \left(c_j + r_j^{(1)} + r_j^{(4)} \right) \leq S \ . \tag{4.74}$$

The coefficients (4.71) determine the basis transformation via (4.49), the first terms read

$$u_1 = \frac{2\pi}{k} \binom{\lambda+1}{3} \mathcal{W}_1 \tag{4.75a}$$

$$u_2 = \frac{8\pi}{k} \binom{\lambda+2}{5} \mathcal{W}_2 + \frac{4\pi}{k} \binom{\lambda+1}{4} \mathcal{W}_1^{(1)} \tag{4.75b}$$

$$\begin{aligned} u_3 = {} & \frac{72\pi}{k} \binom{\lambda+3}{7} \mathcal{W}_3 + \frac{24\pi}{k} \binom{\lambda+2}{6} \mathcal{W}_2^{(1)} + \frac{6\pi}{k} \binom{\lambda+1}{5} \mathcal{W}_1^{(2)} \\ & + \frac{4\pi^2}{3k^2} \binom{\lambda+1}{5} (5\lambda+7) \mathcal{W}_1 \mathcal{W}_1 \end{aligned} \tag{4.75c}$$

$$\begin{aligned} u_4 = {} & \frac{1152\pi}{k} \binom{\lambda+4}{9} \mathcal{W}_4 + \frac{288\pi}{k} \binom{\lambda+3}{8} \mathcal{W}_3^{(1)} + \frac{48\pi}{k} \binom{\lambda+2}{7} \mathcal{W}_2^{(2)} \\ & + \frac{8\pi}{k} \binom{\lambda+1}{6} \mathcal{W}_1^{(3)} + \frac{16\pi^2}{k^2} \binom{\lambda+2}{7} (7\lambda+13) \mathcal{W}_1 \mathcal{W}_2 \\ & + \frac{8\pi^2}{k^2} \binom{\lambda+1}{6} (5\lambda+7) \mathcal{W}_1 \mathcal{W}_1^{(1)} \end{aligned} \tag{4.75d}$$

$$\begin{aligned} u_5 = {} & \frac{28800\pi}{k} \binom{\lambda+5}{11} \mathcal{W}_5 + \frac{5760\pi}{k} \binom{\lambda+4}{10} \mathcal{W}_4^{(1)} + \frac{720\pi}{k} \binom{\lambda+3}{9} \mathcal{W}_3^{(2)} \\ & + \frac{80\pi}{k} \binom{\lambda+2}{8} \mathcal{W}_2^{(3)} + \frac{10\pi}{k} \binom{\lambda+1}{7} \mathcal{W}_1^{(4)} \\ & + \frac{64\pi^2}{5k^2} \binom{\lambda+2}{8} (44 + \lambda(34+7\lambda)) \mathcal{W}_2 \mathcal{W}_2 \\ & + \frac{576\pi^2}{k^2} \binom{\lambda+3}{9} (3\lambda+7) \mathcal{W}_1 \mathcal{W}_3 \\ & + \frac{64\pi^2}{k^2} \binom{\lambda+2}{8} (7\lambda+13) \left(\mathcal{W}_1^{(1)} \mathcal{W}_2 + \mathcal{W}_1 \mathcal{W}_2^{(1)} \right) \\ & + \frac{10\pi^2}{k^2} \binom{\lambda+1}{7} (7\lambda+10) \mathcal{W}_1^{(1)} \mathcal{W}_1^{(1)} + \frac{4\pi^2}{k^2} \binom{\lambda+1}{7} (21\lambda+29) \mathcal{W}_1^{(2)} \mathcal{W}_1 \\ & + \frac{8\pi^3}{63k^3} \binom{\lambda-1}{6} (3843 + \lambda(1717 + 7\lambda(51+5\lambda))) \mathcal{W}_1 \mathcal{W}_1 \mathcal{W}_1 \ . \end{aligned} \tag{4.75e}$$

This result reproduces[11] the basis transformation that was determined in [19] for the first few spins, if one uses the identification of fields given in (3.28) and divides our u's by a factor $-\frac{2\pi}{k}\binom{\lambda+1}{3}$.

5 Conclusions

In the absence of matter couplings, the interactions of higher-spin gauge fields in $D = 2+1$ can be described by Chern-Simons (CS) actions. The asymptotic symmetries of asymptotically-AdS solutions of the field equations are given by the $\mathcal{W}$-algebras that result from the Drinfeld-Sokolov (DS) reduction of the gauge algebras. In this paper we presented a procedure to compute the structure constants of all classical $\mathcal{W}$-algebras that can be obtained from the DS reduction of a (possibly infinite-dimensional) Lie algebra with a non-degenerate Killing form. We used it to discuss some general properties of the resulting $\mathcal{W}$-algebras, and we applied it to a class of infinite-dimensional Lie algebras, denoted by $hs[\lambda]$, that play an important role as higher-spin gauge algebras. A CS action with gauge algebra $hs[\lambda] \oplus hs[\lambda]$ describes indeed the coupling to gravity of a set of symmetric fields $\varphi_{\mu_1 \dots \mu_s}$ with ranks $s = 3, 4, \dots, \infty$. The field content is thus the same as in the gauge sector of Vasiliev's models [8]. The algebra of asymptotic symmetries is a classical centrally extended infinite-dimensional $\mathcal{W}$-algebra, that we denoted by $\mathcal{W}_\infty[\lambda]$. We determined its structure constants in (3.21) in a basis where all its generators $\mathcal{W}_i$ are Virasoro primaries, and where $\{\mathcal{W}_i, \mathcal{W}_j\}$ is a polynomial in the generators of the same order as the minimum of the labels i and j.

For integer $\lambda = N$ the Killing form degenerates and the CS action becomes that of a $sl(N,\mathbb{R}) \oplus sl(N,\mathbb{R})$ theory. The results presented here thus complete the analysis of the asymptotic symmetries of this class of higher-spin theories that we initiated in [14], and provide a closed formula for the structure constants of all classical $\mathcal{W}_N$ algebras in a Virasoro primary basis. For $\lambda = 1/2$ the gauge algebra coincides with the three-dimensional Fradkin-Vasiliev algebra [1, 56], and our formula provides the structure constants of the infinite-dimensional asymptotic $\mathcal{W}$-algebra of [13]. For generic λ it also reproduces the first few structure constants of $\mathcal{W}_\infty[\lambda]$ that were computed in [19]. We eventually presented a way to systematically relate our basis to the non-primary quadratic basis of [36], where $\mathcal{W}_\infty[\lambda]$ first appeared in the context of KP hierarchies.

To stress the relation between $hs[\lambda] \oplus hs[\lambda]$ CS theories and HS gauge theories, in Section 3.1.1 we also discussed how one could express the metric-like fields $\varphi_{\mu_1 \dots \mu_s}$ in terms of the vielbeine and spin connections entering the CS action. We were able to establish this relation up to $s = 5$, and it would be interesting to complete the identification following the lines we proposed. In fact, the relative simplicity of the models we considered could well shed light on the interplay between the disparate approaches that were proposed over the years to tackle the difficult analysis of higher-spin interactions in $D \geq 3+1$. Recovering

[11] Up to the $\mathcal{W}$-algebra automorphism in (3.24).

a non-linear completion of the metric-like formulation of Fronsdal or its generalisations (see e.g. [44]) out of the frame-like formulation of Vasiliev (see e.g. [3, 4]) could be a first important step in this direction. Moreover, this would also allow a reconsideration of our findings along the lines of the original Brown-Henneaux analysis of the asymptotic symmetries of Einstein gravity [15].

Another context where our results could stimulate further developments is the study of higher-spin realisations of the AdS/CFT correspondence. The asymptotic symmetries of the bulk theory should indeed correspond to global symmetries of the boundary CFT. It is still not clear whether the pure higher-spin gauge theories that we discussed admit a CFT dual, but the three-dimensional Vasiliev's models of [8] – describing the coupling to scalar matter of the same gauge fields we considered – are also built upon $hs[\lambda] \oplus hs[\lambda]$ gauge algebras. The suggestive asymptotic $\mathcal{W}$-symmetries of pure three-dimensional higher-spin gauge theories already led Gaberdiel and Gopakumar to conjecture a holographic duality between the large N limit of minimal models with $\mathcal{W}_N \times \mathcal{W}_N$ symmetry and three-dimensional Vasiliev's models [18]. Since $\mathcal{W}$-symmetries are the cornerstone of this conjecture, it would be important to reconsider our analysis – rather closely related to the CS formulation of the dynamics – in order to extend it to Vasiliev's models. In the meantime, we hope that our detailed description of $\mathcal{W}_\infty[\lambda]$ already help further quantitative checks of this proposal beyond those recently presented in [19, 20, 21].

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A Structure constants of $hs[\lambda]$

The structure constants that appear in the definition of the $\star$-product (3.7) can be expressed in terms of those defined in [30] (see also [19]) as

$$f_\lambda\left(\begin{array}{cc|c} k & \ell & i \\ m & n & m+n \end{array}\right) = q^{k+\ell-i-1}\, g^{k-1,\,\ell-1}_{k+\ell-i-1}(m,n;\lambda)\,, \tag{A.1}$$

where q is a normalisation factor that must be equal to $1/4$ for any finite λ, but that is useful to discuss the $\lambda \to \infty$ limit of $hs[\lambda]$ (see e.g. [19]). Moreover, in [30] it was proposed that the functions $g_k^{i,j}(m,n;\lambda)$ are given by

$$g_k^{i,j}(m,n;\lambda) = \frac{1}{2(k+1)!}\,\phi_k^{i,j}(\lambda)\,N_k^{i,j}(m,n)\,, \tag{A.2a}$$

$$N_k^{i,j}(m,n) = \sum_{p=1}^{k+1}(-1)^p\binom{k+1}{p}(2i+2-k)_p[2j+2-p]_{k-p+1}[i+1+m]_{k-p+1}[j+1+n]_p\,, \tag{A.2b}$$

$$\phi_k^{i,j}(\lambda) = \sum_{p=0}^{\lfloor k\rfloor}\prod_{q=1}^{p}\frac{[(2q-3)(2q+1)-4(\lambda^2-1)](k-2q+3)(k/2-q+1)}{q(2i-2q+3)(2j-2q+3)(2i+2j-2k+2q+3)}\,. \tag{A.2c}$$

Here $\lfloor k\rfloor$ denotes the integer part of k, while $(a)_n$ and $[a]_n$ denote respectively the ascending and descending Pochhammer symbols,

$$(a)_n := a(a+1)\ldots(a+n-1)\,, \tag{A.3a}$$

$$[a]_n := a(a-1)\ldots(a-n+1)\,. \tag{A.3b}$$

See also [30] for some alternative rewritings of eqs. (A.2). An expression for the $hs[\lambda]$ structure constants was also provided in [32] in terms of Clebsch-Gordan and generalised Wigner - $6j$ - symbols. See [29] and [33] for the proof that different values of λ lead to algebras which are not isomorphic.

B Structure constants of $\mathcal{W}_\infty[\lambda]$

In this appendix we prove the expression for the structure constants of $\mathcal{W}_\infty[\lambda]$ that we displayed in eq. (3.21). The involved steps closely follow those needed to compute the polynomials in the Virasoro generators that appear in the Drinfeld-Sokolov reduction of a generic algebra. They were presented in eq. (2.62) and we shall begin by proving it, before moving to eq. (3.21). We close this appendix with a proof of eq. (4.71) that, through eq. (4.49), determines the basis transformation needed to connect our result with the quadratic basis of [36].

Proof of eq. (2.62)

In order to prove eqs. (2.62) and (2.63) it is convenient to omit possible colour indices and to consider the gauge parameter $\lambda_+ = \epsilon(\theta)W_\ell^\ell$. Using the truncated covariant derivative of (2.61), each summand in (2.42) then takes the form

$$
\begin{aligned}
(-DRL_-)^n D\lambda_+ \;=\; (-1)^n \Bigg\{ & \partial^{n+1}\epsilon\,(RL_-)^n \\
+ & \sum_{r=1}^{n+1} \left(\frac{2\pi}{k}\right)^r \sum_{i_1=0}^{n-r+1} \sum_{i_2=0}^{(n-r+1)-i_1} \cdots \sum_{i_r=0}^{(n-r+1)-\sum_1^{r-1} i_t} \partial^{i_1}\Big(\mathcal{L}\,\partial^{i_2}\Big(\ldots\Big(\mathcal{L}\,\partial^{i_r}\Big(\mathcal{L}\,\partial^{(n-r+1)-\sum_1^r i_t}\epsilon\Big)\Big)\Big)\Big) \\
\times & (RL_-)^{i_1} L_-(RL_-)^{i_2+1} \ldots L_-(RL_-)^{i_r+1} L_-(RL_-)^{(n-r+1)-\sum_1^r i_t} \Bigg\} W_\ell^\ell\,,
\end{aligned}
\tag{B.1}
$$

where we also omitted the θ dependence in both ordinary and covariant derivatives. One can check this expression by recursion. To this end, let us compute separately the two summands in

$$
(-DRL_-)^{n+1} D\lambda_+ \;=\; -\left\{\partial + \frac{2\pi}{k}\,\mathcal{L}\,L_-\right\}(RL_-)(-DRL_-)^n D\lambda_+\,. \tag{B.2}
$$

The first one reads

$$
\begin{aligned}
-\,\partial(RL_-)(-DRL_-)^n D\lambda_+ \;=\; (-1)^{n+1} \Bigg\{ & \partial^{n+2}\epsilon\,(RL_-)^{n+1} \\
+ & \sum_{r=1}^{n+1} \left(\frac{2\pi}{k}\right)^r \sum_{i_1=1}^{n-r+2} \sum_{i_2=0}^{(n-r+2)-i_1} \cdots \sum_{i_r=0}^{(n-r+2)-\sum_1^{r-1} i_t} \partial^{i_1}\Big(\mathcal{L}\,\partial^{i_2}\Big(\ldots\Big(\mathcal{L}\,\partial^{i_r}\Big(\mathcal{L}\,\partial^{(n-r+2)-\sum_1^r i_t}\epsilon\Big)\Big)\Big)\Big) \\
\times & (RL_-)^{i_1} L_-(RL_-)^{i_2+1} \ldots L_-(RL_-)^{i_r+1} L_-(RL_-)^{(n-r+2)-\sum_1^r i_t} \Bigg\} W_\ell^\ell\,.
\end{aligned}
\tag{B.3}
$$

In order to rebuild (B.1) with $n \to n+1$ two contributions are missing: the terms with $i_1 = 0$ in the first sum over derivatives and the term with $r = n+2$ in the sum over the order of non-linearity. They come from the second summand in (B.2) that can be cast in the form

$$
\begin{aligned}
& -\,\frac{2\pi}{k}\,\mathcal{L}\,L_-(RL_-)(-DRL_-)^n D\lambda_+ \\
& = (-1)^{n+1} \sum_{r=1}^{n+2} \left(\frac{2\pi}{k}\right)^r \sum_{i_2=0}^{n-r+2} \cdots \sum_{i_r=0}^{(n-r+2)-\sum_2^{r-1} i_t} \mathcal{L}\,\partial^{i_2}\Big(\ldots\Big(\mathcal{L}\,\partial^{i_r}\Big(\mathcal{L}\,\partial^{(n-r+2)-\sum_2^r i_t}\epsilon\Big)\Big)\Big) \\
& \times\, L_-(RL_-)^{i_2+1} \ldots L_-(RL_-)^{i_r+1} L_-(RL_-)^{(n-r+2)-\sum_2^r i_t} \Bigg\} W_\ell^\ell
\end{aligned}
\tag{B.4}
$$

and thus gives all terms with $i_1 = 0$. This suffices to conclude because the term with $r = n+2$ does not contain derivatives.

The next step is the elimination of the operators in (B.1) using

$$
L_-\,W_{\ell-p}^\ell \;=\; -\,(2\ell - p)\,W_{\ell-(p+1)}^\ell\,, \tag{B.5a}
$$

$$
(RL_-)^i\,W_{\ell-p}^\ell \;=\; \frac{p!}{(p+i)!}\,W_{\ell-(p+i)}^\ell\,. \tag{B.5b}
$$

Both identities follow from the commutators (2.12). In particular, one can get (B.5b) combining $W^\ell_{\ell-p} = (n+1)^{-1} L_+ W^\ell_{\ell-(p+1)}$ with (2.38). Taking advantage of (B.5) we get

$$\begin{aligned}
&(-1)^r (n+r)!\,(RL_-)^{i_1} L_-(RL_-)^{i_2+1} \dots\, L_-(RL_-)^{i_r+1} L_-(RL_-)^{(n-r+1)-\sum_1^r i_t}\, W^\ell_\ell \qquad \text{(B.6)}\\
&= \prod_{s=0}^{r-1} \left((n-r+1) - \sum_{t=1}^{r-s} i_t + 2s + 1 \right) \left(2\ell - (n-r+1) + \sum_{t=1}^{r-s} i_t - 2s \right) W^\ell_{\ell-(n+r)} \,.
\end{aligned}$$

In order to make contact with eq. (2.62) we can now evaluate the derivatives in (B.1),

$$\begin{aligned}
&\partial^{i_1}\Big(\mathcal{L}\,\partial^{i_2}\Big(\dots\Big(\mathcal{L}\,\partial^{i_r}\Big(\mathcal{L}\,\partial^{(n-r+1)-\sum_1^r i_t}\epsilon\Big)\Big)\Big)\Big) \\
&= \sum_{p_1=0}^{i_1} \sum_{p_2=0}^{i_1+i_2-p_1} \dots \sum_{p_r=0}^{\sum_1^r i_k - \sum_1^{r-1} p_t} \prod_{s=1}^{r} \binom{\sum_1^s i_t - \sum_1^{s-1} p_t}{p_s} \mathcal{L}^{(p_1)} \dots \mathcal{L}^{(p_r)} \epsilon^{(n-r+1-\sum_1^r p_t)} \,, \qquad \text{(B.7)}
\end{aligned}$$

and eventually exchange the sums over i's and p's. Here, as in the main body of the text, an exponent between parentheses denotes the action of the corresponding number of derivatives on the field. The rewriting (B.7) leads to

$$\begin{aligned}
&(-DRL_-)^n D\lambda_+ = \frac{(-1)^n}{n!}\,\epsilon^{(n+1)}\,W^\ell_{\ell-n} \;+\; \sum_{r=1}^{n+1} \frac{(-1)^{n+r}}{(n+r)!} \left(\frac{2\pi}{k}\right)^r \\
&\times \sum_{p_1=0}^{n-r+1} \dots \sum_{p_r=0}^{(n-r+1)-\sum_1^{r-1} p_t} \widetilde{C}[n,r]_{p_1\dots p_r}\, \mathcal{L}^{(p_1)} \dots \mathcal{L}^{(p_r)} \epsilon^{(n-r+1-\sum_1^r p_t)}\; W^\ell_{\ell-(n+r)} \,. \qquad \text{(B.8)}
\end{aligned}$$

The coefficients $\widetilde{C}[n,r]_{p_1\dots p_r}$ are defined by

$$\begin{aligned}
&\widetilde{C}[n,r]_{p_1\dots p_r} = \sum_{i_1=p_1}^{n-r+1} \sum_{i_2=\langle(p_1+p_2)-i_1\rangle_+}^{(n-r+1)-i_1} \dots \sum_{i_r=\langle\sum_1^r p_t - \sum_1^{r-1} i_t\rangle_+}^{(n+r-1)-\sum_1^{r-1} i_t} \prod_{s=1}^{r} \binom{\sum_1^s i_t - \sum_1^{s-1} p_t}{p_s} \qquad \text{(B.9)}\\
&\times \left((n-r+1) - \sum_{t=1}^{r-s+1} i_t + 2s - 1 \right) \left(2\ell - (n-r+1) + \sum_{t=1}^{r-s+1} i_t - 2(s-1) \right),
\end{aligned}$$

where, as in (2.63), we introduced $\langle a \rangle_+ = \max(0,a)$.

To conclude we have to apply the projector P_- to (B.8) and to sum over n as in (2.42). The projection selects the terms in $W^\ell_{-\ell}$ out of (B.8), thus forcing $(n+r) = 2\ell$. We recovered in this way the upper bound $n \leq 2\ell$ already discussed after (2.42). The condition $r \leq n+1$ also induces a lower bound on the number of terms that contribute to (2.42). For $\lambda_+ = \epsilon(\theta) W^\ell_\ell$ one actually has

$$\delta_\lambda a(\theta) \;=\; \sum_{n=\lfloor \ell \rfloor}^{2\ell} P_-\,(-DRL_-)^n\, D\epsilon\, W^\ell_\ell \,, \qquad \text{(B.10)}$$

where $\lfloor \ell \rfloor$ is the integer part of ℓ. Reorganising (B.10) as a sum over the order of non-linearity (e.g. over $r = 2\ell - n$) eventually leads to (2.62).

Proof of eq. (3.21)

The first step of the proof is the natural extension of (B.1). Also in this case we consider the gauge variation induced by a gauge parameter with a definite L_0 eigenvalue, say $\lambda_+ = \epsilon(\theta) W_i^i$. In order to proceed it is convenient to introduce the operators

$$w_a\, x := [\, W^a_{-a}\,,\, x\,] \quad \text{for } x \in hs[\lambda]\,, \tag{B.11}$$

that enable one to cast the $hs[\lambda]$-covariant derivative in the form

$$D \,=\, \partial \,+\, \frac{2\pi}{k} \sum_{a\,=\,1}^{\infty} \mathcal{W}_a(\theta)\, w_a\,. \tag{B.12}$$

As usual we identified $\mathcal{L}$ with $\mathcal{W}_1$ and L_- with w_1. Following (B.3) and (B.4) one can then prove by recursion that

$$\begin{aligned}
\delta_\lambda a \,=\, P_- \sum_{n\,=\,0}^{2i} (-DRL_-)^n D\lambda_+ \,=\, P_- \sum_{n\,=\,0}^{2i} (-1)^n \Bigg\{ \partial^{n+1}\epsilon\,(RL_-)^n + \sum_{r\,=\,1}^{n+1} \sum_{a_1\,=\,1}^{\infty} \cdots \sum_{a_r\,=\,1}^{\infty} \\
\times \left(\frac{2\pi}{k}\right)^r \sum_{q_1\,=\,0}^{n-r+1} \cdots \sum_{q_r\,=\,0}^{(n-r+1)-\sum_1^{r-1} q_t} \partial^{q_1}\Big(\mathcal{W}_{a_1}\partial^{q_2}\Big(\ldots\Big(\mathcal{W}_{a_{r-1}}\partial^{q_r}\Big(\mathcal{W}_{a_r}\partial^{(n-r+1)-\sum_1^r q_t}\epsilon\Big)\Big)\Big)\Big) \\
\times\, (RL_-)^{q_1}\, w_{a_1}(RL_-)^{q_2+1} \ldots\, w_{a_{r-1}}(RL_-)^{q_r+1}\, w_{a_r}(RL_-)^{(n-r+1)-\sum_1^r q_t} \Bigg\} W_i^i\,.
\end{aligned} \tag{B.13}$$

The first contribution in (B.13) gives the central terms that we already discussed in detail in Section 2.4.2. We shall thus often omit it in the following.

The L_0 eigenvalue of the $hs[\lambda]$ generator resulting from the application of the chain of operators on W_i^i can be read off quite easily from (B.13). It is

$$m \,=\, -\left(i \,-\, \sum_{t\,=\,1}^{r} a_t \,-\, n\right) \tag{B.14}$$

since each w_{a_t} insertion lowers it by $-a_t$ and for each n there are n operators (RL_-). On the other hand, as already discussed at the beginning of Section 3.2, the maximum value of the final spin is

$$\ell \,=\, \sum_{t\,=\,1}^{r} a_t \,+\, i \,-\, r\,. \tag{B.15}$$

Since m must satisfy $|m| \leq \ell$, the variable r is bound to obey $r \leq \min(n+1, 2i-n)$. Reversing the order of the sums over r and n this leads to

$$\delta_\lambda a \,=\, \frac{\epsilon^{(2i+1)}}{(2i)!}\, W^i_{-i} \,+\, P_- \sum_{r\,=\,1}^{i} \left(\frac{2\pi}{k}\right)^r \sum_{a_1\,=\,1}^{\infty} \cdots \sum_{a_r\,=\,1}^{\infty} \sum_{n\,=\,r-1}^{2i-r} (-1)^n \Big\{ \ldots \Big\}\, W_i^i\,, \tag{B.16}$$

where the terms between braces have the same structure as those appearing in the second and in the third line of (B.13). Note that in getting (B.16) we already took into account that $hs[\lambda]$ only contains generators with integer spin.

We can now evaluate the action of the operators on W_i^i using (3.8) and (B.5b): with the identification $b_0 = i$ we obtain

$$
\begin{aligned}
&(RL_-)^{q_1}\, w_{a_1} (RL_-)^{q_2+1} \dots\, w_{a_{r-1}} (RL_-)^{q_r+1}\, w_{a_r} (RL_-)^{(n-r+1)-\sum_1^r q_t}\, W_i^i \\
&= \frac{1}{(b_r + \sum_1^r a_t - i + n)!} \sum_{\substack{b_1 = |a_r - b_0|+1 \\ a_r+b_0+b_1 \text{ odd}}}^{a_r+b_0-1} \ \sum_{\substack{b_2 = |a_{r-1} - b_1|+1 \\ a_{r-1}+b_1+b_2 \text{ odd}}}^{a_{r-1}+b_1-1} \dots \sum_{\substack{b_r = |a_1 - b_{r-1}|+1 \\ a_1+b_{r-1}+b_r \text{ odd}}}^{a_1+b_{r-1}-1} \\
&\times \prod_{s=1}^{r} \frac{(n-i+b_s+\sum_{r-s+1}^r a_t - \sum_1^{r-s+1} q_t - r + s)!}{(n-i+b_{s-1}+\sum_{r-s+2}^r a_t - \sum_1^{r-s+1} q_t - r + s)!} \\
&\times f_\lambda \left(\begin{matrix} a_{r-s+1} & b_{s-1} \\ -a_{r-s+1} & i - n - \sum_{r-s+2}^r a_t + \sum_1^{r-s+1} q_t + r - s \end{matrix} \middle| \begin{matrix} b_s \\ \dots \end{matrix} \right) W^{b_r}_{i-n-\sum_1^r a_t} .
\end{aligned}
\tag{B.17}
$$

The final spin is b_r. As a result, we should bring the sum over b_r in the first position among all other summations. In this fashion we can eventually select a particular b_r to read the gauge variation of $\mathcal{W}_{b_r}$. Due to its selected role, in the following we shall define $j = b_r$. Bringing the sum over j in the first position casts the summations over b's in (B.17) in the form

$$
\sum_{\substack{j = \max(1\,,\, M(r,i)) \\ \sum_1^r a_t + i + r \text{ even}}}^{\sum_1^r a_t + i - r} \ \sum_{\substack{b_1 = \max(|a_r - b_0|+1\,,\, M(r-1,j)) \\ a_r+b_0+b_1 \text{ even}}}^{\min\left(a_r+b_0-1\,,\, \sum_1^{r-1} a_t + j - r + 1\right)} \dots \sum_{\substack{b_{r-1} = \max(|a_2 - b_{r-2}|+1\,,\, M(1,j)) \\ a_2+b_{r-1}+b_r \text{ even}}}^{\min(a_2+b_{r-2}-1\,,\, a_1+j-1)}
\tag{B.18}
$$

with

$$
M(s,\ell) :=\ 2 \max\left(\{a_t\}_{t=1}^s\,,\ \ell\,\right) - \sum_{t=1}^{s} a_t - \ell + s\,.
\tag{B.19}
$$

The sum over j commutes with all other summations with the exception of those on a_k. Before performing this last exchange let us evaluate the projector P_-. It forces the L_0 eigenvalue of the generator appearing in (B.17) to coincide with j. Therefore it imposes

$$
n \ =\ (i+j)\ -\ \sum_{t=1}^{r} a_t\,.
\tag{B.20}
$$

Both (B.20) and the extrema of the sum over j only depend on the sum of all a_t. Let us thus introduce $L = \sum_1^r a_k$. For $L \leq i$ the lower bound of the sum over j is given by (B.18) as $j \geq i - L + r$. On the other hand, for $L > i$ the lower bound comes from (B.20) since $L - i + r - 1 > M(r,i)$. In conclusion we are led to consider

$$
\sum_{L=1}^{\infty} \sum_{\{a_t\}} \sum_{\substack{j = |L-i|+r \\ i+j+L+r \text{ even}}}^{L+i-r} \ =\ \sum_{j=1}^{\infty} \sum_{\substack{L = |i-j|+r \\ i+j+L+r \text{ even}}}^{i+j-r} \sum_{\{a_t\}}
\tag{B.21}
$$

where the multiple sum over the a's must be such that $\sum_1^r a_t = L$. Substituting everywhere (B.20) and expanding the derivatives as in (B.7) eventually lead to (3.21).

Proof of eq. (4.71)

We start from (4.70), and first expand the factors of the last product,

$$\begin{aligned}
&\left\{1+\frac{1}{\alpha_1}\left((1+A_2)-\frac{(1+B_1)(1+C_1)(1+A_{j+1})}{(1+B_{j+1})(1+C_{j+1})}\right)\right\}^{-1}\\
&\quad=\left\{1-\frac{1}{\alpha_1(1+B_{j+1})(1+C_{j+1})}\Big[C_1+B_1(1+C_1)+A_{j+1}(1+B_1)(1+C_1)\right.\\
&\qquad\qquad\left.-C_{j+1}-B_{j+1}(1+C_{j+1})-A_2(1+B_{j+1})(1+C_{j+1})\Big]\right\}^{-1} \qquad \text{(B.22)}\\
&\quad=\sum_{r_j^{(1)},\ldots,r_j^{(6)}\geq 0}\left(\alpha_1(1+B_{j+1})(1+C_{j+1})\right)^{-r_j^{(1)}-\cdots-r_j^{(6)}}\\
&\qquad\times\binom{r_j^{(1)}+\cdots+r_j^{(6)}}{r_j^{(1)}}\binom{r_j^{(2)}+\cdots+r_j^{(6)}}{r_j^{(2)}}\cdots\binom{r_j^{(5)}+r_j^{(6)}}{r_j^{(5)}}\\
&\qquad\times[C_1]^{r_j^{(1)}}[B_1(1+C_1)]^{r_j^{(2)}}[A_{j+1}(1+B_1)(1+C_1)]^{r_j^{(3)}}\\
&\qquad\times[-C_{j+1}]^{r_j^{(4)}}[-B_{j+1}(1+C_{j+1})]^{r_j^{(5)}}[-A_2(1+B_{j+1})(1+C_{j+1})]^{r_j^{(6)}}\ . \qquad \text{(B.23)}
\end{aligned}$$

For $j=r$ this expression simplifies to

$$\begin{aligned}
&\left\{1+\frac{1}{\alpha_1}\left((1+A_2)-(1+B_1)(1+C_1)\right)\right\}^{-1}\\
&\quad=\sum_{r_r^{(1)},r_r^{(2)},r_r^{(6)}\geq 0}\alpha_1^{-r_r^{(1)}-r_r^{(2)}-r_r^{(6)}}\\
&\qquad\times\binom{r_r^{(1)}+r_r^{(2)}+r_r^{(6)}}{r_r^{(1)}}\binom{r_r^{(2)}+r_r^{(6)}}{r_r^{(2)}}[C_1]^{r_r^{(1)}}[B_1(1+C_1)]^{r_r^{(2)}}[-A_2]^{r_r^{(6)}}\ . \qquad \text{(B.24)}
\end{aligned}$$

In the next step we expand the powers of $(1+A_1)$, $(1+B_1)$ and $(1+C_1)$ with summation variables a', b, c, respectively, and the powers of $(1+B_{j+1})$ and $(1+C_{j+1})$ with summation variables b_j, c_j with $j = 1, \ldots, r-1$. We obtain

$$
\begin{aligned}
\tilde{C}(\{s_j\}, \{\alpha_j\}, N) \;=\; & \left(\frac{2\pi}{k}\right)^r \prod_{j=1}^{r} (s_j!)^2 (-1)^{s_j} \sum_{\substack{r_j^{(1)},\ldots,r_j^{(6)}\geq 0 \\ j=1,\ldots,r-1}} \; \sum_{r_r^{(1)},r_r^{(2)},r_r^{(6)}} \; \sum_{a',b,c\geq 0} \; \sum_{\substack{b_1,\ldots,b_{r-1}\geq 0 \\ c_1,\ldots,c_{r-1}\geq 0}} \\
& \times \prod_{j=1}^{r-1} \left[(-1)^{r_j^{(4)}+r_j^{(5)}+r_j^{(6)}} \binom{r_j^{(1)}+\cdots+r_j^{(6)}}{r_j^{(1)}} \binom{r_j^{(2)}+\cdots+r_j^{(6)}}{r_j^{(2)}} \cdots \binom{r_j^{(5)}+r_j^{(6)}}{r_j^{(5)}} \right. \\
& \qquad\qquad \left. \times \binom{-1-r_j^{(1)}-\cdots-r_j^{(5)}}{b_j} \binom{-1-r_j^{(1)}-\cdots-r_j^{(4)}}{c_j} \right] \\
& \times (-1)^{r_r^{(6)}} \binom{r_r^{(1)}+r_r^{(2)}+r_r^{(6)}}{r_r^{(1)}} \binom{r_r^{(2)}+r_r^{(6)}}{r_r^{(2)}} \binom{\lambda - S}{a'} \binom{-\lambda + r + S - 1 + \sum_{j=1}^{r-1} r_j^{(3)}}{b} \\
& \times \binom{r + S - 1 + \sum_{j=1}^{r-1}\left(r_j^{(2)} + r_j^{(3)}\right) + r_r^{(2)}}{c} \alpha_1^{-r-r_r^{(1)}-r_r^{(2)}-r_r^{(6)}-\sum_{j=1}^{r-1}\left(r_j^{(1)}+\cdots+r_j^{(6)}\right)} \\
& \times A_1^{a'} B_1^{b+r_r^{(2)}+\sum_{j=1}^{r-1} r_j^{(2)}} C_1^{c+r_r^{(1)}+\sum_{j=1}^{r-1} r_j^{(1)}} A_2^{r_r^{(6)}+\sum_{j=1}^{r-1} r_j^{(6)}} \\
& \times \prod_{j=1}^{r-1} \left(A_{j+1}^{r_j^{(3)}} B_{j+1}^{b_j+r_j^{(5)}} C_{j+1}^{c_j+r_j^{(4)}} \right) \Bigg|_{\substack{\gamma_1^{s_1}\cdots\gamma_r^{s_r} \\ \beta_1^{s_1}\cdots\beta_r^{s_r} \\ \text{non-neg. powers of } \alpha_1}} .
\end{aligned}
\tag{B.25}
$$

We want to extract the coefficients with powers $\beta_j^{s_j}$ and $\gamma_j^{s_j}$. In particular the sum of the exponents of the B_j and of the C_j have to match $S = s_1 + \cdots + s_r$. This fixes the summation variables $r_r^{(1)}$ and $r_r^{(2)}$ to

$$
r_r^{(1)} = S - c - \sum_{j=1}^{r-1} \left(c_j + r_j^{(1)} + r_j^{(4)}\right) \tag{B.26}
$$

$$
r_r^{(2)} = S - b - \sum_{j=1}^{r-1} \left(b_j + r_j^{(2)} + r_j^{(5)}\right) . \tag{B.27}
$$

In the next step we expand the power of $B_1 = \beta_1 + B_2$,

$$
B_1^{S-\sum_{j=1}^{r-1}\left(b_j+r_j^{(5)}\right)} \Big|_{\beta_1^{s_1}} = \binom{S - \sum_{j=1}^{r-1}\left(b_j + r_j^{(5)}\right)}{s_1} B_2^{S-s_1-\sum_{j=1}^{r-1}\left(b_j+r_j^{(5)}\right)} , \tag{B.28}
$$

then the power of $B_2 = \beta_2 + B_3$, and so on, similarly for the C_j.

To extract the coefficients $C(\{s\}, \{p\}, \lambda)$ we also have to project to powers $\alpha_j^{p_j - p_{j-1}}$. The sum of the exponents of the A_j plus the exponent of α_1 has to match p_r, this can be used to fix a' to

$$
a' = r + 2S + p_r - b - c - \sum_{j=1}^{r-1} (b_j + c_j) . \tag{B.29}
$$

Now we expand the power of $A_1 = \alpha_1 + A_2$,

$$A_1^{r+2S+p_r-b-c-\sum_{j=1}^{r-1}(b_j+c_j)} = \sum_{a\geq 0} \binom{r+2S+p_r-b-c-\sum_{j=1}^{r-1}(b_j+c_j)}{a} \times \alpha_1^{r+2S+p_r-a-b-c-\sum_{j=1}^{r-1}(b_j+c_j)} A_2^a\,. \quad \text{(B.30)}$$

The sum of exponents of $A_2, \ldots, A_r$ has to match $p_r - p_1$ leading to the condition

$$r_r^{(6)} = p_r - p_1 - a - \sum_{j=1}^{r-1}\left(r_j^{(3)} + r_j^{(6)}\right)\,, \quad \text{(B.31)}$$

which fixes $r_r^{(6)}$. In the next step the power of A_2 is expanded as

$$A_2^{p_r-p_1-\sum_{i=2}^{r-1} r_i^{(3)}}\Big|_{\alpha_2^{p_2-p_1}} = \binom{p_r-p_1-\sum_{i=2}^{r-1} r_i^{(3)}}{p_2-p_1} A_3^{p_r-p_2-\sum_{i=2}^{r-1} r_i^{(3)}}\,, \quad \text{(B.32)}$$

then the power of A_3, and so on. This leads to the final result (4.71).

C Poisson brackets of $\mathcal{W}_3^{(2)}$, $\mathcal{W}_4^{(2)}$ and $\mathcal{W}_\infty[\lambda]$

In this appendix we present the Poisson brackets of the two examples of $\mathcal{W}$-algebras that we discussed in Section 2.5. We also present the Poisson brackets of $\mathcal{W}_\infty[\lambda]$ for fields of weight $\ell \leq 3$. Imposing $\lambda = 3$ and rescaling them by $N_2(3)$ they give the $\mathcal{W}_3$ algebra, while imposing $\lambda = 4$ and rescaling them by $N_3(4)$ they give the $\mathcal{W}_4$ algebra. In this fashion they allow to compare the $\mathcal{W}_3^{(2)}$ and $\mathcal{W}_4^{(2)}$ algebras with their counterparts associated to a principal $sl(2)$ embedding.

Poisson structure of $\mathcal{W}_3^{(2)}$

Here we present the full $\mathcal{W}_3^{(2)}$ algebra of [55] before implementing the shift (2.75), with the convention that all fields on the right-hand side depend on θ' and $\delta'(\theta-\theta') \equiv \partial_\theta\delta(\theta-\theta')$. Exponents between square brackets denote colour indices, while exponents between parentheses specify the number of derivatives acting on the corresponding object.

$$\left\{\mathcal{W}_0(\theta)\,,\,\mathcal{W}_0(\theta')\right\} = \frac{3k}{4\pi}\,\delta'(\theta-\theta')\,, \quad \text{(C.1a)}$$

$$\left\{\mathcal{W}_0(\theta)\,,\,\mathcal{W}_{\frac{1}{2}}^{[a]}(\theta')\right\} = a\,\frac{3}{2}\,\delta(\theta-\theta')\,\mathcal{W}_{\frac{1}{2}}^{[a]}\,, \quad \text{(C.1b)}$$

$$\left\{\mathcal{W}_0(\theta)\,,\,\mathcal{L}(\theta')\right\} = 0\,, \quad \text{(C.1c)}$$

$$\left\{\mathcal{L}(\theta)\,,\,\mathcal{L}(\theta')\right\} = \delta(\theta-\theta')\,\mathcal{L}' - 2\,\delta'(\theta-\theta')\,\mathcal{L} - \frac{k}{4\pi}\,\delta^{(3)}(\theta-\theta')\,, \quad \text{(C.1d)}$$

$$\left\{\mathcal{L}(\theta)\,,\,\mathcal{W}_{\frac{1}{2}}^{[a]}(\theta')\right\} = \delta(\theta-\theta')\,\mathcal{W}_{\frac{1}{2}}^{[a]\,\prime} - \frac{3}{2}\,\delta'(\theta-\theta')\,\mathcal{W}_{\frac{1}{2}}^{[a]} + a\,\frac{2\pi}{k}\,\delta(\theta-\theta')\,\mathcal{W}_0\mathcal{W}_{\frac{1}{2}}^{[a]}\,, \quad \text{(C.1e)}$$

$$\Big\{\mathcal{W}^{[a]}_{\frac{1}{2}}(\theta)\,,\,\mathcal{W}^{[b]}_{\frac{1}{2}}(\theta')\Big\} \;=\; \delta_{a+b\,,0}\Bigg(-\,\frac{k}{2\pi}\,a\,\delta''(\theta-\theta')\;-\;a\,\delta(\theta-\theta')\,\mathcal{L}$$
$$+\;\delta(\theta-\theta')\,\mathcal{W}_0'\;-\;2\,\delta'(\theta-\theta')\,\mathcal{W}_0\;-\;a\,\frac{2\pi}{k}\,\delta(\theta-\theta')\,\mathcal{W}_0\mathcal{W}_0\Bigg)\,, \tag{C.1f}$$

This algebra is written in a non-primary basis, as can be seen explicitly in (C.1c) and (C.1e). Redefining $\mathcal{L}$ as in (2.69), the correct transformation in (C.1c) is recovered and the distracting term in (C.1e) is removed as discussed in Section 2.5.1.

Poisson structure of $\mathcal{W}_4^{(2)}$

Here we present the full $\mathcal{W}_4^{(2)}$ algebra after the shift (2.75). As before, all fields appearing on the right-hand side are functions of θ' and $\delta'(\theta-\theta')\equiv\partial_\theta\delta(\theta-\theta')$.

$$\Big\{\widehat{\mathcal{L}}(\theta)\,,\,\widehat{\mathcal{L}}(\theta')\Big\} \;=\; \delta(\theta-\theta')\,\widehat{\mathcal{L}}'\;-\;2\,\delta'(\theta-\theta')\,\widehat{\mathcal{L}}\;-\;\frac{k}{4\pi}\,\delta^{(3)}(\theta-\theta')\,, \tag{C.2a}$$

$$\Big\{\widehat{\mathcal{L}}(\theta)\,,\,\mathcal{W}^{[a]}_{\ell}(\theta')\Big\} \;=\; \delta(\theta-\theta')\,\mathcal{W}^{[a]\,\prime}_{\ell}-\;(\ell+1)\,\delta'(\theta-\theta')\,\mathcal{W}^{[a]}_{\ell}\,,\quad \ell=0,1,2\,, \tag{C.2b}$$

$$\Big\{\mathcal{W}_0(\theta)\,,\,\mathcal{W}_0(\theta')\Big\} \;=\; \frac{8k}{3\pi}\,\delta'(\theta-\theta')\,, \tag{C.2c}$$

$$\Big\{\mathcal{W}_0(\theta)\,,\,\mathcal{W}^{[a]}_1(\theta')\Big\} \;=\; a\,\frac{16}{3}\,\delta(\theta-\theta')\,\mathcal{W}^{[a]}_1\,, \tag{C.2d}$$

$$\Big\{\mathcal{W}^{[a]}_1(\theta)\,,\,\mathcal{W}^{[b]}_1(\theta')\Big\} \;=\; \delta_{a+b\,,0}\Bigg(\frac{k}{4\pi}\,\delta^{(3)}(\theta-\theta')$$
$$-\;\frac{3a}{2}\,\delta''(\theta-\theta')\,\mathcal{W}_0\;+\;\frac{3a}{2}\,\delta'(\theta-\theta')\,\mathcal{W}_0'\;-\;\frac{a}{2}\,\delta(\theta-\theta')\,\mathcal{W}_0''$$
$$+\;2\,\delta'(\theta-\theta')\,\widehat{\mathcal{L}}\;-\;\delta(\theta-\theta')\,\widehat{\mathcal{L}}'\;-\;2\,a\,\delta(\theta-\theta')\,\mathcal{W}_2$$
$$-\;\frac{2\pi}{k}\,2\,a\,\delta(\theta-\theta')\,\mathcal{W}_0\widehat{\mathcal{L}}\;+\;\frac{2\pi}{k}\,\frac{21}{16}\,\delta'(\theta-\theta')\,\mathcal{W}_0\mathcal{W}_0$$
$$-\;\frac{2\pi}{k}\,\frac{27}{16}\,\delta(\theta-\theta')\,\mathcal{W}_0\mathcal{W}_0'\;-\;\left(\frac{2\pi}{k}\right)^2\frac{11a}{16}\,\delta(\theta-\theta')\,\mathcal{W}_0\mathcal{W}_0\mathcal{W}_0\Bigg)\,, \tag{C.2e}$$

$$\Big\{\mathcal{W}^{[a]}_1(\theta)\,,\,\mathcal{W}_2(\theta')\Big\} \;=$$
$$-\,\frac{5a}{3}\,\delta''(\theta-\theta')\,\mathcal{W}^{[-a]}_1\;+\;\frac{5a}{6}\,\delta'(\theta-\theta')\,\mathcal{W}^{[-a]\,\prime}_1\;+\;\frac{a}{6}\,\delta(\theta-\theta')\,\mathcal{W}^{[-a]\,\prime\prime}_1$$
$$+\;\frac{2\pi}{k}\,a\,\frac{8}{3}\,\delta(\theta-\theta')\,\widehat{\mathcal{L}}\,\mathcal{W}^{[-a]}_1\;-\;\frac{2\pi}{k}\,\frac{5}{2}\,\delta'(\theta-\theta')\,\mathcal{W}_0\mathcal{W}^{[-a]}_1$$
$$+\;\frac{2\pi}{k}\,\frac{3}{2}\,\delta(\theta-\theta')\,\mathcal{W}_0'\mathcal{W}^{[-a]}_1\;+\;\frac{2\pi}{k}\,\frac{a}{2}\,\delta(\theta-\theta')\,\mathcal{W}_0\mathcal{W}^{[-a]\,\prime}_1$$
$$+\;\left(\frac{2\pi}{k}\right)^2\frac{5a}{4}\,\delta(\theta-\theta')\,\mathcal{W}_0\mathcal{W}_0\mathcal{W}^{[-a]}_1\,, \tag{C.2f}$$

$$
\begin{aligned}
\Big\{\mathcal{W}_2(\theta)\,,\,\mathcal{W}_2(\theta')\Big\} \;=\;& \frac{k}{48\pi}\,\delta^{(5)}(\theta-\theta') \\
&+\;\frac{5}{6}\,\delta^{(3)}(\theta-\theta')\,\widehat{\mathcal{L}}\;-\;\frac{5}{4}\,\delta''(\theta-\theta')\,\widehat{\mathcal{L}}'\;+\;\frac{3}{4}\,\delta'(\theta-\theta')\,\widehat{\mathcal{L}}''\;-\;\frac{1}{6}\,\delta(\theta-\theta')\,\widehat{\mathcal{L}}^{(3)} \\
&+\;\frac{2\pi}{k}\,\frac{8}{3}\,\delta'(\theta-\theta')\,\widehat{\mathcal{L}}\widehat{\mathcal{L}}\;-\;\frac{2\pi}{k}\,\frac{8}{3}\,\delta(\theta-\theta')\,\widehat{\mathcal{L}}\widehat{\mathcal{L}}' \\
&+\;\frac{2\pi}{k}\,\frac{5}{64}\,\delta^{(3)}(\theta-\theta')\,\mathcal{W}_0\mathcal{W}_0\;-\;\frac{2\pi}{k}\,\frac{15}{64}\,\delta''(\theta-\theta')\,\mathcal{W}_0\mathcal{W}_0' \\
&+\;\frac{2\pi}{k}\,\frac{9}{64}\,\delta'(\theta-\theta')\,(\,\mathcal{W}_0'\mathcal{W}_0'\;+\;\mathcal{W}_0\mathcal{W}_0''\,)\;-\;\frac{2\pi}{k}\,\frac{1}{32}\,\delta(\theta-\theta')\,(\,3\,\mathcal{W}_0'\mathcal{W}_0''\;+\;\mathcal{W}_0\mathcal{W}_0'''\,) \\
&+\;\frac{2\pi}{k}\,\frac{1}{2}\,\delta'(\theta-\theta')\,\widehat{\mathcal{L}}\,\mathcal{W}_0\mathcal{W}_0\;+\;\left(\frac{2\pi}{k}\right)^3\frac{3}{128}\,\delta'(\theta-\theta')\,\mathcal{W}_0\mathcal{W}_0\mathcal{W}_0\mathcal{W}_0 \\
&-\;\left(\frac{2\pi}{k}\right)^2\frac{1}{4}\,\delta(\theta-\theta')\,\Big(2\,\widehat{\mathcal{L}}\,\mathcal{W}_0\mathcal{W}_0'\;+\;\widehat{\mathcal{L}}'\,\mathcal{W}_0\mathcal{W}_0\Big) \\
&-\;\left(\frac{2\pi}{k}\right)^3\frac{3}{64}\,\delta(\theta-\theta')\,\mathcal{W}_0\mathcal{W}_0\mathcal{W}_0\mathcal{W}_0' \\
&+\;\frac{2\pi}{k}\,8\,\delta'(\theta-\theta')\,\mathcal{W}_1^{[-1]}\mathcal{W}_1^{[1]}\;-\;\frac{2\pi}{k}\,4\,\delta(\theta-\theta')\,\Big(\mathcal{W}_1^{[-1]}\mathcal{W}_1^{[1]}\Big)'\,. \qquad \text{(C.2g)}
\end{aligned}
$$

Poisson structure of $\mathcal{W}_\infty[\lambda]$: the first brackets

Here we present the first Poisson brackets of $\mathcal{W}_\infty[\lambda]$, i.e. the $\{\mathcal{W}_i,\mathcal{W}_j\}$ with $i,j<4$. As in the previous examples all fields appearing on the right-hand side are functions of θ' and $\delta'(\theta-\theta')\equiv\partial_\theta\delta(\theta-\theta')$. Moreover, N_ℓ denotes the normalisation factor (3.10b).

$$
\Big\{\mathcal{L}(\theta)\,,\,\mathcal{L}(\theta')\Big\} \;=\; \delta(\theta-\theta')\,\mathcal{L}'-2\,\delta'(\theta-\theta')\,\mathcal{L}\;-\;\frac{k}{4\pi}\,\delta^{(3)}(\theta-\theta')\,, \qquad \text{(C.3a)}
$$

$$
\Big\{\mathcal{L}(\theta)\,,\,\mathcal{W}_\ell(\theta')\Big\} \;=\; \delta(\theta-\theta')\,\mathcal{W}_\ell'\;-\;(\ell+1)\,\delta'(\theta-\theta')\,\mathcal{W}_\ell\;, \quad \ell>1\,, \qquad \text{(C.3b)}
$$

$$
\begin{aligned}
\Big\{\mathcal{W}_2(\theta)\,,\,\mathcal{W}_2(\theta')\Big\} \;=\;& -\frac{2N_3}{(N_2)^2}\Big[\,\delta(\theta-\theta')\,\mathcal{W}_3'\;-\;2\,\delta'(\theta-\theta')\,\mathcal{W}_3\Big] \\
-\;\frac{1}{12N_2}&\Big[\,2\,\delta(\theta-\theta')\,\mathcal{L}^{(3)}\;-\;9\,\delta'(\theta-\theta')\,\mathcal{L}''\;+\;15\,\delta''(\theta-\theta')\,\mathcal{L}'\;-\;10\,\delta^{(3)}(\theta-\theta')\,\mathcal{L}\Big] \\
-\;\frac{16\pi}{3kN_2}&\Big[\,\delta(\theta-\theta')\,\mathcal{L}\mathcal{L}'-\;\delta'(\theta-\theta')\,\mathcal{L}^2\Big]\;-\;\frac{k}{48\pi N_2}\,\delta^{(5)}(\theta-\theta')\,, \qquad \text{(C.3c)}
\end{aligned}
$$

$$
\begin{aligned}
\Big\{\mathcal{W}_2(\theta)\,,\,\mathcal{W}_3(\theta')\Big\} \;=\;& -\,\frac{N_4}{N_2N_3}\Big[\,2\,\delta(\theta-\theta')\,\mathcal{W}_4'\;-\;5\,\delta'(\theta-\theta')\,\mathcal{W}_4\Big] \\
-\;\frac{1}{15N_2}&\Big[\,\delta(\theta-\theta')\,\mathcal{W}_2^{(3)}\;-\;6\,\delta'(\theta-\theta')\,\mathcal{W}_2''\;+\;14\,\delta''(\theta-\theta')\,\mathcal{W}_2'\;-\;14\,\delta^{(3)}(\theta-\theta')\,\mathcal{W}_2\Big] \\
-\;\frac{4\pi}{15kN_2}&\Big[\,25\,\delta(\theta-\theta')\,\mathcal{W}_2\,\mathcal{L}'\;+\;18\,\delta(\theta-\theta')\,\mathcal{L}\,\mathcal{W}_2'\;-\;52\,\delta'(\theta-\theta')\,\mathcal{L}\,\mathcal{W}_2\Big]\,, \qquad \text{(C.3d)}
\end{aligned}
$$

$$
\begin{aligned}
\Big\{\mathcal{W}_3(\theta)\,,\,\mathcal{W}_3(\theta')\Big\} \;=\; &-\frac{3N_5}{\left(N_3\right)^2}\Big[\,\delta(\theta-\theta')\,\mathcal{W}_5'\,-\,2\,\delta'(\theta-\theta')\,\mathcal{W}_5\,\Big]\\
+\;\frac{(\lambda^2-19)}{30N_3}\Big[\,&\delta(\theta-\theta')\,\mathcal{W}_3^{(3)}-\,5\,\delta'(\theta-\theta')\,\mathcal{W}_3^{(2)}+\,9\,\delta''(\theta-\theta')\,\mathcal{W}_3'-\,6\,\delta^{(3)}(\theta-\theta')\,\mathcal{W}_3\,\Big]\\
-\;\frac{1}{360N_3}\Big[\,&3\,\delta(\theta-\theta')\,\mathcal{L}^{(5)}-\,20\,\delta'(\theta-\theta')\,\mathcal{L}^{(4)}+\,56\,\delta''(\theta-\theta')\,\mathcal{L}^{(3)}-\,84\,\delta^{(3)}(\theta-\theta')\,\mathcal{L}''\\
&+\,70\,\delta^{(4)}(\theta-\theta')\,\mathcal{L}'-\,28\,\delta^{(5)}(\theta-\theta')\,\mathcal{L}\,\Big]\\
+\;\frac{2\pi(29\lambda^2-284)}{15kN_3}\Big[\,&\delta(\theta-\theta')\,\mathcal{W}_2\mathcal{W}_2'\,-\,\delta'(\theta-\theta')\,\left(\mathcal{W}_2\right)^2\,\Big]\\
-\;\frac{\pi}{90kN_3}\Big[\,&177\,\delta(\theta-\theta')\,\mathcal{L}'\mathcal{L}^{(2)}\,+\,78\,\delta(\theta-\theta')\,\mathcal{L}\mathcal{L}^{(3)}-\,295\,\delta'(\theta-\theta')\,\left(\mathcal{L}'\right)^2\\
&-\,352\,\delta'(\theta-\theta')\,\mathcal{L}\mathcal{L}''+\,588\,\delta''(\theta-\theta')\,\mathcal{L}\mathcal{L}'\,-\,196\,\delta^{(3)}(\theta-\theta')\,\mathcal{L}^2\,\Big]\\
+\;\frac{28\pi(\lambda^2-19)}{15kN_3}\Big[\,&\delta(\theta-\theta')\,\mathcal{W}_3\mathcal{L}'\,+\,\delta(\theta-\theta')\,\mathcal{L}\mathcal{W}_3'\,-\,2\,\delta'(\theta-\theta')\,\mathcal{L}\mathcal{W}_3\Big]\\
-\;\frac{32\pi^2}{5k^2N_3}\Big[\,&3\,\delta(\theta-\theta')\,\mathcal{L}^2\mathcal{L}'\,-\,2\,\delta'(\theta-\theta')\,\mathcal{L}^3\,\Big]\,-\,\frac{k}{1440\pi N_3}\,\delta^{(7)}(\theta-\theta')\,. \qquad \text{(C.3e)}
\end{aligned}
$$

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