

# ELLIPTIC HYPERGEOMETRY OF SUPERSYMMETRIC DUALITIES II. ORTHOGONAL GROUPS, KNOTS, AND VORTICES

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ABSTRACT. We consider Seiberg electric-magnetic dualities for four-dimensional  $\mathcal{N} = 1$  SYM theories with  $SO(N)$  gauge group. For all such theories we construct superconformal indices (SCIs) in terms of the elliptic hypergeometric integrals. Equalities of these indices for dual theories lead both to known special function identities and new highly nontrivial conjectural relations for integrals. In particular, we describe a number of new elliptic beta integrals associated with the  $s$ -confining theories with the spinor matter. Reductions of some dualities from  $SP(2N)$  to  $SO(2N)$  or  $SO(2N + 1)$  gauge groups are described. Interrelation of SCIs and the Witten anomaly is briefly discussed. Possible applications of the elliptic hypergeometric integrals to a two-parameter deformation of the two-dimensional conformal field theory and related matrix models are indicated. Connections of the reduced SCIs with the state integrals of the knot theory, generalized AGT duality for  $(3 + 3)$ -dimensional theories, and the  $2d$  vortex partition function are described.

*Dedicated to D.I. Kazakov on the occasion of his 60th birthday*

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## 1. INTRODUCTION

Gauge field theories play a crucial role in the modern theory of elementary particles. A generalization of the notion of electric-magnetic duality from electrodynamics to non-abelian gauge theories was suggested in the fundamental work of Goddard, Nuyts, and Olive [53]. In the asymptotically free theories the spectrum of elementary excitations in the high energy region is found from the free lagrangian. In the infrared region the interaction becomes strong and one has to pass to the description in terms of collective degrees of freedom (in the usual quantum chromodynamics one should describe formation of the hadrons out of quarks and gluons). The electric-magnetic duality relates these two energy scales and is also referred to as the strong-weak coupling duality transformation. To the present moment consistent consideration of such transformations in four-dimensional space-time has been given only in the maximally extended  $\mathcal{N} = 4$  supersymmetric field theory [88],  $\mathcal{N} = 2$  [103], and  $\mathcal{N} = 1$  [100, 101] theories. In comparison to the dualities for extended supersymmetric theories there exists a whole zoo of different Seiberg dualities for  $\mathcal{N} = 1$  SYM theories (see, e.g., surveys [66, 105]). The problem of their classification using some group-theoretical approach is still open. For a survey of the current status of development of supersymmetric gauge theories, see [102].

Highly nontrivial generalizations of the Witten index called superconformal indices (SCIs) were proposed recently by Kinney et al [75] and Römelsberger [96, 97]. SCIs count BPS states protected by one supercharge and its (superconformal) conjugate which cannot be combined to form long multiplets. They can be considered as twisted partition functions in the Hilbert space of BPS states which are determined by specific matrix integrals over the classical Lie groups. SCI is a conformal manifold invariant [52] which does not change under the marginal deformations [97, 117].

In this paper we continue the systematic study of electric-magnetic dualities for  $\mathcal{N} = 1$  SYM theories and  $s$ -confining theories initiated in [115]. We use for that the theory of elliptic hypergeometric integrals developed by the first author in [107, 108, 109]. The crucial observation on the coincidence of SCIs of supersymmetric field theories with such integrals was done by Dolan and Osborn in [36]. In a sequel of papers [114, 115, 116, 117, 120, 113] we analyzed known supersymmetric dualities, described deep relations between them and the properties of elliptic hypergeometric integrals, and, using these relations, discovered many new dualities. Related questions were considered also in [45, 46]. SCI techniques provides currently the most rigorous mathematical justification of  $\mathcal{N} = 1$  supersymmetric dualities [100, 101], and it serves as a very powerful tool for getting new insights. For instance, it has led to  $\mathcal{N} = 1$  dualities lying outside the conformal window [116], it is useful for consideration of the AdS/CFT correspondence for

gauge groups of infinite [75, 81, 82] and finite [83, 117] rank. It can be applied to theories which are difficult to treat by usual physical tools [120]. Another interesting fact is that  $4d$  SCIs can be reduced to  $3d$  partition functions [70, 38, 67, 55] yielding  $3d$  dualities [37, 48, 62]. Recently in [84] SCIs with the half-BPS superconformal surface operator have been studied. Elliptic hypergeometric integrals are connected with the relativistic Calogero-Sutherland type models where they describe either special wave functions or the normalizations of particular wave functions [110]. In [115] such a connection was conjectured to extend to all SCIs. Elliptic hypergeometric integrals provide a unification of known solvable models of statistical mechanics on two-dimensional ( $2d$ ) lattices [8, 113]. In [113] it was shown that SCIs of the simple gauge group SYM theories have the meaning of partition functions of elementary cells of  $2d$  integrable lattice models. In this picture, the Seiberg duality appears to have the meaning of a generalized Kramers-Wannier duality transformation for partition functions. SCI techniques applies not only to  $4d$  field theories, but also to  $3d$  models [74, 61, 7, 77]. In [77] the equality of superconformal SCIs indices of some  $3d$  dual theories with  $U(1)$  gauge group was proved rigorously for  $N_f = 1, 2$  flavors, and, using these results, in [69] such an equality was established for arbitrary  $N_f$ . Recently the analytical proof of the coincidence for partition functions of some  $3d$  quiver  $\mathcal{N} = 4$  mirror symmetric theories was considered in [11].

There are two different ways of computing SCI inspired by different physical problems. For  $\mathcal{N} = 1$  SYM theories Römelsberger [96, 97] derived it using the operator approach to free superconformal field theories (SCFTs) and suggested that SCIs for Seiberg dual theories coincide. For the asymptotically free theories in the ultraviolet region this is formally justified, so that one can use simply the representation theory for free SCFTs [33]. In [75], Kinney et al derived SCI for  $\mathcal{N} = 4$   $U(N)$ -SYM theory using the localization method and targeting mostly the AdS/CFT correspondence. The localization technique was used also for computing partition functions of  $\mathcal{N} = 2$  SYM theories [86, 87, 89]. In contrast to  $\mathcal{N} = 1$  case, SCIs for extended superconformal field theories can be derived directly from the partition functions by imposing some restriction on the parameters [12]. For related questions concerning counting the BPS operators, see also [10, 34, 42]. One can get  $\mathcal{N} = 2$  and  $\mathcal{N} = 4$  SYM theories out of  $\mathcal{N} = 1$  theories by adjusting the matter fields content and the superpotentials. Analogously, SCIs of extended theories can be obtained from  $\mathcal{N} = 1$  SCIs by appropriate fitting of the set of representations [115].

In this paper we are investigating SCIs for  $4d$   $\mathcal{N} = 1$  theories with orthogonal gauge groups. The most interesting  $SO(N)$ -dualities arise from the matter fields in spinor representation. Dualities without such matter fields can be obtained by reductions from the  $SP(2N)$ -gauge group cases. We discuss also separately the models studied in [65] for  $SO(N)$  gauge group with  $N - 1, N - 2, N - 3$ , and  $N - 4$  number of quarks.

By definition SCIs count gauge invariant operators which saturate the BPS bounds for short and semi-short multiplets.  $\mathcal{N} = 1$  SCFTs are based on the  $SU(2, 2|1)$  space-time symmetry group which is generated by the following set of operators:  $J_i, \bar{J}_i$  – the generators of two  $SU(2)$  subgroups forming the  $4d$  Lorentz group  $SO(3, 1)$ , translations,  $P_\mu$ ,  $\mu = 1, 2, 3, 4$ , special conformal transformations,  $K_\mu$ , the dilations,  $H$ , and also the  $U(1)_R$ -group generator  $R$ . Apart from these bosonic generators there are supercharges  $Q_\alpha, \bar{Q}_{\dot{\alpha}}$  with  $\alpha, \dot{\alpha} = 1, 2$  and their superconformal partners  $S_\alpha, \bar{S}_{\dot{\alpha}}$ . The full set of commutation relations for these operators can be found, for instance, in [115]. Taking a distinguished pair of intrinsically superconformal charges [96], for example,  $Q = \bar{Q}_1$  and  $Q^\dagger = -\bar{S}_1$ , one has

$$\{Q, Q^\dagger\} = 2\mathcal{H}, \quad \mathcal{H} = H - 2\bar{J}_3 - 3R/2. \quad (1.1)$$

In this case the superconformal index is defined by the matrix integral

$$I(p, q, f_k) = \int_G d\mu(g) \operatorname{Tr} \left( (-1)^{\mathcal{F}} p^{\mathcal{R}/2+J_3} q^{\mathcal{R}/2-J_3} \right. \\ \left. \times e^{\sum_a g_a G^a} e^{\sum_k f_k F^k} e^{-\beta \mathcal{H}} \right), \quad \mathcal{R} = R + 2\bar{J}_3, \quad (1.2)$$

where  $\mathcal{F}$  is the fermion number operator and  $d\mu(g)$  is the invariant measure of the gauge group  $G$ . We explicitly singled out the integration over gauge group, though most often it is assumed to be a part of the gauge invariant trace. To calculate the index one should not consider the whole space of states, only zero modes of the operator  $\mathcal{H}$  contribute to the trace because relation (1.1) is preserved by the operators used in (1.2). The fugacities  $g_a, f_k$  correspond to the gauge  $G$  and flavor  $F$  symmetry group generators  $G^a$  and  $F^k$ , respectively.

The 4d index is defined for SYM theories on the  $\mathbf{S}^3 \times \mathbf{S}^1$  manifold (for a latest discussion of such space-time manifestations, see [43, 102]). According to the Römelsberger prescription for building SCIs one should first compute the single particle index, given by the following general formula

$$\begin{aligned} \operatorname{ind}(p, q, z, y) &= \frac{2pq - p - q}{(1-p)(1-q)} \chi_{adj}(z) \\ &+ \sum_j \frac{(pq)^{r_j} \chi_{R_F, j}(y) \chi_{R_G, j}(z) - (pq)^{1-r_j} \chi_{\bar{R}_F, j}(y) \chi_{\bar{R}_G, j}(z)}{(1-p)(1-q)}. \end{aligned} \quad (1.3)$$

Here the first term describes the contribution of gauge fields belonging to the adjoint representation of group  $G$ ; the sum over  $j$  corresponds to the chiral matter superfields  $\Phi_j$  transforming as the gauge group representations  $R_{G, j}$  and the flavor symmetry group representations  $R_{F, j}$  with  $2r_j$  being equal to their  $R$ -charges. The functions  $\chi_{adj}(z)$ ,  $\chi_{R_F, j}(y)$  and  $\chi_{R_G, j}(z)$  are the characters of representations with  $z$  and  $y$  being the maximal torus variables of  $G$  and  $F$  groups, respectively. All the characters needed for this work are explicitly listed in Appendix A. To obtain the full superconformal index, the single particle states index (1.3) is inserted into the ‘‘plethystic’’ exponential averaged over the gauge group:

$$I(p, q, y) = \int_G d\mu(z) \exp \left( \sum_{n=1}^{\infty} \frac{1}{n} \operatorname{ind}(p^n, q^n, z^n, y^n) \right). \quad (1.4)$$

It appears that such matrix integrals are expressed in terms of the new class of special functions of mathematical physics known as elliptic hypergeometric integrals discovered in [107, 108, 109] (see also [111] for a general survey of results on elliptic hypergeometric functions). Their simplest representative – the exactly computable elliptic beta integral [107] is the top level known generalization of the Euler beta integral, the Askey-Wilson and Rahman  $q$ -beta integrals [3]. On the base of a very large number of explicit examples listed in [115], we conjectured that to every supersymmetric duality there corresponds either an exact integration formula for elliptic beta integrals or a nontrivial Weyl group symmetry transformation for the higher order elliptic hypergeometric integrals.

One important remark is in order. It appears that the described index computation algorithm does not impose in advance any constraint on the fugacities, whereas the elliptic hypergeometric integral identities used for establishing equalities of SCIs require neat fitting of parameter constraints for their existence (see below). It would be interesting to find the arguments leading to needed constraints for fugacities directly in formulas (1.3) and (1.4).

The plan of the paper is as follows. In Sect. 2 we discuss reductions of SCIs of  $s$ -confining and interacting SYM field theories from symplectic to orthogonal gauge groups by appropriate restriction of the parameters. In Sect. 3 we describe  $s$ -confining theories for  $\mathcal{N} = 1$  SYM theories with the matter in spinor representation and describe new conjectural exact evaluation formulas for particular elliptic hypergeometric integrals. Sect. 4 discusses a self-dual  $\mathcal{N} = 1$  SYM theory with  $SO(8)$  gauge group, 4 fields in the spinor representation, and 4 quarks in the fundamental representation. In Sect. 5 we derive non-trivial transformation formulas for elliptic hypergeometric integrals following from dualities with non-trivial gauge groups and the spinor matter. In the end of the paper we give some final remarks. In Appendix A we describe characters of representations for unitary, symplectic, and orthogonal groups. In Appendix B we propose an interpretation of some elliptic hypergeometric integrals as correlation functions of an elliptic deformation of  $2d$  CFT and related Virasoro algebra. It is related also to an elliptic generalization of the matrix models connected with  $2d$  CFTs. Appendix C contains a brief description of connections with the knot theory, where it is shown that the state integrals of knots appear from the reduction of elliptic hypergeometric integrals ( $4d$  SCIs) to hyperbolic  $q$ -hypergeometric integrals ( $3d$  partition functions). In the last Appendix D we consider further reduction of a particular  $3d$  partition function to the  $2d$  vortex partition function. This paper can be considered as a second part of the work [115] since we cover several subjects skipped in it. However, there are still some interesting questions touched in [115], but not included in this paper. In particular, we do not discuss SCIs of quiver theories which have attracted recently some interest in [20] (there is also a generalization of an  $s$ -confining duality discovered in [115] to a duality between interacting gauge field theories).

## 2. REDUCTION OF $\mathcal{N} = 1$ DUALITIES FROM SYMPLECTIC TO ORTHOGONAL GAUGE GROUPS

Let us show that known  $\mathcal{N} = 1$  dualities with  $SO(n)$  gauge group without matter in the spinor representation can be derived as consequences of known  $SP(2N)$  gauge group dualities. At the level of SCIs this implication is achieved by a particular restriction of the values of a number of parameters in the corresponding elliptic hypergeometric integrals, as observed first by Dolan and Osborn for the simplest cases [36]. In the present section we discuss such reductions in more detail. The spinor matter  $SO(n)$ -theories will be considered later on.

We start from  $\mathcal{N} = 1$  SYM theory with  $SP(2N)$  gauge group and  $2N_f$  quarks in the fundamental representation having the global symmetry group  $SU(2N_f) \times U(1)_R$ . The matter fields are described in the table below where we indicate their representation types for the gauge and flavor groups, together with hypercharges for the abelian part of the global symmetry group

	$SP(2N)$	$SU(2N_f)$	$U(1)_R$
$Q$	$f$	$f$	$1 - (N + 1)/N_f$

In this and all other tables below we skip the vector superfield  $V$  (or its dual partner  $\tilde{V}$ , except of the confining theories where this field is absent) described by the adjoint representation of  $G$  and singlets of the non-abelian part of the flavor group, and having trivial hypercharges for the abelian global groups.

The dual magnetic theory constructed by Intriligator and Pouliot [64] has the same flavor group and the gauge group  $G = SP(2\tilde{N})$ , where  $\tilde{N} = N_f - N - 2$ , with the matter field content described in the table below

	$SP(2\tilde{N})$	$SU(2N_f)$	$U(1)_R$
$q$	$f$	$\bar{f}$	$(N+1)/N_f$
$M$	1	$T_A$	$2(\tilde{N}+1)/N_f$

where  $f$  ( $\bar{f}$ ) denotes (anti)fundamental representation and  $T_A$  denotes the antisymmetric tensor of the second rank.

The conformal window for this duality is  $3(N+1)/2 < N_f < 3(N+1)$ ; it emerges from the demand that both dual theories are asymptotically free in the one-loop approximation. The Seiberg electric-magnetic duality at the IR (infrared) fixed points of these theories, which is not proven rigorously yet, had the following justifying arguments [101]:

- the 't Hooft anomaly matching conditions are satisfied. They were conjectured in [115, 116] to be a consequence of the so-called total ellipticity condition for the elliptic hypergeometric integrals [111, 112] describing SCIs;
- matching reduction of the number of flavors  $2N_f \rightarrow 2(N_f-1)$ . Integrating out  $2N_f$ ,  $(2N_f-1)$ -th flavor quarks by introducing the mass term to the superpotential of the original theory results in Higgsing the magnetic theory gauge group with decoupling of a number of meson fields. From the elliptic hypergeometric integrals point of view this is realized by restricting in a special way a pair of parameters,  $t_{2N_f}t_{2N_f-1} = pq$  [114, 115];
- matching of the moduli spaces and gauge invariant operators in dual theories. This information is believed to be hidden in the topological meaning of SCIs.

Following the prescription for construction SCIs described above it is easy to obtain SCI for the electric theory [36, 115]

$$I_E^{SP(2N)} = \frac{(p;p)_\infty^N (q;q)_\infty^N}{2^N N!} \int_{\mathbb{T}^N} \frac{\prod_{i=1}^{2N_f} \prod_{j=1}^N \Gamma(t_i z_j^{\pm 1}; p, q)}{\prod_{1 \leq i < j \leq N} \Gamma(z_i^{\pm 1} z_j^{\pm 1}; p, q) \prod_{j=1}^N \Gamma(z_j^{\pm 2}; p, q)} \prod_{j=1}^N \frac{dz_j}{2\pi i z_j}, \quad (2.1)$$

where the parameters satisfy the constrains  $|t_i| < 1$ ,  $i = \overline{1, 2N_f}$ , and the balancing condition

$$\prod_{i=1}^{2N_f} t_i = (pq)^{N_f - N - 1}.$$

Here

$$(z; q)_\infty = \prod_{i=0}^{\infty} (1 - zq^i), \quad |q| < 1,$$

is the standard infinite  $q$ -shifted factorial [3] and

$$\Gamma(z; p, q) = \prod_{i,j=0}^{\infty} \frac{1 - z^{-1} p^{i+1} q^{j+1}}{1 - z p^i q^j}, \quad |p|, |q| < 1, \quad (2.2)$$

is the elliptic gamma function [111].

The dual magnetic theory has SCI of the form

$$I_M^{SP(2\tilde{N})} = \frac{(p;p)_\infty^{\tilde{N}} (q;q)_\infty^{\tilde{N}}}{2^{\tilde{N}} \tilde{N}!} \prod_{1 \leq i < j \leq 2N_f} \Gamma(t_i t_j; p, q) \quad (2.3)$$

$$\times \int_{\mathbb{T}^{\tilde{N}}} \frac{\prod_{i=1}^{2N_f} \prod_{j=1}^{\tilde{N}} \Gamma((pq)^{1/2} t_i^{-1} z_j^{\pm 1}; p, q)}{\prod_{1 \leq i < j \leq \tilde{N}} \Gamma(z_i^{\pm 1} z_j^{\pm 1}; p, q) \prod_{j=1}^{\tilde{N}} \Gamma(z_j^{\pm 2}; p, q)} \prod_{j=1}^{\tilde{N}} \frac{dz_j}{2\pi i z_j},$$

for the magnetic theory.

Römelsberger's conjecture on the equality of SCIs for dual theories

$$I_E^{SP(2N)} = I_M^{SP(2\tilde{N})}$$

was proven in [36] on the basis of the symmetry transformation for integral (2.1) established in [94]. For  $N = 1$  the full symmetry group of SCI is  $W(E_7)$ . The key transformation generating this group was found earlier in [108]. Its physical consequences for multiple dualities have been studied in [114] and the superpotentials for such theories were investigated later in [73]. Altogether the results of [36, 114, 115] gave a new powerful, most rigorous from the mathematical point of view confirmation of the Seiberg duality, complementing the tests mentioned above.

It should be stressed that this and all other equalities of SCIs of dual theories are true or supposed to be true only if the values of parameters in all integrals guarantee that only sequences of poles of the integrands converging to zero are located inside the contour of integration  $\mathbb{T}$  (otherwise one should use the nontrivial analytical continuation procedure for identities to be true in other regions of parameters).

If we introduce an elliptic hypergeometric integral on the  $BC_n$  root system

$$I_n^{(m)}(\underline{x}; p, q) = \frac{(p; p)_\infty^n (q; q)_\infty^n}{2^{nn!}} \int_{\mathbb{T}^n} \prod_{1 \leq i < j \leq n} \frac{1}{\Gamma(z_i^{\pm 1} z_j^{\pm 1}; p, q)} \prod_{j=1}^n \frac{\prod_{i=1}^{2(m+n+2)} \Gamma(x_i z_j^{\pm 1}; p, q)}{\Gamma(z_j^{\pm 2}; p, q)} \prod_{j=1}^n \frac{dz_j}{2\pi i z_j}, \quad (2.4)$$

where  $\prod_{i=1}^{2(m+n+2)} x_i = (pq)^{m+1}$ , then the equality of SCIs is rewritten as [94]

$$I_n^{(m)}(\underline{x}; p, q) = \prod_{1 \leq i < j \leq 2(m+n+2)} \Gamma(x_i x_j; p, q) I_m^{(n)}\left(\frac{\sqrt{pq}}{\underline{x}}; p, q\right). \quad (2.5)$$

Let us consider now the Seiberg duality for  $\mathcal{N} = 1$  SYM theories with orthogonal gauge group [101]. The electric theory matter fields are described in the following table

	$SO(N)$	$SU(N_f)$	$U(1)_R$
$Q$	$f$	$f$	$\frac{N_f - N + 2}{N_f}$

and for the magnetic theory one has

	$SO(\tilde{N})$	$SU(N_f)$	$U(1)_R$
$q$	$f$	$\bar{f}$	$\frac{N-2}{N_f}$
$M$	1	$T_S$	$2\frac{N_f - N + 2}{N_f}$

where  $T_S$  denotes the absolutely symmetric tensor of second rank and  $\tilde{N} = N_f - N + 4$ .

The conformal window [101] for this duality has the form

$$\frac{3}{2}(N-2) < N_f < 3(N-2), \quad (2.6)$$

which guarantees existence of the non-trivial IR fixed points. Actually, one should be accurate with the notion of conformal windows since there are examples [116] of dualities lying outside such windows.

Orthogonal groups  $SO(n)$  are qualitatively different for even  $n = 2N$  (root system  $D_N$ ) and odd  $n = 2N + 1$  (roots system  $B_N$ ). SCIs in the electric theory take the form

$$I_E^{SO(2N)} = \frac{(p; p)_\infty^N (q; q)_\infty^N}{2^{N-1} N!} \int_{\mathbb{T}^N} \frac{\prod_{i=1}^{N_f} \prod_{j=1}^N \Gamma(x_i z_j^{\pm 1}; p, q)}{\prod_{1 \leq i < j \leq N} \Gamma(z_i^{\pm 1} z_j^{\pm 1}; p, q)} \prod_{j=1}^N \frac{dz_j}{2\pi i z_j}, \quad (2.7)$$

and

$$I_E^{SO(2N+1)} = \frac{(p; p)_\infty^N (q; q)_\infty^N}{2^N N!} \prod_{i=1}^{N_f} \Gamma(x_i; p, q) \\ \times \int_{\mathbb{T}^N} \frac{\prod_{i=1}^{N_f} \prod_{j=1}^N \Gamma(x_i z_j^{\pm 1}; p, q)}{\prod_{j=1}^N \Gamma(z_j^{\pm 1}; p, q) \prod_{1 \leq i < j \leq N} \Gamma(z_i^{\pm 1} z_j^{\pm 1}; p, q)} \prod_{j=1}^N \frac{dz_j}{2\pi i z_j}. \quad (2.8)$$

The magnetic theory SCI can be written in the form

$$I_M(\underline{x}; p, q)^{SO(\tilde{N})} = \prod_{1 \leq i < j \leq N_f} \Gamma(x_i x_j; p, q) \prod_{i=1}^{N_f} \Gamma(x_i^2; p, q) I_E\left(\frac{\sqrt{pq}}{\underline{x}}; p, q\right)^{SO(\tilde{N})}. \quad (2.9)$$

To show the duality relation

$$I_E(x; p, q)^{SO(N)} = I_M(x; p, q)^{SO(\tilde{N})} \quad (2.10)$$

one has to restrict the parameters in (2.5) [36]. First we identify

$$I_E(\underline{x}; p, q)^{SO(2n)} = \begin{cases} 2I_n^{\left(\frac{1}{2}(N_f+4)-n\right)}(\underline{x}, \underline{u}; p, q), & N_f \text{ even,} \\ 2I_n^{\left(\frac{1}{2}(N_f+3)-n\right)}(\underline{x}, \underline{v}; p, q), & N_f \text{ odd,} \end{cases} \quad (2.11)$$

where parameters  $\underline{u}$  and  $\underline{v}$  in  $I_n^{(m)}$  are chosen as

$$\underline{u} = \{\pm 1, \pm\sqrt{p}, \pm\sqrt{q}, \pm\sqrt{pq}\}$$

and

$$\underline{v} = \{\pm 1, \pm\sqrt{p}, \pm\sqrt{q}, -\sqrt{pq}\}.$$

Analogously,

$$I_E(\underline{x}; p, q)^{SO(2n+1)} = \begin{cases} \prod_{i=1}^{N_f} \Gamma(x_i; p, q) I_n^{\left(\frac{1}{2}(N_f+4)-n\right)}(\underline{x}, \underline{u}'; p, q), & N_f \text{ even,} \\ \prod_{i=1}^{N_f} \Gamma(x_i; p, q) I_n^{\left(\frac{1}{2}(N_f+5)-n\right)}(\underline{x}, \underline{v}'; p, q), & N_f \text{ odd,} \end{cases} \quad (2.12)$$

where

$$\underline{u}' = \{-1, \pm\sqrt{p}, \pm\sqrt{q}, -\sqrt{pq}\}$$

and

$$\underline{v}' = \{-1, \pm\sqrt{p}, \pm\sqrt{q}, \pm\sqrt{pq}\}.$$

These relations are based on the duplication formula for the elliptic gamma function

$$\Gamma(z^2; p, q) = \Gamma(\pm z, \pm\sqrt{p}z, \pm\sqrt{q}z, \pm\sqrt{pq}z; p, q) \quad (2.13)$$

and the inversion formula

$$\Gamma(z; p, q) \Gamma(pq/z; p, q) = 1.$$

They allow one to reduce the elliptic hypergeometric integrals from  $SP(2n)$ -group to  $SO(2n)$  or  $SO(2n+1)$  and, simultaneously, reduces mesons from  $T_A$ - to  $T_S$ -representation.

The same line of arguments works for checking equality of SCIs for many other known dualities of orthogonal gauge group theories which we list below:

- $SO(N)$  theory with matter fields – the antisymmetric tensor of the second rank (or the adjoint representation) and quarks in the fundamental representation, see [78] for the duality between interacting field theories and [21, 76] for the  $s$ -confining theory;



- $SO(N)$  theory with the symmetric tensor of the second rank field and quarks in the fundamental representation, see [63] for nontrivial dual gauge group case and [21, 76] for the  $s$ -confining theory;
- $SO(N)$  theory with two matter fields – symmetric tensors of the second rank and quarks in the fundamental representation, see [13] for the duality between interacting field theories and [76] for the  $s$ -confining theory;
- $SO(N)$  theory with one matter field, the symmetric tensor of the second rank, and another field, the antisymmetric tensor of the second rank, together with the quarks in the fundamental representation, see [13] for the nontrivial dual gauge group case and [76] for the  $s$ -confining theory.

For brevity we are not presenting explicitly SCIs of these theories and do not indicate how they are related to  $SP(2N)$ -group indices considered in [115] since they are easily obtained by reductions similar to the one described above. Moreover, one can obtain new orthogonal gauge group dualities with the flavor group composed of several  $SP(2m)$ -groups and  $SU(4)$  group after a similar reduction of the duality considered in Sect. 7 of [115] (as well as the related  $s$ -confining theory). The general question why  $SO$ -dualities for theories without spinor matter can be derived from  $SP$ -theories is not understood from the physical point of view yet.

Now we would like to discuss some special cases in more detail to show the peculiarities of such reductions and of the related dualities [65]. The first case we mention is the theory considered in [101, 65] with  $N_f = N$ . The conformal window in this case restricts the parameters to  $N_f = N = 4, 5$ . We are not giving the explicit expressions for SCIs since they can be read straightforwardly from (2.10). Note that here the dual gauge group is  $SO(4)$  which is isomorphic to  $SU(2) \times SU(2)$ .

2.1.  $G = SO(N)$  **with**  $N_f = N - 1$ . This case is known to have three dual pictures [65]: electric, magnetic, and dyonic. First we discuss  $N > 3$  theories and then indicate the peculiarities of self-dual  $SO(3)$ -case.

Let us restrict 6 parameters in SCIs of  $SP(2N)$ -theories (2.1) and (2.3) as  $-1, \pm\sqrt{p}, \pm\sqrt{q}, -\sqrt{pq}$ . This yields SCIs of the electric  $SO(2N + 1)$ -gauge group theory

$$I_E^{SO(2N+1)} = \frac{(p; p)_\infty^N (q; q)_\infty^N}{2^N N!} \prod_{i=1}^{2N} \Gamma(t_i; p, q) \times \int_{\mathbb{T}^N} \prod_{1 \leq i < j \leq N} \frac{1}{\Gamma(z_i^{\pm 1} z_j^{\pm 1}; p, q)} \prod_{j=1}^N \frac{\prod_{m=1}^{2N} \Gamma(t_m z_j^{\pm 1}; p, q)}{\Gamma(z_j^{\pm 1}; p, q)} \frac{dz_j}{2\pi i z_j} \quad (2.14)$$

and of its magnetic dual with the  $SO(3)$  gauge group

$$I_M^{SO(3)} = \prod_{1 \leq m < s \leq 2N} \Gamma(t_m t_s; p, q) \prod_{i=1}^{2N} \Gamma(t_i^2, \frac{\sqrt{pq}}{t_i}; p, q) \frac{(p; p)_\infty (q; q)_\infty}{2} \times \int_{\mathbb{T}} \frac{\prod_{i=1}^{2N} \Gamma(\frac{\sqrt{pq}}{t_i} y^{\pm 1}; p, q)}{\Gamma(y^{\pm 1}; p, q)} \frac{dy}{2\pi i y}, \quad (2.15)$$

the balancing condition here reads  $\prod_{m=1}^{2N} t_m = \sqrt{pq}$ . These expressions can also be obtained from formulas (2.1) and (2.3) with  $N_f = N + 3$ . The moduli space of vacua of the  $SO(3)$ -theory has two non-trivial points leading to two dual theories. One of them is the original  $SO(2N + 1)$ -electric theory, and the second one is the  $SO(2N + 1)$ -dyonic theory, which is

obtained from the electric one by adding a particular term to the superpotential and shifting the theta angle by  $\pi$ . The electric and dyonic theories are related to each other by the “weak-to-weak”  $T$ -duality transformation and, therefore, their superconformal indices are identical,  $I_D \equiv I_E$ . These duality transformations form the permutation group  $S_3$ , a subgroup of the  $SL(2, \mathbb{Z})$ -group, interchanging the three different theories.

The same arguments apply to  $\mathcal{N} = 1$  SYM theory with  $SO(2N)$  gauge group and  $2N - 1$  quarks where SCIs are obtained by restricting seven parameters in (2.1) (taking  $N_f = N + 3$ ) as  $1, \pm\sqrt{p}, \pm\sqrt{q}, \pm\sqrt{pq}$ , one obtains SCI of the electric theory (identically coinciding with the index of the dyonic theory). Substituting the same constraints to (2.3) one obtains SCI of the  $SO(3)$ -magnetic theory. In both cases the balancing condition reads  $\prod_{i=1}^{2N-1} t_i = 1$ , i.e. at least one of the parameters  $t_i$  has modulus greater than 1, which requires an appropriate deformation of the integration contours for separation of relevant sequences of integrand poles.

As to the self-dual case of  $SO(3)$ -gauge group, for it SCIs  $I_E^{SO(3)}$  and  $I_M^{SO(3)}$  depend on two parameters with the balancing condition  $t_1 t_2 = \sqrt{pq}$ . Remarkably, after taking into account the latter constraint, the index  $I_M^{SO(3)}$  becomes identically equal to  $I_E^{SO(3)}$ . So, the electric, magnetic, and dyonic theories differ from each other only by particular terms in the superpotential (governed by the parameter  $e = 0, \pm 1$  in [65]) and have SCIs of identical shape.

**2.2.  $G = SO(N)$  with  $N_f = N - 2$ .** According to Seiberg [101], in this case the dual gauge group is  $SO(2)$ , i.e. the magnetic theory coincides with  $\mathcal{N} = 1$  abelian theory describing the supersymmetric photon with the gauge group  $U(1)$ . This duality can be deduced from the  $SP(2N) \leftrightarrow SP(2(N_f - N - 2))$  duality with  $N_f = N + 3$ . Corresponding SCIs are obtained by imposing appropriate constraints on the parameters, as described above. For the  $SO(2N + 1)$  gauge group we have

$$I_E^{SO(2N+1)} = \frac{(p; p)_\infty^N (q; q)_\infty^N}{2^N N!} \prod_{i=1}^{2N-1} \Gamma(t_i; p, q) \times \int_{\mathbb{T}^N} \prod_{1 \leq i < j \leq N} \frac{1}{\Gamma(z_i^{\pm 1} z_j^{\pm 1}; p, q)} \prod_{j=1}^N \frac{\prod_{m=1}^{2N-1} \Gamma(t_m z_j^{\pm 1}; p, q)}{\Gamma(z_j^{\pm 1}; p, q)} \frac{dz_j}{2\pi i z_j} \quad (2.16)$$

and

$$I_M^{SO(2)} = \prod_{1 \leq m < s \leq 2N-1} \Gamma(t_m t_s; p, q) \prod_{i=1}^{2N-1} \Gamma(t_i^2; p, q) \frac{(p; p)_\infty (q; q)_\infty}{2} \times \int_{\mathbb{T}} \prod_{i=1}^{2N-1} \Gamma\left(\frac{\sqrt{pq}}{t_i} y^{\pm 1}; p, q\right) \frac{dy}{2\pi i y}, \quad (2.17)$$

where it is assumed that  $N \geq 2$ . Here the balancing condition reads  $\prod_{m=1}^{2N-1} t_m = 1$ , so that at least one of the parameters should be of modulus greater than 1. Therefore the integration contours should be deformed appropriately.

For the  $SO(2N)$  gauge group we have

$$I_E^{SO(2N)} = \frac{(p; p)_\infty^N (q; q)_\infty^N}{2^{N-1} N!} \int_{\mathbb{T}^N} \prod_{1 \leq i < j \leq N} \frac{1}{\Gamma(z_i^{\pm 1} z_j^{\pm 1}; p, q)} \prod_{j=1}^N \prod_{m=1}^{2N-2} \Gamma(t_m z_j^{\pm 1}; p, q) \frac{dz_j}{2\pi i z_j} \quad (2.18)$$

and

$$\begin{aligned}
 I_M^{SO(2)} &= \prod_{1 \leq m < s \leq 2N-2} \Gamma(t_m t_s; p, q) \prod_{i=1}^{2N-2} \Gamma(t_i^2; p, q) \frac{(p; p)_\infty (q; q)_\infty}{2} \\
 &\times \int_{\mathbb{T}} \prod_{i=1}^{2N-2} \Gamma\left(\frac{\sqrt{pq}}{t_i} y^{\pm 1}; p, q\right) \frac{dy}{2\pi i y}, \tag{2.19}
 \end{aligned}$$

where the balancing condition is  $\prod_{m=1}^{2N-2} t_m = 1$  and an appropriate deformation of the integration contours is assumed. These indices are well defined only if  $N > 2$ , for  $N = 2$  both expressions diverge and one has to apply appropriate regularization  $t_1 t_2 \neq 1$  and residue calculus to obtain a meaningful limit  $t_1 t_2 \rightarrow 1$ .

Interestingly, both magnetic SCIs are represented by the general well-poised elliptic hypergeometric integrals without the very-well-poisedness condition [111] (which is thus not obligatory for applications to supersymmetric theories).

**2.3.  $G = SO(N)$  with  $N_f = N - 3$ .** We start from the  $s$ -confining  $SP(2N)$ -theory with  $2N + 4$  quarks [64]. In the table below we represent dual fields as gauge invariant combinations of the electric theory degrees of freedom:

	$SP(2N)$	$SU(2N + 4)$	$U(1)_R$
$Q$	$f$	$f$	$2r = \frac{1}{N+2}$
$Q^2$		$T_A$	$2r = \frac{2}{N+2}$

SCIs for these theories are given by the following expressions

$$I_E^{SP(2N)} = \frac{(p; p)_\infty^N (q; q)_\infty^N}{2^N N!} \int_{\mathbb{T}^N} \prod_{1 \leq i < j \leq N} \frac{1}{\Gamma(z_i^{\pm 1} z_j^{\pm 1}; p, q)} \prod_{j=1}^N \frac{\prod_{m=1}^{2N+4} \Gamma(t_m z_j^{\pm 1}; p, q)}{\Gamma(z_j^{\pm 2}; p, q)} \frac{dz_j}{2\pi i z_j} \tag{2.20}$$

and

$$I_M = \prod_{1 \leq m < s \leq 2N+4} \Gamma(t_m t_s; p, q), \tag{2.21}$$

where the balancing condition reads  $\prod_{m=1}^{2N+4} t_m = pq$ .

Now we describe the duality considered in [65] applying the same type reduction of SCIs as mentioned above (see also [36]). For the  $SO(2N + 1)$ -group we have

$$\begin{aligned}
 I_E^{SO(2N+1)} &= \frac{(p; p)_\infty^N (q; q)_\infty^N}{2^N N!} \prod_{i=1}^{2N-2} \Gamma(t_i; p, q) \\
 &\times \int_{\mathbb{T}^N} \prod_{1 \leq i < j \leq N} \frac{1}{\Gamma(z_i^{\pm 1} z_j^{\pm 1}; p, q)} \prod_{j=1}^N \frac{\prod_{m=1}^{2N-2} \Gamma(t_m z_j^{\pm 1}; p, q)}{\Gamma(z_j^{\pm 1}; p, q)} \frac{dz_j}{2\pi i z_j} \tag{2.22}
 \end{aligned}$$

and

$$I_M = \prod_{1 \leq m < s \leq 2N-2} \Gamma(t_m t_s; p, q) \prod_{i=1}^{2N-2} \Gamma\left(t_i^2, \frac{\sqrt{pq}}{t_i}; p, q\right), \tag{2.23}$$

where the balancing condition reads  $\prod_{m=1}^{2N-2} t_m = (pq)^{-1/2}$ , and the integration contour should be deformed appropriately. For the  $SO(2N)$ -group we find

$$I_E^{SO(2N)} = \frac{(p; p)_\infty^N (q; q)_\infty^N}{2^{N-1} N!} \int_{\mathbb{T}^N} \prod_{1 \leq i < j \leq N} \frac{1}{\Gamma(z_i^{\pm 1} z_j^{\pm 1}; p, q)} \prod_{j=1}^N \prod_{m=1}^{2N-3} \Gamma(t_m z_j^{\pm 1}; p, q) \frac{dz_j}{2\pi i z_j} \quad (2.24)$$

and

$$I_M = \prod_{1 \leq m < s \leq 2N-3} \Gamma(t_m t_s; p, q) \prod_{i=1}^{2N-3} \Gamma(t_i^2, \frac{\sqrt{pq}}{t_i}; p, q), \quad (2.25)$$

where the balancing condition is similar to the previous case  $\prod_{m=1}^{2N-3} t_m = (pq)^{-1/2}$ . Extra terms  $\prod_{i=1}^{2N-3} \Gamma(\frac{\sqrt{pq}}{t_i}; p, q)$  appear in (2.25) from the fundamental representation, although the dual gauge group is absent being formally defined as  $SO(1)$ .

2.4.  $G = SO(N)$  **with**  $N_f = N - 4$ . Here we discuss the confining theory studied in [65]. Similar to the previous subsection, the result for  $N_f = N - 4$  can be derived from the equality of (2.20) and (2.29). For  $SO(2N + 1)$ -group we have

$$I_E^{SO(2N+1)} = \frac{(p; p)_\infty^N (q; q)_\infty^N}{2^N N!} \prod_{i=1}^{2N-3} \Gamma(t_i; p, q) \times \int_{\mathbb{T}^N} \prod_{1 \leq i < j \leq N} \frac{1}{\Gamma(z_i^{\pm 1} z_j^{\pm 1}; p, q)} \prod_{j=1}^N \frac{\prod_{m=1}^{2N-3} \Gamma(t_m z_j^{\pm 1}; p, q)}{\Gamma(z_j^{\pm 1}; p, q)} \frac{dz_j}{2\pi i z_j} \quad (2.26)$$

and

$$I_M = \prod_{1 \leq m < s \leq 2N-3} \Gamma(t_m t_s; p, q) \prod_{i=1}^{2N-3} \Gamma(t_i^2; p, q), \quad (2.27)$$

where the balancing condition reads  $\prod_{m=1}^{2N-3} t_m = (pq)^{-1}$ . For  $SO(2N)$ -group we obtain

$$I_E^{SO(2N)} = \frac{(p; p)_\infty^N (q; q)_\infty^N}{2^{N-1} N!} \int_{\mathbb{T}^N} \prod_{1 \leq i < j \leq N} \frac{1}{\Gamma(z_i^{\pm 1} z_j^{\pm 1}; p, q)} \prod_{j=1}^N \prod_{m=1}^{2N-4} \Gamma(t_m z_j^{\pm 1}; p, q) \frac{dz_j}{2\pi i z_j} \quad (2.28)$$

and

$$I_M = \prod_{1 \leq m < s \leq 2N-4} \Gamma(t_m t_s; p, q) \prod_{i=1}^{2N-4} \Gamma(t_i^2; p, q), \quad (2.29)$$

where the balancing condition reads  $\prod_{m=1}^{2N-4} t_m = (pq)^{-1}$ .

**2.5. Connection to the Witten anomaly.** The even-dimensional (in particular, four-dimensional) theories have triangle anomalies associated with the global currents. For odd-dimensional field theories these anomalies are absent and this fact plays a negative role in searching odd-dimensional dual supersymmetric field theories (because of the absence of powerful 't Hooft anomaly matching conditions). That is why derivation of  $3d$  partition functions from  $4d$  indices discovered in [37] plays an important role in finding  $3d$  dualities, since this procedure inherits the information hidden in higher dimensional anomaly matching conditions.

However, apart from the global triangle anomalies there is a non-perturbative anomaly found by Witten [122], which is associated with the fact that the fourth homotopy group is non-trivial

for some gauge groups. For examples, it was found that an  $SU(2)$  gauge group theory with odd number of fermions is not well defined because  $\pi^4(SU(2)) = \mathbb{Z}_2$ . The same argument applies to supersymmetric field theories. Therefore it is important to understand how this anomaly manifests itself in SCIs and we analyze this question below.

We start from  $4d \mathcal{N} = 1$  SYM theory with  $SU(2)$  gauge group and 6 chiral superfields – an example of the  $s$ -confining theory. The confining phase contains baryons  $M_{ij}$  lying in the absolutely antisymmetric representation of the global symmetry group  $SU(6)$ . Corresponding SCIs were discussed in [36, 115] and they are given by the left- and right-hand sides of the elliptic beta-integral [107]. So, the electric SCI has the form

$$I_E(s_1, \dots, s_6) = \frac{(p; p)_\infty (q; q)_\infty}{2} \int_{\mathbb{T}} \frac{\prod_{i=1}^6 \Gamma(s_i z^{\pm 1}; p, q)}{\Gamma(z^{\pm 2}; p, q)} \frac{dz}{2\pi i z}, \quad (2.30)$$

with the balancing condition  $\prod_{i=1}^6 s_i = pq$ . Changing the integration variable  $z \rightarrow -z$  we see that  $I_E(s_1, \dots, s_6) = I_E(-s_1, \dots, -s_6)$ . The magnetic SCI is  $I_M = \prod_{1 \leq j < k \leq 6} \Gamma(s_j s_k; p, q) = I_E$ .

Let us set  $s_6 = \sqrt{pq}$ . From the reflection equation for the elliptic gamma function one has  $\Gamma(\sqrt{pq} z^{\pm 1}; p, q) = 1$ . Therefore the reduced SCI takes the form

$$I_{E1}(s_1, \dots, s_5) = \frac{(p; p)_\infty (q; q)_\infty}{2} \int_{\mathbb{T}} \frac{\prod_{i=1}^5 \Gamma(s_i z^{\pm 1}; p, q)}{\Gamma(z^{\pm 2}; p, q)} \frac{dz}{2\pi i z}, \quad (2.31)$$

where the balancing condition is  $\prod_{i=1}^5 s_i = \sqrt{pq}$ . According to the prescription for constructing SCIs, this expression describes the SCI for  $\mathcal{N} = 1$  SYM theory with  $SU(2)$  gauge group and 5 quarks forming a fundamental representation of the flavor group  $SU(5)$  and having the  $R$ -charges  $2r = 2/5$ . (The situation looks as if one of the quarks has been integrated out.) As to the magnetic SCI, it takes the form

$$I_{M1}(s_1, \dots, s_5) = \prod_{1 \leq i < j \leq 5} \Gamma(s_i s_j; p, q) \prod_{i=1}^5 \Gamma(\sqrt{pq} s_i; p, q)$$

and describes a confined theory of two types of mesons – the antisymmetric tensor representation  $T_A$  of the group  $SU(5)$  with the  $R$ -charge  $4/5$  and the fundamental representation of  $SU(5)$  with the  $R$ -charge  $7/5$ . As a consequence of the superconformal algebra the canonical dimension of the latter field is bigger than 1, i.e. the unitarity is broken.

So, the electric theory has the Witten anomaly and the magnetic theory is not unitary. Despite of the non-physical properties, these theories are presumably dual to each other since all known duality tests are valid for them, including the equality of SCIs. A natural question is whether SCI feels in any way this anomaly ambiguity or not? As argued in [122], physical observables in this anomalous theory should vanish due to the cancellation induced by the “large” gauge transformations which change the sign of the fermion determinant. This means that SCI should vanish as well as the gauge invariant object. However, the SCI we use was computed basically from the free field theory (in a sense, perturbatively), and the non-perturbative effect of the large gauge transformation do not enter it, yielding a nonzero result.

Still, we believe that SCIs catch this effect. For instance, in the above confining theory with 5 quarks  $I_{E1}(s_1, \dots, s_5) \neq I_{E1}(-s_1, \dots, -s_5)$ , since the balancing condition is not preserved by the reflections  $s_j \rightarrow -s_j$ . There is an ambiguity in reducing the number of quarks – one can choose  $s_6 = -\sqrt{pq}$  and obtain SCI of the same shape (2.31), but with the balancing condition having the different sign  $\prod_{i=1}^5 s_i = -\sqrt{pq}$ . We interpret this ambiguity in reductions together with the breaking of the reflection symmetry  $s_j \rightarrow -s_j$  as manifestations of the Witten anomaly.

For instance, if we choose in the elliptic beta integral  $s_6 = \sqrt{pq}$  and  $s_5 = -\sqrt{pq}$ , we obtain the relation

$$\begin{aligned} I_{E2} &= \frac{(p; p)_\infty (q; q)_\infty}{2} \int_C \frac{\prod_{k=1}^4 \Gamma(s_k z^{\pm 1}; p, q)}{\Gamma(z^{\pm 2}; p, q)} \frac{dz}{2\pi i z} \\ &= I_{M2} = 2(-p; p)_\infty (-q; q)_\infty \prod_{1 \leq j < k \leq 4} \Gamma(s_j s_k; p, q) \prod_{k=1}^4 \Gamma(pq s_k^2; p^2, q^2), \end{aligned}$$

where  $\prod_{k=1}^4 s_k = -1$  and the contour  $C$  is chosen appropriately. (There is a misprint in the corresponding equality given before formula (4.9) in [111] – the infinite products independent on  $s_j$  were combined there in an erroneous way.) If we interpret this relation as the equality of superconformal indices for some confining theory with four quarks, then the Witten anomaly is absent and, indeed,  $I_{E2}(s_1, \dots, s_4) = I_{E2}(-s_1, \dots, -s_4)$ . The physical meaning of this duality is not quite clear since the standard Römelsberger prescription does not apply to it. Namely, the electric theory has four quarks, but some nontrivial topological contributions to SCI are present leading to the non-standard balancing condition indicating on a non-marginal deformation of the standard four quarks electric theory.

It is natural to expect that the discrete symmetries also should give some contribution to SCIs, which is not analyzed yet appropriately. We would like to suggest a modification of the general formula for computing SCIs to the non-standard balancing condition cases. Namely, the single particle index (1.3) for such theories should be written as

$$\begin{aligned} ind(p, q, z, y) &= \frac{2pq - p - q}{(1-p)(1-q)} \chi_{adj}(z) \\ &+ \sum_j \frac{x_\theta (pq)^{r_j} \chi_{R_F, j}(y) \chi_{R_G, j}(z) - (pq)^{1-r_j} \chi_{\bar{R}_F, j}(y) \chi_{\bar{R}_G, j}(z)}{x_\theta (1-p)(1-q)}, \end{aligned} \quad (2.32)$$

where  $x_\theta = e^{\pi i/N_f}$  and the general formula (1.4) should be used with the powers  $x_\theta^n$  in the plethystic exponential. The term  $x_\theta$  looks like a contribution coming from the discrete “large” gauge transformations.

**2.6.  $SO/SP$  gauge group theories with small number of flavors.** Here we consider relations between  $\mathcal{N} = 1$  SYM theories with orthogonal and symplectic gauge groups with small number of flavors. Take the dualities for  $SP(2)$  gauge group theory with 8 quarks. This model was suggested in [23] and studied in detail in [114], where it was argued that there are in total 72 dual theories having specific physical manifestations [73].

Electric theory SCI is described by an elliptic analogue of the Euler-Gauss hypergeometric function introduced in [108, 109]

$$I(t_1, \dots, t_8; p, q) = \kappa \int_{\mathbb{T}} \frac{\prod_{j=1}^8 \Gamma(t_j z^{\pm 1}; p, q)}{\Gamma(z^{\pm 2}; p, q)} \frac{dz}{z} \quad (2.33)$$

with the constraints  $|t_j| < 1$  for eight complex variables  $t_1, \dots, t_8 \in \mathbb{C}$ , the balancing condition  $\prod_{j=1}^8 t_j = (pq)^2$ , and  $\kappa = (p; p)_\infty (q; q)_\infty / 4\pi i$ . This function obeys the following symmetry transformation derived in [108]

$$I(t_1, \dots, t_8; p, q) = \prod_{1 \leq j < k \leq 4} \Gamma(t_j t_k; p, q) \Gamma(t_{j+4} t_{k+4}; p, q) I(s_1, \dots, s_8; p, q), \quad (2.34)$$

where complex variables  $s_j$ ,  $|s_j| < 1$ , are connected with  $t_j$ ,  $j = 1, \dots, 8$ , as follows

$$\begin{aligned} s_j &= \rho^{-1} t_j, \quad j = 1, 2, 3, 4, \quad s_j = \rho t_j, \quad j = 5, 6, 7, 8, \\ \rho &= \sqrt{\frac{t_1 t_2 t_3 t_4}{pq}} = \sqrt{\frac{pq}{t_5 t_6 t_7 t_8}}. \end{aligned} \quad (2.35)$$

This fundamental relation taken together with the evident  $S_8$ -permutational group of symmetries in parameters  $t_j$  generates the Weyl group  $W(E_7)$  [94].

Let us apply the following constraint on the parameters

$$t_{3,4,5,6,7,8} = \{\pm\sqrt{p}, \pm\sqrt{q}, -1, -\sqrt{pq}\}.$$

The initial electric SCI takes the form

$$I_E = \kappa \int_{\mathbb{T}} \frac{\prod_{i=1}^2 \Gamma(t_i z^{\pm 1}; p, q)}{\Gamma(z^{\pm 1}; p, q)} \frac{dz}{2\pi i z}, \quad (2.36)$$

where  $t_1 t_2 = \sqrt{pq}$ , while in the magnetic SCIs  $S_8$ -symmetry is explicitly broken and we can get various inequivalently looking expressions. Let us split the initial 8 parameters into two sets

$$\{\pm\sqrt{q}, -\sqrt{pq}, s_1\} \quad \text{and} \quad \{\pm\sqrt{p}, -1, s_2\}$$

for which  $\rho = \sqrt{s_1}(q/p)^{1/4}$ . In terms of the parameters

$$s_{1,2,3,4} = \rho^{-1} \{\pm\sqrt{q}, -\sqrt{pq}, s_1\} \quad \text{and} \quad s_{5,6,7,8} = \rho \{\pm\sqrt{p}, -1, s_2\}$$

the magnetic SCI takes the form

$$I_M = \kappa \int_{\mathbb{T}} \frac{\prod_{i=1}^8 \Gamma(s_i z^{\pm 1}; p, q)}{\Gamma(z^{\pm 2}; p, q)} \frac{dz}{2\pi i z}. \quad (2.37)$$

After multiplication of both  $I_E$  and  $I_M$  by  $\prod_{i=1,2} \Gamma(s_i; p, q)$ , on the electric side we obtain the SCI for  $\mathcal{N} = 1$  SYM with  $SO(3)$  gauge group and on the magnetic side we have SCI of a  $\mathcal{N} = 1$  SYM theory with  $SP(2)$  gauge group with the fugacities chosen in a special way. This relation can be generalized to arbitrary number of colors  $N$  and to the theories discussed before Sect. 2.1.

### 3. $S$ -CONFINING THEORIES WITH THE SPINOR MATTER

In this chapter we systematically consider all known  $s$ -confining theories with  $SO(N)$ -gauge groups and the matter in spinor representation [22]. The upper parts of the tables contain information on the charges and field representation types of the electric models (except of the vector superfield). The lower part of the tables describes the  $s$ -confining phase of the theory. The models with the rank of the gauge group smaller than 6 are not considered because of different isomorphisms for orthogonal groups:  $SO(6) \simeq SU(4)$ ,  $SO(5) \simeq SP(4)$ ,  $SO(4) \simeq SU(2) \times SU(2)$ ,  $SO(3) \simeq SU(2)$ , and  $SO(2) \simeq U(1)$ .

For the orthogonal group  $SO(2N)$  there are two types of spinor representations: the proper spinor representation, which we denote as  $s$ , and its complex conjugate which is denoted as  $c$ , both representations have dimension  $2^{N-1}$ . For gauge group  $SO(2N+1)$  there exists only the spinor representation  $s$  which has the dimension  $2^N$ . Characters of the corresponding representations can be found in Appendix A.

#### 3.1. Confinement for $SO(7)$ gauge group.

3.1.1. *SU(6) flavor symmetry group.* The matter field content is

	$SO(7)$	$SU(6)$	$U(1)_R$
$S$	$s$	$f$	$2r = \frac{1}{6}$
$S^2$		$T_S$	$\frac{1}{3}$
$S^4$		$\bar{T}_A$	$\frac{2}{3}$

Corresponding SCIs have the form

$$I_E = \frac{(p; p)_\infty^3 (q; q)_\infty^3}{2^3 3!} \int_{\mathbb{T}^3} \frac{\prod_{i=1}^6 \Gamma(s_i (z_1 z_2 z_3)^{\pm 1}; p, q) \prod_{j=1}^3 \Gamma(s_j (z_j^{-2} z_1 z_2 z_3)^{\pm 1}; p, q)}{\prod_{j=1}^3 \Gamma(z_j^{\pm 2}; p, q) \prod_{1 \leq j < k \leq 3} \Gamma(z_j^{\pm 2} z_k^{\pm 2}; p, q)} \prod_{j=1}^3 \frac{dz_j}{2\pi i z_j}, \quad (3.1)$$

where  $|s_i| < 1$ , and

$$I_M = \prod_{i=1}^6 \Gamma(s_i^2; p, q) \prod_{1 \leq i < j \leq 6} \Gamma(s_i s_j, (pq)^{\frac{1}{2}} s_i^{-1} s_j^{-1}; p, q) \quad (3.2)$$

with the balancing condition  $\prod_{i=1}^6 s_i = (pq)^{1/2}$ .

To justify the suggested equality for the elliptic hypergeometric integrals  $I_E = I_M$  we checked the total ellipticity condition [112, 115] which is satisfied in this case, as well as for all other integral identities given below. One simple check of the limiting relation is easy to perform. Namely, in the limit  $p = q = 0$  and  $s_{2,3,4,5} = 0$  it is easy to see that the above equality for the elliptic hypergeometric integrals reduces to

$$\int_{\mathbb{T}^3} \frac{\prod_{1 \leq i < j \leq 3} (1 - z_i^{\pm 2} z_j^{\pm 2}) \prod_{j=1}^3 (1 - z_j^{\pm 2})}{(1 - (s_1 z_1 z_2 z_3)^{\pm 1}) \prod_{j=1}^3 (1 - s_1 (z_j^{-2} z_1 z_2 z_3)^{\pm 1})} \prod_{j=1}^3 \frac{dz_j}{2\pi i z_j} = \frac{2^3 3!}{1 - s_1^2}, \quad (3.3)$$

which is verified by direct residue calculus. To obtain this result one should carefully take into account the balancing condition. In computing this limit one should use the following reduction for the elliptic gamma function

$$\Gamma(z; p, q) \xrightarrow{p=0} \frac{1}{(z; q)_\infty} \xrightarrow{q=0} \frac{1}{1 - z}.$$

3.1.2. *SU(5)  $\times$  U(1) flavor group.* The matter content is

	$SO(7)$	$SU(5)$	$U(1)$	$U(1)_R$
$S$	$s$	$f$	1	0
$Q$	$f$	1	-5	1
$Q^2$		1	-10	2
$S^2$		$T_S$	2	0
$S^4$		$\bar{f}$	4	0
$S^2 Q$		$T_A$	-3	1
$S^4 Q$		$\bar{f}$	-1	1

Corresponding SCIs are

$$I_E = \frac{(p; p)_\infty^3 (q; q)_\infty^3}{2^3 3!} \Gamma(x; p, q) \int_{\mathbb{T}^3} \frac{\prod_{j=1}^3 \Gamma(x z_j^{\pm 2}; p, q)}{\prod_{j=1}^3 \Gamma(z_j^{\pm 2}; p, q) \prod_{1 \leq j < k \leq 3} \Gamma(z_j^{\pm 2} z_k^{\pm 2}; p, q)} \times \prod_{i=1}^5 \Gamma(s_i (z_1 z_2 z_3)^{\pm 1}; p, q) \prod_{j=1}^3 \Gamma(s_i \left(\frac{z_j^2}{z_1 z_2 z_3}\right)^{\pm 1}; p, q) \prod_{j=1}^3 \frac{dz_j}{2\pi i z_j}, \quad (3.4)$$



where  $|s_i| < 1$ , and

$$I_M = \Gamma(x^2; p, q) \prod_{i=1}^5 \Gamma\left(\frac{\sqrt{pq}}{s_i x}, \frac{\sqrt{pq}}{s_i}, s_i^2; p, q\right) \prod_{1 \leq i < j \leq 5} \Gamma(s_i s_j, x s_i s_j; p, q) \quad (3.5)$$

with the balancing condition  $x \prod_{i=1}^5 s_i = \sqrt{pq}$ .

Again, this  $s$ -confining duality predicts the exact integration formula  $I_E = I_M$ . Similar to the previous case, this identity is easily checked in the limit  $p = q = 0$  and  $s_{2,3,4} = 0$ , which leads to the relation

$$\int_{\mathbb{T}^3} \frac{\prod_{1 \leq i < j \leq 3} (1 - z_i^{\pm 2} z_j^{\pm 2}) \prod_{j=1}^3 (1 - z_j^{\pm 2})}{(1 - (s_1 z_1 z_2 z_3)^{\pm 1}) \prod_{j=1}^3 (1 - s_1 (z_j^{-2} z_1 z_2 z_3)^{\pm 1}) (1 - x z_j^{\pm 2})} \prod_{j=1}^3 \frac{dz_j}{2\pi i z_j} = \frac{2^3 3!}{(1 - s_1^2)(1 + x)}, \quad (3.6)$$

verified by the direct residue calculus.

3.1.3.  $SU(4) \times SU(2) \times U(1)$  flavor group. The matter content is

	$SO(7)$	$SU(4)$	$SU(2)$	$U(1)$	$U(1)_R$
$S$	$s$	$f$	1	1	0
$Q$	$f$	1	$f$	-2	$\frac{1}{2}$
$Q^2$		1	$T_S$	-4	1
$S^2$		$T_S$	1	2	0
$S^2 Q$		$T_A$	$f$	0	$\frac{1}{2}$
$S^2 Q^2$		$T_A$	1	-2	1
$S^4$		1	1	4	0
$S^4 Q$		1	$f$	2	$\frac{1}{2}$

Corresponding SCIs

$$I_E = \frac{(p; p)_\infty^3 (q; q)_\infty^3}{2^3 3!} \prod_{i=1}^2 \Gamma(t_i; p, q) \int_{\mathbb{T}^3} \frac{\prod_{i=1}^2 \prod_{j=1}^3 \Gamma(t_i z_j^{\pm 2}; p, q)}{\prod_{j=1}^3 \Gamma(z_j^{\pm 2}; p, q) \prod_{1 \leq j < k \leq 3} \Gamma(z_j^{\pm 2} z_k^{\pm 2}; p, q)} \times \prod_{i=1}^4 \Gamma(s_i (z_1 z_2 z_3)^{\pm 1}; p, q) \prod_{j=1}^3 \Gamma\left(s_i \left(\frac{z_j^2}{z_1 z_2 z_3}\right)^{\pm 1}; p, q\right) \prod_{j=1}^3 \frac{dz_j}{2\pi i z_j}, \quad (3.7)$$

where  $|s_i|, |t_j| < 1$ , and

$$I_M = \Gamma(s, t; p, q) \prod_{i=1}^2 \Gamma(st_i, t_i^2; p, q) \prod_{i=1}^4 \Gamma(s_i^2; p, q) \prod_{1 \leq i < j \leq 4} \Gamma(s_i s_j, t s_i s_j; p, q) \prod_{k=1}^2 \Gamma(s_i s_j t_k; p, q), \quad (3.8)$$

with  $s = \prod_{i=1}^4 s_i, t = \prod_{i=1}^2 t_i$  and the balancing condition  $st = \sqrt{pq}$ .

3.1.4.  $SU(3) \times SU(3) \times U(1)$  flavor group. The matter content is

	$SO(7)$	$SU(3)$	$SU(3)$	$U(1)$	$U(1)_R$
$S$	$s$	$f$	1	1	0
$Q$	$f$	1	$f$	-1	$\frac{1}{3}$
$Q^2$		1	$T_S$	-2	$\frac{2}{3}$
$S^2$		$T_S$	1	2	0
$S^2Q$		$\bar{f} = T_A$	$f$	1	$\frac{1}{3}$
$S^2Q^2$		$\bar{f}$	$\bar{f}$	0	$\frac{2}{3}$
$S^2Q^3$		$T_S$	1	-1	1

Corresponding SCIs are

$$I_E = \frac{(p; p)_\infty^3 (q; q)_\infty^3}{2^3 3!} \prod_{i=1}^3 \Gamma(t_i; p, q) \int_{\mathbb{T}^3} \frac{\prod_{i=1}^3 \prod_{j=1}^3 \Gamma(t_i z_j^{\pm 2}; p, q)}{\prod_{j=1}^3 \Gamma(z_j^{\pm 2}; p, q) \prod_{1 \leq j < k \leq 3} \Gamma(z_j^{\pm 2} z_k^{\pm 2}; p, q)} \\ \times \prod_{i=1}^3 \Gamma(s_i (z_1 z_2 z_3)^{\pm 1}; p, q) \prod_{i,j=1}^3 \Gamma(s_i \left(\frac{z_j^2}{z_1 z_2 z_3}\right)^{\pm 1}; p, q) \prod_{j=1}^3 \frac{dz_j}{2\pi i z_j}, \quad (3.9)$$

where  $|s_i|, |t_j| < 1$ , and

$$I_M = \prod_{i=1}^3 \Gamma(s_i^2, t_i^2, ts_i^2; p, q) \prod_{i,j=1}^3 \Gamma(sts_i^{-1} t_j^{-1}, ss_i^{-1} t_j; p, q) \prod_{1 \leq i < j \leq 3} \Gamma(s_i s_j, t_i t_j, ts_i s_j; p, q), \quad (3.10)$$

with  $s = \prod_{i=1}^3 s_i, t = \prod_{i=1}^3 t_i$ , and the balancing condition  $st = \sqrt{pq}$ .

### 3.2. $G = SO(8)$ .

3.2.1.  $SU(4) \times SU(3) \times U(1)$  flavor group. The matter content is

	$SO(8)$	$SU(4)$	$SU(3)$	$U(1)$	$U(1)_R$
$Q$	$f$	$f$	1	3	$\frac{1}{4}$
$S$	$s$	1	$f$	-4	0
$Q^2$		$T_S$	1	6	$\frac{1}{2}$
$S^2$		1	$T_S$	-8	0
$S^2Q^2$		$T_A$	$\bar{f}$	-2	$\frac{1}{2}$
$S^2Q^4$		1	$T_S$	4	1

Corresponding SCIs are

$$I_E = \frac{(p; p)_\infty^4 (q; q)_\infty^4}{2^3 4!} \int_{\mathbb{T}^4} \frac{\prod_{i=1}^4 \prod_{j=1}^4 \Gamma(s_i z_j^{\pm 2}; p, q)}{\prod_{1 \leq j < k \leq 4} \Gamma(z_j^{\pm 2} z_k^{\pm 2}; p, q)} \\ \times \prod_{i=1}^3 \Gamma(t_i (z_1 z_2 z_3 z_4)^{\pm 1}; p, q) \prod_{i=1}^3 \prod_{1 \leq j < k \leq 4} \Gamma(t_i \frac{z_j^2 z_k^2}{z_1 z_2 z_3 z_4}; p, q) \prod_{j=1}^4 \frac{dz_j}{2\pi i z_j}, \quad (3.11)$$

where  $|s_i|, |t_j| < 1$ , and

$$I_M = \prod_{i=1}^3 \Gamma(t_i^2, st_i^2; p, q) \prod_{i=1}^4 \Gamma(s_i^2; p, q) \prod_{1 \leq i < j \leq 3} \Gamma(t_i t_j, st_i t_j; p, q) \\ \times \prod_{1 \leq i < j \leq 4} \Gamma(s_i s_j; p, q) \prod_{1 \leq i < j \leq 4} \prod_{k=1}^3 \Gamma(ts_i s_j t_k^{-1}; p, q), \quad (3.12)$$

with  $s = \prod_{i=1}^4 s_i, t = \prod_{i=1}^3 t_i$ , and the balancing condition  $st = \sqrt{pq}$ .

Again, a simple check of the equality of these integrals is obtained in the limit  $p = q = 0$  and  $s_{2,3,4} = t_2 = 0$  leading to the following integral evaluation

$$\int_{\mathbb{T}^4} \frac{\prod_{1 \leq i < j \leq 4} (1 - z_i^{\pm 2} z_j^{\pm 2})}{(1 - t_1 (z_1 z_2 z_3 z_4)^{\pm 1}) \prod_{1 \leq i < j \leq 4} (1 - t_1 z_i^2 z_j^2 / z_1 z_2 z_3 z_4) \prod_{j=1}^4 (1 - s_1 z_j^{\pm 2})} \prod_{j=1}^4 \frac{dz_j}{2\pi i z_j} = \frac{2^3 4!}{(1 - s_1^2)(1 - t_1^2)}, \quad (3.13)$$

checked by direct residue calculus.

3.2.2.  $SU(4) \times SU(2) \times U(1)_1 \times U(1)_2$  flavor group. The matter content is

	$SO(8)$	$SU(4)$	$SU(2)$	$U(1)_1$	$U(1)_2$	$U(1)_R$
$Q$	$f$	$f$	1	1	0	$\frac{1}{4}$
$S$	$s$	1	$f$	-2	1	0
$S'$	$c$	1	1	0	-2	0
$Q^2$		$T_S$	1	2	0	$\frac{1}{2}$
$S^2$		1	$T_S$	-4	2	0
$S'^2$		1	1	0	-4	0
$S^2 Q^2$		$T_A$	1	-2	2	$\frac{1}{2}$
$S^2 Q^4$		1	$T_S$	0	2	1
$S'^2 Q^4$		1	1	4	-4	1
$SS'Q$		$f$	$f$	-1	-1	$\frac{1}{4}$
$SS'Q^3$		$\bar{f}$	$f$	1	-1	$\frac{3}{4}$

Corresponding SCIs are

$$I_E = \frac{(p; p)_\infty^4 (q; q)_\infty^4}{2^3 4!} \int_{\mathbb{T}^4} \frac{\prod_{i,j=1}^4 \Gamma(s_i z_j^{\pm 2}; p, q)}{\prod_{1 \leq j < k \leq 4} \Gamma(z_j^{\pm 2} z_k^{\pm 2}; p, q)} \prod_{i=1}^2 \Gamma(t_i (z_1 z_2 z_3 z_4)^{\pm 1}; p, q) \times \prod_{i=1}^2 \prod_{1 \leq j < k \leq 4} \Gamma\left(t_i \frac{z_j^2 z_k^2}{z_1 z_2 z_3 z_4}; p, q\right) \prod_{j=1}^4 \Gamma\left(u \left(\frac{z_j^2}{z_1 z_2 z_3 z_4}\right)^{\pm 1}; p, q\right) \prod_{j=1}^4 \frac{dz_j}{2\pi i z_j}, \quad (3.14)$$

where  $|s_i|, |t_j|, |u| < 1$ , and

$$I_M = \Gamma(u^2, su^2, t, st; p, q) \prod_{i=1}^2 \Gamma(t_i^2, st_i^2; p, q) \prod_{i=1}^4 \prod_{j=1}^2 \Gamma(us_i t_j, uss_i^{-1} t_j; p, q) \times \prod_{i=1}^4 \Gamma(s_i^2; p, q) \prod_{1 \leq i < j \leq 4} \Gamma(s_i s_j, ts_i s_j; p, q), \quad (3.15)$$

with  $s = \prod_{i=1}^4 s_i, t = \prod_{i=1}^2 t_i$ , and the balancing condition  $stu = \sqrt{pq}$ .

3.2.3.  $SU(3) \times SU(3) \times U(1)_1 \times U(1)_2$  flavor group. The matter content is

	$SO(8)$	$SU(3)$	$SU(3)$	$U(1)_1$	$U(1)_2$	$U(1)_R$
$Q$	$f$	1	1	0	6	1
$S$	$s$	$f$	1	1	-1	0
$S'$	$c$	1	$f$	-1	-1	0
$Q^2$		1	1	0	12	2
$S^2$		$T_S$	1	2	-2	0
$S'^2$		1	$T_S$	-2	-2	0
$SS'Q$		$f$	$f$	0	4	1
$S^3S'Q$		1	$f$	2	2	1
$SS'^3Q$		$f$	1	-2	2	1
$S^2S'^2$		$\bar{f}$	$\bar{f}$	0	-4	0

Corresponding SCIs are

$$\begin{aligned}
I_E &= \frac{(p;p)_\infty^4 (q;q)_\infty^4}{2^3 4!} \int_{\mathbb{T}^4} \frac{\prod_{j=1}^4 \Gamma(uz_j^{\pm 2}; p, q)}{\prod_{1 \leq j < k \leq 4} \Gamma(z_j^{\pm 2} z_k^{\pm 2}; p, q)} \prod_{i=1}^3 \Gamma(s_i (z_1 z_2 z_3 z_4)^{\pm 1}; p, q) \\
&\times \prod_{i=1}^3 \prod_{1 \leq j < k \leq 4} \Gamma\left(s_i \frac{z_j^2 z_k^2}{z_1 z_2 z_3 z_4}; p, q\right) \prod_{i=1}^3 \prod_{j=1}^4 \Gamma\left(t_i \left(\frac{z_j^2}{z_1 z_2 z_3 z_4}\right)^{\pm 1}; p, q\right) \prod_{j=1}^4 \frac{dz_j}{2\pi i z_j}, \quad (3.16)
\end{aligned}$$

where  $|s_i|, |t_j|, |u| < 1$ , and

$$\begin{aligned}
I_M &= \Gamma(u^2; p, q) \prod_{i=1}^3 \Gamma(s_i^2, t_i^2, sut_i, tus_i; p, q) \\
&\times \prod_{i,j=1}^3 \Gamma(us_i t_j, sts_i^{-1} t_j^{-1}; p, q) \prod_{1 \leq i < j \leq 3} \Gamma(s_i s_j, t_i t_j; p, q), \quad (3.17)
\end{aligned}$$

with  $s = \prod_{i=1}^3 s_i, t = \prod_{i=1}^3 t_i$ , and the balancing condition  $stu = \sqrt{pq}$ .

In the limit  $p = q = 0$  and  $s_{2,3} = t_2 \rightarrow 0$  one can check the equality of these elliptic hypergeometric integrals reducing to the relation

$$\begin{aligned}
&\int_{\mathbb{T}^4} \frac{\prod_{1 \leq i < j \leq 4} (1 - z_i^{\pm 2} z_j^{\pm 2})}{(1 - s_1 (z_1 z_2 z_3 z_4)^{\pm 1}) \prod_{1 \leq i < j \leq 4} (1 - s_1 z_i^2 z_j^2 / z_1 z_2 z_3 z_4) \prod_{j=1}^4 (1 - uz_j^{\pm 2})} \\
&\times \frac{1}{\prod_{j=1}^4 (1 - t_1 (z_j^2 / z_1 z_2 z_3 z_4)^{\pm 1})} \prod_{j=1}^4 \frac{dz_j}{2\pi i z_j} = \frac{2^3 4!}{(1 - s_1^2)(1 - t_1^2)(1 - u^2)(1 - s_1 t_1 u)}, \quad (3.18)
\end{aligned}$$

verified by direct residue calculus.

3.2.4.  $SU(3) \times SU(2)_1 \times SU(2)_2 \times U(1)_1 \times U(1)_2$  flavor group. The matter content is

	$SO(8)$	$SU(3)$	$SU(2)_1$	$SU(2)_2$	$U(1)_1$	$U(1)_2$	$U(1)_R$
$Q$	$f$	$f$	1	1	0	4	0
$S$	$s$	1	$f$	1	1	-3	$\frac{1}{4}$
$S'$	$c$	1	1	$f$	-1	-3	$\frac{1}{4}$
$Q^2$		$T_S$	1	1	0	8	0
$S^2$		1	$T_S$	1	2	-6	$\frac{1}{2}$
$S'^2$		1	1	$T_S$	-2	-6	$\frac{1}{2}$
$SS'Q$		$f$	$f$	$f$	0	-2	$\frac{1}{2}$
$S^2Q^2$		$\overline{f}$	1	1	2	2	$\frac{1}{2}$
$S'^2Q^2$		$\overline{f}$	1	1	-2	2	$\frac{1}{2}$
$SS'Q^3$		1	$f$	$f$	0	6	$\frac{1}{2}$
$S^2S'^2$		1	1	1	0	-12	1
$S^2S'^2Q^2$		$\overline{f}$	1	1	0	-4	1

Corresponding SCIs are

$$\begin{aligned}
 I_E = & \frac{(p; p)_\infty^4 (q; q)_\infty^4}{2^{34}!} \int_{\mathbb{T}^4} \frac{\prod_{i=1}^3 \prod_{j=1}^4 \Gamma(s_i z_j^{\pm 2}; p, q)}{\prod_{1 \leq j < k \leq 4} \Gamma(z_j^{\pm 2} z_k^{\pm 2}; p, q)} \prod_{i=1}^2 \Gamma(t_i (z_1 z_2 z_3 z_4)^{\pm 1}; p, q) \\
 & \times \prod_{i=1}^2 \prod_{1 \leq j < k \leq 4} \Gamma\left(t_i \frac{z_j^2 z_k^2}{z_1 z_2 z_3 z_4}; p, q\right) \prod_{i=1}^2 \prod_{j=1}^4 \Gamma\left(u_i \left(\frac{z_j^2}{z_1 z_2 z_3 z_4}\right)^{\pm 1}; p, q\right) \prod_{j=1}^4 \frac{dz_j}{2\pi i z_j}, \quad (3.19)
 \end{aligned}$$

where  $|s_i|, |t_j|, |u| < 1$ , and

$$\begin{aligned}
 I_M = & \Gamma(t, u, tu; p, q) \prod_{i=1}^3 \Gamma(s_i^2; p, q) \prod_{i=1}^2 \Gamma(t_i^2, u_i^2; p, q) \prod_{i,j=1}^2 \Gamma(st_i u_j; p, q) \\
 & \times \prod_{1 \leq i < j \leq 3} \Gamma(s_i s_j; p, q) \prod_{i=1}^3 \Gamma(st u s_i^{-1}, st s_i^{-1}, s u s_i^{-1}; p, q) \prod_{i=1}^2 \prod_{j,k=1}^2 \Gamma(s_i t_j u_k; p, q), \quad (3.20)
 \end{aligned}$$

with  $s = \prod_{i=1}^3 s_i$ ,  $t = \prod_{i=1}^2 t_i$ ,  $u = \prod_{i=1}^2 u_i$ , and the balancing condition  $stu = \sqrt{pq}$ .

### 3.3. $G = SO(9)$ .

3.3.1.  $SU(4)$  flavor group. The matter content is

	$SO(9)$	$SU(4)$	$U(1)_R$
$S$	$s$	$f$	$\frac{1}{8}$
$S^2$		$T_S$	$\frac{1}{4}$
$S^4$		$T_{AASS}$	$\frac{1}{2}$
$S^6$		$T_S$	$\frac{3}{4}$

where  $T_{AASS}$  represents the fourth rank tensor representation symmetric in two indices and antisymmetric in other two indices, whose character is given by the formula

$$\chi_{T_{AASS}, SU(4)}(\underline{s}) = \sum_{1 \leq i < j \leq 4} s_i^2 s_j^2 + \sum_{i=1}^4 \sum_{1 \leq j < k \leq 4; j, k \neq i} s_i^2 s_j s_k + 2.$$

Corresponding SCIs are

$$I_E = \frac{(p; p)_\infty^4 (q; q)_\infty^4}{2^{44}!} \int_{\mathbb{T}^4} \frac{\prod_{i=1}^4 \Gamma(s_i z^{\pm 1}; p, q) \prod_{i,j=1}^4 \Gamma(s_i (\frac{z_j^2}{z})^{\pm 1}; p, q)}{\prod_{i=1}^4 \Gamma(z_i^{\pm 2}; p, q) \prod_{1 \leq j < k \leq 4} \Gamma(z_j^{\pm 2} z_k^{\pm 2}; p, q)} \\ \times \prod_{i=1}^4 \prod_{1 \leq j < k \leq 4} \Gamma(s_i \frac{z_j^2 z_k^2}{z}; p, q) \prod_{j=1}^4 \frac{dz_j}{2\pi i z_j}, \quad (3.21)$$

where  $z = z_1 z_2 z_3 z_4$ ,  $|s_i|, |t_j|, |u| < 1$ , and

$$I_M = \Gamma^2(s; p, q) \prod_{i=1}^4 \Gamma(s_i^2, s s_i^2; p, q) \\ \times \prod_{1 \leq i < j \leq 4} \Gamma(s_i s_j, s s_i s_j, s_i^2 s_j^2; p, q) \prod_{i=1}^4 \prod_{1 \leq j < k \leq 4; j, k \neq i} \Gamma(s_i^2 s_j s_k; p, q), \quad (3.22)$$

with  $s = \prod_{i=1}^4 s_i$  and the balancing condition  $s^2 = \sqrt{pq}$ .

3.3.2.  $SU(3) \times SU(2) \times U(1)$  flavor group. The matter content is

	$SO(9)$	$SU(3)$	$SU(2)$	$U(1)$	$U(1)_R$
$S$	$s$	$f$	1	1	0
$Q$	$f$	1	$f$	-3	$\frac{1}{2}$
$Q^2$		1	$T_S$	-6	1
$S^2 Q$		$T_S$	$f$	-1	$\frac{1}{2}$
$S^2$		$T_S$	1	2	0
$S^4$		$\overline{T_S}$	1	4	0
$S^2 Q^2$		$\overline{f}$	1	-4	1
$S^4 Q^2$		$f$	1	-2	1
$S^4 Q$		$f$	$f$	1	$\frac{1}{2}$

Corresponding SCIs are

$$I_E = \frac{(p; p)_\infty^4 (q; q)_\infty^4}{2^{44}!} \prod_{i=1}^2 \Gamma(t_i; p, q) \int_{\mathbb{T}^4} \frac{\prod_{i=1}^2 \prod_{j=1}^4 \Gamma(t_i z_j^{\pm 2}; p, q) \prod_{i=1}^3 \Gamma(s_i z^{\pm 1}; p, q)}{\prod_{j=1}^4 \Gamma(z_j^{\pm 2}; p, q) \prod_{1 \leq j < k \leq 4} \Gamma(z_j^{\pm 2} z_k^{\pm 2}; p, q)} \\ \times \prod_{i=1}^3 \prod_{j=1}^4 \Gamma(s_i (z_j^2 z^{-1})^{\pm 1}; p, q) \prod_{i=1}^3 \prod_{1 \leq j < k \leq 4} \Gamma(s_i z_j^2 z_k^2 z^{-1}; p, q) \prod_{j=1}^4 \frac{dz_j}{2\pi i z_j}, \quad (3.23)$$

where  $z = z_1 z_2 z_3 z_4$ ,  $|s_i| < 1$ , and

$$I_M = \prod_{i=1}^2 \Gamma(t_i^2; p, q) \prod_{i=1}^3 \Gamma(s_i^2, s t s_i, s^2 s_i^{-1}, s t s_i^{-1}; p, q) \prod_{i=1}^3 \prod_{j=1}^2 \Gamma(s_i^2 t_j, s s_i t_j; p, q) \\ \times \Gamma(t; p, q) \prod_{1 \leq i < j \leq 3} \Gamma(s_i s_j, s^2 s_i^{-1} s_j^{-1}; p, q) \prod_{1 \leq i < j \leq 3} \prod_{k=1}^2 \Gamma(s_i s_j t_k; p, q), \quad (3.24)$$

with  $s = \prod_{i=1}^3 s_i$ ,  $t = \prod_{i=1}^2 t_i$ , and the balancing condition  $s^2 t = \sqrt{pq}$ .

3.3.3.  $SU(2) \times SU(4) \times U(1)$  flavor group. The matter content is

	$SO(9)$	$SU(2)$	$SU(4)$	$U(1)$	$U(1)_R$
$S$	$s$	$f$	1	1	$\frac{1}{4}$
$Q$	$f$	1	$f$	-1	0
$Q^2$		1	$T_S$	-2	0
$S^2Q$		$T_S$	$f$	1	$\frac{1}{2}$
$S^2$		$T_S$	1	2	$\frac{1}{2}$
$S^2Q^3$		1	$\bar{f}$	-1	$\frac{1}{2}$
$S^2Q^2$		1	$T_A$	0	$\frac{1}{2}$
$S^4Q^3$		1	$\bar{f}$	1	1
$S^2Q^4$		$T_S$	1	-2	$\frac{1}{2}$
$S^4$		1	1	4	1

Corresponding SCIs are

$$\begin{aligned}
 I_E &= \frac{(p;p)_\infty^4 (q;q)_\infty^4}{2^{44}!} \prod_{i=1}^4 \Gamma(t_i; p, q) \int_{\mathbb{T}^4} \frac{\prod_{i,j=1}^4 \Gamma(t_i z_j^{\pm 2}; p, q) \prod_{i=1}^2 \Gamma(s_i z^{\pm 1}; p, q)}{\prod_{j=1}^4 \Gamma(z_j^{\pm 2}; p, q) \prod_{1 \leq j < k \leq 4} \Gamma(z_j^{\pm 2} z_k^{\pm 2}; p, q)} \\
 &\times \prod_{i=1}^2 \prod_{j=1}^4 \Gamma(s_i (z_j^2 z^{-1})^{\pm 1}; p, q) \prod_{i=1}^2 \prod_{1 \leq j < k \leq 4} \Gamma(s_i z_j^2 z_k^2 z^{-1}; p, q) \prod_{j=1}^4 \frac{dz_j}{2\pi i z_j}, \quad (3.25)
 \end{aligned}$$

where  $z = z_1 z_2 z_3 z_4$ ,  $|s_i| < 1$ , and

$$\begin{aligned}
 I_M &= \Gamma(s, st, s^2; p, q) \prod_{i=1}^2 \Gamma(s_i^2, ts_i^2; p, q) \prod_{i=1}^4 \Gamma(t_i^2, st_i^2, stt_i^{-1}, s^2 tt_i^{-1}; p, q) \\
 &\times \prod_{i=1}^2 \prod_{j=1}^4 \Gamma(s_i^2 t_j; p, q) \prod_{1 \leq i < j \leq 4} \Gamma(t_i t_j, st_i t_j; p, q), \quad (3.26)
 \end{aligned}$$

with  $s = \prod_{i=1}^2 s_i$ ,  $t = \prod_{i=1}^4 t_i$ , and the balancing condition  $s^2 t = \sqrt{pq}$ .

3.4.  $G = SO(10)$ .

3.4.1.  $SU(4) \times U(1)$  flavor group. The matter content is

	$SO(10)$	$SU(4)$	$U(1)$	$U(1)_R$
$S$	$s$	$f$	1	0
$Q$	$f$	1	-8	1
$Q^2$		1	-16	2
$S^2Q$		$T_S$	-6	1
$S^4$		$T_{AASS}$	4	0
$S^6Q$		$T_S$	-2	1

Corresponding SCIs are

$$\begin{aligned}
 I_E &= \frac{(p;p)_\infty^5 (q;q)_\infty^5}{2^{45}!} \int_{\mathbb{T}^5} \frac{\prod_{i=1}^4 \Gamma(s_i z; p, q) \prod_{i=1}^4 \prod_{j=1}^5 \Gamma(s_i z_j^2 z^{-1}; p, q)}{\prod_{1 \leq j < k \leq 5} \Gamma(z_j^{\pm 2} z_k^{\pm 2}; p, q)} \\
 &\times \prod_{i=1}^4 \prod_{1 \leq j < k \leq 5} \Gamma(s_i z z_j^{-2} z_k^{-2}; p, q) \prod_{j=1}^5 \Gamma(t z_j^{\pm 2}; p, q) \prod_{j=1}^5 \frac{dz_j}{2\pi i z_j}, \quad (3.27)
 \end{aligned}$$

where  $z = z_1 z_2 z_3 z_4 z_5$ ,  $|s_i|, |t| < 1$ , and

$$I_M = \prod_{i=1}^4 \Gamma(ts_i^2, sts_i^2; p, q) \prod_{1 \leq i < j \leq 4} \Gamma(ts_i s_j, sts_i s_j, s_i^2 s_j^2; p, q) \\ \times \Gamma(t^2; p, q) \Gamma^2(s; p, q) \prod_{i=1}^4 \prod_{1 \leq j < k \leq 4, j, k \neq i} \Gamma(s_i^2 s_j s_k; p, q), \quad (3.28)$$

with  $s = \prod_{i=1}^4 s_i$  and the balancing condition  $s^2 t = \sqrt{pq}$ .

A simple check of the equality of these integrals is obtained in the limit  $p = q = 0$  and  $s_{2,3} = 0$  reducing it to the relation

$$\int_{\mathbb{T}^5} \frac{\prod_{1 \leq i < j \leq 5} (1 - z_i^{\pm 2} z_j^{\pm 2})}{(1 - s_1 z_1 z_2 z_3 z_4 z_5) \prod_{1 \leq i < j \leq 5} (1 - s_1 z_i^{-2} z_j^{-2} z_1 z_2 z_3 z_4 z_5) \prod_{j=1}^4 (1 - t z_j^{\pm 2})} \\ \times \frac{1}{\prod_{j=1}^5 (1 - s_1 z_i^2 / z_1 z_2 z_3 z_4 z_5)} \prod_{j=1}^5 \frac{dz_j}{2\pi i z_j} = \frac{2^{45}!}{(1 - s_1^2 t)(1 - t^2)}, \quad (3.29)$$

verified by the direct residue calculus.

3.4.2.  $SU(3) \times SU(3) \times U(1)$  flavor group. The matter content is

	$SO(10)$	$SU(3)$	$SU(3)$	$U(1)$	$U(1)_R$
$S$	$s$	$f$	1	1	0
$Q$	$f$	1	$f$	-2	$\frac{1}{3}$
$Q^2$		1	$T_S$	-4	$\frac{2}{3}$
$S^2 Q$		$T_S$	$f$	0	$\frac{1}{3}$
$S^2 Q^3$		$\bar{f}$	1	-4	1
$S^4$		$\bar{T}_S$	1	4	0
$S^4 Q^2$		$f$	$\bar{f}$	0	$\frac{2}{3}$

Corresponding SCIs are

$$I_E = \frac{(p; p)_\infty^5 (q; q)_\infty^5}{2^{45}!} \int_{\mathbb{T}^5} \frac{\prod_{i=1}^3 \Gamma(s_i z; p, q) \prod_{i=1}^3 \prod_{j=1}^5 \Gamma(s_i z_j^2 z^{-1}; p, q)}{\prod_{1 \leq j < k \leq 5} \Gamma(z_j^{\pm 2} z_k^{\pm 2}; p, q)} \\ \times \prod_{i=1}^3 \prod_{1 \leq j < k \leq 5} \Gamma(s_i z z_j^{-2} z_k^{-2}; p, q) \prod_{i=1}^3 \prod_{j=1}^5 \Gamma(t_i z_j^{\pm 2}; p, q) \prod_{j=1}^5 \frac{dz_j}{2\pi i z_j}, \quad (3.30)$$

where  $z = z_1 z_2 z_3 z_4 z_5$ ,  $|s_i|, |t_j| < 1$ , and

$$I_M = \prod_{i=1}^3 \Gamma(t_i^2, s^2 s_i^{-2}, sts_i^{-1}; p, q) \prod_{i,j=1}^3 \Gamma(s_i^2 t_j, sts_i t_j^{-1}; p, q) \\ \times \prod_{1 \leq i < j \leq 3} \Gamma(t_i t_j, s^2 s_i^{-1} s_j^{-1}; p, q) \prod_{k=1}^3 \prod_{1 \leq i < j \leq 3} \Gamma(s_i s_j t_k; p, q), \quad (3.31)$$

with  $s = \prod_{i=1}^3 s_i$ ,  $t = \prod_{i=1}^3 t_i$ , and the balancing condition  $s^2 t = \sqrt{pq}$ .



3.4.3.  $SU(2) \times SU(5) \times U(1)$  flavor group. The matter content is

	$SO(10)$	$SU(2)$	$SU(5)$	$U(1)$	$U(1)_R$
$S$	$s$	$f$	1	5	$\frac{1}{4}$
$Q$	$f$	1	$f$	-4	0
$Q^2$		1	$T_S$	-8	0
$S^2Q$		$T_S$	$f$	6	$\frac{1}{2}$
$S^2Q^3$		1	$\overline{T}_A$	-2	$\frac{1}{2}$
$S^2Q^5$		$T_S$	1	-10	$\frac{1}{2}$
$S^4$		1	1	20	1
$S^4Q^4$		1	$\overline{f}$	4	1

Corresponding SCIs are

$$\begin{aligned}
 I_E &= \frac{(p;p)_\infty^5 (q;q)_\infty^5}{245!} \int_{\mathbb{T}^5} \frac{\prod_{i=1}^2 \Gamma(s_i z; p, q) \prod_{i=1}^2 \prod_{j=1}^5 \Gamma(s_i z_j^2 z^{-1}; p, q)}{\prod_{1 \leq j < k \leq 5} \Gamma(z_j^{\pm 2} z_k^{\pm 2}; p, q)} \\
 &\times \prod_{i=1}^2 \prod_{1 \leq j < k \leq 5} \Gamma(s_i z z_j^{-2} z_k^{-2}; p, q) \prod_{i,j=1}^5 \Gamma(t_i z_j^{\pm 2}; p, q) \prod_{j=1}^5 \frac{dz_j}{2\pi i z_j}, \tag{3.32}
 \end{aligned}$$

where  $z = z_1 z_2 z_3 z_4 z_5$ ,  $|s_i|, |t_j| < 1$ , and

$$\begin{aligned}
 I_M &= \Gamma(st, s^2; p, q) \prod_{i=1}^2 \Gamma(ts_i^2; p, q) \prod_{i=1}^5 \Gamma(t_i^2, st_i, s^2 t_i^{-1}; p, q) \\
 &\times \prod_{i=1}^2 \prod_{j=1}^5 \Gamma(s_i^2 t_j; p, q) \prod_{1 \leq i < j \leq 5} \Gamma(t_i t_j, stt_i^{-1} t_j^{-1}; p, q), \tag{3.33}
 \end{aligned}$$

with  $s = \prod_{i=1}^2 s_i$ ,  $t = \prod_{i=1}^5 t_i$ , and the balancing condition  $s^2 t = \sqrt{pq}$ .

3.4.4.  $SU(3) \times U(1)_1 \times U(1)_2$  flavor group. The matter content is

	$SO(10)$	$SU(3)$	$U(1)_1$	$U(1)_2$	$U(1)_R$
$S$	$s$	$f$	1	0	0
$\overline{S}$	$c$	1	-3	1	0
$Q$	$f$	1	0	-2	1
$Q^2$		1	0	-4	2
$S^2Q$		$T_S$	2	-2	1
$S\overline{S}$		$f$	-2	1	0
$S^3\overline{S}Q$		$T_{AS}$	0	-1	1
$S^2\overline{S}^2$		$T_S$	-4	2	0
$S^4$		$\overline{T}_S$	4	0	0
$S^5\overline{S}$		$T_S$	2	1	0
$S^4\overline{S}^2Q$		$f$	-2	0	1
$\overline{S}^2Q$		1	-6	0	1
$S^3\overline{S}^3Q^2$		1	-6	-1	2

where  $T_{AS}$  stands for the rank three tensor representation which is totally symmetric in 1st and 2nd indices and totally antisymmetric in 2nd and 3rd indices.

Corresponding SCIs are

$$\begin{aligned}
I_E &= \frac{(p;p)_\infty^5 (q;q)_\infty^5}{2^{45}!} \int_{\mathbb{T}^5} \frac{\prod_{i=1}^3 \Gamma(s_i z; p, q) \prod_{i=1}^3 \prod_{j=1}^5 \Gamma(s_i z_j^2 z^{-1}; p, q)}{\prod_{1 \leq j < k \leq 5} \Gamma(z_j^{\pm 2} z_k^{\pm 2}; p, q)} \\
&\quad \times \prod_{i=1}^3 \prod_{1 \leq j < k \leq 5} \Gamma(s_i z z_j^{-2} z_k^{-2}; p, q) \Gamma(t z^{-1}; p, q) \prod_{j=1}^5 \Gamma(t z z_j^{-2}; p, q) \\
&\quad \times \prod_{1 \leq j < k \leq 5} \Gamma(t z_j^2 z_k^2 z^{-1}; p, q) \prod_{j=1}^5 \Gamma(u z_j^{\pm 2}; p, q) \prod_{j=1}^5 \frac{dz_j}{2\pi i z_j}, \tag{3.34}
\end{aligned}$$

where  $z = z_1 z_2 z_3 z_4 z_5$ ,  $|s_i|, |t|, |u| < 1$ , and

$$\begin{aligned}
I_M &= \Gamma(u^2, t^2 u, st^3 u^2; p, q) \Gamma^2(stu; p, q) \prod_{i=1}^3 \Gamma(ts_i, us_i^2, t^2 s_i^2, s^2 s_i^{-2}, sts_i^2, st^2 us_i; p, q) \\
&\quad \times \prod_{1 \leq i < j \leq 3} \Gamma(us_i s_j, t^2 s_i s_j, sts_i s_j, s^2 s_i^{-1} s_j^{-1}; p, q) \prod_{i,j=1; i \neq j}^3 \Gamma(tus_i^2 s_j; p, q), \tag{3.35}
\end{aligned}$$

with  $s = \prod_{i=1}^3 s_i$  and the balancing condition  $s^2 t^2 u = \sqrt{pq}$ .

3.4.5.  $SU(2) \times SU(3) \times U(1)_1 \times U(1)_2$  flavor group. The matter content is

	$SO(10)$	$SU(2)$	$SU(3)$	$U(1)_1$	$U(1)_2$	$U(1)_R$
$S$	$s$	$f$	1	1	1	0
$\bar{S}$	$c$	1	1	-2	1	$\frac{1}{2}$
$Q$	$f$	1	$f$	0	-2	0
$Q^2$		1	$T_S$	0	-4	0
$S^2 Q$		$T_S$	$f$	2	0	0
$\bar{S}^2 Q$		1	$f$	-4	0	1
$S\bar{S}$		$f$	1	-1	2	$\frac{1}{2}$
$S^2 \bar{S}^2$		$T_S$	1	-2	4	1
$S^2 Q^3$		1	1	2	-4	0
$S^3 \bar{S} Q$		$f$	$f$	1	2	$\frac{1}{2}$
$S^4$		1	1	4	4	0
$S\bar{S} Q^2$		$f$	$\bar{f}$	-1	-2	$\frac{1}{2}$
$S^2 \bar{S}^2 Q^2$		1	$\bar{f}$	-2	0	1
$S^3 \bar{S} Q^3$		$f$	1	1	-2	$\frac{1}{2}$

Corresponding SCIs are

$$\begin{aligned}
I_E &= \frac{(p;p)_\infty^5 (q;q)_\infty^5}{2^{45}!} \int_{\mathbb{T}^5} \frac{\prod_{i=1}^2 \Gamma(s_i z; p, q) \prod_{i=1}^2 \prod_{j=1}^5 \Gamma(s_i z_j^2 z^{-1}; p, q)}{\prod_{1 \leq j < k \leq 5} \Gamma(z_j^{\pm 2} z_k^{\pm 2}; p, q)} \\
&\quad \times \prod_{i=1}^2 \prod_{1 \leq j < k \leq 5} \Gamma(s_i z z_j^{-2} z_k^{-2}; p, q) \Gamma(t z^{-1}; p, q) \prod_{j=1}^5 \Gamma(t z z_j^{-2}; p, q) \\
&\quad \times \prod_{1 \leq j < k \leq 5} \Gamma(t z_j^2 z_k^2 z^{-1}; p, q) \prod_{j=1}^5 \Gamma(u z_j^{\pm 2}; p, q) \prod_{j=1}^5 \frac{dz_j}{2\pi i z_j}, \tag{3.36}
\end{aligned}$$

where  $z = z_1 z_2 z_3 z_4 z_5$ ,  $|s_i|, |t|, |u_j| < 1$ , and

$$\begin{aligned}
 I_M = & \Gamma(s^2, su, st^2; p, q) \prod_{i=1}^2 \Gamma(ts_i, t^2 s_i^2, stus_i; p, q) \prod_{i=1}^3 \Gamma(su_i, u_i^2, t^2 u_i, st^2 u u_i^{-1}; p, q) \\
 & \times \prod_{i=1}^2 \prod_{j=1}^3 \Gamma(s_i^2 u_j, sts_i u_j, tus_i u_j^{-1}; p, q) \prod_{1 \leq i < j \leq 3} \Gamma(u_i u_j; p, q), \tag{3.37}
 \end{aligned}$$

with  $s = \prod_{i=1}^3 s_i$  and the balancing condition  $s^2 t^2 u = \sqrt{pq}$ .

3.4.6.  $SU(5) \times U(1)_1 \times U(1)_2$  flavor group. The matter content is

	$SO(10)$	$SU(5)$	$U(1)_1$	$U(1)_2$	$U(1)_R$
$S$	$s$	1	1	5	$\frac{1}{4}$
$\bar{S}$	$c$	1	-1	5	$\frac{1}{4}$
$Q$	$f$	$f$	0	-4	0
$Q^2$		$T_S$	0	-8	0
$S^2 Q$		$f$	2	6	$\frac{1}{2}$
$\bar{S}^2 Q$		$f$	-2	6	$\frac{1}{2}$
$S\bar{S}$		1	0	10	$\frac{1}{2}$
$S^2 Q^5$		1	2	-10	$\frac{1}{2}$
$\bar{S}^2 Q^5$		1	-2	-10	$\frac{1}{2}$
$S\bar{S}Q^2$		$T_A$	0	2	$\frac{1}{2}$
$S\bar{S}Q^4$		$\bar{f}$	0	-6	$\frac{1}{2}$
$S^2 \bar{S}^2$		1	0	20	1
$S^2 \bar{S}^2 Q^4$		$\bar{f}$	0	4	1

Corresponding SCIs are

$$\begin{aligned}
 I_E = & \frac{(p; p)_\infty^5 (q; q)_\infty^5}{2^{45}!} \int_{\mathbb{T}^5} \frac{\Gamma(sz; p, q) \prod_{j=1}^5 \Gamma(sz_j^2 z^{-1}; p, q)}{\prod_{1 \leq j < k \leq 5} \Gamma(z_j^{\pm 2} z_k^{\pm 2}; p, q)} \\
 & \times \prod_{1 \leq j < k \leq 5} \Gamma(szz_j^{-2} z_k^{-2}; p, q) \Gamma(tz^{-1}; p, q) \prod_{j=1}^5 \Gamma(tzz_j^{-2}; p, q) \\
 & \times \prod_{1 \leq j < k \leq 5} \Gamma(tz_j^2 z_k^2 z^{-1}; p, q) \prod_{i,j=1}^5 \Gamma(u_i z_j^{\pm 2}; p, q) \prod_{j=1}^5 \frac{dz_j}{2\pi i z_j}, \tag{3.38}
 \end{aligned}$$

where  $z = z_1 z_2 z_3 z_4 z_5$ ,  $|s|, |t|, |u_i| < 1$ , and

$$\begin{aligned}
 I_M = & \Gamma(st, su, tu, s^2 t^2; p, q) \prod_{i=1}^5 \Gamma(u_i^2, s^2 u_i, t^2 u_i, stuu_i^{-1}, s^2 t^2 uu_i^{-1}; p, q) \\
 & \times \prod_{1 \leq i < j \leq 5} \Gamma(u_i u_j, stu_i u_j; p, q), \tag{3.39}
 \end{aligned}$$

with  $u = \prod_{i=1}^5 u_i$  and the balancing condition  $s^2 t^2 u = \sqrt{pq}$ .

3.4.7.  $SU(2)_1 \times SU(2)_2 \times U(1)_1 \times U(1)_2$  flavor group. The matter content is

	$SO(10)$	$SU(2)_1$	$SU(2)_2$	$U(1)_1$	$U(1)_2$	$U(1)_R$
$S$	$s$	$f$	1	1	1	0
$\bar{S}$	$c$	1	$f$	-1	1	0
$Q$	$f$	1	1	0	-8	1
$Q^2$		1	1	0	-16	2
$S^2Q$		$T_S$	1	2	-6	1
$\bar{S}^2Q$		1	$T_S$	-2	-6	1
$S\bar{S}$		$f$	$f$	0	2	0
$S^4$		1	1	4	4	0
$\bar{S}^4$		1	1	-4	4	0
$S^2\bar{S}^2$		$T_S$	$T_S$	0	4	0
$S^3\bar{S}Q$		$f$	$f$	2	-4	1
$S\bar{S}^3Q$		$f$	$f$	-2	-4	1
$S^2\bar{S}^2Q^2$		1	1	0	-12	2
$S^4\bar{S}^2Q$		$T_S$	1	2	-2	1
$S^2\bar{S}^4Q$		1	$T_S$	-2	-2	1
$S^3\bar{S}^3$		$f$	$f$	0	6	0
$S^6\bar{S}^2$		1	1	4	8	0
$S^2\bar{S}^6$		1	1	-4	8	0

Corresponding SCIs are

$$\begin{aligned}
I_E = & \frac{(p; p)_\infty^5 (q; q)_\infty^5}{2^{45}!} \int_{\mathbb{T}^5} \frac{\prod_{i=1}^2 \Gamma(s_i z, t_i z^{-1}; p, q) \prod_{i=1}^2 \prod_{j=1}^5 \Gamma(s_i z_j^2 z^{-1}, t_i z z_j^{-2}; p, q)}{\prod_{1 \leq j < k \leq 5} \Gamma(z_j^{\pm 2} z_k^{\pm 2}; p, q)} \\
& \times \prod_{i=1}^2 \prod_{1 \leq j < k \leq 5} \Gamma(s_i z z_j^{-2} z_k^{-2}, t_i z_j^2 z_k^2 z^{-1}; p, q) \prod_{j=1}^5 \Gamma(u z_j^{\pm 2}; p, q) \prod_{j=1}^5 \frac{dz_j}{2\pi i z_j}, \quad (3.40)
\end{aligned}$$

where  $z = z_1 z_2 z_3 z_4 z_5$ ,  $|s_i|, |t_i|, |u| < 1$ , and

$$\begin{aligned}
I_M = & \Gamma(s^2, t^2, u^2, st, su, tu, s^3 t, st^3, stu^2, s^2 tu, st^2 u; p, q) \\
& \times \prod_{i=1}^2 \Gamma(us_i^2, ut_i^2, st_i^2, ts_i^2, stus_i^2, stut_i^2; p, q) \prod_{i,j=1}^2 \Gamma(s_i t_j, s_i^2 t_j^2, sus_i t_j, tus_i t_j, sts_i t_j; p, q), \quad (3.41)
\end{aligned}$$

with  $s = \prod_{i=1}^3 s_i$  and the balancing condition  $s^2 t^2 u = \sqrt{pq}$ .

3.5.  $G = SO(11)$ .

3.5.1.  $SU(6) \times U(1)$  flavor group. The matter content is

	$SO(11)$	$SU(6)$	$U(1)$	$U(1)_R$
$S$	$s$	1	3	$\frac{1}{4}$
$Q$	$f$	$f$	-2	0
$Q^2$		$T_S$	-4	0
$S^2Q^2$		$T_A$	2	$\frac{1}{2}$
$S^2Q^5$		$\bar{f}$	-4	$\frac{1}{2}$
$S^4$		1	12	1
$S^4Q^5$		$\bar{f}$	2	1
$S^2Q$		$f$	4	$\frac{1}{2}$
$S^2Q^6$		1	-6	$\frac{1}{2}$

Corresponding SCIs are

$$\begin{aligned}
 I_E = & \frac{(p;p)_\infty^5 (q;q)_\infty^5}{2^5 5!} \prod_{i=1}^6 \Gamma(t_i; p, q) \int_{\mathbb{T}^5} \frac{\Gamma(sz^{\pm 1}; p, q) \prod_{j=1}^5 \Gamma(s(z_j^2 z^{-1})^{\pm 1}; p, q)}{\prod_{j=1}^5 \Gamma(z_j^{\pm 2}; p, q) \prod_{1 \leq j < k \leq 5} \Gamma(z_j^{\pm 2} z_k^{\pm 2}; p, q)} \\
 & \times \prod_{1 \leq j < k \leq 5} \Gamma(s(z z_j^{-2} z_k^{-2})^{\pm 1}; p, q) \prod_{i=1}^6 \prod_{j=1}^5 \Gamma(t_i z_j^{\pm 2}; p, q) \prod_{j=1}^5 \frac{dz_j}{2\pi i z_j}, \quad (3.42)
 \end{aligned}$$

where  $z = z_1 z_2 z_3 z_4 z_5$ ,  $|s|, |t_i| < 1$ , and

$$I_M = \Gamma(s^4, s^2 t; p, q) \prod_{i=1}^6 \Gamma(s^2 t_i, t_i^2, s^2 t t_i^{-1}, s^4 t t_i^{-1}; p, q) \prod_{1 \leq i < j \leq 6} \Gamma(t_i t_j, s^2 t_i t_j; p, q), \quad (3.43)$$

with  $t = \prod_{i=1}^6 t_i$  and the balancing condition  $s^4 t = \sqrt{pq}$ .

3.5.2.  $SU(2)_1 \times SU(2)_2 \times U(1)$  flavor group. This  $s$ -confining theory was obtained in [80]. The matter content is

	$SO(11)$	$SU(2)_1$	$SU(2)_2$	$U(1)$	$U(1)_R$
$S$	$s$	$f$	1	1	0
$Q$	$f$	1	$f$	-4	$\frac{1}{2}$
$Q^2$		1	$T_S$	-8	1
$S^2Q^2$		$T_S$	1	-6	1
$S^2Q$		$T_S$	$f$	-2	$\frac{1}{2}$
$S^2$		1	1	2	0
$S^4$		$T_{4S}$	1	4	0
$S^{4'}$		1	1	4	0
$S^4Q^2$		1	$T_S$	-4	1
$S^4Q^{2'}$		$T_S$	1	-4	1
$S^4Q$		$T_S$	$f$	0	$\frac{1}{2}$
$S^6Q^2$		$T_S$	1	-2	1
$S^6Q$		$T_S$	$f$	2	$\frac{1}{2}$
$S^8$		1	1	8	0
$S^8Q$		1	$f$	4	$\frac{1}{2}$
$S^4Q$		1	$f$	0	$\frac{1}{2}$
$S^6$		1	1	6	0

Corresponding SCIs are

$$I_E = \frac{(p; p)_\infty^5 (q; q)_\infty^5}{2^5 5!} \prod_{i=1}^2 \Gamma(t_i; p, q) \int_{\mathbb{T}^5} \frac{\prod_{i=1}^2 \Gamma(s_i z^{\pm 1}; p, q) \prod_{i=1}^2 \prod_{j=1}^5 \Gamma(s_i (z_j^2 z^{-1})^{\pm 1}; p, q)}{\prod_{j=1}^5 \Gamma(z_j^{\pm 2}; p, q) \prod_{1 \leq j < k \leq 5} \Gamma(z_j^{\pm 2} z_k^{\pm 2}; p, q)} \\ \times \prod_{i=1}^2 \prod_{1 \leq j < k \leq 5} \Gamma(s_i (z z_j^{-2} z_k^{-2})^{\pm 1}; p, q) \prod_{i=1}^2 \prod_{j=1}^5 \Gamma(t_i z_j^{\pm 2}; p, q) \prod_{j=1}^5 \frac{dz_j}{2\pi i z_j}, \quad (3.44)$$

where  $z = z_1 z_2 z_3 z_4 z_5$ ,  $|s|, |t_i| < 1$ , and

$$I_M = \Gamma(s, t, st, s^3, s^3 t, s^4; p, q) \Gamma^2(s, s^2 t; p, q) \prod_{i=1}^2 \Gamma(t_i^2, st_i, ts_i^2, s^2 t_i, ss_i^2; p, q) \\ \times \prod_{i=1}^2 \Gamma(s^2 t_i, s^2 t_i^2, sts_i^2, s^2 ts_i^2, s^3 t_i, s_i^4, s^4 t_i; p, q) \prod_{i,j=1}^2 \Gamma(s_i^2 t_j, ss_i^2 t_j, s^2 s_i^2 t_j; p, q), \quad (3.45)$$

with  $s = \prod_{i=1}^2 s_i$ ,  $t = \prod_{i=1}^2 t_i$ , and the balancing condition  $s^4 t = \sqrt{pq}$ .

### 3.6. $G = SO(12)$ .

3.6.1.  $SU(7) \times U(1)$  flavor group. The matter content is

	$SO(12)$	$SU(7)$	$U(1)$	$U(1)_R$
$S$	$s$	1	7	$\frac{1}{4}$
$Q$	$f$	$f$	-4	0
$Q^2$		$T_S$	-8	0
$S^2 Q^2$		$T_A$	6	$\frac{1}{2}$
$S^2 Q^6$		$\bar{f}$	-10	$\frac{1}{2}$
$S^4$		1	28	1
$S^4 Q^6$		$\bar{f}$	4	1

Corresponding SCIs are

$$I_E = \frac{(p; p)_\infty^6 (q; q)_\infty^6}{2^5 6!} \int_{\mathbb{T}^6} \frac{\Gamma(sz^{\pm 1}; p, q)}{\prod_{1 \leq j < k \leq 6} \Gamma(z_j^{\pm 2} z_k^{\pm 2}; p, q)} \\ \times \prod_{1 \leq j < k \leq 6} \Gamma(s (z z_j^{-2} z_k^{-2})^{\pm 1}; p, q) \prod_{i=1}^7 \prod_{j=1}^6 \Gamma(t_i z_j^{\pm 2}; p, q) \prod_{j=1}^6 \frac{dz_j}{2\pi i z_j}, \quad (3.46)$$

where  $z = z_1 z_2 z_3 z_4 z_5 z_6$ ,  $|s|, |t_i| < 1$ , and

$$I_M = \Gamma(s^4; p, q) \prod_{i=1}^7 \Gamma(t_i^2, s^2 t_i^{-1}, s^4 t_i^{-1}; p, q) \prod_{1 \leq i < j \leq 6} \Gamma(t_i t_j, s^2 t_i t_j; p, q), \quad (3.47)$$

with  $t = \prod_{i=1}^7 t_i$  and the balancing condition  $s^4 t = \sqrt{pq}$ .

3.6.2.  $SU(2) \times SU(3) \times U(1)$  flavor group. The matter content is

	$SO(12)$	$SU(2)$	$SU(3)$	$U(1)$	$U(1)_R$
$S$	$s$	$f$	1	3	$\frac{1}{8}$
$Q$	$f$	1	$f$	-8	0
$Q^2$		1	$T_S$	-16	0
$S^2$		1	1	6	$\frac{1}{4}$
$S^2Q^2$		$T_S$	$\bar{f}$	-10	$\frac{1}{4}$
$S^4$		$T_{4S}$	1	12	$\frac{1}{2}$
$S^4Q^2$		1	$T_S$	-4	$\frac{1}{2}$
$S^4Q^{2'}$		$T_S$	$\bar{f}$	-4	$\frac{1}{2}$
$S^6$		1	1	18	$\frac{3}{4}$
$S^6Q^2$		$T_S$	$\bar{f}$	2	$\frac{3}{4}$
$S^8Q^2$		1	$T_S$	8	1

Corresponding SCIs are

$$\begin{aligned}
 I_E &= \frac{(p; p)_\infty^6 (q; q)_\infty^6}{2^5 6!} \int_{\mathbb{T}^6} \frac{\prod_{i=1}^2 \Gamma(s_i z^{\pm 1}; p, q)}{\prod_{1 \leq j < k \leq 6} \Gamma(z_j^{\pm 2} z_k^{\pm 2}; p, q)} \\
 &\times \prod_{i=1}^2 \prod_{1 \leq j < k \leq 6} \Gamma(s_i (z z_j^{-2} z_k^{-2})^{\pm 1}; p, q) \prod_{i=1}^3 \prod_{j=1}^6 \Gamma(t_i z_j^{\pm 2}; p, q) \prod_{j=1}^6 \frac{dz_j}{2\pi i z_j}, \quad (3.48)
 \end{aligned}$$

where  $z = z_1 z_2 z_3 z_4 z_5 z_6$ ,  $|s_i|, |t_i| < 1$ , and

$$\begin{aligned}
 I_M &= \Gamma(s, s^2, s^3; p, q) \prod_{i=1}^2 \Gamma(ss_i^2, s_i^4; p, q) \prod_{i=1}^3 \Gamma(t_i^2, s^2 t_i^2, stt_i^{-1}, s^2 t_i^{-1}, s^4 t_i^2, s^3 t_i^{-1}; p, q) \\
 &\times \prod_{i=1}^2 \prod_{j=1}^3 \Gamma(sts_i^2 t_j^{-1}, s^2 ts_i^2 t_j^{-1}, ts_i^2 t_j^{-1}; p, q) \prod_{1 \leq i < j \leq 3} \Gamma(t_i t_j, s^2 t_i t_j, s^4 t_i t_j; p, q), \quad (3.49)
 \end{aligned}$$

with  $s = \prod_{i=1}^2 s_i$ ,  $t = \prod_{i=1}^3 t_i$ , and the balancing condition  $s^4 t = \sqrt{pq}$ .

3.6.3.  $SU(3) \times U(1)_1 \times U(1)_2$  flavor group. The matter content is

	$SO(12)$	$SU(3)$	$U(1)_1$	$U(1)_2$	$U(1)_R$
$S$	$s$	1	1	3	$\frac{1}{8}$
$S'$	$c$	1	-1	3	$\frac{1}{8}$
$Q$	$f$	$f$	0	-8	0
$Q^2$		$T_S$	0	-16	0
$SS'Q^3$		1	0	-18	$\frac{1}{4}$
$S^2Q^2$		$\bar{f}$	2	-10	$\frac{1}{4}$
$S'^2Q^2$		$\bar{f}$	-2	-10	$\frac{1}{4}$
$SS'Q$		$f$	0	-2	$\frac{1}{4}$
$S^4$		1	4	12	$\frac{1}{2}$
$S'^4$		1	-4	12	$\frac{1}{2}$
$S^2S'^2$		1	0	12	$\frac{1}{2}$
$S^3S'Q^3$		1	2	-12	$\frac{1}{2}$
$SS'^3Q^3$		1	-2	-12	$\frac{1}{2}$
$S^2S'^2Q^2$		$T_S$	0	-4	$\frac{1}{2}$
$S^2S'^2Q^2'$		$\bar{f}$	0	-4	$\frac{1}{2}$
$S^3S'Q$		$f$	2	4	$\frac{1}{2}$
$SS'^3Q^2$		$f$	-2	4	$\frac{1}{2}$
$S^3S'^3Q^3$		1	0	-6	$\frac{3}{4}$
$S^3S'^3Q$		$f$	0	10	$\frac{3}{4}$
$S^4S'^2Q^2$		$\bar{f}$	2	2	$\frac{3}{4}$
$S^2S'^4Q^2$		$\bar{f}$	-2	2	$\frac{3}{4}$
$S^4S'^4$		1	0	24	1
$S^4S'^4Q^2$		$\bar{f}$	0	8	1

Corresponding SCIs are

$$\begin{aligned}
I_E &= \frac{(p; p)_\infty^6 (q; q)_\infty^6}{2^{56} 6!} \int_{\mathbb{T}^6} \frac{\Gamma(sz^{\pm 1}; p, q)}{\prod_{1 \leq j < k \leq 6} \Gamma(z_j^{\pm 2} z_k^{\pm 2}; p, q)} \prod_{1 \leq j < k \leq 6} \Gamma(s(zz_j^{-2} z_k^{-2})^{\pm 1}; p, q) \\
&\times \prod_{i=1}^6 \Gamma(t(z_i^2 z^{-1})^{\pm 1}; p, q) \prod_{1 \leq i < j < k \leq 6} \Gamma(t(z_i^2 z_j^2 z_k^2 z^{-1})^{\pm 1}; p, q) \prod_{i=1}^3 \prod_{j=1}^6 \Gamma(u_i z_j^{\pm 2}; p, q) \prod_{j=1}^6 \frac{dz_j}{2\pi i z_j}, \quad (3.50)
\end{aligned}$$

where  $z = z_1 z_2 z_3 z_4 z_5 z_6$ ,  $|s|, |t|, |u_i| < 1$ , and

$$\begin{aligned}
I_M &= \Gamma(stu, s^4, t^4, s^2 t^2, st^3 u, s^3 t u, s^3 t^3 u, s^4 t^4; p, q) \prod_{1 \leq i < j \leq 3} \Gamma(u_i u_j, s^2 t^2 u_i u_j; p, q) \\
&\times \prod_{i=1}^3 \Gamma(s^3 t u_i, st^3 u_i, s^3 t^3 u_i, s^4 t^2 u u_i^{-1}, s^2 t^4 u u_i^{-1}, s^4 t^4 u u_i^{-1}; p, q) \\
&\times \prod_{i=1}^3 \Gamma(u_i^2, s^2 u u_i^{-1}, t^2 u u_i^{-1}, st u_i, s^2 t^2 u_i^2, s^2 t^2 u u_i^{-1}; p, q), \quad (3.51)
\end{aligned}$$

with  $u = \prod_{i=1}^3 u_i$  and the balancing condition  $(st)^4 u = \sqrt{pq}$ .

3.7.  $G = SO(13)$ .



3.7.1.  $SU(4) \times U(1)$  flavor group. The matter content is

	$SO(13)$	$SU(4)$	$U(1)$	$U(1)_R$
$S$	$s$	1	1	$\frac{1}{8}$
$Q$	$f$	$f$	-2	0
$Q^2$		$T_S$	-4	0
$S^2Q^3$		$\bar{f}$	-4	$\frac{1}{4}$
$S^2Q^2$		$T_A$	-2	$\frac{1}{4}$
$S^4Q^4$		1	-4	$\frac{1}{2}$
$S^4Q^3$		$\bar{f}$	-2	$\frac{1}{2}$
$S^4Q^2$		$T_S$	0	$\frac{1}{2}$
$S^4Q$		$f$	2	$\frac{1}{2}$
$S^4$		1	4	$\frac{1}{2}$
$S^6Q^3$		$\bar{f}$	0	$\frac{3}{4}$
$S^6Q^2$		$T_A$	2	$\frac{3}{4}$
$S^8Q^3$		$\bar{f}$	2	1
$S^8$		1	8	1

Corresponding SCIs are

$$\begin{aligned}
 I_E &= \frac{(p; p)_\infty^6 (q; q)_\infty^6}{2^{66}!} \prod_{i=1}^4 \Gamma(t_i; p, q) \int_{\mathbb{T}^6} \frac{\Gamma(sz^{\pm 1}; p, q) \prod_{j=1}^6 \Gamma(s(z_j^2 z^{-1})^{\pm 1}; p, q)}{\prod_{j=1}^6 \Gamma(z_j^{\pm 2}; p, q) \prod_{1 \leq j < k \leq 6} \Gamma(z_j^{\pm 2} z_k^{\pm 2}; p, q)} \quad (3.52) \\
 &\times \prod_{1 \leq j < k \leq 6} \Gamma(s(z z_j^{-2} z_k^{-2})^{\pm 1}; p, q) \prod_{1 \leq i < j < k \leq 6} \Gamma(s z_i^2 z_j^2 z_k^2 z^{-1}; p, q) \prod_{i=1}^4 \prod_{j=1}^6 \Gamma(t_i z_j^{\pm 2}; p, q) \prod_{j=1}^6 \frac{dz_j}{2\pi i z_j},
 \end{aligned}$$

where  $z = z_1 z_2 z_3 z_4 z_5 z_6$ ,  $|s|, |t_i| < 1$ , and

$$\begin{aligned}
 I_M &= \Gamma(s^4, s^4 t, s^8; p, q) \prod_{i=1}^4 \Gamma(t_i^2, s^2 t t_i^{-1}, s^4 t t_i^{-1}, s^4 t_i^2, s^4 t_i, s^6 t t_i^{-1}, s^8 t t_i^{-1}; p, q) \\
 &\times \prod_{1 \leq i < j \leq 4} \Gamma(t_i t_j, s^2 t_i t_j, s^4 t_i t_j, s^6 t_i t_j; p, q), \quad (3.53)
 \end{aligned}$$

with  $t = \prod_{i=1}^4 t_i$  and the balancing condition  $s^8 t = \sqrt{pq}$ .

3.8.  $G = SO(14)$ .

3.8.1.  $SU(5) \times U(1)$  flavor group. The matter content is

	$SO(14)$	$SU(5)$	$U(1)$	$U(1)_R$
$S$	$s$	1	5	$\frac{1}{8}$
$Q$	$f$	$f$	-8	0
$Q^2$		$T_S$	-16	0
$S^2Q^3$		$\bar{T}_A$	-14	$\frac{1}{4}$
$S^4Q^2$		$T_S$	4	$\frac{1}{2}$
$S^4Q^4$		$\bar{f}$	-12	$\frac{1}{2}$
$S^6Q^3$		$\bar{T}_A$	6	$\frac{3}{4}$
$S^8$		1	40	1
$S^8Q^4$		$\bar{f}$	8	1

Corresponding SCIs are

$$I_E = \frac{(p; p)_\infty^7 (q; q)_\infty^7}{2^6 7!} \int_{\mathbb{T}^7} \frac{\Gamma(sz; p, q) \prod_{j=1}^7 \Gamma(sz_j^2 z^{-1}; p, q)}{\prod_{1 \leq j < k \leq 7} \Gamma(z_j^{\pm 2} z_k^{\pm 2}; p, q)} \quad (3.54)$$

$$\times \prod_{1 \leq j < k \leq 7} \Gamma(s z z_j^{-2} z_k^{-2}; p, q) \prod_{1 \leq i < j < k \leq 7} \Gamma(s z_i^2 z_j^2 z_k^2 z^{-1}; p, q) \prod_{i=1}^5 \prod_{j=1}^7 \Gamma(t_i z_j^{\pm 2}; p, q) \prod_{j=1}^7 \frac{dz_j}{2\pi i z_j},$$

where  $z = z_1 z_2 z_3 z_4 z_5 z_6 z_7$ ,  $|s|, |t_i| < 1$ , and

$$I_M = \Gamma(s^8; p, q) \prod_{i=1}^5 \Gamma(t_i^2, s^4 t_i^{-1}, s^8 t_i^{-1}, s^4 t_i^2; p, q) \prod_{1 \leq i < j \leq 5} \Gamma(t_i t_j, s^2 t_i^{-1} t_j^{-1}, s^4 t_i t_j, s^6 t_i^{-1} t_j^{-1}; p, q), \quad (3.55)$$

with  $t = \prod_{i=1}^5 t_i$  and the balancing condition  $s^8 t = \sqrt{pq}$ .

To summarize, the formulas of this section lead to conjectures for exact evaluations of elliptic hypergeometric integrals on  $B_N$  and  $D_N$  root systems constructed from the characters of various representations, necessarily including the spinor representation, which require now rigorous mathematical proofs.

#### 4. SELF-DUAL THEORIES WITH THE SPINOR MATTER

We start by presenting a basic example of a self-dual  $\mathcal{N} = 1$  SYM theory based on the orthogonal gauge group with some number of fields in spinor representation. It was considered first in [23], further examples have been described in [32, 71]. First we consider the theory with  $SO(8)$  gauge group, 4 quarks in the fundamental representation, and 4 fields in spinor representation having the flavor group  $SU(4)_l \times SU(4)_r \times U(1)_B$ . The matter content of a theory is described in the table below

	$SO(8)$	$SU(4)_l$	$SU(4)_r$	$U(1)_B$	$U(1)_R$
$S$	$s$	$f$	1	1	$\frac{1}{4}$
$Q$	$f$	1	$f$	-1	$\frac{1}{4}$

In [23] there were found 5 theories dual to the original electric theory. We reconsidered these theories using the SCI technique and found that there are, actually, only 3 dual theories. Other theories have the fields which can be integrated out and, in particular, their contribution to 't Hooft anomaly matching conditions is trivial (none). The matter fields of dual theories are listed below in the table, where the double lines separate dual theories.

	$SO(8)$	$SU(4)_l$	$SU(4)_r$	$U(1)_B$	$U(1)_R$
$s$	$s$	$\bar{f}$	1	1	$\frac{1}{4}$
$v$	$f$	1	$f$	-1	$\frac{1}{4}$
$M$	1	$T_S$	1	2	$\frac{1}{2}$
$N$	1	$T_S$	1	-2	$\frac{3}{2}$
$s$	$s$	$f$	$\frac{1}{\bar{f}}$	1	$\frac{1}{4}$
$v$	$f$	1	$\bar{f}$	-1	$\frac{1}{4}$
$\widetilde{M}$	1	1	$T_S$	-2	$\frac{1}{2}$
$\widetilde{N}$	1	1	$T_S$	2	$\frac{3}{2}$
$s$	$s$	$\bar{f}$	$\frac{1}{\bar{f}}$	1	$\frac{1}{4}$
$v$	$f$	1	$\bar{f}$	-1	$\frac{1}{4}$
$M$	1	$T_S$	1	2	$\frac{1}{2}$
$N$	1	$T_S$	1	-2	$\frac{3}{2}$
$\widetilde{M}$	1	1	$T_S$	-2	$\frac{1}{2}$
$\widetilde{N}$	1	1	$T_S$	2	$\frac{3}{2}$

Corresponding SCIs are given by the following expressions

$$\begin{aligned}
 I_E &= \frac{(p; p)_\infty^4 (q; q)_\infty^4}{2^{34}!} \int_{\mathbb{T}^4} \frac{\prod_{i,j=1}^4 \Gamma(s_i z_j^{\pm 2}; p, q)}{\prod_{1 \leq i < j \leq 4} \Gamma(z_i^{\pm 2} z_j^{\pm 2}; p, q)} \\
 &\times \prod_{i=1}^4 \Gamma(t_i Z^{\pm 1}; p, q) \prod_{i=1}^4 \prod_{1 \leq j < k \leq 4} \Gamma(t_i z_j^2 z_k^2 Z^{-1}; p, q) \prod_{j=1}^4 \frac{dz_j}{2\pi i z_j},
 \end{aligned} \tag{4.1}$$

where  $Z = z_1 z_2 z_3 z_4$  and the balancing condition reads  $\prod_{i=1}^4 s_i t_i = pq$ .

Magnetic SCIs are

$$\begin{aligned}
 I_M^{(1)} &= \prod_{i=1}^4 \Gamma(t_i^2, S t_i^2; p, q) \frac{(p; p)_\infty^4 (q; q)_\infty^4}{2^{34}!} \int_{\mathbb{T}^4} \frac{\prod_{i,j=1}^4 \Gamma(s_i z_j^{\pm 2}; p, q)}{\prod_{1 \leq i < j \leq 4} \Gamma(z_i^{\pm 2} z_j^{\pm 2}; p, q)} \\
 &\times \prod_{i=1}^4 \Gamma\left(\frac{\sqrt{T}}{t_i} Z^{\pm 1}; p, q\right) \prod_{i=1}^4 \prod_{1 \leq j < k \leq 4} \Gamma\left(\frac{\sqrt{T}}{t_i} z_j^2 z_k^2 Z^{-1}; p, q\right) \prod_{j=1}^4 \frac{dz_j}{2\pi i z_j},
 \end{aligned} \tag{4.2}$$

for the first magnetic theory;

$$\begin{aligned}
 I_M^{(2)} &= \prod_{i=1}^4 \Gamma(s_i^2, T s_i^2; p, q) \frac{(p; p)_\infty^4 (q; q)_\infty^4}{2^{34}!} \int_{\mathbb{T}^4} \frac{\prod_{i,j=1}^4 \Gamma\left(\frac{\sqrt{S}}{s_i} z_j^{\pm 2}; p, q\right)}{\prod_{1 \leq i < j \leq 4} \Gamma(z_i^{\pm 2} z_j^{\pm 2}; p, q)} \\
 &\times \prod_{i=1}^4 \Gamma(t_i Z^{\pm 1}; p, q) \prod_{i=1}^4 \prod_{1 \leq j < k \leq 4} \Gamma(t_i z_j^2 z_k^2 Z^{-1}; p, q) \prod_{j=1}^4 \frac{dz_j}{2\pi i z_j},
 \end{aligned} \tag{4.3}$$

for the second magnetic theory;

$$\begin{aligned}
 I_M^{(3)} &= \prod_{i=1}^4 \Gamma(s_i^2, t_i^2, T s_i^2, S t_i^2; p, q) \frac{(p; p)_\infty^4 (q; q)_\infty^4}{2^{34}!} \int_{\mathbb{T}^4} \frac{\prod_{i,j=1}^4 \Gamma\left(\frac{\sqrt{S}}{s_i} z_j^{\pm 2}; p, q\right)}{\prod_{1 \leq i < j \leq 4} \Gamma(z_i^{\pm 2} z_j^{\pm 2}; p, q)} \\
 &\times \prod_{i=1}^4 \Gamma\left(\frac{\sqrt{T}}{t_i} Z^{\pm 1}; p, q\right) \prod_{i=1}^4 \prod_{1 \leq j < k \leq 4} \Gamma\left(\frac{\sqrt{T}}{t_i} z_j^2 z_k^2 Z^{-1}; p, q\right) \prod_{j=1}^4 \frac{dz_j}{2\pi i z_j},
 \end{aligned} \tag{4.4}$$

for the third magnetic theory.

The situation with other self-dual theories is not so clear, e.g. the self-duality of [23, 71] based on  $SO(12)$  gauge group with one field in the spinor representation and 8 quarks in the fundamental representation seems to be incorrect. First, the representations and charges of the dual quarks and spinor representation fields are not changed. Second, the fields  $M_4$  and  $M_8$  (taken from the second section of [71]) can be integrated out and their contributions to anomalies cancel out thus bringing oneself back to the original theory.

## 5. SEIBERG DUALITIES FOR $SO(N)$ GAUGE GROUP WITH THE SPINOR MATTER

5.1.  $G = SO(5)$  **and**  $F = SU(N_f) \times SO(4) \times U(1)$ . A duality with the  $SU(N_f) \times SU(4) \times U(1)$  flavor group was studied in [32]. It was claimed that the corresponding duality can be derived from a more general duality, which we shall consider later in Sect. 5.8. Using the SCI technique we show that this statement is incorrect. In our language, the duality of Sect. 5.8 reduces to the duality discussed below which is based on the  $SO(4)$ -flavor subgroup instead of  $SU(4)$ .

Our electric theory is represented by the following table

	$SO(5)$	$SU(N_f)$	$SO(4)$	$U(1)$	$U(1)_R$
$Q$	$f$	$f$	1	-1	$1 - \frac{3}{N_f+2}$
$S$	$s$	1	$f$	$\frac{N_f}{2}$	$1 - \frac{3}{N_f+2}$

while the magnetic theory is

	$SU(N_f)$	$SU(N_f)$	$SO(5) \simeq SP(4)$	$U(1)$	$U(1)_R$
$q$	$\bar{f}$	$\bar{f}$	1	1	$\frac{3}{N_f+2} - \frac{1}{N_f}$
$q'$	$f$	1	1	$-N_f$	$-1 + \frac{6}{N_f+2} + \frac{1}{N_f}$
$w$	$T_S$	1	1	0	$\frac{2}{N_f}$
$t$	$\bar{f}$	1	$f$	0	$1 - \frac{1}{N_f}$
$Y$	1	$f$	1	$N_f - 1$	$3 - \frac{9}{N_f+2}$
$M$	1	$T_S$	1	-2	$2 - \frac{6}{N_f+2}$
$N$	1	1	$f$	$N_f$	$2 - \frac{6}{N_f+2}$

The indices are

$$\begin{aligned}
I_E = & \frac{(p; p)_\infty^2 (q; q)_\infty^2}{2^2 2!} \prod_{i=1}^{N_f} \Gamma(s_i; p, q) \int_{\mathbb{T}^2} \frac{\Gamma(tu_1^{\pm 1} u_2^{\pm 1} (z_1 z_2)^{\pm 1}, tu_1^{\pm 1} u_2^{\pm 1} (z_1 z_2^{-1})^{\pm 1}; p, q)}{\Gamma(z_1^{\pm 2} z_2^{\pm 2}, z_1^{\pm 2}, z_2^{\pm 2}; p, q)} \\
& \times \prod_{i=1}^{N_f} \prod_{j=1}^2 \Gamma(s_i z_j^{\pm 2}; p, q) \prod_{j=1}^2 \frac{dz_j}{2\pi i z_j}, \tag{5.1}
\end{aligned}$$

where the balancing condition is  $s^2 t^4 = (pq)^{N_f-1}$  with  $s = \prod_{i=1}^{N_f} s_i$ , and

$$\begin{aligned}
 I_M &= \frac{(p; p)_\infty^{N_f-1} (q; q)_\infty^{N_f-1}}{N_f!} \Gamma((pq)^{\frac{N_f-1}{2}} s^{-1}; p, q) \prod_{j=1,2} \Gamma((pq)^{\frac{N_f-1}{2}} s^{-1} u_j^{\pm 1}; p, q) \quad (5.2) \\
 &\times \prod_{1 \leq i < j \leq N_f} \Gamma(s_i s_j; p, q) \prod_{i=1}^{N_f} \Gamma(s_i^2; p, q) \prod_{i=1}^{N_f} \Gamma((pq)^{\frac{N_f-1}{2}} s^{-1} s_i; p, q) \\
 &\times \int_{\mathbb{T}^{N_f-1}} \prod_{1 \leq i < j \leq N_f} \frac{\Gamma((pq)^{\frac{1}{N_f}} y_i y_j; p, q)}{\Gamma(y_i y_j^{-1}, y_i^{-1} y_j; p, q)} \prod_{i=1}^{N_f} \Gamma((pq)^{\frac{1}{N_f}} y_i^2; p, q) \\
 &\times \prod_{i,j=1}^{N_f} \Gamma((pq)^{\frac{N_f-1}{2N_f}} s_i^{-1} y_j^{-1}; p, q) \prod_{j=1}^{N_f} \Gamma((pq)^{\frac{1}{2}(2+\frac{1}{N_f}-N_f)} s y_j, (pq)^{\frac{N_f-1}{2N_f}} y_j^{-1}; p, q) \\
 &\times \prod_{i=1,2} \prod_{j=1}^{N_f} \Gamma((pq)^{\frac{N_f-1}{2N_f}} u_i^{\pm 1} y_j^{-1}; p, q) \prod_{j=1}^{N_f-1} \frac{dy_j}{2\pi i y_j},
 \end{aligned}$$

where  $\prod_{j=1}^{N_f} y_j = 1$ . These SCIs are obtained by a reduction from the indices of the duality presented in Sect. 5.8 after the restriction of parameters described there explicitly.

A simple explanation of the inconsistency of the duality of [32] consists in the mismatch of the number of independent fugacities (parameters) in the dual indices, for the  $SU(4)$ -flavor subgroup there will be an extra parameter in the electric theory in comparison with the magnetic one. In principle, as described in [114], the integrands entering indices may have different number of parameters, but there should be some additional multipliers to the integrals which cancel contribution of these extra parameters.

**5.2.  $SO(7)$  gauge group with  $N_f$  fundamentals.** The  $\mathcal{N} = 1$  SYM electric theory described in this section was historically the first model including a matter field in the spinor representation with known dual theory. It was discovered by Pouliot [90], and it is based on  $SO(7)$  gauge group with the following matter content

	$SO(7)$	$SU(N_f)$	$U(1)_R$
$Q$	$s$	$f$	$1 - \frac{5}{N_f}$

where  $s$  means the spinor representation. Pouliot found the following dual magnetic theory

	$SU(N_f - 4)$	$SU(N_f)$	$U(1)_R$
$q$	$f$	$f$	$\frac{5}{N_f} - \frac{1}{N_f-4}$
$w$	$T_S$	1	$\frac{2}{N_f-4}$
$M$	1	$T_S$	$2 - \frac{10}{N_f}$

where the number of flavors is constrained by the conformal window  $6 \leq N_f \leq 15$ .

According to this duality one should have equality of the following SCIs/integrals

$$I_E = \frac{(p; p)_\infty^3 (q; q)_\infty^3}{2^3 3!} \int_{\mathbb{T}^3} \frac{\prod_{i=1}^{N_f} \Gamma(t_i (z_1 z_2 z_3)^{\pm 1}; p, q) \prod_{i=1}^{N_f} \prod_{j=1}^3 \Gamma(t_i \left(\frac{z_j^2}{z_1 z_2 z_3}\right)^{\pm 1}; p, q)}{\prod_{1 \leq i < j \leq 3} \Gamma(z_i^{\pm 2} z_j^{\pm 2}; p, q) \prod_{j=1}^3 \Gamma(z_j^{\pm 2}; p, q)} \prod_{j=1}^3 \frac{dz_j}{2\pi i z_j}, \quad (5.3)$$

with  $|t|, |t_j| < 1$ , and the balancing condition  $\prod_{m=1}^{N_f} t_m = (pq)^{(N_f-5)/2}$ , and

$$I_M = \prod_{1 \leq i < j \leq N_f} \Gamma(t_i t_j; p, q) \prod_{j=1}^{N_f} \Gamma(t_j^2; p, q) \frac{(p; p)_\infty^{N_f-5} (q; q)_\infty^{N_f-5}}{(N_f-4)!} \int_{\mathbb{T}^{N_f-5}} \prod_{j=1}^{N_f-5} \frac{dy_j}{2\pi i y_j} \\ \times \prod_{1 \leq i < j \leq N_f-4} \frac{\Gamma((pq)^{\frac{1}{N_f-4}} y_i y_j; p, q)}{\Gamma(y_i y_j^{-1}, y_i^{-1} y_j; p, q)} \prod_{j=1}^{N_f-4} \Gamma((pq)^{\frac{1}{N_f-4}} y_j^2; p, q) \prod_{i=1}^{N_f} \Gamma(S^{\frac{1}{N_f-4}} t_i^{-1} y_j^{-1}; p, q), \quad (5.4)$$

where  $\prod_{j=1}^{N_f-4} y_j = 1$ .

To stress the non-trivial character of the dualities for the orthogonal gauge group with spinor matter and promote them, we describe the duality for  $\mathcal{N} = 1$  SYM theory with the  $G_2$  gauge group proposed in [90]. Pouliot's idea to derive  $G_2$ -group model consists in the following:  $G_2$  is a subgroup of  $SO(7)$  and the corresponding duality can be obtained from the  $SO(7)$ -group case with  $N_f$  fields in the spinor representation after giving masses to some mesons or integrating out one of the quarks. In our language one should calculate accurately the limit  $t_F \rightarrow 1$  in the electric and magnetic SCIs. In the magnetic SCI one has the diverging multiplier in front of the integral  $\Gamma(t_F^2; p, q)$ . In the left hand-side one has a divergency as well coming from the poles approaching the integration contour. Note also that for the magnetic SCI the limit  $t_F \rightarrow 1$  is problematic only for the indicated multiplier. Therefore we can plug  $t_F = 1$  in other terms and see that the rank of the magnetic gauge group is not changed, i.e. no Higgs mechanism applies from the physical point of view.

On the left-hand side one should use the residue calculus. Let us slightly deform the contour  $\mathbb{T}$  for the 3rd integration variable and pick up the residues of the poles at  $z_3 = z_1 z_2 t_F^{\pm 1}$  and their reciprocals.

Now divide both electric and magnetic SCIs by  $\Gamma(t_F^2; p, q)$  and take the limit  $t_F \rightarrow 1$ . Then the electric index can be rewritten as the integral (13.3) of [115] with  $N_f$  replaced by  $N_f - 1$  which describes SCI of the  $G_2$  gauge group theory with  $N_f - 1$  fundamental quarks:

$$I_E = \frac{(p; p)_\infty^2 (q; q)_\infty^2}{2^2 3} \prod_{m=1}^{N_f-1} \Gamma(t_m; p, q) \int_{\mathbb{T}^2} \frac{\prod_{k=1}^3 \prod_{m=1}^{N_f-1} \Gamma(t_m z_k^{\pm 1}; p, q)}{\prod_{1 \leq j < k \leq 3} \Gamma(z_j^{\pm 1} z_k^{\pm 1}; p, q)} \prod_{k=1}^2 \frac{dz_k}{2\pi i z_k}, \quad (5.5)$$

where  $z_1 z_2 z_3 = 1$  and the balancing condition reads  $\prod_{m=1}^{N_f-1} t_m = (pq)^{(N_f-5)/2}$ . The magnetic index (5.4) reduces to the integral (13.4) of [115]:

$$I_M = \frac{(p; p)_\infty^{N_f-5} (q; q)_\infty^{N_f-5}}{(N_f-4)!} \prod_{1 \leq j < k \leq N_f-1} \Gamma(t_j t_k; p, q) \prod_{j=1}^{N_f-1} \Gamma(t_j^2; p, q) \quad (5.6) \\ \times \int_{\mathbb{T}^{N_f-5}} \prod_{1 \leq j < k \leq N_f-4} \frac{\Gamma((pq)^{r_s} z_j z_k; p, q)}{\Gamma(z_j^{-1} z_k, z_j z_k^{-1}; p, q)} \prod_{j=1}^{N_f-4} \Gamma((pq)^{r_s} z_j^2; p, q) \\ \times \prod_{j=1}^{N_f-4} \Gamma((pq)^{(1-r_s)/2} z_j^{-1}; p, q) \prod_{k=1}^{N_f-1} \Gamma((pq)^{(1-r_s)/2} t_k^{-1} z_j^{-1}; p, q) \prod_{j=1}^{N_f-5} \frac{dz_j}{2\pi i z_j},$$

where  $\prod_{j=1}^{N_f-4} z_j = 1$ . Another possibility of deriving this  $G_2$ -duality out of the standard Seiberg duality for  $SU(3)$ -gauge group has been described in [115].

5.3.  $G = SO(7)$  **and**  $F = SU(N_f) \times U(1)$ . This duality was considered in [19]. The electric theory is represented in the following table

	$SO(7)$	$SU(N_f)$	$U(1)$	$U(1)_R$
$Q$	$f$	$f$	$-1$	$\frac{N_f-4}{N_f}$
$S$	$s$	$1$	$N_f$	$0$

while the magnetic theory is

	$SU(N_f - 3)$	$SU(N_f)$	$U(1)$	$U(1)_R$
$q$	$f$	$\bar{f}$	$\frac{2N_f-3}{N_f-3}$	$\frac{3(N_f-4)}{N_f(N_f-3)}$
$q'$	$f$	$1$	$\frac{N_f}{N_f-3}$	$\frac{N_f-4}{N_f-3}$
$w$	$\bar{T}_S$	$1$	$-\frac{2N_f}{N_f-3}$	$\frac{2}{N_f-3}$
$M$	$1$	$T_S$	$-2$	$\frac{2(N_f-4)}{N_f}$
$L$	$1$	$1$	$2N_f$	$0$

where  $5 \leq N_f \leq 13$ .

The superconformal index for the electric theory is

$$\begin{aligned}
 I_E = & \frac{(p; p)_\infty^3 (q; q)_\infty^3}{2^3 3!} \prod_{i=1}^{N_f} \Gamma(s_i; p, q) \int_{\mathbb{T}^3} \frac{\Gamma(tz^{\pm 1}; p, q) \prod_{j=1}^3 \Gamma(t(z_j^2 z^{-1})^{\pm 1}; p, q)}{\prod_{j=1}^3 \Gamma(z_j^{\pm 2}; p, q) \prod_{1 \leq i < j \leq 3} \Gamma(z_i^{\pm 2} z_j^{\pm 2}; p, q)} \\
 & \times \prod_{i=1}^{N_f} \prod_{j=1}^3 \Gamma(s_i z_j^{\pm 2}; p, q) \prod_{j=1}^3 \frac{dz_j}{2\pi i z_j}
 \end{aligned} \tag{5.7}$$

where  $z = z_1 z_2 z_3$  and the balancing condition reads  $st = (pq)^{\frac{1}{2}(N_f-4)}$  with  $s = \prod_{i=1}^{N_f} s_i$ . In the magnetic theory we have

$$\begin{aligned}
 I_M = & \Gamma(t^2; p, q) \prod_{i=1}^{N_f} \Gamma(s_i^2; p, q) \prod_{1 \leq i < j \leq N_f} \Gamma(s_i s_j; p, q) \frac{(p; p)_\infty^{N_f-4} (q; q)_\infty^{N_f-4}}{(N_f-3)!} \\
 & \times \int_{\mathbb{T}^{N_f-4}} \prod_{1 \leq i < j \leq N_f-3} \frac{\Gamma(s^{\frac{2}{(N_f-4)(N_f-3)}} t^{-\frac{2(N_f-5)}{(N_f-3)(N_f-4)}} y_i^{-1} y_j^{-1}; p, q)}{\Gamma(y_i y_j^{-1}, y_i^{-1} y_j; p, q)} \prod_{i=1}^{N_f} \prod_{j=1}^{N_f-3} \Gamma((st^2)^{\frac{1}{N_f-3}} s_i^{-1} y_j; p, q) \\
 & \times \prod_{j=1}^{N_f-3} \Gamma((st^2)^{\frac{1}{N_f-3}} y_j, s^{\frac{2}{(N_f-4)(N_f-3)}} t^{-\frac{2(N_f-5)}{(N_f-3)(N_f-4)}} y_i^{-2}; p, q) \prod_{j=1}^{N_f-4} \frac{dy_j}{2\pi i y_j},
 \end{aligned} \tag{5.8}$$

where  $\prod_{j=1}^{N_f-3} y_j = 1$ .

5.4.  $G = SO(7)$  **and**  $F = SU(N_f) \times SU(2) \times U(1)$ . This duality was considered in [19]. The electric theory is represented in the following table

	$SO(7)$	$SU(N_f)$	$SU(2)$	$U(1)$	$U(1)_R$
$Q$	$f$	$f$	$1$	$-2$	$1 - 5/N_f$
$S$	$s$	$1$	$f$	$N_f$	$1$

while the magnetic theory is

	$SU(N_f - 2)$	$SU(N_f)$	$SO(3)$	$U(1)$	$U(1)_R$
$q$	$f$	$\bar{f}$	1	2	$\frac{2(N_f-5)}{N_f(N_f-2)}$
$q'$	$f$	1	$f$	0	$\frac{N_f-3}{N_f-2}$
$\tilde{q}$	$\bar{f}$	1	1	$-2N_f$	$-\frac{N_f-3}{N_f-2}$
$w$	$\bar{T}_S$	1	1	0	$\frac{2}{N_f-2}$
$M$	1	$T_S$	1	-4	$2 - 10/N_f$
$L$	1	1	$f$	$2N_f$	2
$N$	1	$f$	1	$2(N_f - 1)$	$3 - 5/N_f$

where  $4 \leq N_f \leq 12$ .

The superconformal index for the electric theory is

$$\begin{aligned}
I_E = & \frac{(p; p)_\infty^3 (q; q)_\infty^3}{2^3 3!} \prod_{i=1}^{N_f} \Gamma(s_i; p, q) \int_{\mathbb{T}^3} \frac{\Gamma(yx^{\pm 1} z^{\pm 1}; p, q) \prod_{j=1}^3 \Gamma(yx^{\pm 1} (z_j^2 z^{-1})^{\pm 1}; p, q)}{\prod_{j=1}^3 \Gamma(z_j^{\pm 2}; p, q) \prod_{1 \leq i < j \leq 3} \Gamma(z_i^{\pm 2} z_j^{\pm 2}; p, q)} \\
& \times \prod_{i=1}^{N_f} \prod_{j=1}^3 \Gamma(s_i z_j^{\pm 2}; p, q) \prod_{j=1}^3 \frac{dz_j}{2\pi i z_j}
\end{aligned} \tag{5.9}$$

where  $z = z_1 z_2 z_3$  and the balancing condition reads  $sy^2 = (pq)^{\frac{1}{2}(N_f-3)}$  with  $s = \prod_{i=1}^{N_f} s_i$ . In the magnetic theory we have

$$\begin{aligned}
I_M = & \Gamma(y^2, y^2 x^{\pm 1}; p, q) \prod_{i=1}^{N_f} \Gamma(s_i^2, y^2 s_i; p, q) \prod_{1 \leq i < j \leq N_f} \Gamma(s_i s_j; p, q) \frac{(p; p)_\infty^{N_f-3} (q; q)_\infty^{N_f-3}}{(N_f - 2)!} \\
& \times \int_{\mathbb{T}^{N_f-3}} \prod_{1 \leq i < j \leq N_f-2} \frac{\Gamma((pq)^{\frac{1}{N_f-2}} y_i^{-1} y_j^{-1}; p, q)}{\Gamma(y_i y_j^{-1}, y_i^{-1} y_j; p, q)} \prod_{i=1}^{N_f} \prod_{j=1}^{N_f-2} \Gamma((pq)^{\frac{N_f-5}{2(N_f-2)}} s_i^{-1} y_j; p, q) \\
& \times \prod_{j=1}^{N_f-2} \Gamma((pq)^{\frac{1}{N_f-2}} y_j^{-2}, (pq)^{\frac{N_f-1}{2(N_f-2)}} y^{-1} y_j^{-1}, (pq)^{\frac{N_f-3}{2(N_f-2)}} y_j, (pq)^{\frac{N_f-3}{2(N_f-2)}} x^{\pm 1} y_j; p, q) \prod_{j=1}^{N_f-3} \frac{dy_j}{2\pi i y_j},
\end{aligned} \tag{5.10}$$

where  $\prod_{j=1}^{N_f-2} y_j = 1$ .

5.5.  $G = SO(8)$  and  $F = SU(N_f) \times U(1)$ . This duality was considered in [91]. The electric theory is represented in the following table

	$SU(N_f - 4)$	$SU(N_f)$	$U(1)$	$U(1)_R$
$Q$	$\bar{f}$	$f$	$2N_f - 4$	$\frac{6(N_f-5)}{(N_f+1)(N_f-4)}$
$S$	$T_S$	1	$-2N_f$	$\frac{12}{(N_f+1)(N_f-4)}$

while the magnetic theory is

	$SO(8)$	$SU(N_f)$	$U(1)$	$U(1)_R$
$q$	$f$	$\bar{f}$	$4 - N_f$	$\frac{N_f-5}{N_f+1}$
$p$	$s$	1	$N_f(N_f - 4)$	$\frac{N_f-5}{N_f+1}$
$M$	1	$T_S$	$2N_f - 8$	$\frac{12}{N_f+1}$
$U$	1	1	$-2N_f(N_f - 4)$	$\frac{12}{N_f+1}$



where  $6 \leq N_f \leq 16$ .

The superconformal index for the electric theory is

$$\begin{aligned}
 I_E &= \frac{(p; p)_\infty^{N_f-5} (q; q)_\infty^{N_f-5}}{(N_f-4)!} \int_{\mathbb{T}^{N_f-5}} \prod_{1 \leq i < j \leq N_f-4} \frac{\Gamma(uz_i z_j; p, q)}{\Gamma(z_i z_j^{-1}, z_i^{-1} z_j; p, q)} \\
 &\times \prod_{j=1}^{N_f-4} \Gamma(uz_j^2; p, q) \prod_{i=1}^{N_f} \prod_{j=1}^{N_f-4} \Gamma(s_i z_j^{-1}; p, q) \prod_{j=1}^{N_f-5} \frac{dz_j}{2\pi i z_j}
 \end{aligned} \tag{5.11}$$

where  $\prod_{j=1}^{N_f-4} z_j = 1$  and the balancing condition reads  $su^{N_f-2} = (pq)^3$  with  $s = \prod_{i=1}^{N_f} s_i$ . In the magnetic theory we have

$$\begin{aligned}
 I_M &= \Gamma(u^{n-4}; p, q) \prod_{1 \leq i < j \leq N_f} \Gamma(us_i s_j; p, q) \prod_{i=1}^{N_f} \Gamma(us_i^2; p, q) \\
 &\times \frac{(p; p)_\infty^4 (q; q)_\infty^4}{2^3 4!} \int_{\mathbb{T}^4} \prod_{j=1}^4 \frac{dz_j}{2\pi i z_j} \frac{\Gamma(s^{\frac{1}{6}} u^{-\frac{1}{3}(N_f-5)} z^{\pm 1}; p, q)}{\prod_{1 \leq i < j \leq 4} \Gamma(z_i^{\pm 2} z_j^{\pm 2}; p, q)} \\
 &\times \prod_{1 \leq i < j \leq 4} \Gamma(s^{\frac{1}{6}} u^{-\frac{1}{3}(N_f-5)} z_i^2 z_j^2 z^{-1}; p, q) \prod_{i=1}^{N_f} \prod_{j=1}^4 \Gamma(s^{\frac{1}{6}} u^{\frac{1}{6}(N_f-5)} s_i^{-1} z_j^{\pm 2}; p, q),
 \end{aligned} \tag{5.12}$$

where  $z = z_1 z_2 z_3 z_4$ .

5.6.  $G = SO(8)$  and  $F = SU(N_f) \times U(1)_1 \times U(1)_2$ . This duality was considered in [19]. The electric theory is represented in the following table

	$SO(8)$	$SU(N_f)$	$U(1)_1$	$U(1)_2$	$U(1)_R$
$Q$	$f$	$f$	$-2$	$0$	$1 - 6/N_f$
$S$	$s$	$1$	$N_f$	$1$	$1$
$S'$	$c$	$1$	$N_f$	$-1$	$1$

while the magnetic theory is

	$SU(N_f-3)$	$SU(N_f)$	$U(1)_1$	$U(1)_2$	$U(1)_R$
$q$	$f$	$\bar{f}$	$2$	$0$	$\frac{5N_f-18}{N_f(N_f-3)}$
$q'$	$f$	$1$	$0$	$2$	$\frac{N_f-4}{N_f-3}$
$q''$	$f$	$1$	$0$	$-2$	$\frac{N_f-4}{N_f-3}$
$\tilde{q}$	$\bar{f}$	$1$	$-2N_f$	$0$	$-\frac{N_f-4}{N_f-3}$
$w$	$\bar{T}_S$	$1$	$0$	$0$	$\frac{2}{N_f-3}$
$M$	$1$	$T_S$	$-4$	$0$	$2 - 12/N_f$
$L_1$	$1$	$1$	$2N_f$	$2$	$2$
$L_2$	$1$	$1$	$2N_f$	$-2$	$2$
$N$	$1$	$f$	$2(N_f-1)$	$0$	$3 - 6/N_f$

where  $5 \leq N_f \leq 15$ .

The superconformal index for the electric theory is

$$I_E = \frac{(p; p)_\infty^4 (q; q)_\infty^4}{2^3 4!} \int_{\mathbb{T}^4} \frac{\Gamma(tz^{\pm 1}; p, q) \prod_{1 \leq i < j \leq 4} \Gamma(tz_i^2 z_j^2 z^{-1}; p, q)}{\prod_{1 \leq i < j \leq 4} \Gamma(z_i^{\pm 2} z_j^{\pm 2}; p, q)} \\ \times \prod_{j=1}^4 \Gamma(u(z_j^2 z^{-1})^{\pm 1}; p, q) \prod_{i=1}^{N_f} \prod_{j=1}^4 \Gamma(s_i z_j^{\pm 2}; p, q) \prod_{j=1}^4 \frac{dz_j}{2\pi i z_j}, \quad (5.13)$$

where  $z = z_1 z_2 z_3 z_4$  and the balancing condition reads  $stu = (pq)^{\frac{1}{2}(N_f-4)}$  with  $s = \prod_{i=1}^{N_f} s_i$ . In the magnetic theory we have

$$I_M = \Gamma(u^2, t^2; p, q) \prod_{i=1}^{N_f} \Gamma(s_i^2, tus_i; p, q) \prod_{1 \leq i < j \leq N_f} \Gamma(s_i s_j; p, q) \frac{(p; p)_\infty^{N_f-4} (q; q)_\infty^{N_f-4}}{(N_f-3)!} \\ \times \int_{\mathbb{T}^{N_f-4}} \prod_{1 \leq i < j \leq N_f-3} \frac{\Gamma((pq)^{\frac{1}{N_f-3}} y_i^{-1} y_j^{-1}; p, q)}{\Gamma(y_i y_j^{-1}, y_i^{-1} y_j; p, q)} \prod_{i=1}^{N_f} \prod_{j=1}^{N_f-3} \Gamma((pq)^{\frac{N_f-4}{2(N_f-3)}} s_i^{-1} y_j; p, q) \\ \times \prod_{j=1}^{N_f-3} \Gamma((pq)^{\frac{1}{N_f-3}} y_j^{-2}, (pq)^{\frac{N_f-2}{2(N_f-3)}} (tu)^{-1} y_j^{-1}, (pq)^{\frac{N_f-4}{2(N_f-3)}} (tu^{-1})^{\pm 1} y_j; p, q) \prod_{j=1}^{N_f-4} \frac{dy_j}{2\pi i y_j}, \quad (5.14)$$

where  $\prod_{j=1}^{N_f-3} y_j = 1$ .

5.7.  $G = SO(9)$  **and**  $F = SU(N_f) \times U(1)$ . This duality was considered in [19]. The electric theory is represented in the following table

	$SO(9)$	$SU(N_f)$	$U(1)$	$U(1)_R$
$Q$	$f$	$f$	$-2$	$1 - 5/N_f$
$S$	$s$	$1$	$N_f$	$0$

while the magnetic theory is

	$SU(N_f - 4)$	$SU(N_f)$	$U(1)$	$U(1)_R$
$q$	$f$	$\bar{f}$	$2$	$\frac{4(N_f-5)}{N_f(N_f-4)}$
$q'$	$f$	$1$	$0$	$\frac{N_f-5}{N_f-4}$
$\tilde{q}$	$\bar{f}$	$1$	$-2N_f$	$\frac{N_f-3}{N_f-4}$
$w$	$\bar{T}_S$	$1$	$0$	$\frac{2}{N_f-4}$
$M$	$1$	$T_S$	$-4$	$2 - 10/N_f$
$L$	$1$	$1$	$2N_f$	$0$
$N$	$1$	$f$	$2(N_f - 1)$	$1 - 5/N_f$

where  $6 \leq N_f \leq 18$ . The electric theory superconformal index is

$$I_E = \prod_{i=1}^{N_f} \Gamma(s_i; p, q) \frac{(p; p)_\infty^4 (q; q)_\infty^4}{2^4 4!} \int_{\mathbb{T}^4} \frac{\Gamma(tz^{\pm 1}; p, q) \prod_{1 \leq i < j \leq 4} \Gamma(tz_i^2 z_j^2 z^{-1}; p, q)}{\prod_{1 \leq i < j \leq 4} \Gamma(z_i^{\pm 2} z_j^{\pm 2}; p, q)} \\ \times \frac{\prod_{j=1}^4 \Gamma(t(z_j^2 z^{-1})^{\pm 1}; p, q)}{\prod_{j=1}^4 \Gamma(z_j^{\pm 2}; p, q)} \prod_{i=1}^{N_f} \prod_{j=1}^4 \Gamma(s_i z_j^{\pm 2}; p, q) \prod_{j=1}^4 \frac{dz_j}{2\pi i z_j}, \quad (5.15)$$

where  $z = z_1 z_2 z_3 z_4$  and the balancing condition reads  $st^2 = (pq)^{\frac{1}{2}(N_f-5)}$  with  $s = \prod_{i=1}^{N_f} s_i$ . In the magnetic theory we have

$$\begin{aligned}
 I_M &= \Gamma(t^2; p, q) \prod_{i=1}^{N_f} \Gamma(s_i^2, t^2 s_i; p, q) \prod_{1 \leq i < j \leq N_f} \Gamma(s_i s_j; p, q) \frac{(p; p)_\infty^{N_f-5} (q; q)_\infty^{N_f-5}}{(N_f - 4)!} \\
 &\times \int_{\mathbb{T}^{N_f-5}} \prod_{1 \leq i < j \leq N_f-4} \frac{\Gamma((pq)^{\frac{1}{N_f-4}} y_i^{-1} y_j^{-1}; p, q)}{\Gamma(y_i y_j^{-1}, y_i^{-1} y_j; p, q)} \prod_{i=1}^{N_f} \prod_{j=1}^{N_f-4} \Gamma((pq)^{\frac{N_f-5}{2(N_f-4)}} s_i^{-1} y_j; p, q) \\
 &\times \prod_{j=1}^{N_f-4} \Gamma((pq)^{\frac{1}{N_f-4}} y_j^{-2}, (pq)^{\frac{N_f-3}{2(N_f-4)}} t^{-2} y_j^{-1}, (pq)^{\frac{N_f-5}{2(N_f-4)}} y_j; p, q) \prod_{j=1}^{N_f-5} \frac{dy_j}{2\pi i y_j}, \quad (5.16)
 \end{aligned}$$

where  $\prod_{j=1}^{N_f-4} y_j = 1$ .

5.8.  **$SO(10)$  gauge group with  $N_f f + s$  matter.** This duality was considered in [92]. The electric theory is represented in the following table

	$SO(10)$	$SU(N_f)$	$U(1)$	$U(1)_R$
$Q$	$f$	$f$	$-1$	$1 - \frac{8}{N_f+2}$
$P$	$s$	$1$	$\frac{N_f}{2}$	$1 - \frac{8}{N_f+2}$

while the magnetic theory is

	$SU(N_f - 5)$	$SU(N_f)$	$U(1)$	$U(1)_R$
$w$	$T_S$	$1$	$0$	$\frac{2}{N_f-5}$
$q$	$\bar{f}$	$\bar{f}$	$1$	$\frac{8}{N_f+2} - \frac{1}{N_f-5}$
$q'$	$f$	$1$	$-N_f$	$-1 + \frac{16}{N_f+2} + \frac{1}{N_f-5}$
$M$	$1$	$T_S$	$-2$	$2 - \frac{16}{N_f+2}$
$Y$	$1$	$f$	$N_f - 1$	$3 - \frac{24}{N_f+2}$

where  $7 \leq N_f \leq 21$ .

The SCIs are

$$\begin{aligned}
 I_E &= \frac{(p; p)_\infty^5 (q; q)_\infty^5}{245!} \int_{\mathbb{T}^5} \frac{\Gamma(tZ; p, q) \prod_{j=1}^5 \Gamma(tz_j^2 Z^{-1}; p, q) \prod_{1 \leq i < j \leq 5} \Gamma(tZ z_i^{-2} z_j^{-2}; p, q)}{\prod_{1 \leq i < j \leq 5} \Gamma(z_i^{\pm 2} z_j^{\pm 2}; p, q)} \\
 &\times \prod_{i=1}^{N_f} \prod_{j=1}^5 \Gamma(s_i z_j^{\pm 2}; p, q) \prod_{j=1}^5 \frac{dz_j}{2\pi i z_j}, \quad (5.17)
 \end{aligned}$$

where  $st^2 = (pq)^{\frac{N_f}{2}-3}$ ,  $s = \prod_{i=1}^{N_f} s_i$ ,  $Z = z_1 z_2 z_3 z_4 z_5$ , and

$$\begin{aligned}
I_M &= \prod_{1 \leq i < j \leq N_f} \Gamma(s_i s_j; p, q) \prod_{i=1}^{N_f} \Gamma(s_i^2, t^2 s_i; p, q) \frac{(p; p)_\infty^{N_f-6} (q; q)_\infty^{N_f-6}}{(N_f - 5)!} \\
&\times \int_{\mathbb{T}^{N_f-6}} \prod_{1 \leq i < j \leq N_f-5} \frac{\Gamma((pq)^{\frac{1}{N_f-5}} y_i y_j; p, q)}{\Gamma(y_i^{-1} y_j, y_i y_j^{-1}; p, q)} \prod_{i=1}^{N_f-5} \Gamma((pq)^{\frac{1}{N_f-5}} y_i^2, (pq)^{\frac{N_f-4}{2(N_f-5)}} t^{-2} y_i; p, q) \\
&\times \prod_{i=1}^{N_f} \prod_{j=1}^{N_f-5} \Gamma((pq)^{\frac{N_f-6}{2(N_f-5)}} s_i^{-1} y_j^{-1}; p, q) \prod_{j=1}^{N_f-6} \frac{dy_j}{2\pi i y_j},
\end{aligned} \tag{5.18}$$

where  $\prod_{j=1}^{N_f-5} y_j = 1$ .

We checked that the total ellipticity condition [112] for the equality  $I_E = I_M$  holds true. An interesting fact is that fixing  $s_1 = 1$  and  $t = \sqrt{pq}$  in both integrals, we come to SCIs of the original Seiberg duality [101], namely, of the duality between  $SO(9)$  and  $SO(N_f - 5)$  gauge theories with  $N_f$  quarks in the fundamental representation. A connection between these dualities was understood first from the physical point of view in [92], and our observation is that the SCIs are connected as well after imposing appropriate constraints. The residue calculus similar to that of [24] should be applied to the electric theory. In the limit  $s_1 \rightarrow 1$  the integration contour is pinched by the poles coming from the term  $\prod_{j=1}^5 \Gamma(s_1 z_j^{\pm 2}; p, q)$ . Picking up residues of the poles at  $z_j = s_1^{\pm 1/2}$  we come to the SCI of  $\mathcal{N} = 1$  SYM theory with  $SO(9)$  gauge group and  $N_f$  quarks in the fundamental representation. In the magnetic (dual) theory SCI we have the multiplier  $\Gamma(t^2 s_1; p, q)$  vanishing in the discussed limit and further steps are a little tricky. For  $N_f > 5$  and  $N_f$  odd it is convenient first to rescale  $y_i \rightarrow (pq)^{-1/2(N_f-4)} y_i$ ,  $i = \overline{1, N_f - 5}$ . Then the first residue comes from the pole at  $y_j = \sqrt{pq}$ , and other residues come from the poles  $y_{2i+1} = y_{2i}$ ,  $i = \overline{1, (N_f - 5)/2}$ . Accurately computing all these sequential residues one can verify that the resulting integral describes SCI of the magnetic dual theory with  $SO(N_f - 5)$  gauge group with  $N_f$  quarks in the fundamental representation with the gauge singlet baryon field in the  $T_S$ -representation of the flavor group  $SU(N_f)$ .

There is another nice reduction of dual theories observed in a remarkable paper [92]. If we take  $N_f = 8$  then we can obtain  $S$ -duality for  $\mathcal{N} = 2$  SYM theory with  $SU(2)$  gauge group and 4 hypermultiplets studied in detail in [103]. From the mathematical point of view we need to apply the following constraints in (5.17) and (5.18)

$$s_1 s_5 = 1, \quad s_2 s_6 = 1, \quad s_3 s_7 = 1, \quad s_4 = 1$$

and then compute the residues of poles at  $z_1 = s_1, z_2 = s_2, z_3 = s_3, z_4 = s_4$  (and all their permutations) which leads to the equality of reduced SCIs

$$\begin{aligned}
I'_E &= \frac{(p; p)_\infty (q; q)_\infty}{2} \int_{\mathbb{T}} \frac{\Gamma(s_8 z^{\pm 2}, \sqrt{pq} s_8^{-\frac{1}{2}} (s_1 s_2 s_3)^{\pm \frac{1}{2}} z^{\pm 1}; p, q)}{\Gamma(z^{\pm 2}; p, q)} \\
&\times \prod_{i=1}^3 \Gamma(\sqrt{pq} s_8^{-\frac{1}{2}} (s_i (s_1 s_2 s_3)^{-\frac{1}{2}}))^{\pm 1} z^{\pm 1}; p, q) \frac{dz}{2\pi i z},
\end{aligned} \tag{5.19}$$

and

$$\begin{aligned}
 I'_M &= \frac{(p; p)_\infty (q; q)_\infty}{2} \int_{\mathbb{T}} \frac{\Gamma(s_8 z^{\pm 2}; p, q) \Gamma^2(\sqrt{pq} s_8^{-\frac{1}{2}} z^{\pm 1}; p, q)}{\Gamma(z^{\pm 2}; p, q)} \\
 &\quad \times \prod_{i=1}^3 \Gamma(\sqrt{pq} s_8^{-\frac{1}{2}} s_i^{\pm 1} z^{\pm 1}; p, q) \frac{dz}{2\pi i z}.
 \end{aligned} \tag{5.20}$$

The equality  $I'_E = I'_M$  is a particular case of the identity obtained in [15] with

$$b = s_8, \quad t_4 = \sqrt{pq} s_8^{-\frac{1}{2}}, \quad t_i = \sqrt{pq} s_8^{-\frac{1}{2}} s_i, \quad i = 1, 2, 3.$$

One can reduce also the duality considered in this section to the dualities studied in [32]. If we give vacuum expectation values to  $k$  fundamental quarks in the electric theory, it breaks the gauge group on the electric side to  $SO(10 - k)$  while in the magnetic side the gauge group remains the same [32], see Section 5.1 for particular example when  $k = 5$ . But these dualities from [32] should be considered very accurately since as we have shown in Sect. 5.1 instead of  $SU(4)$  flavor symmetry group one should have  $SO(4)$  symmetry group. From the SCIs point of view we should restrict some of the parameters to form the divergency  $\propto \Gamma(1; p, q)$  in (5.18). Appearance of such a term in the magnetic index requires the residue calculus on the electric side. For example, the model of subsection 3.1 from [32] where  $SU(4)$  is changed to  $SO(4)$  and considered in Sect. 5.1 is obtained from (5.17) and (5.18) by taking in these expressions the following limits

$$s_{N_f-4} s_{N_f-3} = s_{N_f-2} s_{N_f-1} = s_{N_f} = 1, \tag{5.21}$$

with the subsequent replacement  $N_f \rightarrow N_f + 5$  and identification  $s_{N_f-2} = u_1, s_{N_f-4} = u_2$ .

A more general duality was proposed in [9, 72] having on the electric side the same  $SO(10)$  gauge group with  $N_f$  vectors and  $N_k$  spinors. The magnetic dual side was conjectured to be a quiver gauge theory with  $SU$  and  $SP$  gauge groups. Again, as in Sect. 5.1, we have not found evidence for this duality from the SCIs technique point of view, except of the obvious cases when the dual gauge group contains only one simple component or when the theory  $s$ -confines, in which cases one obtains known dualities. Anyway, we are not considering quiver gauge theories in this work, so we leave open the detailed analysis of repairing the general duality of [9, 72]. As in Sect. 5.1, perhaps this question may be settled by a reduction from an even more general duality for  $SO(N)$ ,  $N > 10$ , gauge group.

## 6. CONCLUSION

In [114, 115], we initiated the classification of elliptic hypergeometric integrals on different root systems and described all known examples of such integrals for  $A_N$ ,  $BC_N$ , and  $G_2$  root systems in association with  $\mathcal{N} = 1$  supersymmetric dualities. In [117], for all irreducible root systems we described such integrals associated with  $\mathcal{N} = 4$  SYM theories; there are also two more particular examples associated with  $\mathcal{N} = 1$  SYM  $E_6$  and  $F_4$  gauge group theories. In the present paper we described many new elliptic hypergeometric integrals which admit either exact evaluation or have nontrivial symmetry transformation properties. These identities are based on equalities of superconformal indices induced by supersymmetric dualities for  $\mathcal{N} = 1$  SYM theories with orthogonal gauge group and the spinor matter.

We have described all known cases when the  $BC_n$ -elliptic hypergeometric integrals and corresponding physical dualities with the symplectic gauge groups are reduced to the integrals/dualities for orthogonal groups by a restriction of parameters entering the integrals. In

particular, we described confining  $SO(N)$  gauge theories with  $N - 4$  and  $N - 3$  quarks in the fundamental representation, as well as the case of  $N - 2$  such fields leading to the Coloumb branch theory. An interesting particular example corresponds to  $N - 1$  flavors leading to three dual theories [65]: electric, magnetic and dyonic, for which we considered peculiarities of SCIs (which are obtained by a reduction of SCIs of gauge theories with the symplectic gauge group). Note that in this way one can obtain a number of new physical dualities for orthogonal gauge groups, as consequences of known symplectic gauge group dualities.

Remarkably, there are elliptic hypergeometric integrals for the  $B_N$  and  $D_N$  root systems which (currently) cannot be obtained from integrals on the  $BC_N$  root system — they come from SCIs for  $\mathcal{N} = 1$  SYM  $SO(2N + 1)$  or  $SO(2N)$  gauge group theories with the matter in spinor representation. Description of this type of integrals is the main result of the present paper. Physical dualities of the corresponding gauge theories lead to the conjectures on the equality of respective SCIs. The latter conjectural identities for elliptic hypergeometric integrals use characters of the spinor representations, and they were not predicted by the mathematical developments prior to the supersymmetric duality ideas intervention. All of them require now rigorous mathematical proofs.

In addition to SCIs for  $\mathcal{N} = 1$  dualities considered in this paper, one can investigate SCIs for electric-magnetic dualities for extended supersymmetric field theories: the quiver  $\mathcal{N} = 2$  SYM theories with  $SO/SP$  gauge groups [117] or the  $SP/SO$ -groups duality [53] in  $\mathcal{N} = 4$  SYM theory [45, 117]. Note that SCIs for extended supersymmetric theories can be obtained from SCIs of  $\mathcal{N} = 1$  theories by adjusting the matter content appropriately together with the hypercharges, as described in [115, 117]. From the physical point of view one should fix also appropriately the superpotentials.

As described in [115], one of the physical applications of the elliptic hypergeometric integral identities uses the reduction  $p = q = 0$ , which yields the Hilbert-Molien series counting gauge invariant operators [93, 56]. Another interesting application of our identities is connected with the Seiberg type dualities for three dimensional super-Yang-Mills and Chern-Simons theories with orthogonal gauge groups [37]. Actually, this yields the most efficient way of obtaining three-dimensional dualities. Technically, the reduction to  $3d$  theories is obtained after the following parametrization of the integration variables  $z_j$ , global symmetry fugacities  $t_i$ , and bases  $p, q$ ,

$$z_j = e^{2\pi i r \alpha_j}, \quad s_i = e^{2\pi i r g_i}, \quad p = e^{2\pi i \omega_1 r}, \quad q = e^{2\pi i \omega_2 r},$$

with the subsequent limit  $r \rightarrow 0$ . As a result,  $4d$  superconformal indices defined on  $\mathbf{S}^3 \times \mathbf{S}^1$  reduce to partition functions on the squashed three-sphere  $\mathbf{S}^3$  [67, 55]. In this limit the elliptic gamma function is reduced to the hyperbolic gamma function. It is thus natural to expect that all dualities considered in [68] can be recovered by a reduction from the  $4d$  SCIs considered in the present paper. For more details on the resulting hyperbolic  $q$ -hypergeometric integrals, see Appendix C.

As an aside result, in Appendix B below we are discussing possible connection of the elliptic hypergeometric integrals with an elliptic deformation of the Virasoro algebra. We propose that the three- and four-point correlation functions of a hypothetical elliptic  $2d$  CFT are given by a particular elliptic beta integral and its higher order generalization. However, it is rather hard to construct a deformation of the  $2d$  CFT primary fields leading to such results.

In Appendix C we describe an analytical reduction of SCIs for  $4d$  supersymmetric field theories to state integrals for knots living in  $3d$  space. A connection to the generalized  $(3 + 3)$ -dimensional AGT-dualities, where one deals with the  $3d$  non-supersymmetric CS theory on some

nontrivial manifold and  $3d$  supersymmetric theory on the squashed three-sphere, is discussed as well. The latter connection emerges due to the equality of their partition functions defined in terms of the mentioned hyperbolic integrals. We show that the equality for these partition functions follows from the hyperbolic reduction of  $4d$  SCIs of  $\mathcal{N} = 1$  supersymmetric theories with the subsequent restriction of the fugacities. As shown already in [37], the same type of reduction leads to the  $3d$  analogs of Seiberg dualities for  $3d$  (SYM or CS theories). So, altogether this gives a nice interpretation of the same equality of integrals emerging as special limiting cases of SCIs for  $4d$  theories: one is the  $3d$  generalization of the Seiberg duality and the other one is the generalization of AGT duality to  $(3 + 3)$ -dimensional theories which definitely deserves a separate study.

In Appendix D we show that further reduction of  $3d$  partition functions results in the vortex partition function for  $2d$  supersymmetric field theories. This observation leads us to the connection of  $4d$  SCIs for  $\mathcal{N} = 1$  SYM theories with  $4d$  partition functions for  $\mathcal{N} = 2$  SYM theories in some limits. We stress that all our computations are performed analytically (i.e., we describe exact equalities of the compared functions in all admissible domains of values of the parameters).

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#### APPENDIX A. CHARACTERS OF REPRESENTATIONS OF CLASSICAL GROUPS

In this and the next Appendix we present general results for characters of the Lie group representations used in the paper. For the  $SU(N)$  group representations, the characters, depending on the maximal torus variables  $x = (x_1, \dots, x_N)$  subject to the constraint  $\prod_{i=1}^N x_i = 1$ , are the well known Schur polynomials

$$s_{\underline{\lambda}}(x) = s_{(\lambda_1, \dots, \lambda_N)}(x) = \frac{\det [x_i^{\lambda_j + N - j}]}{\det [x_i^{N - j}]}, \quad (\text{A.1})$$

where  $\underline{\lambda}$  is the partition ordered as  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N$ . One can assume that  $\lambda_N = 0$  without loss of generality.

Let us list explicitly the simplest characters. For the fundamental and antifundamental representations:

$$\chi_{SU(N),f}(x) = s_{(1,0,\dots,0)}(x) = \sum_{i=1}^N x_i, \quad \chi_{SU(N),\bar{f}} = s_{(1,\dots,1,0)}(x) = \chi_{SU(N),f}(x^{-1}).$$

The adjoint representation:

$$\chi_{SU(N),adj}(x) = s_{(2,1,\dots,1,0)}(x) = \sum_{1 \leq i, j \leq N} x_i x_j^{-1} - 1.$$

The absolutely anti-symmetric tensor representation of rank two:

$$\chi_{SU(N),T_A}(x) = s_{(1,1,0,\dots,0)}(x) = \sum_{1 \leq i < j \leq N} x_i x_j, \quad \chi_{SU(N),\bar{T}_A} = \chi_{SU(N),T_A}(x^{-1}).$$

The symmetric representation:

$$\chi_{SU(N),T_S}(x) = s_{(2,0,\dots,0)}(x) = \sum_{1 \leq i < j \leq N} x_i x_j + \sum_{i=1}^N x_i^2, \quad \chi_{SU(N),\bar{T}_S}(x) = \chi_{SU(N),T_S}(x^{-1}).$$

The absolutely anti-symmetric tensor representation of rank three:

$$\chi_{SU(N),T_{3A}}(x) = s_{(1,1,1,0,\dots,0)}(x) = \sum_{1 \leq i < j < k \leq N} x_i x_j x_k.$$

The absolutely symmetric tensor representation of rank three:

$$\chi_{SU(N),T_{3S}}(x) = s_{(3,0,\dots,0)}(x) = \sum_{1 \leq i < j < k \leq N} x_i x_j x_k + \sum_{i,j=1, i \neq j}^N x_i^2 x_j + \sum_{i=1}^N x_i^3.$$

In the mixed case, we have

$$\chi_{SU(N),T_{AS}}(x) = s_{(2,1,0,\dots,0)}(x) = 2 \sum_{1 \leq i < j < k \leq N} x_i x_j x_k + \sum_{i,j=1, i \neq j}^N x_i^2 x_j.$$

The Weyl characters for  $SP(2N)$ -group are given by the determinant

$$\tilde{s}_{(\lambda_1, \dots, \lambda_N)}(x) = \frac{\det \left[ x_i^{\lambda_j + N - j + 1} - x_i^{-\lambda_j - N + j - 1} \right]}{\det \left[ x_i^{N - j + 1} - x_i^{-N + j - 1} \right]}, \quad (\text{A.2})$$

with  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N \geq 0$ . For the fundamental and antifundamental representations

$$\chi_{SP(2N),f}(x) = \chi_{SP(2N),\bar{f}}(x) = \tilde{s}_{(1,0,\dots,0)}(x) = \sum_{i=1}^N x_i^{\pm 1},$$

where we use the conventions  $x_i^{\pm 1} = x_i + x_i^{-1}$ .

For the adjoint representation

$$\chi_{SP(2N),adj}(x) = \tilde{s}_{(2,0,\dots,0)}(x) = \sum_{1 \leq i < j \leq N} x_i^{\pm 1} x_j^{\pm 1} + \sum_{i=1}^N x_i^{\pm 2} + N,$$

where we use the conventions  $x_i^{\pm 1} x_j^{\pm 1} = x_i x_j + x_i x_j^{-1} + x_i^{-1} x_j + x_i^{-1} x_j^{-1}$ .

For the absolutely anti-symmetric representation

$$\chi_{SP(2N),T_A}(x) = \tilde{s}_{(1,1,0,\dots,0)}(x) = \sum_{1 \leq i < j \leq N} x_i^{\pm 1} x_j^{\pm 1} + N - 1.$$

The invariant measure for the unitary group  $SU(N)$  weighted with an arbitrary symmetric function  $f(z)$ , where  $z = (z_1, \dots, z_N)$ ,  $\prod_{j=1}^N z_j = 1$ , has the form

$$\int_{SU(N)} f(z) d\mu(z) = \frac{1}{N!} \int_{\mathbb{T}^{N-1}} \Delta(z) \Delta(z^{-1}) f(z) \prod_{j=1}^{N-1} \frac{dz_j}{2\pi i z_j}, \quad (\text{A.3})$$



where  $\Delta(z)$  is the Vandermonde determinant

$$\Delta(z) = \prod_{1 \leq i < j \leq N} (z_i - z_j).$$

The invariant measure for the symplectic group  $SP(2N)$  weighted with any symmetric function  $f(z)$ ,  $z = (z_1, \dots, z_N)$ , has the form

$$\int_{SP(2N)} f(z) d\mu(z) = \frac{(-1)^N}{2^N N!} \int_{\mathbb{T}^N} \prod_{j=1}^N (z_j - z_j^{-1})^2 \Delta(z + z^{-1})^2 f(z) \prod_{j=1}^N \frac{dz_j}{2\pi i z_j}. \quad (\text{A.4})$$

Orthogonal groups  $SO(N)$  with even and odd  $N$  have substantially different properties and should be considered separately. For instance, the invariant measure of the group  $SO(2N)$  with any symmetric function  $f(z)$ ,  $z = (z_1, \dots, z_N)$ , is:

$$\int_{SO(2N)} f(z) d\mu(z) = \frac{1}{2^{N-1} N!} \int_{\mathbb{T}^N} \Delta(z + z^{-1})^2 f(z) \prod_{j=1}^N \frac{dz_j}{2\pi i z_j}, \quad (\text{A.5})$$

and for the group  $SO(2N+1)$  one has

$$\int_{SO(2N+1)} f(z) d\mu(z) = \frac{(-1)^N}{2^N N!} \int_{\mathbb{T}^N} \prod_{j=1}^N \left( z_j^{\frac{1}{2}} - z_j^{-\frac{1}{2}} \right)^2 \Delta(z + z^{-1})^2 f(z) \prod_{j=1}^N \frac{dz_j}{2\pi i z_j}. \quad (\text{A.6})$$

The characters for spinor representations of  $SO(N)$  groups are described most conveniently by the expressions involving square roots of  $z_j$ -variables which are not analytical. To overcome this obstacle we just double the root lengths which results in the replacement in the characters the variables  $z_j$  by  $z_j^2$  and in the integrals over  $z_j$  we keep the same integration contour  $\mathbb{T}$ , the unit circle with positive orientation.

Before passing to the description of particular representation characters, we note that the adjoint representation for orthogonal groups coincides with the  $T_A$ -representation.

**$SO(2N)$  group.** The characters are expressed in terms of  $N$  independent variables  $z_i$ ,  $i = 1, \dots, N$ . For the fundamental representation one has

$$\chi_{f,SO(2N)} = \sum_{i=1}^N z_i^{\pm 1}. \quad (\text{A.7})$$

The  $T_S$ -representation character is

$$\chi_{T_S,SO(2N)} = \sum_{1 \leq i < j \leq N} z_i^{\pm 1} z_j^{\pm 1} + \sum_{i=1}^N z_i^{\pm 2} + N - 1, \quad (\text{A.8})$$

the  $T_A$ -representation character is

$$\chi_{T_A,SO(2N)} = \sum_{1 \leq i < j \leq N} z_i^{\pm 1} z_j^{\pm 1} + N. \quad (\text{A.9})$$

The needed spinor representation characters are listed case by case. For  $SO(2N)$  groups there are two types of inequivalent spinors, denoted as  $s$  and  $c$ . For  $SO(8)$ , the spinor representation  $s$  and  $c$  are 8-dimensional, self-conjugate, and their characters have the form

$$\chi_{s,SO(8)} = z^{\pm 1} + z^{-1} \sum_{1 \leq i < j \leq 4} z_i z_j, \quad (\text{A.10})$$

where  $z = \sqrt{z_1 z_2 z_3 z_4}$ . For  $SO(10)$ , the  $s$ -representation is 16-dimensional, it is complex conjugate to  $c$  (so that the character for  $c$  can be obtained from the  $s$ -character by the substitution  $z \rightarrow 1/z$ ). Its character is

$$\chi_{s,SO(10)} = z + z^{-1} \sum_{j=1}^5 z_j + z \sum_{1 \leq i < j \leq 5} z_i^{-1} z_j^{-1}, \quad (\text{A.11})$$

where  $z = \sqrt{z_1 z_2 z_3 z_4 z_5}$ . For  $SO(12)$ , the  $s$ ,  $c$ -representations are 32-dimensional, self-conjugate, and have the character

$$\chi_{s,SO(12)} = z^{\pm 1} + z^{-1} \sum_{j=1}^6 z_j + z \sum_{j=1}^6 z_j^{-1}, \quad (\text{A.12})$$

where  $z = \sqrt{z_1 z_2 z_3 z_4 z_5 z_6}$ . For  $SO(14)$ , the  $s$ -representation is 64-dimensional, it is complex-conjugate to  $c$ , and its character is

$$\chi_{s,SO(14)} = z + z^{-1} \sum_{j=1}^7 z_j + z \sum_{1 \leq i < j \leq 7} z_i^{-1} z_j^{-1} + z^{-1} \sum_{1 \leq i < j < k \leq 7} z_i z_j z_k, \quad (\text{A.13})$$

where  $z = \sqrt{z_1 z_2 z_3 z_4 z_5 z_6 z_7}$ .

$SO(2N+1)$  **group**. All the characters are expressed in terms of  $N$  independent variables  $z_i, i = 1, \dots, N$ . The fundamental representation character is

$$\chi_{f,SO(2N+1)} = \sum_{i=1}^N z_i^{\pm 1} + 1. \quad (\text{A.14})$$

The character for  $T_S$ -representation is

$$\chi_{T_S,SO(2N+1)} = \sum_{1 \leq i < j \leq N} z_i^{\pm 1} z_j^{\pm 1} + \sum_{i=1}^N z_i^{\pm 2} + \sum_{i=1}^N z_i^{\pm 1} + N, \quad (\text{A.15})$$

the character for the  $T_A$ -representation is

$$\chi_{T_A,SO(2N+1)} = \sum_{1 \leq i < j \leq N} z_i^{\pm 1} z_j^{\pm 1} + \sum_{i=1}^N z_i^{\pm 1} + N. \quad (\text{A.16})$$

The spinor representation characters are given for the lowest rank groups only. For  $SO(7)$ , the spinor representation is 8-dimensional and its character is

$$\chi_{s,SO(7)} = z^{\pm 1} + z^{-1} \sum_{j=1}^3 z_j + z \sum_{j=1}^3 z_j^{-1}, \quad (\text{A.17})$$

where  $z = \sqrt{z_1 z_2 z_3}$ . For  $SO(9)$ , the spinor representation is 16-dimensional and its character is

$$\chi_{s,SO(9)} = z^{\pm 1} + z^{-1} \sum_{j=1}^4 z_j + z \sum_{j=1}^4 z_j^{-1} + z^{-1} \sum_{1 \leq i < j \leq 4} z_i z_j, \quad (\text{A.18})$$

where  $z = \sqrt{z_1 z_2 z_3 z_4}$ . For  $SO(11)$ , the spinor representation is 32-dimensional and its character is

$$\chi_{s,SO(11)} = z^{\pm 1} + z^{-1} \sum_{j=1}^5 z_j + z \sum_{j=1}^5 z_j^{-1} + z^{-1} \sum_{1 \leq i < j \leq 5} z_i z_j + z \sum_{1 \leq i < j \leq 5} (z_i z_j)^{-1}, \quad (\text{A.19})$$

where  $z = \sqrt{z_1 z_2 z_3 z_4 z_5}$ . For  $SO(13)$ , the spinor representation is 64-dimensional and its character is

$$\chi_{s,SO(13)} = z^{\pm 1} + z^{-1} \sum_{j=1}^6 z_j + z \sum_{j=1}^6 z_j^{-1} + z^{-1} \sum_{1 \leq i < j \leq 6} z_i z_j + z \sum_{1 \leq i < j \leq 6} (z_i z_j)^{-1} + z^{-1} \sum_{1 \leq i < j < k \leq 6} z_i z_j z_k, \quad (\text{A.20})$$

where  $z = \sqrt{z_1 z_2 z_3 z_4 z_5 z_6}$ .

## APPENDIX B. MATRIX MODELS AND AN ELLIPTIC DEFORMATION OF THE $2d$ CFT

Main inspiration for this section comes from paper [99], where a  $q$ -deformed  $2d$  CFT and the corresponding matrix model description in terms of the Jackson integrals was proposed. From the elliptic hypergeometric integrals' point of view there is a natural way to propose a generalization of CFT to the elliptic and different  $q$ -deformed levels.  $q$ -extensions of the Virasoro algebra have been considered already some time ago [41, 79, 4] (see also [6, 44, 5] for a recent discussion). Here we propose expressions for the three- and four-point correlation functions presumably associated with new hypothetical  $q$ -deformations and an elliptic deformation of the  $2d$  CFT.

**Elliptic Selberg integral.** Let us describe the Selberg integral, the basic integral appearing in calculations of the three-point correlation function in  $2d$  CFT, and its various extensions. The following elliptic generalization of the Selberg integral attached to the root system  $BC_N$  was discovered in [24, 25]:

$$\begin{aligned} & \frac{(p; p)_\infty^N (q; q)_\infty^N}{2^N N!} \int_{\mathbb{T}^N} \prod_{1 \leq i < j \leq N} \frac{\Gamma(t z_i^{\pm 1} z_j^{\pm 1}; p, q)}{\Gamma(z_i^{\pm 1} z_j^{\pm 1}; p, q)} \prod_{j=1}^N \frac{\prod_{i=1}^6 \Gamma(t_i z_j^{\pm 1}; p, q)}{\Gamma(z_j^{\pm 2}; p, q)} \prod_{j=1}^N \frac{dz_j}{2\pi i z_j} \\ & = \prod_{j=1}^N \left( \frac{\Gamma(t^j; p, q)}{\Gamma(t; p, q)} \prod_{1 \leq i < k \leq 6} \Gamma(t^{j-1} t_i t_k; p, q) \right), \end{aligned} \quad (\text{B.1})$$

where  $|t|, |t_j| < 1$ , and

$$t^{2(N-1)} \prod_{i=1}^6 t_i = pq \quad (\text{B.2})$$

is the balancing condition. This integral describes the  $\mathcal{N} = 1$   $s$ -confining SYM theory with  $SP(2N)$  gauge group, one chiral superfield in the  $T_A$ -representation of  $SP(2N)$ , and 6 quarks [115]. This physical application provides a matrix model interpretation of the formula (B.1). Note also that this integral describes the normalization of a particular eigenstate of a relativistic Calogero-Sutherland type model [110].

We *postulate* that the chiral part of the three-point correlation function of a hypothetical elliptic deformation of  $2d$  CFT based on an elliptic extension of the Virasoro algebra is given by integral (B.1), admitting the exact evaluation. This proposition fits the fact that in all known variations of  $2d$  CFT the three-point function is computable exactly. Note that in [44] a simple elliptic deformation of the free bosonic field algebra was proposed, but its relevance to our construction is not clear, in particular, the number and meaning of the parameters  $t_j$  are not evident in this case.

**$q$ -Selberg integral.** Different reductions of the elliptic hypergeometric integrals were systematically investigated in [95] (see also [16]). First we reduce the integral (B.1) to the trigonometric level, and then to the standard Selberg integral. The limit  $p \rightarrow 0$  is not straightforward due to the balancing condition which we get rid of by substituting in (B.1)  $t_6 = pq/(t^{2(N-1)}T)$ , where  $T = \prod_{i=1}^5 t_i$ , and obtain

$$\begin{aligned} & \frac{(p; p)_\infty^N (q; q)_\infty^N}{2^N N!} \int_{\mathbb{T}^N} \prod_{1 \leq i < j \leq N} \frac{\Gamma(tz_i^{\pm 1} z_j^{\pm 1}; p, q)}{\Gamma(z_i^{\pm 1} z_j^{\pm 1}; p, q)} \prod_{j=1}^N \frac{\prod_{i=1}^5 \Gamma(t_i z_j^{\pm 1}; p, q)}{\Gamma(t^{2(N-1)} T z_j^{\pm 1}; p, q) \Gamma(z_j^{\pm 2}; p, q)} \prod_{j=1}^N \frac{dz_j}{2\pi i z_j} \\ &= \prod_{j=1}^N \left( \frac{\Gamma(t^j; p, q)}{\Gamma(t; p, q)} \prod_{1 \leq i < k \leq 5} \Gamma(t^{j-1} t_i t_k; p, q) \prod_{i=1}^5 \frac{1}{\Gamma(t^{N+j-2} T/t_i; p, q)} \right), \end{aligned} \quad (\text{B.3})$$

where we take into the account that

$$\prod_{j=1}^N \Gamma(t^{2(N-1)-j+1} T/t_i; p, q) = \prod_{j=1}^N \Gamma(t^{N+j-2} T/t_i; p, q).$$

Now we can set  $t_5 = 0$  and obtain the trigonometric  $q$ -Selberg integral derived by Gustafson in [54]

$$\begin{aligned} & \frac{1}{2^N N!} \int_{\mathbb{T}^N} \prod_{1 \leq i < j \leq N} \frac{(z_i^{\pm 1} z_j^{\pm 1}; q)_\infty}{(t z_i^{\pm 1} z_j^{\pm 1}; q)_\infty} \prod_{j=1}^N \frac{(z_j^{\pm 2}; q)_\infty}{\prod_{i=1}^4 (t_i z_j^{\pm 1}; q)_\infty} \prod_{j=1}^N \frac{dz_j}{2\pi i z_j} \\ &= \prod_{j=1}^N \left( \frac{(t; q)_\infty}{(t^j; q)_\infty (q; q)_\infty} (t^{N+j-2} t_1 t_2 t_3 t_4; q)_\infty \prod_{1 \leq i < k \leq 4} \frac{1}{(t^{j-1} t_i t_k; q)_\infty} \right). \end{aligned} \quad (\text{B.4})$$

Again, as above, we *postulate* that the three-point correlation function of a hypothetical  $2d$  CFT based on a (yet unknown)  $q$ -deformed Virasoro algebra is given by the function (B.4). Note that it is described by the standard contour integral and not the Jackson  $q$ -integral, as discussed in [99].

**Reduction to the Selberg integral.** To obtain the Selberg integral from (B.4) one should carefully take the limit  $q \rightarrow 1^-$ . To simplify the left-hand side of formula (B.4) we use the relation

$$\lim_{q \rightarrow 1^-} \frac{(q^a z; q)_\infty}{(z; q)_\infty} = (1-z)^{-a}, \quad (\text{B.5})$$

and the duplication formula

$$(z^2; q)_\infty = (\pm z, \pm q^{\frac{1}{2}} z; q)_\infty.$$

To simplify the right-hand side expression we replace infinite products by the Jackson  $q$ -gamma function

$$\Gamma_q(x) = \frac{(q; q)_\infty}{(q^x; q)_\infty} (1-q)^{1-x}, \quad \Gamma_q(x) \underset{q \rightarrow 1}{=} \Gamma(x). \quad (\text{B.6})$$

Now we denote the parameters entering (B.4) as

$$t = q^\gamma, \quad t_1 = q^{\alpha - \frac{1}{2}}, \quad t_2 = -q^{\beta - \frac{1}{2}}, \quad t_3 = q^{\frac{1}{2}}, \quad t_4 = -q^{\frac{1}{2}}. \quad (\text{B.7})$$

On the left-hand side of (B.4) we change also the integration variables  $z_j = e^{i\theta_j}$  and denote  $x_i = (1 + \cos \theta_i)/2$ .

Finally, for fixed  $\alpha, \beta, \gamma$ , we can take safely the limit  $q \rightarrow 1^-$ , which brings us to the standard Selberg integral [3]

$$\begin{aligned} & \int_0^1 \dots \int_0^1 \prod_{j=1}^N x_j^{\alpha-1} (1-x_j)^{\beta-1} \prod_{1 \leq i < j \leq N} |x_i - x_j|^{2\gamma} dx_1 \dots dx_N \\ &= \prod_{j=1}^N \frac{\Gamma(\alpha + (j-1)\gamma) \Gamma(\beta + (j-1)\gamma) \Gamma(1+j\gamma)}{\Gamma(\alpha + \beta + (n+j-2)\gamma) \Gamma(1+\gamma)}, \end{aligned} \quad (\text{B.8})$$

where the integral converges for

$$\Re\alpha, \Re\beta > 0, \quad \Re\gamma > -\min\left(\frac{1}{N}, \frac{\Re\alpha}{N-1}, \frac{\Re\beta}{N-1}\right). \quad (\text{B.9})$$

Expression (B.8) defines the  $\beta$ -deformed matrix integral and gives the three-point function of the standard undeformed  $2d$  CFT, see, e.g., Sect. 4.1 of [99].

**A higher order elliptic Selberg integral.** A two parameter extension of the elliptic Selberg integral (B.1) is given by the integral

$$\begin{aligned} V(t_1, \dots, t_8; t; p, q) &= \frac{(p; p)_\infty^N (q; q)_\infty^N}{2^N N!} \int_{\mathbb{T}^N} \prod_{1 \leq i < j \leq N} \frac{\Gamma(tz_i^{\pm 1} z_j^{\pm 1}; p, q)}{\Gamma(z_i^{\pm 1} z_j^{\pm 1}; p, q)} \\ &\times \prod_{j=1}^N \frac{\prod_{i=1}^8 \Gamma(t_i z_j^{\pm 1}; p, q)}{\Gamma(z_j^{\pm 2}; p, q)} \prod_{j=1}^N \frac{dz_j}{2\pi i z_j}, \end{aligned} \quad (\text{B.10})$$

where the balancing condition reads  $t^{2(N-1)} \prod_{i=1}^8 t_i = (pq)^2$ . The symmetry transformation properties of this integral were found in [108, 94]. We are not presenting them here explicitly for brevity (for  $N = 1$  they are described by formula (2.34)). We *postulate* that integral (B.10) defines the four-point correlation function for a hypothetical elliptic deformation of the  $2d$  CFT, the corresponding  $s$ - $t$ -channels duality is described then by known symmetries of (B.10).

Again, taking appropriately the (trigonometric) limit  $p \rightarrow 0$  we can come to the two parameter extension of the  $q$ -Selberg integral with further degeneration to the rational level [95]. For arbitrary  $N$  and a special choice of one of the parameters, there emerges the  ${}_2F_1$ -hypergeometric function describing the chiral part of the four point correlation function (see formula (4.9) in [99]). General  ${}_2F_1$ -function is obtained also for  $N = 1$ , we skip explicit description of these well known results. In [79], the four point correlation function of a  $q$ -deformed CFT was connected to a  $q$ -analog of the  ${}_2F_1$ -hypergeometric function. We conjecture that an appropriate elliptic analog of the latter correlation function will be expressed in terms of the  $V$ -function of [111] given by  $N = 1$  case of (B.10). Apart from the mentioned limit  $p \rightarrow 0$ , there exists a different degenerating limit for the elliptic Selberg integral to the hyperbolic  $q$ -hypergeometric level [26], which was discussed recently in detail in [37] where one of the resulting integrals was interpreted as the partition function of a particular  $3d$   $\mathcal{N} = 2$  supersymmetric field theory model.

### APPENDIX C. CONNECTION TO THE KNOT THEORY

In this Appendix we discuss the connection of partition functions for some  $3d$  supersymmetric field theories and non-supersymmetric CS theories with the complexified gauge groups to topological invariants of the knot theory [57, 58, 31, 27, 29]. In [37], the theory of hyperbolic  $q$ -hypergeometric integrals has been exploited for checking and searching for  $3d$  supersymmetric

field theory dualities. Earlier it was proposed in [57] that the state integrals for knots are also defined in terms of such integrals. In an independent approach to state integrals [27], Dimofte proposed a new expression for the figure-eight knot state integral and conjectured that it coincides with the one of [57]. Using the approach of [37] we prove here this conjecture, as well as some other similar identities needed in [28].

The hyperbolic  $q$ -hypergeometric integrals can be rigorously obtained as reductions of the elliptic hypergeometric integrals [95] (for an earlier formal approach see, e.g., [26], and for a detailed explicit analysis of reducing many integrals see [14]). The reduction procedure inherits certain pieces of the *unique* symmetry properties of the original integrals and yields *many* nontrivial identities at the hyperbolic level. The resulting hyperbolic integrals and identities emerge in various physical problems. Here we stress that they describe partition functions for  $3d$  supersymmetric theories living on the squashed three-sphere and the state integrals for the knots. As the most recent example of their relevance, we mention a generalization of the AGT duality [1] to the duality inspired by the  $(3 + 3)$ -dimensional theories [60, 118, 29, 28], with the non-supersymmetric CS theory living on a  $3d$  manifold  $\mathcal{M}$  on the one side and  $3d$   $\mathcal{N} = 2$  supersymmetric theory living on the squashed sphere on the other side.

**The figure-eight knot.** We start from the notation for hyperbolic gamma function used in [113, 37]. This function appeared in [39] under the name “noncompact quantum dilogarithm”. For  $q = e^{2\pi i \frac{\omega_1}{\omega_2}}$  and  $\tilde{q} = e^{-2\pi i \frac{\omega_2}{\omega_1}}$  with  $|q| < 1$  we define

$$\gamma(u; \omega_1, \omega_2) = \frac{(e^{2\pi i u/\omega_1} \tilde{q}; \tilde{q})_\infty}{(e^{2\pi i u/\omega_2}; q)_\infty}.$$

The hyperbolic gamma function is defined as

$$\gamma^{(2)}(u; \omega_1, \omega_2) = e^{-\pi i B_{2,2}(u)/2} \gamma(u; \omega_1, \omega_2),$$

where  $B_{2,2}(u; \omega_1, \omega_2)$  is the second order Bernoulli polynomial,

$$B_{2,2}(u; \omega_1, \omega_2) = \frac{u^2}{\omega_1 \omega_2} - \frac{u}{\omega_1} - \frac{u}{\omega_2} + \frac{\omega_1}{6\omega_2} + \frac{\omega_2}{6\omega_1} + \frac{1}{2}.$$

For  $\text{Re}(\omega_1), \text{Re}(\omega_2) > 0$  and  $0 < \text{Re}(u) < \text{Re}(\omega_1 + \omega_2)$  one has the following integral representation

$$\gamma^{(2)}(u; \omega_1, \omega_2) = \exp \left( -\text{PV} \int_{\mathbb{R}} \frac{e^{ux}}{(1 - e^{\omega_1 x})(1 - e^{\omega_2 x})} \frac{dx}{x} \right),$$

where ‘PV’ means the principal value integral.

Different notations and names for slight modifications of this function are used in the literature, most of them were explicitly described in Appendix A of [113]. In [31], the following “quantum dilogarithm” is employed

$$\Phi(z; \tau) = \frac{(-e(z + \tau/2); e(\tau))_\infty}{(-e((z - 1/2)/\tau); e(-1/\tau))_\infty}, \quad (\text{C.1})$$

where  $e(x) = e^{2\pi i x}$ . One can easily find by comparison that

$$\Phi(z; \tau) = \gamma \left( \frac{\omega_1 + \omega_2}{2} + z\omega_2; \omega_1, \omega_2 \right)^{-1}, \quad \tau = \frac{\omega_1}{\omega_2}. \quad (\text{C.2})$$

Consider the so-called state integral for the figure eight knot  $\mathbf{4}_1$  which was found first by Hikami in [57] and studied further in [58, 31, 27, 29]. We stick to the notation of paper [31]

where this integral is given by formula (4.46) and has the form

$$I = \frac{e^{2\pi i u/\hbar+u}}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \frac{\Phi((p-u)/2\pi i; \hbar/\pi i)}{\Phi(-(p+u)/2\pi i; \hbar/\pi i)} e^{-2pu/\hbar} dp. \quad (\text{C.3})$$

This integral describes also the partition function of non-supersymmetric CS theory with the complexified gauge group  $SL(2, \mathbb{C})$  living on the  $3d$  manifold  $\mathcal{M} = \mathbf{S}^3 \setminus \mathbf{4}_1$  [31].

Denoting  $\omega_1 = b, \omega_2 = b^{-1}, \tau = b^2$ , and changing the variables

$$p \rightarrow 2\pi i p, \quad u \rightarrow 2\pi i u, \quad \hbar \rightarrow \pi i \tau$$

in (C.3), we obtain

$$I = e^{2\pi i(2+b^2)u/b^2} \int_{-i\infty}^{i\infty} \frac{\Phi(p-u; b^2)}{\Phi(-p-u; b^2)} e^{-8\pi i p u/b^2} dp, \quad (\text{C.4})$$

where we drop the multiplier  $2\pi/i\sqrt{2\pi\hbar}$  in front of the integral. Using relation (C.2), we can write

$$I = e^{2\pi i(2+b^2)u/b^2} \int_{-i\infty}^{i\infty} \frac{\gamma\left(\frac{b+1/b}{2} - \frac{p+u}{b}; b, b^{-1}\right)}{\gamma\left(\frac{b+1/b}{2} + \frac{p-u}{b}; b, b^{-1}\right)} e^{-8\pi i p u/b^2} dp. \quad (\text{C.5})$$

We apply the inversion formula  $\gamma(u, b+1/b-u; b, b^{-1}) = e^{\pi i B_{2,2}(u; b, b^{-1})}$  to move the denominator  $\gamma$ -function to the numerator and pass from the  $\gamma$ -function to the  $\gamma^{(2)}$ -function. This yields another form of the integral:

$$I = e^{2\pi i(2+b^2)u/b^2} \int_{-i\infty}^{i\infty} \gamma^{(2)}\left(\frac{b+1/b}{2} - \frac{p \pm u}{b}; b, b^{-1}\right) e^{-8\pi i p u/b^2} \times e^{\pi i(B_{2,2}((b+1/b)/2-(p+u)/b) + B_{2,2}((b+1/b)/2-(p-u)/b) - 2B_{2,2}((b+1/b)/2+(p-u)/b))/2} dp, \quad (\text{C.6})$$

or, after the simplification,

$$I = e^{2\pi i(2+b^2)u/b^2} \int_{-i\infty}^{i\infty} \gamma^{(2)}\left(\frac{b+1/b}{2} - \frac{p \pm u}{b}; b, b^{-1}\right) e^{-6\pi i p u/b^2} dp. \quad (\text{C.7})$$

Let us take now integral (6.77) from [27] (in the suggested there normalization without the multiplier  $2^{-1/2}e^{(4\pi^2-h^2)/24h^2}$ ). After changing the notation in it similar to the integral  $I$ , we come to the following expression

$$\tilde{I} = e^{-2\pi i u} \int_{-i\infty}^{i\infty} \gamma^{(2)}\left(\frac{b+1/b}{2} - \frac{p \pm u}{b}; b, b^{-1}\right) e^{6\pi i p u/b^2} dp. \quad (\text{C.8})$$

One can see that the difference between expressions (C.7) and (C.8) is in the coefficients in front of the integrals and in the sign of the exponent of the integrand.

Let us take now the  $n = 1$  case of the integral  $II_{n,(3,3)*a}^1(\mu; -; \lambda; \tau)$  defined on page 218 of [14]. Replacing the integration variable  $x \rightarrow p/b$  in it and changing slightly its normalizing multiplier, we come to the integral

$$Z_E(\mu_1, \mu_2, \sigma) = \int_{-i\infty}^{i\infty} \prod_{i=1}^2 \gamma^{(2)}(\mu_i - p/b; b, b^{-1}) e^{\pi i \sigma p/b} dp, \quad (\text{C.9})$$

where  $\mu_1, \mu_2$ , and  $\sigma$  are some free parameters. We denote this expression as  $Z_E(\mu_1, \mu_2, \sigma)$  not accidentally, it appears that  $Z_E$  describes the partition function of a particular  $3d$  supersymmetric field theory.

Remarkably, our original integral of interest  $I$  (C.7) is a special subcase of (C.9), which is obtained after imposing the constraints

$$\mu_1 = (b + 1/b)/2 - u/b, \quad \mu_2 = (b + 1/b)/2 + u/b, \quad \sigma = -6u/b. \quad (\text{C.10})$$

Using the results of [37], we see that expression (C.7) with arbitrary  $\mu_1, \mu_2, \sigma$  describes the partition function of  $3d \mathcal{N} = 2$  theory living on the squashed three-sphere with the  $U(1)$  gauge group and two quarks, which is referred to as the “electric theory”. The global symmetry group is  $SU(2) \times U(1)_A \times U(1)_R$ . The matter content with the corresponding charges is presented in the following table

	$U(1)$	$SU(2)$	$U(1)_A$	$U(1)_R$
$q$	-1	$f$	1	$\frac{1}{2}$

We do not discuss the vector superfield having well known properties.

The integral (C.9) has the transformation formula described in Theorem 5.6.20 of [14]:

$$\begin{aligned} & e^{\pi i(4\mu_1\mu_2 - \mu_3^2 + (b+1/b)\mu_3 - (b+1/b)^2/4)/2 + \pi i(b^2 + 1/b^2)/24} Z_E(\mu_1, \mu_2, 2\mu_3 - \mu_1 - \mu_2) \\ &= Z_M(\mu_1, \mu_2, \mu_3, \lambda), \end{aligned} \quad (\text{C.11})$$

where

$$Z_M(\mu_1, \mu_2, \mu_3; \lambda) = \int_{-i\infty}^{i\infty} \prod_{i=1}^3 \gamma^{(2)}(\mu_i - p/b; b, b^{-1}) e^{\pi i \lambda p/b - 3\pi i p^2/2b^2} dp, \quad (\text{C.12})$$

with  $\mu_3$  being a new parameter introduced through the balancing condition

$$\sum_{i=1}^3 \mu_i = \lambda - \frac{b + 1/b}{2}.$$

This condition relates fugacities associated with the  $SU(3)$  flavor group acting on quarks and the Fayet-Illiopoulos term  $\lambda$ .

Expression (C.12) represents the partition function of a “magnetic theory” defined as the  $3d \mathcal{N} = 2$  CS theory with  $U(1)_{3/2}$  gauge group and 3 quarks. The global symmetry group of the magnetic theory is  $SU(3) \times U(1)_A \times U(1)_R$ . Note that the flavor groups of the electric and magnetic theories differ although the number of independent variables is the same for both statistical sums. The matter fields together with the corresponding charges are presented in the table below

	$U(1)_{3/2}$	$SU(3)$	$U(1)_A$	$U(1)_R$
$q$	-1	$f$	1	$\frac{1}{2}$

The duality between these two  $3d$  theories is one of very many dualities not considered in [37] due to their abundance.

Now we can easily prove the equality of two forms of the figure-eight state integrals (C.7) and (C.8),  $I = \tilde{I}$ . Evidently, expression (C.12) is symmetric in parameters  $\mu_1, \mu_2$ , and  $\mu_3$ . If we substitute in the left-hand side of (C.11) restrictions (C.10), we obtain the integral  $I$  up to some factor. Now we permute the parameters in the left-hand side  $(\mu_1, \mu_2, \mu_3) \rightarrow (\mu_3, \mu_1, \mu_2)$  (which is permitted because of the identity) and substitute anew the same restrictions (C.10). As a result we obtain the integral  $\tilde{I}$  up to the same multiplier as before. Equating both expressions, we prove that  $I = \tilde{I}$ .

Moreover, we can use further this permutational symmetry and replace in the left-hand side of (C.11)  $(\mu_1, \mu_2, \mu_3) \rightarrow (\mu_2, \mu_3, \mu_1)$ , and impose the constraints (C.10). As a result we come to



one more form of the figure-eight state integral

$$I = \tilde{I} = \hat{I} := e^{2\pi i u(1-6u)/b^2} \int_{-i\infty}^{i\infty} \gamma^{(2)} \left( \frac{b+1/b}{2} - \frac{3u+p}{b}, \frac{b+1/b}{2} + \frac{u-p}{b}; b, b^{-1} \right) dp, \quad (\text{C.13})$$

which was not considered in [57, 58, 31, 27].

The last comment before passing to the description of connections with the elliptic hypergeometric integrals is the following. As observed in [29], there is an extension of the AGT duality [1] to the situation when the 6-dimensional space-time is decomposed as a  $(3+3)d$  manifold with the duality relation between the complexified CS theories living on some  $3d$  manifold  $\mathcal{M}$  and  $3d$  supersymmetric field theories. Our equality of partition functions (C.11) gives an explicit example of such a duality. In this example we see that the CS theory with  $SL(2, \mathbb{C})$  gauge group on  $\mathcal{M} = \mathbf{S}^3 \setminus \mathbf{4}_1$  is dual to the  $3d$   $\mathcal{N} = 2$  theory with  $U(1)$  gauge group and two flavors, which is also dual to the  $3d$   $\mathcal{N} = 2$  CS theory with  $U(1)_{3/2}$  gauge group and three flavors, as described above.

Now we are coming to the main point of this section, namely, to derivation of the identities presented above from the theory of elliptic hypergeometric integrals. Identity (C.11) arises from the reduction of a transformation formula for the elliptic extension of the Gauss hypergeometric function  $V$  [108] defined by formula (2.33). From the physical point of view elliptic hypergeometric integrals describe superconformal indices for  $4d$  supersymmetric field theories. Analogously to [49], we can claim that important ingredients of the knot theory are coming from the  $4d$  supersymmetric field theories. In the considered example, the state integral model for the figure-eight knot is obtained from  $4d$   $\mathcal{N} = 1$  SYM theory with  $SP(2)$  gauge group and 8 quarks, which was studied in detail in [114].

The  $V$ -function obeys symmetry transformation (2.34). First, we reduce it to the level of hyperbolic  $q$ -hypergeometric integrals by means of the reparametrization of variables

$$y = e^{2\pi i r z}, \quad t_i = e^{2\pi i r \mu_i}, \quad i = 1, \dots, 8, \quad p = e^{2\pi i b r}, \quad q = e^{2\pi i r/b},$$

and the subsequent limit  $r \rightarrow 0$ . In this limit the elliptic gamma function has the asymptotics [98]

$$\Gamma(e^{2\pi i r z}; e^{2\pi i r b}, e^{2\pi i r/b}) \underset{r \rightarrow 0}{=} e^{-\pi i(z-(b+1/b))/24r} \gamma^{(2)}(z; b, b^{-1}). \quad (\text{C.14})$$

Using it in the reduction, one obtains the integral lying on the top of a list of integrals obtained from the reduction of the  $V$ -function

$$I_h(\mu_1, \dots, \mu_8) = \int_{-i\infty}^{i\infty} \frac{\prod_{i=1}^8 \gamma^{(2)}(\mu_i \pm z; b, b^{-1})}{\gamma^{(2)}(\pm 2z; b, b^{-1})} dz, \quad (\text{C.15})$$

with the balancing condition  $\sum_{i=1}^8 \mu_i = 2(b+1/b)$ .

It has symmetry transformation formula descending from the elliptic one

$$I_h(\mu_1, \dots, \mu_8) = \prod_{1 \leq i < j \leq 4} \gamma^{(2)}(\mu_i + \mu_j; b, b^{-1}) \prod_{5 \leq i < j \leq 8} \gamma^{(2)}(\mu_i + \mu_j; b, b^{-1}) I_h(\nu_1, \dots, \nu_8), \quad (\text{C.16})$$

where

$$\nu_i = \mu_i + \xi, \quad i = 1, 2, 3, 4, \quad \nu_i = \mu_i - \xi, \quad i = 5, 6, 7, 8,$$

and the parameter  $\xi$  is

$$2\xi = \sum_{i=5}^8 \mu_i - b - 1/b = b + 1/b - \sum_{i=1}^4 \mu_i.$$

To get the desired transformation formula (C.11) one should use the following asymptotic formulas

$$\begin{aligned} \lim_{u \rightarrow \infty} e^{\pi i B_{2,2}(u)} \gamma^{(2)}(u) &= 1, \quad \text{for } \arg b < \arg u < \arg 1/b + \pi, \\ \lim_{u \rightarrow \infty} e^{-\pi i B_{2,2}(u)} \gamma^{(2)}(u) &= 1, \quad \text{for } \arg b - \pi < \arg u < \arg 1/b \end{aligned} \quad (\text{C.17})$$

when some of the parameters go to infinity.

The proof by van de Bult presented in [14] is rather bulky. Starting from the key transformation formula (C.16) one has to pass step by step from one level of complexity to another one in the list of integrals obtained from  $I_h$  by diminishing of the number of independent parameters. Therefore we are not presenting this proof here explicitly although it is very straightforward.

**The trefoil knot.** Let us apply the same procedure to the state integral model of the trefoil knot described by formula (6.59) in [27] (where we omit a coefficient in front of the integral):

$$J = \int_{-\infty}^{\infty} \Phi\left(-\frac{p}{2\pi i}; \frac{\hbar}{\pi i}\right) \Phi\left(\frac{p-c}{2\pi i}; \frac{\hbar}{\pi i}\right) e^{pu/2\hbar} dp. \quad (\text{C.18})$$

After rewriting the latter expression as in the figure-eight knot case (replacing  $p \rightarrow 2\pi ip$ ,  $c \rightarrow 2\pi ic$ ,  $\hbar \rightarrow \pi i\tau$ , etc), we come to the integral

$$\begin{aligned} J &= e^{\pi i(1+b^4-6c^2)/12b^2} \\ &\times \int_{-i\infty}^{i\infty} \gamma^{(2)}\left(\frac{b+1/b}{2} + \frac{p}{b}, \frac{b+1/b}{2} - \frac{p-c}{b}; b, b^{-1}\right) e^{-\pi ip^2/b^2 + 3\pi ipc/b^2} dp. \end{aligned} \quad (\text{C.19})$$

Consider now the integral  $II_{1,(3,3)a}^1(\mu, \nu; \lambda)$  on page 218 in [14]. We choose the integration variable in it  $z = p/b$ , impose the constraints  $\mu = (b+1/b)/2$ ,  $\nu = (b+1/b)/2 + c/b$ ,  $\lambda = 3c/b$ , and denote the resulting function as  $\tilde{Z}_E(\mu, \nu, \lambda)$ . According to Theorem 5.6.19 of [14], it obeys the following transformation formula:

$$\tilde{Z}_E(\mu, \nu, \lambda) = \tilde{Z}_M(\mu + \sigma', \nu - \sigma') e^{\pi i(\lambda^2 + (\mu + \nu)^2 - 2(b+1/b)(\mu + \nu))/4}, \quad (\text{C.20})$$

where  $4\sigma' = \nu - \mu - \lambda$  and

$$\tilde{Z}_E(\mu, \nu, \lambda) = \int_{-i\infty}^{i\infty} \gamma^{(2)}(\mu - z, \nu + z; b, b^{-1}) e^{\pi i\lambda z - \pi iz^2} dz, \quad (\text{C.21})$$

describing the partition function of a  $3d \mathcal{N} = 2$  CS theory with  $U(1)_1$  gauge group and two quarks. On the right-hand side one has

$$\tilde{Z}_M(\alpha, \beta) = \frac{1}{2} \int_{-i\infty}^{i\infty} \frac{\gamma^{(2)}(\alpha \pm y, \beta \pm y; b, b^{-1})}{\gamma^{(2)}(\pm 2y; b, b^{-1})} e^{-4\pi iy^2} dy, \quad (\text{C.22})$$

which is the partition function of a  $3d \mathcal{N} = 2$  CS theory with  $SU(2)_8$  gauge group and two quarks. Comparing with [27], we see that (C.21) coincides with the product wavefunction in the transformed basis. To get the state integral model for the trefoil knot one has to specify

$$\mu + \nu = b + 1/b.$$

Then expression (C.18) simplifies (due to the inversion formula) becoming a Gaussian integral which is easily evaluated (or  $c = 0$  in (C.18)). Again, one can use equality (C.20) for the connection of  $3d$  complexified CS theory living on  $\tilde{\mathcal{M}} = \mathbf{S}^3 \setminus \mathbf{3}_1$  with  $3d$  supersymmetric field theories.

**Some other integrals.** In the rest of this section we would like to consider some other hyperbolic integrals which appear in this context [40, 17, 18] and describe their connection to the elliptic hypergeometric integrals. There is a nice Fourier transformation formula for the hyperbolic gamma function [17, 40] (in particular, in [27] it is given by formula (6.54)):

$$J_E = \int_{-i\infty}^{i\infty} \gamma^{(2)}(\mu - z/b; b, b^{-1}) e^{\pi i(2\lambda z/b - z^2/b^2)/2} dz, \quad (\text{C.23})$$

where one should restrict the parameters to match the notations in [27]

$$\mu = \frac{b + 1/b}{2}, \quad \lambda = 2x.$$

Expression (C.23) can be found in [14] where it is defined as the integral  $II_{1,(3,2)_a}^0(\mu; \lambda)$ . It is an example of the computable integrals with the evaluation formula  $J_E = J_M$ , which is given by Theorem 5.6.8 in [14], with

$$J_M = \gamma^{(2)}((b + 1/b)/4 + \lambda/2 - \mu/2; b, b^{-1}) \times e^{\pi i(-3\mu^2 + (\lambda - (b+1/b)/2)^2 + 2\mu(3\lambda + (b+1/b)/2))/4 - \pi i(b^2 + 1/b^2)/24}. \quad (\text{C.24})$$

To see the coincidence with formula (6.54) from [27] one should take into account the inversion formula for the hyperbolic gamma functions.

The equality  $J_E = J_M$  defines one of the simplest examples of dualities between two  $3d$  supersymmetric field theories. The electric theory is a  $3d$   $\mathcal{N} = 2$  CS theory with  $U(1)_{1/2}$  gauge group and one quark  $Q$ , while the magnetic theory is just a free  $3d$   $\mathcal{N} = 2$  theory of one chiral field  $X$ . Again such dualities were skipped in [37] because of their abundance, where for brevity only the first steps of the reduction procedure from  $4d$  SCIs to  $3d$  partition functions were considered explicitly. The identities presented in this section lie further in the reduction hierarchy of the elliptic hypergeometric integrals to hyperbolic integrals.

In contrast to the first two examples considered in this section, the equality  $J_E = J_M$  is obtained from the reduction of SCI for  $4d$   $\mathcal{N} = 1$  SYM with  $SU(2)$  gauge group and 6 quarks from the physical point of view, and from the mathematical point of view it is obtained as the reduction of the elliptic beta integral [107].

The equality of partition functions considered in [60] (later also discussed in [118, 29]) and given by formula (3.11) in [60] for  $b = 1$  (it describes a  $3d$  mass-deformed  $T[SU(2)]$  supersymmetric field theory) and generalized to  $b \neq 1$  in [118] (see formula (4.10)) is also obtained as a reduction of the  $V$ -function identities considered in [14]. The equality of the statistical sums of the initial theory and the mirror dual is taken from the work [18], where it was proven using the Fourier transformation formula [40]. The partition function of the  $3d$  mass-deformed  $T[SU(2)]$  SYM theory coincides with the integral  $II_{1,(2,2)}^1(\mu_1, \mu_2, \nu_1, \nu_2; \lambda)$  from [14] (again we take  $\omega_1 = b, \omega_2 = 1/b$ ):

$$K(\mu_1, \mu_2, \nu_1, \nu_2, \lambda) = \int_{-i\infty}^{i\infty} \prod_{i=1}^2 \gamma^{(2)}(\mu_i - z, \nu_i + z; b, b^{-1}) e^{\pi i \lambda z} dz, \quad (\text{C.25})$$

where one should restrict the parameters to obtain the expression from [60] as follows

$$\mu_1 = \nu_1 = \frac{b + 1/b}{4} - \frac{m}{2} + \mu, \quad \mu_2 = \nu_2 = \frac{b + 1/b}{4} - \frac{m}{2} - \mu, \quad \lambda = -4\xi.$$

Integral (C.25) has the transformation formula described in Theorem 5.6.17 of [14]:

$$K(\mu_1, \mu_2, \nu_1, \nu_2, \lambda) = \tilde{K}(\sigma_1, \dots, \sigma_4) e^{\pi i(4\tilde{\sigma}^2 - 2\mu_1\mu_2 - 2\nu_1\nu_2)/2} \\ \times \gamma^{(2)}((\pm\lambda - \mu_1 - \mu_2 - \nu_1 - \nu_2)/2 + b + 1/b; b, b^{-1}), \quad (\text{C.26})$$

where

$$\tilde{K}(\sigma_1, \dots, \sigma_4) = \frac{1}{2} \int_{-i\infty}^{i\infty} \frac{\prod_{i=1}^4 \gamma^{(2)}(\sigma_i \pm y; b, b^{-1})}{\gamma^{(2)}(\pm 2y; b, b^{-1})} e^{-2\pi i y^2} dy, \quad (\text{C.27})$$

and

$$\sigma_1 = \mu_1 + \tilde{\sigma}, \quad \sigma_2 = \mu_2 + \tilde{\sigma}, \quad \sigma_3 = \nu_1 - \tilde{\sigma}, \quad \sigma_4 = \nu_2 - \tilde{\sigma},$$

with

$$4\tilde{\sigma} = \nu_1 + \nu_2 - \mu_1 - \mu_2 - \lambda.$$

There is a transformation formula for the integral  $\tilde{K}$  described in Theorem 5.6.14 (for  $n = 1$ ) in [14]:

$$\tilde{K}(\sigma_1, \dots, \sigma_4) = \tilde{K}(\rho_1, \dots, \rho_4) \prod_{1 \leq i < j \leq 4} \gamma^{(2)}(\sigma_i + \sigma_j; b, b^{-1}) e^{-\pi i(b+1/b)\xi}, \quad (\text{C.28})$$

where

$$2\xi = b + 1/b - \sum_{i=1}^4 \sigma_i, \quad \rho_i = \sigma_i + \xi, \quad i = 1, 2, 3, 4.$$

Combining together formula (C.26), symmetry transformation (C.28) and, finally, again (C.26) (taking into account that (C.27) is symmetric in all the parameters  $\sigma_i$ ), one gets the symmetry transformation

$$K((b + 1/b)/4 - m/2 \pm \mu, (b + 1/b)/4 - m/2 \pm \mu, -4\xi) \gamma^{(2)}(-m; b, b^{-1}) \\ = K(m/2 \pm \xi, m/2 \pm \xi, -4\mu) \gamma^{(2)}(m; b, b^{-1}), \quad (\text{C.29})$$

which coincides with (3.22) from [60] for  $b = 1$  and (4.10) from [118] for general  $b$  (generalizing to arbitrary parameters  $\mu_1, \mu_2, \nu_1, \nu_2$  one obtains formula (A.31) from [18]). Described symmetry transformation formulas allow one to derive more identities apart from (C.29), which should be explored separately. Here our aim was to show that all the known examples of the equalities of partition functions from the literature are obtained as reductions of the identities for the elliptic hypergeometric integrals (actually, here we have discussed only the reduction of the elliptic beta integral and the  $V$ -function). There is also an interesting connection of the mass-deformed  $T[SU(2)]$  theory partition function with the Liouville theory [60], where it coincides with the  $S$ -duality kernel connecting conformal blocks [119].

A comment concerning the latter identity is in order. Partition function for the  $3d$  mass-deformed  $T[SU(2)]$  theory can be obtained from the reduction of either the  $V$ -function, which is the SCI of  $4d$   $\mathcal{N} = 1$  SYM with  $SU(2)$  gauge group and 8 quarks, or from the SCI of the  $4d$   $\mathcal{N} = 2$  SYM theory with  $SU(2)$  gauge group and 4 hypermultiplets [48].

We want to conclude this section by stating that the arguments given above are quite general and are applied to any state integral model. Other examples for different knots presented in [58] are obtained from the reduction of SCIs of  $4d$   $\mathcal{N} = 1$  quiver supersymmetric field theories and coincide with the partition functions of  $3d$   $\mathcal{N} = 2$  supersymmetric field theories in which we restrict the fugacities associated with the matter content of the theory. The results of this section may be useful for a better understanding of a generalization of the AGT duality [1], connecting  $4d$  and  $2d$  theories, to the theory connecting  $3d$  CS theories living on some manifold  $\mathcal{M}$  to  $3d$   $\mathcal{N} = 2$  supersymmetric field theories [60, 118, 29, 28].

APPENDIX D. REDUCTION TO THE  $2d$  VORTEX PARTITION FUNCTION

Dimensional reductions of field theories are usually considered directly at the level of physical degrees of freedom. As discussed in the previous Appendix, often it is easier to make such reductions at the level of collective objects such as partition or correlation functions and topological indices. In particular, partition functions of the field theories on a squashed three-sphere can be derived from  $4d$  SCIs [37] (the case of ordinary  $\mathbf{S}^3$  corresponds to the limit  $\omega_1 = \omega_2^{-1} \rightarrow 1$ ). An obvious question is whether one can proceed further and reduce  $3d$  partition functions to  $2d$  statistical sums? The squashed three-sphere is isomorphic to  $\mathbf{S}^2 \times \mathbf{S}^1$  and by taking the radius of  $\mathbf{S}^1$  to zero one reduces this manifold to  $\mathbf{S}^2$ , which is a two-dimensional space-time. What one obtains in the end of this reduction is the vortex partition function for a  $2d$  supersymmetric sigma-model. The vortex partition function is the object of recent active studies [104, 30, 50, 123]. Its relation to the  $3d$  Omega background is discussed in [29]. From the mathematical point of view the  $4d/3d$  correspondence of [37] is described by the reduction of elliptic hypergeometric integrals to the hyperbolic  $q$ -hypergeometric integrals (see, e.g., [26, 95]). Here we proceed with further reduction to the rational level [95] described by the integrals employing elementary functions and the standard gamma function. In [84], it was found that introducing into  $4d$  SCI of the surface operators leads to the  $2d$   $(4, 4)$  SCFT coupled to the  $4d$  theory; here we obtain a more complete  $2d$  picture. A different type of  $2d$  partition function associated with SCIs of  $\mathcal{N} = 2$  theories was considered recently in [47].

Let us discuss first the reduction of  $4d$  SCIs to  $3d$  partition functions on the example of Intriligator-Pouliot duality [64]. As shown in Sect. 2 above and in [36], one can derive SCIs of  $4d$   $\mathcal{N} = 1$  SYM theories with the orthogonal gauge groups from the corresponding  $SP(2N)$ -SCIs. But we can reduce the latter  $4d$  SCIs to  $3d$  partition functions along the lines of [37]. This results in  $3d$  dualities for both the SYM [2] and CS [51] theories and both  $SP(2N)$  and  $U(N)$  gauge groups. From the mathematical point of view, it is a consequence of the reduction of elliptic hypergeometric integrals to the hyperbolic level classified again in [14]. We would like to stress that the  $4d$  SCIs and dualities are defined as a rule by a unique relation for elliptic hypergeometric integrals, and at the  $3d$ -level one obtains the whole web of dualities/SCIs both for SYM and CS theories based on different gauge groups.

More technically, we start from the electric theory of [64] presented in the beginning of Sect. 2. The SCI for this theory is given by expression (2.1) as described in [36, 115]. Applying the reduction of elliptic hypergeometric integrals to the hyperbolic integrals [26, 95] one finds the following item among the hyperbolic integrals stemming from (2.1) [14]:

$$Z = \frac{1}{N!} \int_{C^N} \prod_{1 \leq i < j \leq N} \frac{1}{\gamma^{(2)}(\pm(z_i - z_j); \omega_1, \omega_2)} \times e^{2\pi i(\lambda+1/2)(\omega_1+\omega_2) \sum_{j=1}^N z_j/\omega_1\omega_2} \prod_{i,j=1}^N \gamma^{(2)}(\mu_i - z_j, \nu_i + z_j; \omega_1, \omega_2) \prod_{j=1}^N \frac{dz_j}{i\sqrt{\omega_1\omega_2}}, \quad (\text{D.1})$$

where  $C$  is the Mellin-Barnes type integration contour.

In [121], Willett and Yakov showed that this integral describes the partition function [67, 55] of the electric theory for Aharony duality [2], which is a  $3d$   $\mathcal{N} = 2$  SYM theory living on the squashed three-sphere with  $U(N)$  gauge group,  $N_f = N$  left quarks forming the fundamental representation of  $U(N)$ ,  $N_f = N$  right quarks forming the antifundamental representation of  $U(N)$ , and additional singlets  $V_{\pm}$ . Moreover, in [121] the coincidence with the magnetic theory partition function was proved using the transformation formula for (D.1) from [14].

In (D.1), the parameters  $z_j, j = 1, \dots, N$ , are the fugacities associated with the gauge group  $U(N)$ ,  $\lambda$  is associated with the Fayet-Iliopoulos term (the coefficient  $4(\lambda + 1/2)(\omega_1 + \omega_2)$  is introduced for a convenience). Parameters  $\mu_i, \nu_i, i = 1, \dots, N$ , are the fugacities of  $SU(N) \times SU(N)$  non-abelian global symmetry group, which are normalized by taking into account the abelian part of the global symmetry  $U(1)_A \times U(1)_J \times U(1)_R$ .

Let us study the limit  $\omega_2 \rightarrow \infty$  using the hyperbolic gamma function asymptotics

$$\gamma^{(2)}(z; \omega_1, \omega_2) \underset{\omega_2 \rightarrow \infty}{=} \left( \frac{\omega_2}{2\pi\omega_1} \right)^{\frac{1}{2}-z} \frac{\Gamma(z/\omega_1)}{\sqrt{2\pi}},$$

where  $\Gamma(z)$  is the usual gamma function. 3d partition function (D.1) in this limit becomes

$$\begin{aligned} Z^{lim} &= \frac{\omega_2^{N/2}}{N! \omega_1^{3N/2}} \left( \frac{\omega_2}{\omega_1} \right)^{-\sum_{i=1}^N (\mu_i + \nu_i)} \int_{\mathcal{C}^N} \prod_{1 \leq i < j \leq N} \frac{1}{\Gamma(\pm(z_i - z_j)/\omega_1)} \\ &\times e^{2\pi i(\lambda+1/2)\sum_{j=1}^N z_j/\omega_1} \prod_{i,j=1}^N \Gamma((\mu_i - z_j)/\omega_1, (\nu_i + z_j)/\omega_1) \prod_{j=1}^N \frac{dz_j}{2\pi i}. \end{aligned} \quad (D.2)$$

One can see that this integral coincides up to some normalization factor with the function appearing after formula (2.6) in [50]

$$\begin{aligned} Z^{vortex} &= \int_{\mathcal{C}^N} \prod_{1 \leq i < j \leq N} \frac{1}{\Gamma(\pm(z_i - z_j)/\omega_1)} e^{2\pi i(\lambda+1/2)\sum_{j=1}^N z_j/\omega_1} \\ &\times \prod_{i,j=1}^N \Gamma((\mu_i - z_j)/\omega_1, (\nu_i + z_j)/\omega_1) \prod_{j=1}^N \frac{dz_j}{2\pi i}, \end{aligned} \quad (D.3)$$

where we take  $N_f = N$ . The multiplier  $\prod_{i \neq j} \Gamma((a_i - a_j)/\omega_1)$  standing in front of the integral in [50] is not relevant for our discussion and is omitted.

Expression (D.3) defines the vortex partition function for  $2d$   $(2, 2)$  supersymmetric field theory with  $U(N)$  gauge group and  $N_f = N$  flavors. This formula is derived from the representation of vortex partition function as a sum over the Young diagrams which is the usual representation for partition functions of  $4d$   $\mathcal{N} = 2$  SYM theories [86, 87]. In an interesting paper [104] it was pointed out that taking the limit  $\omega_2 \rightarrow \infty$  in the partition function for  $4d$   $\mathcal{N} = 2$  SYM theory one gets the  $2d$  vortex partition function (more precisely, one should also normalize the variable associated with the instanton parameter to compensate additional divergences emerging in the limit  $\omega_2 \rightarrow \infty$ ). In [50], it was realized that the sum over instantons (sums over Young diagrams) can be rewritten as a single contour integral, which leads to a better understanding of this function from the mathematical point of view.

This observation can be generalized to any number of flavors  $N_f$  appearing in [50] by starting from the partition function for  $3d$   $\mathcal{N} = 2$  SYM theory with  $U(N)$  gauge group,  $N_f \neq N$  flavors, and looking at the same limit  $\omega_2 \rightarrow \infty$  accompanied by pulling some of the parameters to infinity (i.e., by integrating out some of the quarks). Technically, one should use the asymptotic expansion of the gamma function

$$\Gamma(x) \rightarrow \sqrt{2\pi} e^{-x} x^{x-1/2}, \quad x \rightarrow \infty.$$

We conclude by several remarks on the importance of the observation made in this section. First, this reduction may be very useful for checking a  $2d$  analog of Seiberg's duality which was recently proposed and studied in [106, 59]. Second, this reduction is close to the one studied in

the literature on connections of  $3d$  Chern-Simons theories with  $2d$  supersymmetric field theories [30] linking vortex partition function to the BPS invariants of the dual geometries. Finally, the last, perhaps the most important, remark is the connection of  $4d$  SCIs for  $\mathcal{N} = 1$  SYM theories with  $4d$  partition functions for  $\mathcal{N} = 2$  SYM theories in the discussed above limit.

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