Phase Transition in Dually Weighted Colored Tensor Models

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Tensor models are a generalization of matrix models (their graphs being dual to higher-dimensional triangulations) and, in their colored version, admit a $1/N$ expansion and a continuum limit. We introduce a new class of colored tensor models with a modified propagator which allows us to associate weight factors to the faces of the graphs, *i.e.* to the bones (or hinges) of the triangulation, where curvature is concentrated. They correspond to dynamical triangulations in three and higher dimensions with generalized amplitudes. We solve analytically the leading order in $1/N$ of the most general model in arbitrary dimensions. We then show that a particular model, corresponding to dynamical triangulations with a non-trivial measure factor, undergoes a third-order phase transition in the continuum characterized by a jump in the susceptibility exponent.

Keywords: 1/N expansion of random tensor models, critical behavior, dynamical triangulation

I. INTRODUCTION

Statistical models of fluctuating geometry are a generous source of results and ideas for physics and mathematics. A particularly attractive feature of many such models is that they can be thought as providing either a regularization or a fundamental description of quantum gravity [1]. Dynamical Triangulations (DT) [2, 3] are one of the most studied examples, and have been very successful in two dimensions, where they are related to the large-N limit of matrix models [4, 5], and whose link to non-critical string theory in the continuum limit is well-understood [6].

Higher-dimensional models of DT have not been equally successful in providing a sensible continuum limit for quantum gravity [7], leading to degenerate geometries at large scales [8] with a first-order phase transition separating them [9, 10]. However one cannot exclude that some unknown essential ingredient was missing in the models analyzed so far. A non-local modification of DT, which goes under the name of causal dynamical triangulations or CDT [11] has produced substantial evidence for the emergence of an extended geometry at large scale [12, 13], and signs of a second-order phase transition [14], hinting at the possibility to attain a good continuum limit. Recently it has also been suggested [15] that the effects of a non-trivial measure factor were overlooked in the past and could potentially lead to an improvement of the large scale behavior of DT models.

Tensor models [16–18] and group field theories [19] are the generalization of matrix models to higher dimensions. In particular, the colored tensor models and group field theories [20–22]

generate graphs dual to orientable [23] pseudo-manifolds in any dimension. Much progress has been done in understanding these models. Various power counting estimates and bounds of graph amplitudes in tensor models and group field theories have been obtained [29–35]. The symmetries of tensor models have been analyzed either with the help of n-ary algebras [36–38], or, in a more quantum field theoretical approach through Ward-Takahashi identities [39]. The relation between symmetries of group field theories and the diffeomorphism symmetry of the resulting triangulation has been explored [40, 41]. Solutions of the classical equations of motions [42, 43] have been derived and some of them interpreted as matter fields on non commutative spaces. Most importantly, for our purposes, the colored tensor models have been shown to possess an additional (and welcomed) feature as compared to non-colored models: their amplitudes are such that a $1/N$ expansion is possible, with the leading order encoding a sum over a class of colored triangulations of the D-sphere [24–26]. This discovery led to the possibility of new analytical investigations of DT models and their continuum limit in $D \geq 3$ dimensions [27, 28].

The link between tensor models and DT leads straight to a daunting question: will any colored tensor model admit a richer continuum limit? The question is twofold, as at first instance one can wonder whether the coloring will already suffice to generate new universality classes in these models. The results of [27] indicate that, as far as the critical exponents are concerned, the color alone is not enough and the continuum limit is strongly reminiscent of branched polymers[1]. The next question is then whether it is possible to modify the tensor models in such a way that new, appealing, phases would appear (like in CDT) and/or a second-order phase transition would occur between phases.

The dually weighted matrix models introduced by Kazakov et al. [44] are an ideal candidate for introducing relevant modifications of a two-dimensional DT model via a local modification of the corresponding matrix model (see also [45–47]). For example, one can impose the non-local condition on foliations that characterizes two-dimensional CDT just by a suitable modification of the propagator of a two-matrices model [48].

In the present work we extend the idea of dually weighted matrix models to colored tensor models (which by lack of fantasy we call dually weighted colored tensor models) and solve them analytically at leading order in $1/N$. We then consider a particular example corresponding to a DT model initially proposed in [49] and recently claimed to exhibit a new phase with promising geometrical properties [15]. In the large-N limit, equivalent to weak-coupling from the gravitational point of view, we find a third-order phase transition and a new critical behavior. We compute explicitly the susceptibility exponent associated to the different phases. Although, as we are in a different regime[2], we cannot make direct contact to the results of [15], our results show that a non-trivial continuum limit is possible in these models and open a small window of hope on non-causal DT models.

This paper is organized as follows. In Sec. II we introduce the new models, discuss their DT interpretation and derive a Schwinger-Dyson equation relating the free energy and the connected two-point function of our new tensor models. In Sec. III we derive, in the large-N limit, a set of self-consistency equations which allow to exactly solve (at leading order in $1/N$) the model. In Sec. IV we specialize to a particular model and study its phase transition.

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[1] The reader should keep in mind however that no other characteristic of branched polymers, like say the Hausdorff or spectral dimension has so far been reproduced for the colored tensor models.

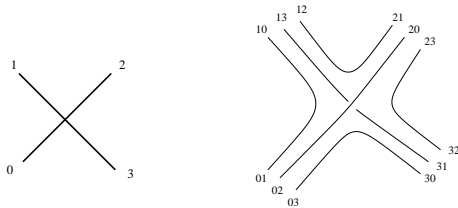[2] And have not computed the spectral dimension of the emerging geometry.

FIG. 1. The index structure of a $D = 3$ vertex, in a "coarse grained" view, showing only the tensor color, and in a detailed view, showing the strands (indices should be read clockwise).

II. DUALLY WEIGHTED COLORED TENSOR MODELS

In this section we introduce a modification of the independent identically distributed (i.i.d.) colored tensor model[3] of [20–22, 24–27], which we baptize "dually weighted colored tensor model".

We denote $\vec{n}_i$, for $i = 0, \ldots, D$, the D-tuple of integers $\vec{n}_i = (n_{ii-1}, \ldots, n_{i0}, \ n_{iD}, \ldots, n_{ii+1})$, with $n_{ik} = 1, \ldots, N$. This N is the size of the tensors and the large N limit defined in [24–26] represents the limit of infinite size tensors. We set $n_{ij} = n_{ji}$. Let $\bar{\psi}^i_{\vec{n}_i}$, $\psi^i_{\vec{n}_i}$, with $i = 0, \ldots, D$, be $D + 1$ couples of complex conjugated tensors with D indices[4]. The dually weighted colored tensor model in dimension D is defined by the partition function

$$e^{N^D E} = Z_N(\lambda, \bar{\lambda}) = \int d\bar{\psi} \, d\psi \; e^{-S(\psi, \bar{\psi})} \ ,$$

$$S(\psi, \bar{\psi}) = \sum_{i=0}^{D} \sum_{\vec{p}_i, \vec{n}_i} \psi^i_{\vec{p}_i} \Big(\prod_j (C^{-1})_{p_{ij} \bar{n}_{ij}} \Big) \bar{\psi}^i_{\vec{n}_i}$$

$$+ \frac{\lambda}{N^{D(D-1)/4}} \sum_n \prod_{i=0}^{D} \psi^i_{\vec{n}_i} + \frac{\bar{\lambda}}{N^{D(D-1)/4}} \sum_{\bar{n}} \prod_{i=0}^{D} \bar{\psi}^i_{\vec{n}_i} \ . \tag{1}$$

$\sum_n$ denotes the sum over all indices n_{ij} from 1 to N. The index structure of a vertex is represented in Fig. 1. The quadratic part is chosen such that Gaussian correlations, i.e. the propagator of the model, is (no summation over i, the propagator between fields of different color is zero)

$$\langle \bar{\psi}^i_{\vec{n}_i} \psi^i_{\vec{p}_i} \rangle_0 = \prod_j C_{\bar{n}_{ij} p_{ij}} \ , \qquad \langle \psi^i_{\vec{p}_i} \bar{\psi}^i_{\vec{n}_i} \rangle_0 = \prod_j C_{\bar{n}_{ij} p_{ij}} = \prod_j C^T_{p_{ij} \bar{n}_{ij}} \ , \tag{2}$$

where C^T denotes the transposed matrix. Obviously, when $C_{\bar{n}_{ij} p_{ij}} = \delta_{\bar{n}_{ij} p_{ij}}$ the model reduces to the i.i.d. one.

Graphs are made of vertices, (colored) lines and "faces". Like in matrix models, the indices of the tensors are associated to "strands" (the solid line in the detailed view of the vertex in figure 1). A index n_{ij} (associated to the strand common to the half lines of color i and j of the vertex) is identified by the two colors, i and j. The colors are conserved along the lines, hence the "faces" (closed strands) of the graph are identified by couples of colors. A graph is dual to a triangulation[5]: its vertices are dual to D simplices, its lines to $D - 1$ simplices and its faces to $D - 2$ simplices. When computing amplitudes of graphs, one picks up the trace of the alternate product of C and C^T along each face of a graph. We consider only matrices C which do not modify the scaling in $1/N$ of the amplitude, i.e. such that $\lim_{N \to \infty} \frac{\mathrm{Tr}[(CC^T)^q]}{N}$ is finite for all q. We *do not* impose for

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[3] The epithet i.i.d. refers to the fact that in such models the free Gaussian measure weights independently and identically each component of each tensor. In the new models we are introducing this is not the case.

[4] We denote the indices of the $\bar{\psi}$ field by $\bar{n}$. This does not denote a complex conjugation, being merely a book keeping device.

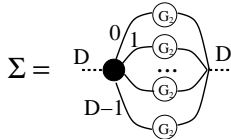[5] More precisely to an abstract finite simplicial pseudo-manifold [21].

FIG. 2. The self energy Σ at leading order in terms of the connected two-point function G_2. The labels $0, 1, \ldots D$ denote the colors of various lines.

the moment any further restrictions on C, in particular C is not required to be hermitian. Using the counting of faces of a colored graph established in [26], the amplitude writes

$$A(\mathcal{G}) = (\lambda\bar\lambda)^p \Big(\prod_{(ij),\rho} \frac{\text{Tr}[(CC^T)^{p^{ij}_{(\rho)}}]}{N} \Big) N^{D - \frac{2}{(D-1)!}\omega(\mathcal{G})} , \tag{3}$$

with $2p^{ij}_{(\rho)}$ the number of vertices of the ρ'th face of colors ij. The number $\omega(\mathcal{G}) \geq 0$ is called the degree of the graph $\mathcal{G}$ [26], and the leading order graphs are those of degree 0, which are dual to triangulations of D-spheres [25]. Crucially for the study of the leading sector in the large N limit is the flowing fact [27]: at leading order in $1/N$ the self energy Σ (i.e. one particle irreducible amputated two-point function) factors into the convolution of D connected two-point functions G_2, one for each color. Such graphs (represented schematically in Fig. 2) are called melons[6]. Denoting $g = \lambda\bar\lambda$, the free energy of the model is a function $E(C_{\bar pn}, g)$ of the coupling g and the matrix entries $C_{\bar pn}$.

For $C = \mathbb{I}$ the i.i.d. model is in direct correspondence with DT in D dimensions, just like matrix models are with DT in two dimensions. Each graph is dual to a triangulation, and denoting N_k the number of k-simplices, its amplitude rewrites

$$A(\mathcal{G}) = e^{\kappa_{D-2} N_{D-2} - \kappa_D N_D} , \tag{4}$$

where κ_{D-2} and κ_D are (with $g = \lambda\bar\lambda$),

$$\kappa_{D-2} = \ln N, \quad \text{and} \quad \kappa_D = \frac{1}{2}\Big(\frac{1}{2}D(D-1)\ln N - \ln(g)\Big). \tag{5}$$

The grand canonical partition function of DT is given by

$$Z_{DT} = \sum_T \frac{1}{s(T)} e^{\kappa_{D-2} N_{D-2} - \kappa_D N_D} , \tag{6}$$

where the sum is over all D-dimensional triangulations T, and $s(T)$ is the order of the automorphism group of T. The grand canonical partition function of DT equals the free energy of the tensor model, $Z_{DT} = E$.

The large-N limit corresponds to $\kappa_{D-2} \to \infty$, with $\kappa_D = \frac{1}{4}D(D-1)\kappa_{D-2} - \frac{1}{2}\ln g$. As the DT action can be interpreted as a Regge action for equilateral simplices, one has $\kappa_{D-2} \sim 1/G$ thus the large-N limit amounts to the weak coupling limit, with vanishing (bare) Newton's constant. Note that this is true also for matrix models, as in two dimensions the Regge action just gives the Euler character of the surface, and the limit $G \to 0$ is the planar limit. As in that case, in order to keep a finite G one would have to perform some kind of double scaling limit.

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[6] This is stark contrast with the $D = 2$ case of usual matrix models. Indeed, the $1/N$ expansion in that case selects all planar graphs, not only the melonic ones. As shown in [27] the results leading to the factorization of the self energy explicitly break down in $D < 3$.

For $C \neq \mathbb{I}$ the amplitudes associated to triangulations will be in general different, thus defining new models of DT. In particular, the effect of the modified propagator is to associate a weight factor to each $(D-2)$-subsimplex of the triangulation, also called a bone (or hinge) in Regge calculus. As in Regge's construction it is precisely on the bones that curvature resides, the dually weighted colored tensor models can be a precious tool for the study of DT with more a complicated curvature dependence of the action.

A. The connected two-point function

The full connected two-point function of the dually weighted model,

$$\langle \bar{\psi}^i_{\vec{n}_i} \psi^i_{\vec{p}_i} \rangle = \prod_j P_{\bar{n}_{ij} p_{ij}}(g, C) \;, \tag{7}$$

is of course independent of the color i and factored along strands. More subtly, the contribution of the strand of color ij to the two-point function is in fact independent of j. This is due to the fact that the action is invariant under a transformation which permutes any two strands ij and ik in the field ψ^i and permutes the strands on all other fields (and also the fields between themselves) to restore the connectivity of the vertex.

We are now going to show that knowledge of the full connected two-point function suffices to compute the derivatives of the free energy. First, we observe that from

$$\frac{1}{Z} \sum_{\vec{\bar{n}}_i} \int \frac{\delta}{\delta \bar{\psi}_{\vec{\bar{n}}_i}} \left(\bar{\psi}_{\vec{\bar{n}}_i} e^{-S} \right) = 0 \;, \tag{8}$$

it follows that

$$N^D - \sum_{\vec{\bar{n}}_i, \vec{p}_i} \prod_j (C^{-1})_{p_{ij} \bar{n}_{ij}} \langle \bar{\psi}^i_{\vec{\bar{n}}_i} \psi^i_{\vec{p}_i} \rangle + \frac{1}{Z} \bar{\lambda} \partial_{\bar{\lambda}} Z = N^D - \prod_j \text{Tr}[C^{-1} P] + \bar{\lambda} N^D \partial_{\bar{\lambda}} E = 0 \;, \tag{9}$$

which, recalling that E is a function only of g, can be rewritten as

$$\bar{\lambda} \partial_{\bar{\lambda}} E = g \partial_g E = \left[\frac{\text{Tr}[C^{-1} P]}{N} \right]^D - 1 \;. \tag{10}$$

Furthermore, we have

$$\frac{\partial E}{\partial C_{\bar{n}p}} = -N^{-D} \sum_{\vec{a}_i, \vec{b}_i} \left(\prod_j \frac{\partial (C^{-1})_{a_{ij} \bar{b}_{ij}}}{\partial C_{\bar{n}p}} \right) \left\langle \psi^i_{\vec{a}_i} \bar{\psi}^i_{\vec{b}_i} \right\rangle \;, \tag{11}$$

which, using $\partial_{C_{\bar{n}p}} (C_{\bar{c}a} C^{-1}_{a\bar{b}}) = \delta_{\bar{n}\bar{c}} C^{-1}_{p\bar{b}} + C_{\bar{c}a} \partial_{C_{\bar{n}p}} C^{-1}_{a\bar{b}} = 0$, yields

$$\begin{aligned} \frac{\partial E}{\partial C_{\bar{n}p}} &= N^{-D} \sum_{\vec{a}_i, \vec{b}_i} \left(\prod_j (C^{-1}_{a_{ij} \bar{n}} C^{-1}_{p\bar{b}_{ij}}) \right) \left\langle \psi^i_{\vec{a}_i} \bar{\psi}^i_{\vec{b}_i} \right\rangle = N^{-D} \prod_j (C^{-1} P C^{-1})_{p\bar{n}} \\ &= \left(\frac{(C^{-1} P C^{-1})_{p\bar{n}}}{N} \right)^D \;. \end{aligned} \tag{12}$$

Solving the model consists therefore in determining P. We will do this in the next section.

III. THE LEADING ORDER IN $1/N$

At leading order only melonic two-point functions contribute [27]. They are characterized, as we already mentioned, by the fact that the self energy of the model $\Sigma = \langle \bar{\psi}^i_{\bar{n}_i} \psi^i_{\bar{p}_i} \rangle_{1PI,amputated}$ is given by the convolution (respecting the strand structure) of connected two-point functions. Supplementing this by the classical Schwinger-Dyson equation relating the full two-point function G_2, the self energy Σ and the propagator C of a field theory

$$G_2 = C \frac{1}{1 - \Sigma C} \ , \tag{13}$$

yields the system of equations

$$\sum_{\vec{q}_i} \left[\langle \bar{\psi}^i_{\bar{n}_i} \psi^i_{\vec{q}_i} \rangle \left(\prod_j \delta_{q_{ij} p_{ij}} - \sum_{\bar{\vec{r}}_i} \langle \psi^i_{\vec{q}_i} \bar{\psi}^i_{\bar{\vec{r}}_i} \rangle_{1PI,amputated} \prod_j C_{\bar{r}_{ij} p_{ij}} \right) \right] = \prod_j C_{\bar{n}_{ij} p_{ij}} \ ,$$

$$\langle \psi^i_{\vec{q}_i} \bar{\psi}^i_{\bar{\vec{r}}_i} \rangle_{1PI,amputated} = g N^{-\frac{D(D-1)}{2}} \sum_{\vec{q}_j, \bar{\vec{r}}_j \neq \vec{q}_i, \bar{\vec{r}}_i} \prod_{j \neq i} \langle \psi^j_{\vec{q}_j} \bar{\psi}^j_{\bar{\vec{r}}_j} \rangle \ . \tag{14}$$

Substituting the connected two-point function and performing the sums leads to

$$\prod_j P_{\bar{n}_j p_j} - g N^{-\frac{D(D-1)}{2}} [\mathrm{Tr}(PP^T)]^{\frac{D(D-1)}{2}} \prod_j (PP^T C)_{\bar{n}_j p_j} = \prod_j C_{\bar{n}_j p_j} \ , \tag{15}$$

where the index i has been erased as it plays no role. Surprising as it might be, equation (15) can be solved analytically. To do so, we first introduce a matrix $X = C^{-1}P$, we multiply by $X_{p_j b_j}$ from the right and by $P^{-1}_{a_j \bar{n}_j}$ from the left. Summing the resulting expression over all $\bar{n}_j$ and p_j (with range from 1 to N) we get

$$\prod_j X_{a_j b_j} = \prod_j \delta_{a_j b_j} + g N^{-\frac{D(D-1)}{2}} [\mathrm{Tr}(PP^T)]^{\frac{D(D-1)}{2}} \prod_j (P^T P)_{a_j b_j} \ . \tag{16}$$

To solve for X, we first take $a_j = b_j$ for all j and sum, obtaining

$$[\mathrm{Tr}(X)]^D = N^D + g N^{-\frac{D(D-1)}{2}} [\mathrm{Tr}(PP^T)]^{\frac{D(D+1)}{2}} \ , \tag{17}$$

and then we take $a_j = b_j$ for all but one j and sum, obtaining

$$X_{ab} [\mathrm{Tr}(X)]^{D-1} = \delta_{ab} N^{D-1} + g N^{-\frac{D(D-1)}{2}} [\mathrm{Tr}(PP^T)]^{\frac{D(D+1)}{2}-1} (P^T P)_{ab} \ . \tag{18}$$

Combining the two equations we have

$$X = \frac{N^{D-1}\mathbb{I} + g N^{-\frac{D(D-1)}{2}} [\mathrm{Tr}(PP^T)]^{\frac{D(D+1)}{2}-1} P^T P}{\left(N^D + g N^{-\frac{D(D-1)}{2}} [\mathrm{Tr}(PP^T)]^{\frac{D(D+1)}{2}} \right)^{\frac{D-1}{D}}} \ , \tag{19}$$

and we finally get the following expression for C as a function of P,

$$C = P \frac{\left(1 + g\alpha^{D(D+1)}\right)^{\frac{D-1}{D}}}{\mathbb{I} + g\alpha^{D(D+1)-2} P^T P} \ , \qquad C^T C = \frac{\left(1 + g\alpha^{D(D+1)}\right)^{2\frac{D-1}{D}}}{\left(\mathbb{I} + g\alpha^{D(D+1)-2} P^T P\right)^2} P^T P \ , \tag{20}$$

where $\alpha^2 = \frac{1}{N}\text{Tr}(PP^T) = \frac{1}{N}\text{Tr}(P^TP)$. As C^TC is a function of P^TP, the two commute, and equation (20) can be written as a quadratic equation for P^TP in terms of C^TC, whose physical solution is obtained by choosing the sign of the root that gives $P^TP = 0$ when $C^TC = 0$:

$$P^TP = \frac{1}{2g^2\alpha^{2D(D+1)-4}C^TC}\left[\left(1 + g\alpha^{D(D+1)}\right)^{2\frac{D-1}{D}}\mathbb{I} - 2g\alpha^{D(D+1)-2}C^TC\right.$$
$$\left. - \left(1 + g\alpha^{D(D+1)}\right)^{2\frac{D-1}{D}}\sqrt{\mathbb{I} - \frac{4g\alpha^{D(D+1)-2}}{\left(1 + g\alpha^{D(D+1)}\right)^{2\frac{D-1}{D}}}C^TC}\right], \quad (21)$$

Finally, expanding the square root in Taylor series we can write

$$P^TP = \sum_{q=1}\frac{1}{q+1}\binom{2q}{q}\frac{\left[g\alpha^{D(D+1)-2}\right]^{q-1}}{\left[\left(1 + g\alpha^{D(D+1)}\right)^{2\frac{D-1}{D}}\right]^q}(C^TC)^q , \quad (22)$$

$$\alpha^2 = \sum_{q=1}\frac{1}{q+1}\binom{2q}{q}\frac{\left[g\alpha^{D(D+1)-2}\right]^{q-1}}{\left[\left(1 + g\alpha^{D(D+1)}\right)^{2\frac{D-1}{D}}\right]^q}\frac{\text{Tr}[(C^TC)^q]}{N} . \quad (23)$$

Equation (22) determines P^TP in terms of C^TC and α, whereas (23) implicitly defines α in terms of the traces of powers of C^TC yielding the analytic solution at leading order in $1/N$ of the dually weighted colored tensor model. Note that the derivative of the free energy with respect to g is

$$g\partial_g E = \left[\frac{1}{N}\text{Tr}[C^{-1}P]\right]^D - 1 = \left(1 + gN^{-\frac{D(D+1)}{2}}[\text{Tr}(PP^T)]^{\frac{D(D+1)}{2}}\right) - 1 = g\alpha^{D(D+1)} , \quad (24)$$

hence studying only the self-consistency equation (23) for α suffices to study the critical behavior of the model. For $C = \mathbb{I}$ a straightforward computation leads from (23) to $\alpha^D = 1 + g\alpha^{D(D+1)}$, reproducing the solution of the i.i.d. model found in [27].

IV. PHASE PORTRAIT OF A PARTICULAR MODEL

In order to draw the phase portrait of a specific model it is more convenient to denote $g\partial_g E = g\alpha^{D(D+1)} \equiv U$ and write the self-consistency equation (23) in terms of U as

$$U = \sum_{q=1}\frac{1}{(q+1)}\binom{2q}{q}\left[\frac{U^{1-\frac{2}{D(D+1)}}}{(1+U)^{2\frac{D-1}{D}}}g^{\frac{2}{D(D+1)}}\right]^q\frac{\text{Tr}[(C^TC)^q]}{N} . \quad (25)$$

In the remainder of this paper we deal with the model defined by a covariance C such that

$$\text{Tr}[(C^TC)^q] = Nq^{-\beta} . \quad (26)$$

Such a choice corresponds to the DT amplitude

$$A(\mathcal{G}) = e^{\kappa_{D-2}N_{D-2} - \kappa_D N_D}\prod_i q_i^{-\beta} , \quad (27)$$

where the product is over all $(D-2)$-dimensional simplices (bones) of the triangulation, with q_i being the number of D-simplices to which the bone i belongs. The DT amplitude (27) was studied via numerical simulations in [49], and more recently in [15], where it was argued that a new phase

appears, for large enough β, with promising geometrical properties. We will now solve analytically in the large-N limit the corresponding dually weighted colored tensor model defined by (26) .

First note that a matrix C satisfying (26) exists in the large-N limit. To determine C we can for instance diagonalize it and write N equations for the N eigenvalues, corresponding to the traces up to power N. These equations will always have roots in the complex domain[7], thus we obtain in general a non-hermitian matrix C which satisfies (26) up to $q = N$. Alternatively we can impose a weaker condition $\text{Tr}[(C^T C)^q] = Nq^{-\beta} + O(1)$ for every $q \geq 1$, and solve for the eigenvalue distribution in the large-N limit, using standard techniques from matrix models. This way the spectrum of $C^T C$ can be chosen to be real.

The self-consistency equation (25) (and its physical initial condition) is now

$$U = S\big(\beta, z(g, U)\big) \, , \qquad S(\beta, z) = \sum_{q=1}^{\infty} \frac{1}{q^{\beta}(q+1)} \binom{2q}{q} z^q$$

$$z(g, U) = \frac{U^{1 - \frac{2}{D(D+1)}}}{\left(1 + U\right)^{2\frac{D-1}{D}}} g^{\frac{2}{D(D+1)}} \, , \qquad g(\beta, 0) = 0 \, . \tag{28}$$

We will denote $g(\beta, U)$ the solution of the equation (28).

The series $S(\beta, z)$ has radius of convergence $z_b = \frac{1}{4}$ for all values of β, that is, it converges for all g and U under the curve

$$z\big(g_b(U), U\big) = \frac{1}{4} \Rightarrow g_b(U) = \frac{(1 + U)^{D^2 - 1}}{4^{\frac{D(D+1)}{2}} U^{\frac{D(D+1)}{2} - 1}} \, . \tag{29}$$

Consider first the case the case $\beta = 0$. We have

$$U = S(0, z) = \frac{1 - 2z - \sqrt{1 - 4z}}{2z} \Rightarrow g(0, U) = \frac{U}{(1 + U)^{D+1}} \, . \tag{30}$$

The function $g(0, U)$ has an unique maximum in $U = \frac{1}{D}$. The curves $g(0, U)$ and $g_b(U)$ intersect for $U = 1$, irrespective of D. The reader can check that $g_b(U)$ is strictly decreasing for $U < 1$. The two curves are represented in Fig. 3a. At the critical point $U = \frac{1}{D}, g_c(0) = g(0, \frac{1}{D}) = \frac{D^D}{(D+1)^{D+1}}$ the function U becomes critical, as

$$(g - g_c) \sim (U - \frac{1}{D})^2 \Rightarrow U - \frac{1}{D} \sim (g - g_c)^{\frac{1}{2}} \Rightarrow E \sim (g - g_c)^{\frac{3}{2}} \, . \tag{31}$$

This is the standard branched polymer phase, obtained in the i.i.d. colored tensor model setting in [27].

We now slowly turn on β. The derivative of $S(\beta, z)$ w.r.t. z is $\left(\frac{\partial S}{\partial z}\right)_{\beta} = \frac{1}{z} S(\beta - 1, z)$. The derivatives of the solution $g(\beta, U)$ of the equation (28) are

$$U = S(\beta, z(g, U)) \Rightarrow dU = \partial_{\beta} S d\beta + \partial_z S \Big[\partial_U z \, dU + \partial_g z \, dg \Big] \, , \tag{32}$$

$$\left(\frac{\partial g}{\partial U}\right)_{\beta} = \frac{1 - \partial_z S \partial_U z}{\partial_z S \partial_g z} = \frac{1 - S(\beta - 1, z)\frac{D(D+1) - 2 - S(\beta, z)D(D-1)}{D(D+1)S(\beta, z)\left(1 + S(\beta, z)\right)}}{S(\beta - 1, z)\frac{2}{D(D+1)g}} \, ,$$

$$\left(\frac{\partial g}{\partial \beta}\right)_U = -\frac{\partial_{\beta} S}{\partial_z S \partial_g z} \, ,$$

---

[7] Actually it can be shown that for given N the solution is unique up to permutations of the eigenvalues.
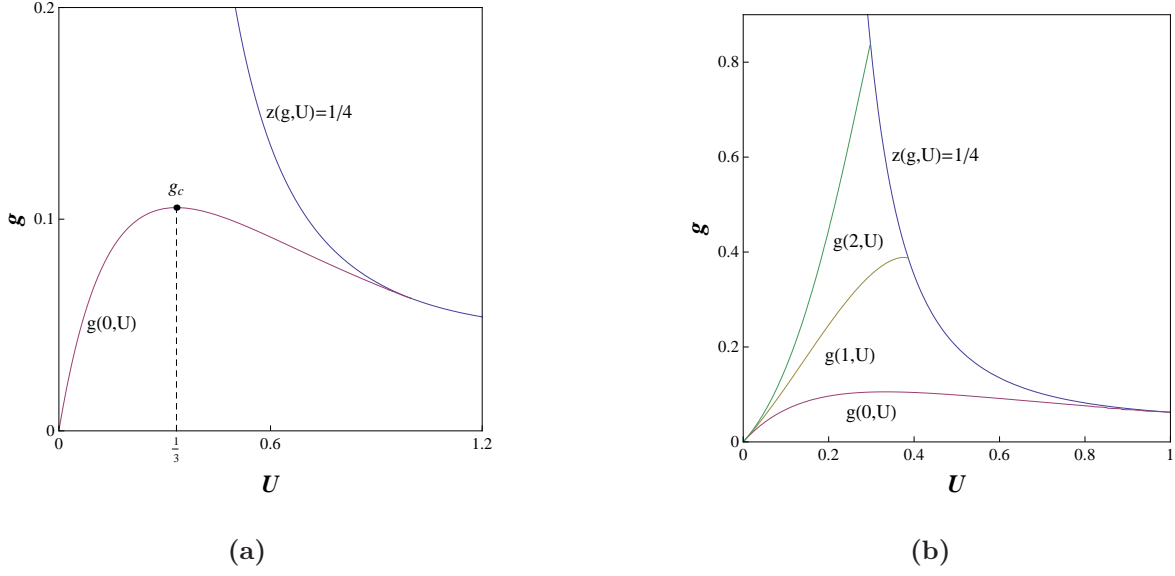
(a)

(b)

FIG. 3. The curve $g(\beta, U)$, for $D = 3$ and for $\beta = 0$ **(a)** and $\beta = 0, 1, 2$ **(b)**, together with the critical curve $z\big(g_b(U), U\big) = 1/4$.

where $z = z\big(g(\beta, U), U\big)$. The function $g(\beta, U)$ becomes critical for $(\frac{\partial g}{\partial U})_\beta = 0$, *i.e.* at $U = U_c^<(\beta)$ solution of

$$1 - \partial_z S\big|_{\beta, z = z\big(g(\beta, U), U\big)} \partial_U z\big|_{g = g(\beta, U), U} = 0 , \tag{33}$$

thus the critical curve $g_c^<(\beta)$ is characterized by

$$g_c^<(\beta) = g\big(\beta, U_c^<(\beta)\big) , \quad z_c^<(\beta) = z\big(g_c^<(\beta), U_c^<(\beta)\big) , \quad U_c^<(\beta) = S\big(\beta, z_c^<(\beta)\big) . \tag{34}$$

Equation (33) translates in parametrized form in terms of $z_c^<(\beta)$ (dropping the argument β to simplify notations) as

$$1 - S(\beta - 1, z_c^<)\frac{D(D+1) - 2 - S(\beta, z_c^<)D(D-1)}{D(D+1)S(\beta, z_c^<)\big(1 + S(\beta, z_c^<)\big)} = 0 . \tag{35}$$

Using (34) , with (32) and (33), we obtain

$$\frac{dg_c^<}{d\beta} = \left(\frac{\partial g}{\partial \beta}\right)_U\Big|_{\beta, U_c^<(\beta)} = -\frac{\partial_\beta S}{\partial_z S \partial_g z}\Big|_{\beta, U_c^<(\beta)} > 0 , \tag{36}$$

thus the critical curve $g_c^<(\beta)$ is increasing with β. As $\left(\frac{\partial^2 g}{\partial U^2}\right)_{U = U_c} < 0$ for all β, the critical behavior remains that of (31).

However this holds only for small enough β. Indeed we find that for $\beta > \beta_c > 1$, $g(\beta, U)$ exits the analyticity domain of $S(\beta, z)$ before it can reach its first maximum. Some curves $g(\beta, U)$ for increasing values of β are represented in Fig. 3b.

For $\beta > \beta_c$ both $\frac{\partial g}{\partial U}$ and $\frac{\partial U}{\partial g}$ remain finite all the way up to (and including) the boundary $g_b(U)$. The self consistency equation becomes

$$\left(\frac{\partial U}{\partial g}\right)_\beta = \frac{S(\beta - 1, z)\frac{2}{D(D+1)g}}{1 - S(\beta - 1, z)\frac{D(D+1) - 2 - S(\beta, z)D(D-1)}{D(D+1)S(\beta, z)\big(1 + S(\beta, z)\big)}} , \tag{37}$$
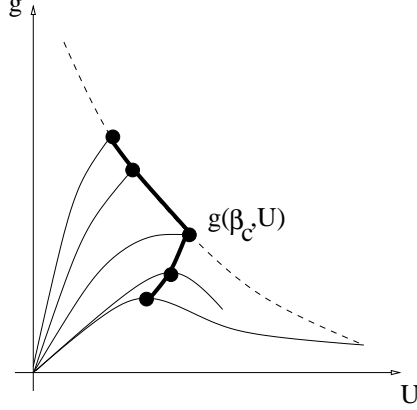
FIG. 4. A schematic representation of the phase transition and the critical line $g(\beta, U_c(\beta))$ (in bold), intersecting the line $g(\beta_c, U)$.

and the denominator stays finite. Close to $z = \frac{1}{4}$, the non-analytic behavior is given by $S(\beta-1, z) \sim (1 - 4z)^{\beta-1+\frac{1}{2}}$, hence

$$\left(\frac{\partial U}{\partial g}\right)_{\text{non-analytic}} \sim (g - g_b)^{\beta-1+\frac{1}{2}} \Rightarrow E_{\text{non-analytic}} \sim (g - g_b)^{\beta+\frac{3}{2}} . \tag{38}$$

Note that, as $\beta_c > 1$, for $\beta > \beta_c$ the non-analytic part of the free energy E is always preceded by an analytic part with non-zero linear and quadratic terms.

It follows that for $\beta > \beta_c$ the critical curve $g_c^>(\beta)$ is defined by

$$g_c^>(\beta) \text{ solution of } z(g, S(\beta, 1/4)) = \frac{1}{4} , \tag{39}$$

that is $g_c^>(\beta) = g_b\big(S(\beta, 1/4)\big)$. Along the critical curve $g_c^>(\beta)$ we have

$$U_c^>(\beta) = S(\beta, 1/4) , \qquad g_c^>(\beta) = g_b\big(U_c^>(\beta)\big) , \qquad z_c^>(\beta) = \frac{1}{4} . \tag{40}$$

The derivative of $g_c^>(\beta)$ computes using the implicit function theorem as

$$\frac{dg_c^>}{d\beta} = -\frac{\partial_U z \partial_\beta S}{\partial_g z} \bigg|_{\beta, U_c^>(\beta)} > 0 , \tag{41}$$

as $U_c^>(\beta) < 1$.

The critical β_c corresponds to the point where the two critical curves $g_c^<(\beta)$ and $g_c^>(\beta)$ meet, that is

$$g_c^<(\beta_c) = g_c^>(\beta_c) , \quad U_c^<(\beta) = U_c^>(\beta) , \quad z_c^<(\beta) = z_c^>(\beta) , \tag{42}$$

which translates, using equation (35), (40) and (42), as the single equation

$$S(\beta_c - 1, 1/4) = \frac{D(D+1)S(\beta_c, 1/4)[1 + S(\beta_c, 1/4)]}{D(D+1) - 2 - S(\beta_c, 1/4)D(D-1)} . \tag{43}$$

Numerically we find $\beta_c \approx 1.162$ for $D = 3$, $\beta_c \approx 1.216$ for $D = 4$, and $\beta_c \approx 1.134$ for $D = \infty$. A schematic portrait of the transition is presented in Fig. 4.

The critical behavior is different at $\beta = \beta_c$, as at $g = g_b = g_c$ we have at the same time $\left(\frac{\partial g}{\partial U}\right)_\beta = 0$, and a non-analytic behavior of $S(\beta, z)$ with exponent $\beta_c + \frac{1}{2} < 2$, hence

$$(g - g_c) \sim (U - U_c)^{\beta_c + \frac{1}{2}} \Rightarrow U - U_c \sim (g - g_c)^{\frac{1}{\beta_c + \frac{1}{2}}} \Rightarrow E \sim (g - g_c)^{\frac{\beta_c + \frac{3}{2}}{\beta_c + \frac{1}{2}}} . \tag{44}$$

In conclusion the susceptibility exponent, defined by $E_{\text{non-analytic}} \sim (g - g_c)^{2-\gamma}$, is

$$\gamma = \begin{cases} \frac{1}{2} & \text{for } \beta < \beta_c , \\ \frac{\beta_c - \frac{1}{2}}{\beta_c + \frac{1}{2}} & \text{for } \beta = \beta_c , \\ \frac{1}{2} - \beta & \text{for } \beta > \beta_c . \end{cases} \tag{45}$$

We observe that for a large range of values of β, *i.e.* for $\beta < \beta_c$, universality holds: the critical exponent γ is independent of β. On the contrary, for $\beta \geq \beta_c$ we have a one-parameter family of different critical behaviors. A possible interpretation of such unusual behavior is that for β sufficiently large the measure term starts behaving as non-local or long-range interaction, for which universality is not expected to hold.

A. Order of the phase transition

Introducing the canonical DT partition function via

$$Z_{DT}(\kappa_D, \kappa_{D-2}) = \sum_{N_D} e^{-\kappa_D N_D} Z_{DT,\text{can}}(N_D, \kappa_{D-2}) , \tag{46}$$

one has that the thermodynamic limit for the DT free energy is given by

$$F_\infty = \lim_{N_D \to \infty} \frac{1}{N_D} \ln Z_{DT,\text{can}}(N_D, \kappa_{D-2}) \sim -\ln g_c(\beta) . \tag{47}$$

The order of the phase transition around β_c is then assessed by studying the discontinuity of $g_c(\beta)$ or its derivative at β_c. Combining eq. (36), with (41) and using eq. (33) we obtain

$$\frac{dg_c^>}{d\beta}(\beta_c) = \frac{dg_c^<}{d\beta}(\beta_c) , \tag{48}$$

thus the phase transition is higher than first order!

To check the precise order of the phase transition we re-parametrize the self consistency equation by eliminating U

$$H(g, U) = \frac{U^{1 - \frac{2}{D(D+1)}}}{\left(1 + U\right)^{2\frac{D-1}{D}}} g^{\frac{2}{D(D+1)}} , \qquad S(\beta, z) = \sum_{q=1}^{\infty} \frac{1}{q^\beta (q+1)} \binom{2q}{q} z^q ,$$
$$z = H\big(g, S(\beta, z)\big) , \qquad g(\beta, 0) = 0 , \tag{49}$$

and denote its solution $g(\beta, z)$. We therefore have

$$dz = \partial_g H dg + \partial_U H \big(\partial_\beta S d\beta + \partial_z S dz\big) ,$$
$$\left(\frac{\partial g}{\partial z}\right)_\beta = \frac{1 - \partial_z S \partial_U H}{\partial_g H} , \qquad \left(\frac{\partial g}{\partial \beta}\right)_z = -\frac{\partial_U H \partial_\beta S}{\partial_g H} . \tag{50}$$

The two critical curves are

$$g_c^<(\beta) = g(\beta, z_c^<(\beta)) \text{ with } z_c^<(\beta) \text{ solution of } 1 - \partial_z S \partial_U H = 0 ,$$

$$g_c^>(\beta) = g(\beta, \tfrac{1}{4}) \ \text{ with } \ z_c^>(\beta) = \frac{1}{4}, \ \text{ i.e. } \ g_c^>(\beta) \ \text{ solution of } \ \frac{1}{4} = H(g, S(\beta, 1/4)) \ . \tag{51}$$

and meet at β_c, when

$$z_c^<(\beta_c) = \frac{1}{4} \qquad g_c^>(\beta_c) = g_c^<(\beta_c) \ . \tag{52}$$

It follows that the derivatives of the critical couplings are

$$
\begin{aligned}
\frac{dg_c^<}{d\beta} &= \left(\frac{\partial g}{\partial \beta}\right)_z = -\frac{\partial_U H \partial_\beta S}{\partial_g H}\Bigg|_{\beta, z_c^<(\beta)} = -\frac{\partial_U H\Big(g_c^<(\beta), S\big(\beta, z_c^<(\beta)\big)\Big)\partial_\beta S\big(\beta, z_c^<(\beta)\big)}{\partial_g H\Big(g_c^<(\beta), S\big(\beta, z_c^<(\beta)\big)\Big)} \ , \\
\frac{dg_c^>}{d\beta} &= -\frac{\partial_U H \partial_\beta S}{\partial_g H}\Bigg|_{\beta, 1/4} = -\frac{\partial_U H\Big(g_c^>(\beta), S\big(\beta, 1/4\big)\Big)\partial_\beta S\big(\beta, 1/4\big)}{\partial_g H\Big(g_c^>(\beta), S\big(\beta, 1/4\big)\Big)} \ ,
\end{aligned}
\tag{53}
$$

which, as we already knew, are finite and continuous at the critical point. The new parametrization comes in handy once we move to higher order derivatives.

All the derivatives of $g_c^>$ are finite at $\beta = \beta_c$ (note that $H(g, U)$ is analytic for $g > 0, U > 0$, hence its derivatives just go to constants at $\beta_c, z = \frac{1}{4}$), and the derivatives of $g_c^<$ will differ from them due to terms involving the derivatives of $z_c^<$

$$\lim_{\Delta\beta \to 0^+} \left(\frac{d^2 g_c^<}{d\beta^2}\bigg|_{\beta_c - \Delta\beta} - \frac{d^2 g_c^>}{d\beta^2}\bigg|_{\beta_c + \Delta\beta}\right) \sim \frac{dz_c^<}{d\beta} \ . \tag{54}$$

We thus only have to check at which order the derivatives of $z_c^<$ are non-zero and/or singular. In order to evaluate $\frac{dz_c^<}{d\beta}$ we use

$$d(\partial_z S \partial_U H) = 0 \Rightarrow \frac{dz^<}{d\beta} = -\frac{\partial_{\beta z} S \partial_U H - \partial_{gU} H \frac{\partial_\beta S}{\partial_g H} + \partial_z S \partial_\beta S \partial_{UU} H}{\partial_{zz} S \partial_U H + (\partial_z S)^2 \partial_{UU} H} \ . \tag{55}$$

The singular behavior comes from approaching the convergence radius of $S(\beta, z)$, where we have

$$S(\beta, \tfrac{1}{4} - \Delta z) \approx S(\beta, \tfrac{1}{4}) - 4S(\beta - 1, \tfrac{1}{4})\Delta z + \frac{\Gamma(-\beta - \tfrac{1}{2})}{\sqrt{\pi}}\Delta z^{\beta + \frac{1}{2}} \ , \tag{56}$$

from which we deduce that at $\beta = \beta_c$ the first singularity is in

$$\partial_{zz} S \sim \Delta z^{\beta_c - \frac{3}{2}} \Rightarrow \frac{dz^<}{d\beta} \sim \Delta z^{\frac{3}{2} - \beta_c} \ . \tag{57}$$

Finally we use the self-consistency equation and the expansion of $S(\beta, z)$ to get $\Delta z \sim \Delta\beta^{\frac{1}{\beta_c - \frac{1}{2}}}$, and

$$\frac{dz^<}{d\beta} \sim \Delta\beta^{\frac{\frac{3}{2} - \beta_c}{\beta_c - \frac{1}{2}}} \xrightarrow{\Delta\beta \to 0} 0 \ \Rightarrow \ \frac{d^2 g_c^<}{d\beta^2}(\beta_c) = \frac{d^2 g_c^>}{d\beta^2}(\beta_c) , \tag{58}$$

while

$$\lim_{\Delta\beta \to 0^+} \left(\frac{d^3 g_c^<}{d\beta^3}\bigg|_{\beta_c - \Delta\beta} - \frac{d^3 g_c^>}{d\beta^3}\bigg|_{\beta_c + \Delta\beta}\right) \sim \frac{d^2 z^<}{d\beta^2} \sim \Delta\beta^{\frac{2 - 2\beta_c}{\beta_c - \frac{1}{2}}} \xrightarrow{\Delta\beta \to 0} \infty , \tag{59}$$

thus the transition is third order.

B. Comparison to branched polymers

The β-dependent behavior we found is reminiscent of that of certain models of branched polymers (BP) [50, 51]. Such models are defined by the partition function

$$Z_{BP}(\mu) = \sum_N e^{-\mu N} \bar{Z}_{BP}(N) \,, \tag{60}$$

where

$$\bar{Z}_{BP}(N) = \sum_{BP_N} \prod_{i=1}^N p(n_i) \tag{61}$$

is the canonical partition function for rooted trees BP_N with N vertices, and n_i is the degree of the vertex i. Making the choice

$$p(n) = n^{-\alpha} \,, \tag{62}$$

one finds a critical behavior[8] $Z_{BP}(\mu) \sim (\mu - \mu_c)^{1-\gamma}$, with the susceptibility exponent γ given by

$$\gamma = \begin{cases} \frac{1}{2} & \text{for } \alpha < \alpha_c \,, \\ \frac{\alpha_c - 2}{\alpha_c - 1} \simeq 0.3237 & \text{for } \alpha = \alpha_c = 2.4787 \,, \\ 2 - \alpha & \text{for } \alpha > \alpha_c \,. \end{cases} \tag{63}$$

Comparing to the DT model we studied here, we see that we get qualitatively the same kind of behavior (up to the precise value of the critical coupling) if we identify $\alpha = \beta + 3/2$. The extra $3/2$ is coming from the asymptotic scaling of the factor $\frac{1}{q+1}\binom{2q}{q} \sim q^{-3/2} 4^q$, and can be understood as the natural entropy factor of the triangulations with trivial measure. The precise value of the critical coupling differs in the two cases (as for any D we have $\beta_c + 3/2 > \alpha_c$), but this comes as no surprise as critical couplings are generically non-universal quantities. For example one can adjust its value, without affecting anything else in (63), along the lines of [51]: introducing an additional weight t for the the number of vertices on the last generation of branches of the tree, one finds that the critical point changes with t, while the susceptibility exponent above and below the transition remains unaltered. For that particular modification one finds that decreasing t leads to an increase in α_c, and that for $\alpha_c > 3$ the value of γ at the critical point remains frozen at $\gamma = 1/2$.

It is very tempting to exploit the apparent connection to BP in order to extract other properties of the model. For example, a number of results are known about the Hausdorff dimension d_H [52] and spectral dimension d_S [53, 54] of BP. In particular, following [54] we could conjecture that, for $\gamma > 0$ (that is for $\beta \leq \beta_c$ in our model)

$$d_H = \frac{1}{\gamma} \,, \quad d_S = \frac{2}{1+\gamma} \,, \tag{64}$$

while for $\gamma < 0$ (i.e. $\beta > \beta_c$)

$$d_H = \infty \,, \quad d_S = 2 \,. \tag{65}$$

For $\beta < \beta_c$ we have the standard BP phase with $d_H = 2$, in agreement with the numerical results for DT [8]. For $\beta > \beta_c$ we enter a new phase, which doesn't seem to have been observed yet in

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[8] Note that as a result of the presence of a root, the partition function for rooted BP is interpreted as the derivative of a "non-rooted" model.

simulations of DT. From the BP point of view it is understood that for $\beta > \beta_c$ the dominating trees are actually short bushes, with the limit of $\beta \to \infty$ eventually represented by a bush with all the branches attached to a unique vertex. Given this intuition, and $d_H = \infty$, one could hypothesize that the new phase corresponds to the crumpled phase of DT (which at $\beta = 0$ is only visible at small κ_{D-2}). However, as the crumpled phase has $\gamma = -\infty$, such an interpretation can only make sense for $\beta \to \infty$.

Interestingly at $\beta = \beta_c$ we could get $2 < d_H < \infty$, as $\gamma > 0$. One then could possibly tweak the model so that the value of β_c at the critical point fixes either $d_H = D$ or $d_S = D$.[9]

However the reader should be warned that it is not at all obvious whether the parallel with BP should be extended to the Hausdorff and spectral dimensions. Sharing one critical exponent is of course not a sufficient reason to believe that all critical exponents are common. The relation between melon graphs and trees was already made evident in [27], where it was used as an exact bijection for the combinatorial counting. On the other hand, and most importantly, the notion of neighborhood in a triangulation seems to be very poorly represented by the abstract trees associated to melons. Nevertheless it is intriguing that a strong relation between DT (for $\beta = 0$) in the weak coupling phase and BP is supported both by numerical simulations [8] (supporting for example $d_H = 1/\gamma = 2$) and by theoretical arguments [55]. Whether or not the link can be clarified further in the context of the colored tensor models is an open issue.

V. FINAL DISCUSSION AND OUTLOOK

In this paper we have introduced and solved a new class of colored tensor models. The new models, dubbed dually weighted colored tensor models, allow to associate arbitrary weights to the faces of the graphs, *i.e.* to the bones of the dual triangulation. Choosing the weights as in (26), we have explicitly computed the susceptibility exponent of the model, and showed that in the continuum limit the model admits two phases separated by a third order phase transition. The results are reminiscent of certain models of branched polymers, with which the susceptibility exponent shares the same qualitative behavior. However, we have not computed the Hausdorff dimension for our model, and we have at the moment no reason to believe that it agrees with that of the BP.

We believe that our results are very important for the DT approach to quantum gravity. First of all, this is to our knowledge the first time that a phase transition in the continuum limit is accessed by analytical means in dimensions higher than two. Furthermore, it is the first time that a *third-order* phase transition is observed in DT, which opens up the possibility of obtaining a continuum limit with $2 < d_H < \infty$, without the causality condition employed in CDT [11–14]. The link to the numerical results of [15] is unclear, and deserves further exploration, either by pushing the simulations to larger κ_{D-2} and larger volumes, or by trying to extend the analytical tools to finite κ_{D-2}.

It should also be mentioned that the BP models we have discussed in Sec. IV B have been mapped and generalized to a balls-in-boxes models [56–58], which has been interpreted as a mean field model of DT. Indeed the model successfully reproduces the first-order transition (at $\beta = 0$ and $\kappa_{D-2} = \kappa_{D-2}^c$) between a crumpled and a BP phase, as in the DT simulations [9, 10], and in the $\{\beta, \kappa_{D-2}\}$ plane it shows a phase diagram which is reminiscent of the one found in [49] (and more recently revisited in [15]). In the light of that, the new phase we observed might extend at finite κ_{D-2} into a phase similar to the "condensed phase" of [58] or the "crinkled phase" of [49].

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[9] The relations (64) imply that branched polymers can have $d_S = d_H$ only for $\gamma = 1$, thus it seems unlikely to be able to fit both dimensions to D simultaneously.

Indeed our result of a negative and β-dependent exponent γ seems compatible with the findings of [49].

In this work we have concentrated on a specific choice of weights, but the solution provided in Sec. III can be used for any other choice of matrix C. For example, we could consider

$$\mathrm{Tr}[(C^T C)^q] = N q^{-\beta} e^{\mu q^n} . \tag{66}$$

The case $n \geq 1$ is not very interesting. The modification either shifts the critical point ($n = 1$) or makes the series in (25) always convergent (resp. divergent) for $n > 1$, $\mu < 0$ (resp. $n > 1$, $\mu > 0$). An interesting choice from the DT point of view is $n = -1$, corresponding to the DT amplitude

$$A(\mathcal{G}) = e^{\kappa_{D-2} N_{D-2} - \kappa_D N_D + \sum_i \frac{\mu}{q_i}} \prod_i q_i^{-\beta} . \tag{67}$$

Such amplitude corresponds to the addition of an R^2 term to the Regge action [59]. For $n < 0$ (66) the asymptotic of the series in (25) are unaffected, and the large-N limit of the model would be the same as the one we have found for $\mu = 0$. This is in agreement with the findings of [59], according to which the inclusion of higher-derivative terms does not significantly affect the phase diagram of DT. Finally, for $0 < n < 1$ we have $\gamma = -\infty$ for $\beta > \beta_c$. In such case the similarity between the $\beta > \beta_c$ phase and the crumpled phase of DT seems stronger than what we have outlined in Sec. IV B. It would be interesting to study this model further and check whether the phase transition remains third-order or it becomes second or first order .

Another option would be to modify the model by choosing C such that at the critical point g_c the first $m > 1$ derivatives of the coupling become zero, rather than just the first one. In all likelihood this would still relate to some type of branched polymers, like the multicritical BP studied in [52].

The most important open problems in our view are to study whether the connection to branched polymers extends to other critical exponents, and in particular to the effective dimension of the triangulations. Even more important would be to access finite κ_{D-2}, maybe via some sort of double scaling limit.

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