

Path integral approach to the full Dicke model with dipole-dipole interaction

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Abstract

We consider the full Dicke spin-boson model composed by a single bosonic mode and an ensemble of N identical two-level atoms with different couplings for the resonant and anti-resonant interaction terms, and incorporate a dipole-dipole interaction between the atoms. Assuming that the system is in thermal equilibrium with a reservoir at temperature β^{-1} , we compute the free energy in the thermodynamic limit $N \rightarrow \infty$ in the saddle-point approximation to the path integral and determine the critical temperature for the superradiant phase transition. In the zero temperature limit, we recover the critical coupling of the quantum phase transition, presented in the literature.

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1 Introduction

At low temperatures a gas of N slow moving molecules with a few accessible electronic states interacting with the vacuum electromagnetic field in a cavity may be considered as a single quantum mechanical system with a ground state which determines the macroscopic properties of the gas and a set of excited states emerging as collective matter-field modes. In conditions where the dependence on the center of mass coordinates of the molecules may be omitted and for the simplest case of two level atoms, the system may be approximately described by Dicke [1] and generalized Dicke models which are written in terms of equivalent spin variables for the atoms. At zero temperature the properties of these models can be investigated by solving the quantum mechanical problem in a suitable approximation. In the original Dicke model [1] it is furthermore assumed that the atoms are confined to a small volume and that their direct interaction may be disregarded. The field is coupled to the total equivalent spin operator and enhanced rates for spontaneous radiation may be computed in the dipolar approximation for particular, so called superradiant states. In a modern view this is recognized as a hint of the existence of a quantum phase transition for this system [2]. A direct determination of this transition was obtained in Ref. [3] performing the diagonalization of the bimodal effective Hamiltonian resulting of applying a Holstein-Primakoff transformation [4, 5] in the thermodynamic limit and computing the critical coupling for which the boson expectation value vanish. In this work the energy of the excited collective modes was also computed and the relation of the quantum phase transition and the entanglement properties of the collective states was discussed. This connection is particularly interesting due to the wide interest that have attracted the applications to communications and quantum computing of entangled systems [6]. In this context efforts are focused in how to create and also how to detect entangled states in many-body systems [7, 8, 9], but many authors have also investigated the relation between entanglement of collective modes and quantum phase transitions [10, 11, 12, 13].

An alternative approach to the study of this model, which in fact was pursued much earlier is to explore the thermodynamical properties of the system at finite temperature by computing the partition function and the free energy. This was done first in the rotating wave approximation by Hepp and Lieb [14, 15] and Wang and Hioe [16] who determined the critical temperature for the superradiant phase transition in the Dicke model. In Ref. [17] a generalized Dicke model which include both the resonant and anti-resonant interaction terms was considered. The results in this works were confirmed in [18, 19] using different methods.

The use of the path integral approach for the computation of the thermodynamical properties of these models was advocated by Popov and their collaborators [20, 21, 22, 23, 24] and produce results which are consistent with those obtained by other techniques in a very efficient way. The path integral formalism provides in particular a simple method to compute the spectrum of the system under consideration in either of it phases using the semiclassical expansion of the partition function [22, 25]. In Refs. [26, 27] these methods where used to determine the partition function and critical temperature of the generalized Dicke model and of the full Dicke model which emerges

when different couplings for the resonant and anti-resonant terms are introduced. Then in Ref. [28] the spectrum of these models was determined by computing the quadratic approximation of the path integral. The zero temperature limit of the energies, in the case when the resonant and anti-resonant couplings are the same, were shown to correspond to the result in [3].

When dipole-dipole interactions are included the procedure using the Holstein Primakoff transformation is still useful and the critical coupling corresponding to the zero temperature quantum phase transition as well as the energy of the excited states may be computed [29, 30]. See also Refs. [31, 32, 33].

In this paper we compute the free energy in this generalized situation using functional methods and determine the critical temperature. Our result show a dependence of the critical coupling on the dipole-dipole coupling which in the zero temperature limit reproduce consistently the quantum phase transition point computed in [29]. The organization of the paper is as follows: In section II we discuss the interacting Hamiltonian of the model. In section III the functional integral for the model is performed and the critical temperature is determined. In section IV we present the free energy of the model. Finally, section V contains our conclusions. We use $k_B = c = \hbar = 1$

2 The Hamiltonian and the fermionic model

We consider the full Dicke model with dipole-dipole interaction. This model contemplates resonant and anti-resonant coupling terms between the atoms and the bosonic field with different couplings g_1 and g_2 . The Hamiltonian for this system, with the dipole-dipole interaction included reads

$$\begin{aligned}
 H &= \frac{\lambda}{N} \sum_{i \neq j}^N \sigma_i^+ \sigma_j^- + \frac{\Omega}{2} \sum_{i=1}^N \sigma_i^z + \omega_0 b^\dagger b \\
 &+ \frac{g_1}{\sqrt{N}} \sum_{i=1}^N (b \sigma_i^+ + b^\dagger \sigma_i^-) + \frac{g_2}{\sqrt{N}} \sum_{i=1}^N (b \sigma_i^- + b^\dagger \sigma_i^+). \tag{1}
 \end{aligned}$$

In the above equation the operators σ_i^1, σ_i^2 and $\sigma_i^z = \sigma_i^3$ satisfy the commutation relations $[\sigma_i^p, \sigma_i^q] = 2 \epsilon^{pqr} \sigma_i^r$ for $p, q, r = 1, 2, 3$ and we use through the paper the operators $\sigma_i^\pm = \frac{1}{2}(\sigma_i^1 \pm i \sigma_i^2)$. These operators satisfy $[\sigma_i^+, \sigma_i^-] = \sigma_{(j)}^z$ and $[\sigma_i^z, \sigma_i^\pm] = \pm 2 \sigma_i^\pm$. The boson annihilation and creation operators b and b^\dagger satisfy the usual commutation relation rules. Each two-level atom interact with all other atoms of the ensemble with the same coupling strength λ/N , and the summation is over all the atoms. Note that this model is related to models describing spin systems with long range interaction [34, 35, 36].

To proceed, let us define an auxiliary model to be called the fermionic full Dicke model in terms of fermionic raising and lowering operators $\alpha_i^\dagger, \alpha_i, \beta_i^\dagger$ and β_i , that satisfy the anti-commutator relations $\alpha_i \alpha_j^\dagger + \alpha_j^\dagger \alpha_i = \delta_{ij}$ and $\beta_i \beta_j^\dagger + \beta_j^\dagger \beta_i = \delta_{ij}$. We can also define the following bilinear

combination of fermionic operators, $\alpha_i^\dagger \alpha_i - \beta_i^\dagger \beta_i$, $\alpha_i^\dagger \beta_i$ and $\beta_i^\dagger \alpha_i$ which obey the same commutation relations as the pseudo-spin operators $\sigma_{(i)}^z$, $\sigma_{(i)}^+$ and $\sigma_{(i)}^-$ with the correspondence given by,

$$\sigma_i^z \longrightarrow \alpha_i^\dagger \alpha_i - \beta_i^\dagger \beta_i, \quad (2)$$

$$\sigma_i^+ \longrightarrow \alpha_i^\dagger \beta_i, \quad (3)$$

$$\sigma_i^- \longrightarrow \beta_i^\dagger \alpha_i. \quad (4)$$

Substituting these relations in the Hamiltonian (1) we obtain the Hamiltonian H_F of the auxiliary fermionic full Dicke model with dipole-dipole coupling,

$$\begin{aligned} H_F = & \frac{\lambda}{N} \sum_{ij} \alpha_i^\dagger \beta_i \beta_j^\dagger \alpha_j + \frac{\Omega}{2} \sum_{i=1}^N (\alpha_i^\dagger \alpha_i - \beta_i^\dagger \beta_i) \\ & + \omega_0 b^\dagger b + \frac{g_1}{\sqrt{N}} \sum_{i=1}^N (b \alpha_i^\dagger \beta_i + b^\dagger \beta_i^\dagger \alpha_i) + \frac{g_2}{\sqrt{N}} \sum_{i=1}^N (b \beta_i^\dagger \alpha_i + b^\dagger \alpha_i^\dagger \beta_i). \end{aligned} \quad (5)$$

The Hamiltonians H and H_F are defined in different Hilbert spaces, since each operator σ_i^α in H acts on a two-dimensional Hilbert space, and each of the fermionic operators α_i^\dagger , α_i , β_i^\dagger and β_i , in H_F acts on a four-dimensional Fock space. Nevertheless the partition function Z of the Dicke model may be written as a trace in the space of the fermionic operators due to the the following useful relation whose demonstration follows the one given by Popov for the Dicke model [21]

$$Z = Tr(\exp(-\beta H)) = i^N Tr\left(\exp\left(-\beta H_F - \frac{i\pi}{2} N_F\right)\right). \quad (6)$$

In this relation H is given by Eq. (1), H_F is given by Eq. (5) and the operator N_F is defined by

$$N_F = \sum_{i=1}^N (\alpha_i^\dagger \alpha_i + \beta_i^\dagger \beta_i). \quad (7)$$

It should be clear that the traces in Eq. (6) for each Hamiltonian are taken over their respective spaces.

3 The partition function in the path integral approach

As mentioned in the introduction the critical temperature for the Dicke and the full Dicke models may be computed using functional methods [27, 28]. In this section we consider the corresponding computation for the full Dicke model with dipole-dipole interaction. This problem was addressed

earlier in Ref. [37], but within the approximation scheme proposed there no variation of the critical temperature was detected. In this section we prove that the introduction of the dipole-dipole interaction indeed modify the temperature for transition from normal to superradiant phase. Concerning the path integral approach that we use, we note that in this case due to the dipole-dipole interaction term the path integral is non Gaussian. To overcome this difficulty we introduce an auxiliary field in the path integral and perform a series expansion..

At finite temperature the Euclidean action S associated to the fermionic Dicke model is given by

$$S = \int_0^\beta d\tau \left(b^*(\tau) \partial_\tau b(\tau) + \sum_{i=1}^N \left(\alpha_i^*(\tau) \partial_\tau \alpha_i(\tau) + \beta_i^*(\tau) \partial_\tau \beta_i(\tau) \right) \right) - \int_0^\beta d\tau H_F(\tau), \quad (8)$$

where the Hamiltonian density $H_F(\tau)$ is obtained from Eq. (5). In this case it takes the form

$$\begin{aligned} H_F(\tau) &= \omega_0 b^*(\tau) b(\tau) + \frac{\Omega}{2} \sum_{i=1}^N \left(\alpha_i^*(\tau) \alpha_i(\tau) - \beta_i^*(\tau) \beta_i(\tau) \right) + \\ &+ \frac{\lambda}{N} \sum_{i \neq j}^N \alpha_i^*(\tau) \beta_i(\tau) \beta_j^*(\tau) \alpha_j(\tau) + \frac{g_1}{\sqrt{N}} \sum_{i=1}^N \left(\alpha_i^*(\tau) \beta_i(\tau) b(\tau) + \beta_i^*(\tau) \alpha_i(\tau) b^*(\tau) \right) + \\ &+ \frac{g_2}{\sqrt{N}} \sum_{i=1}^N \left(\beta_i^*(\tau) \alpha_i(\tau) b(\tau) + \alpha_i^*(\tau) \beta_i(\tau) b^*(\tau) \right). \end{aligned} \quad (9)$$

Now we consider the formal ratio of the partition function of the dipole-dipole full Dicke model and the partition function of the free Dicke model,

$$\frac{Z}{Z_0} = \frac{\int [d\eta(\alpha, \beta, b)] \exp \left(S - \frac{i\pi}{2\beta} \int_0^\beta n(\tau) d\tau \right)}{\int [d\eta(\alpha, \beta, b)] \exp \left(S_0 - \frac{i\pi}{2\beta} \int_0^\beta n(\tau) d\tau \right)}, \quad (10)$$

where the function $n(\tau)$ in the above expression is defined by

$$n(\tau) = \sum_{i=1}^N \left(\alpha_i^*(\tau) \alpha_i(\tau) + \beta_i^*(\tau) \beta_i(\tau) \right). \quad (11)$$

In Eq. (10) $S = S(b, b^*, \alpha, \alpha^\dagger, \beta, \beta^\dagger)$ is the Euclidean action of the full Dicke model given by Eq. (8), $S_0 = S_0(b, b^*, \alpha, \alpha^\dagger, \beta, \beta^\dagger)$ is the free Euclidean action obtained by taking $\lambda = g_1 = g_2 = 0$ and $[d\eta(\alpha, \beta, b)]$ is the functional measure. The functional integrals in Eq. (10), have to be done in the space of complex functions $b^*(\tau)$ and $b(\tau)$ and Grassmann variables $\alpha_i^*(\tau)$, $\alpha_i(\tau)$, $\beta_i^*(\tau)$ and $\beta_i(\tau)$. In thermal equilibrium in the imaginary time formalism, the integration variables in Eq. (10) obey periodic boundary conditions for the Bose field, i. e., $b(\beta) = b(0)$ and anti-periodic boundary conditions for Grassmann variables, i. e., $\alpha_i(\beta) = -\alpha_i(0)$ and $\beta_i(\beta) = -\beta_i(0)$.

To advance in the computation of the quotient (10), we perform now the following decoupling transformation

$$\begin{aligned}\alpha_i(\tau) &\rightarrow e^{\frac{i\pi}{2\beta}\tau} \alpha_i(\tau), & \alpha_i^*(\tau) &\rightarrow e^{-\frac{i\pi}{2\beta}\tau} \alpha_i^*(\tau), \\ \beta_i(\tau) &\rightarrow e^{\frac{i\pi}{2\beta}\tau} \beta_i(\tau), & \beta_i^*(\tau) &\rightarrow e^{-\frac{i\pi}{2\beta}\tau} \beta_i^*(\tau).\end{aligned}\tag{12}$$

which makes the coefficient of the term proportional to $n(\tau)$ in the action to vanish. We should remark that the price to pay for this benefit is that the boundary conditions are modified. In Eq. (14), the Bose field still obeys periodic boundary conditions, i. e., $b(\beta) = b(0)$, but the Fermi fields obey now the following boundary conditions:

$$\begin{aligned}\alpha_i(\beta) &= i \alpha_i(0), & \alpha_i^*(\beta) &= -i \alpha_i^*(0), \\ \beta_i(\beta) &= i \beta_i(0), & \beta_i^*(\beta) &= -i \beta_i^*(0).\end{aligned}\tag{13}$$

After this procedure we obtain

$$\frac{Z}{Z_0} = \frac{\int [d\eta(\alpha, \beta, b)] e^S}{\int [d\eta(\alpha, \beta, b)] e^{S_0}}.\tag{14}$$

In order to obtain the effective action of the bosonic mode we must integrate over the Grassmann Fermi fields in Eq. (14). The problem that we now are confronting is that in the action given by Eq. (9) there is a quartic term corresponding to the dipole-dipole interaction and the integral is non Gaussian. Although it is not possible to integrate this term directly, we can use an auxiliary field to circumvent this difficult and express the path integral as a series expansion. Since terms quadratic in the Grassman variables obey Bose statistics we have $(\alpha_i(\tau) \beta_i^*(\tau)) (\alpha_j(\tau) \beta_j^*(\tau)) = (\alpha_j(\tau) \beta_j^*(\tau)) (\alpha_i(\tau) \beta_i^*(\tau))$, and we can use following functional identity

$$\begin{aligned}& e^{-\frac{\lambda}{N} \sum_{i,j=1}^N \alpha_i^*(\tau) \beta_i(\tau) \beta_j^*(\tau) \alpha_j(\tau)} \\ &= N_0 \int [d\eta(r)] e^{\int_0^\beta d\tau \left(r^*(\tau) r(\tau) + \sqrt{\frac{\lambda}{N}} \sum_{i=1}^N r(\tau) \beta_i^*(\tau) \alpha_i(\tau) + \sqrt{\frac{\lambda}{N}} \sum_{i=1}^N r^*(\tau) \alpha_i^*(\tau) \beta_i(\tau) \right)},\end{aligned}\tag{15}$$

where $[d\eta(r)]$ is the functional measure for the functional integral with respect the fields $r(\tau)$ and $r^*(\tau)$. The normalization factor is defined by $N_0^{-1} = \int [d\eta(r)] \exp \left(\int_0^\beta d\tau r^*(\tau) r(\tau) \right)$, and the fields $r(\tau)$ and $r^*(\tau)$ satisfy the boundary conditions $r(0) = r(\beta)$ and $r^*(0) = r^*(\beta)$. Substituting this last expression in Z , we obtain

$$Z = N_0 \int [d\eta(r)] [d\eta(\alpha, \beta, b)] e^{S_r}.\tag{16}$$

The new action S_r can be separated into a free action $S_0(b, r)$ for the bosons and the field $r(\tau)$ and a Gaussian fermionic part in the form,

$$S_r = S_0(b, r) + \int_0^\beta d\tau \sum_{i=1}^N \rho_i^\dagger(\tau) M_r(b^*, b) \rho_i(\tau),\tag{17}$$

with

$$S_0(b, r) = \int_0^\beta d\tau \left(b^*(\tau) (\partial_\tau - \omega_0) b(\tau) + r^*(\tau) r(\tau) \right), \quad (18)$$

In (17), the column matrix $\rho_i(\tau)$ arranges the Grassmann Fermi fields in the form

$$\begin{aligned} \rho_i(\tau) &= \begin{pmatrix} \beta_i(\tau) \\ \alpha_i(\tau) \end{pmatrix}, \\ \rho_i^\dagger(\tau) &= \begin{pmatrix} \beta_i^*(\tau) & \alpha_i^*(\tau) \end{pmatrix} \end{aligned} \quad (19)$$

and the matrix $M_r(b^*, b)$ is given in terms of the operators $L = \partial_\tau + \Omega/2$ and $L_* = \partial_\tau - \Omega/2$ by

$$\begin{pmatrix} L & -N^{-1/2} \left(g_1 b^*(\tau) + g_2 b(\tau) - \sqrt{\lambda} r(\tau) \right) \\ -N^{-1/2} \left(g_1 b(\tau) + g_2 b^*(\tau) - \sqrt{\lambda} r^*(\tau) \right) & L_* \end{pmatrix}, \quad (20)$$

From the form of S_r we see that the fermionic part of the functional integral given by Eq. (16) is Gaussian and can be done directly. Therefore we obtain

$$Z = N_0 \int [d\eta(r)] [d\eta(b)] e^{S_0(b,r)} \left(\det M_r(b^*, b) \right)^N, \quad (21)$$

where in this case, $[d\eta(b)]$ is the functional measure only for the bosonic field. With the help of the following property for matrices with operator components

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det (AD - ACA^{-1}B), \quad (22)$$

and using also determinant properties, we finally obtain,

$$\begin{aligned} \det M_r(b^*, b) &= \\ \det (LL_*) \det \left(1 - N^{-1} L_*^{-1} (g_1 b + g_2 b^* - \sqrt{\lambda} r^*) L^{-1} (g_1 b^* + g_2 b - \sqrt{\lambda} r) \right). \end{aligned} \quad (23)$$

Substituting Eq. (21) and Eq. (23) in Eq. (14), we have

$$\frac{Z}{Z_0} = \frac{Z_A}{\int [d\eta(r)] [d\eta(b)] e^{S_0(b,r)}}, \quad (24)$$

with Z_A defined by

$$Z_A = \int [d\eta(r)] [d\eta(b)] \exp \left(S_0(b, r) + N \operatorname{tr} \ln \left(1 - N^{-1} L_*^{-1} (g_1 b + g_2 b^* - \sqrt{\lambda} r^*) L^{-1} (g_1 b^* + g_2 b - \sqrt{\lambda} r) \right) \right). \quad (25)$$

We have arrived at an effective action in terms of two bosonic modes interacting in a highly non trivial way. Since we are interested in knowing the asymptotic behavior of the quotient $\frac{Z}{Z_0}$ in the thermodynamic limit, i. e., $N \rightarrow \infty$ we analyze the asymptotic behavior of the last defined expression Z_A . First, to make the dependence in N more explicit, we rescale the bosonic fields by $b \rightarrow \sqrt{N} b$, $b^* \rightarrow \sqrt{N} b^*$, $r \rightarrow \sqrt{N} r$ and $r^* \rightarrow \sqrt{N} r^*$. We get

$$Z_A = A(N) \int [d\eta(r)] [d\eta(b)] \exp \left(N \Phi(b, b^*, r, r^*) \right), \quad (26)$$

with the function $\Phi(b, b^*, r, r^*)$ defined by

$$\Phi(b, b^*, r, r^*) = S_0(b, r) + \operatorname{tr} \ln \left(1 - L_*^{-1} (g_1 b + g_2 b^* - \sqrt{\lambda} r^*) L^{-1} (g_1 b^* + g_2 b - \sqrt{\lambda} r) \right). \quad (27)$$

The term $A(N)$ in Eq. (26) comes from transforming the functional measures $[d\eta(r)]$ and $[d\eta(b)]$ under scaling the bosonic field by $b \rightarrow \sqrt{N} b$, $b^* \rightarrow \sqrt{N} b^*$, $r \rightarrow \sqrt{N} r$ and $r^* \rightarrow \sqrt{N} r^*$ and will be reabsorbed when we transform back to the original variable.

The asymptotic behavior of the functional integral appearing in Eq. (26) when $N \rightarrow \infty$, is obtained using the method of steepest descent [38]. We expand the function $\Phi(b, b^*, r, r^*)$ around stationary points $b(\tau) = b_0(\tau)$, $b^*(\tau) = b_0^*(\tau)$, $r(\tau) = r_0(\tau)$ and $r^*(\tau) = r_0^*(\tau)$. The stationary points are solution of the equations

$$\begin{aligned} \frac{\delta \Phi(b, b^*, r, r^*)}{\delta b(\tau)} \Big|_{b_0, b_0^*, r_0, r_0^*} = 0 & \quad , \quad \frac{\delta \Phi(b, b^*, r, r^*)}{\delta b^*(\tau)} \Big|_{b_0, b_0^*, r_0, r_0^*} = 0 & \quad , \\ \frac{\delta \Phi(b, b^*, r, r^*)}{\delta r(\tau)} \Big|_{b_0, b_0^*, r_0, r_0^*} = 0 & \quad , \quad \frac{\delta \Phi(b, b^*, r, r^*)}{\delta r^*(\tau)} \Big|_{b_0, b_0^*, r_0, r_0^*} = 0 & \quad . \end{aligned} \quad (28)$$

which for the system at hand are constant functions, $b(\tau) = b_0$, $b^*(\tau) = b_0^*$, $r(\tau) = r_0$ and $r^*(\tau) = r_0^*$ that fulfill the following system of algebraic equations,

$$\omega_0 b_0 = \left(g_1 (g_1 b_0 + g_2 b_0^* - \sqrt{\lambda} r_0^*) + g_2 (g_1 b_0^* + g_2 b_0 - \sqrt{\lambda} r_0) \right) \frac{1}{\Omega_\Delta} \tanh \left(\frac{\beta}{2} \Omega_\Delta \right), \quad (29)$$

$$\omega_0 b_0^* = \left(g_1 (g_1 b_0^* + g_2 b_0 - \sqrt{\lambda} r_0) + g_2 (g_1 b_0 + g_2 b_0^* - \sqrt{\lambda} r_0^*) \right) \frac{1}{\Omega_\Delta} \tanh \left(\frac{\beta}{2} \Omega_\Delta \right), \quad (30)$$

$$r_0 = \sqrt{\lambda} (g_1 b_0^* + g_2 b_0 - \sqrt{\lambda} r_0) \frac{1}{\Omega_\Delta} \tanh\left(\frac{\beta}{2}\Omega_\Delta\right), \quad (31)$$

$$r_0^* = \sqrt{\lambda} (g_1 b_0 + g_2 b_0^* - \sqrt{\lambda} r_0^*) \frac{1}{\Omega_\Delta} \tanh\left(\frac{\beta}{2}\Omega_\Delta\right), \quad (32)$$

with $\Omega_\Delta = \sqrt{\Omega^2 + 4 |g_1 b_0 + g_2 b_0^* - \sqrt{\lambda} r_0^*|^2}$. Substituting Eqs. (31) and (32) in Eqs. (29) and (30) we have also

$$\begin{aligned} \sqrt{\lambda} \omega_0 b_0 &= g_1 r_0^* + g_2 r_0, \\ \sqrt{\lambda} \omega_0 b_0^* &= g_1 r_0 + g_2 r_0^*. \end{aligned} \quad (33)$$

Substituting Eq. (33) in the Eq. (29) and imposing $b_0 \neq 0$ we finally obtain that

$$\frac{\omega_0}{(g_1 + g_2)^2 - \omega_0 \lambda} = \frac{1}{\Omega_\Delta} \tanh\left(\frac{\beta}{2}\Omega_\Delta\right). \quad (34)$$

This last equation allows us to calculate $|b_0|$. Substituting Eq. (34) in Eq. (29) shows that b_0 is real and in consequence r_0 is real. The critical temperature is characterized by the condition $b_0 = 0$. Therefore we have

$$\frac{\omega_0 \Omega}{(g_1 + g_2)^2 - \omega_0 \lambda} = \tanh\left(\frac{\beta_c}{2}\Omega\right). \quad (35)$$

At zero temperature this result reproduces the critical coupling of the quantum phase transition discussed in Ref. [29]. We stress that the introduction of the dipole-dipole interaction modifies the critical temperature of the transition from the fluorescent to the superradiant phase.

4 The free energy

Let us complete the computation of the asymptotic behavior of the functional integral, appearing in Eq. (26), in the thermodynamic limit, $N \rightarrow \infty$. We consider the two first leading terms in the functional integral appearing in Eq. (26) coming from the expansion of $\Phi(b, b^*, r, r^*)$ around the

maximal value b_0, b_0^*, r_0 and r_0^* this expansion is given by

$$\Phi(b, b^*, r, r^*) = \Phi(b_0, b_0^*, r_0, r_0^*) + \frac{1}{2} \int_0^\beta d\tau_1 d\tau_2$$

$$(b(\tau_1) - b_0, b^*(\tau_1) - b_0^*, r(\tau_1) - r_0, r^*(\tau_1) - r_0^*) M_\Phi \begin{pmatrix} b(\tau_2) - b_0 \\ b^*(\tau_2) - b_0^* \\ r(\tau_2) - r_0 \\ r^*(\tau_2) - r_0^* \end{pmatrix}, \quad (36)$$

the matrix M_Φ , is given by

$$M_\Phi = \begin{pmatrix} \frac{\delta^2 \Phi(b, b^*, r, r^*)}{\delta b(\tau_1) \delta b(\tau_2)} & \frac{\delta^2 \Phi(b, b^*, r, r^*)}{\delta b(\tau_1) \delta b^*(\tau_2)} & \frac{\delta^2 \Phi(b, b^*, r, r^*)}{\delta b(\tau_1) \delta r(\tau_2)} & \frac{\delta^2 \Phi(b, b^*, r, r^*)}{\delta b(\tau_1) \delta r^*(\tau_2)} \\ \frac{\delta^2 \Phi(b, b^*, r, r^*)}{\delta b^*(\tau_1) \delta b(\tau_2)} & \frac{\delta^2 \Phi(b, b^*, r, r^*)}{\delta b^*(\tau_1) \delta b^*(\tau_2)} & \frac{\delta^2 \Phi(b, b^*, r, r^*)}{\delta b^*(\tau_1) \delta r(\tau_2)} & \frac{\delta^2 \Phi(b, b^*, r, r^*)}{\delta b^*(\tau_1) \delta r^*(\tau_2)} \\ \frac{\delta^2 \Phi(b, b^*, r, r^*)}{\delta r(\tau_1) \delta b(\tau_2)} & \frac{\delta^2 \Phi(b, b^*, r, r^*)}{\delta r(\tau_1) \delta b^*(\tau_2)} & \frac{\delta^2 \Phi(b, b^*, r, r^*)}{\delta r(\tau_1) \delta r(\tau_2)} & \frac{\delta^2 \Phi(b, b^*, r, r^*)}{\delta r(\tau_1) \delta r^*(\tau_2)} \\ \frac{\delta^2 \Phi(b, b^*, r, r^*)}{\delta r^*(\tau_1) \delta b(\tau_2)} & \frac{\delta^2 \Phi(b, b^*, r, r^*)}{\delta r^*(\tau_1) \delta b^*(\tau_2)} & \frac{\delta^2 \Phi(b, b^*, r, r^*)}{\delta r^*(\tau_1) \delta r(\tau_2)} & \frac{\delta^2 \Phi(b, b^*, r, r^*)}{\delta r^*(\tau_1) \delta r^*(\tau_2)} \end{pmatrix} \Big|_{b_0, b_0^*, r_0, r_0^*}. \quad (37)$$

Substituting this expansion given by Eq. (36) in Eq. (26) we have that

$$Z_A = e^{N\Phi(b_0^*, b_0)} \int [d\eta(r)] [d\eta(b)]$$

$$\exp \left(\frac{1}{2} \int_0^\beta d\tau_1 d\tau_2 (b(\tau_1), b^*(\tau_1), r(\tau_1), r^*(\tau_1)) M_\Phi \begin{pmatrix} b(\tau_2) \\ b^*(\tau_2) \\ r(\tau_2) \\ r^*(\tau_2) \end{pmatrix} \right). \quad (38)$$

To obtain the last expression, we applied the transformation $b(\tau) \rightarrow (b(\tau) + b_0)/\sqrt{N}$, $b^*(\tau) \rightarrow (b^*(\tau) + b_0^*)/\sqrt{N}$, $r(\tau) \rightarrow (r(\tau) + r_0)/\sqrt{N}$ and $r^*(\tau) \rightarrow (r^*(\tau) + r_0^*)/\sqrt{N}$ to the fields in the functional integral. Terms of higher order in the fields in the expansion given by Eq. (36) generates terms in Eq. (38) suppressed by powers of $1/\sqrt{N}$. Therefore in the thermodynamic limit the Eq. (36) is a good approximation.

The free energy $F = -\frac{1}{N\beta} \ln Z$, in the thermodynamic limit, $N \rightarrow \infty$ is obtained from Eq. (24) using Eq. (38). In the normal phase $F - F_0 = 0$, where F_0 is the free energy for the non-interacting model. Finally, in the superradiant phase we have that

$$F - F_0 = \frac{\omega_0 (\Omega_\Delta^2 - \Omega^2)}{4((g_1 + g_2)^2 - \lambda \omega_0)} - \frac{1}{\beta} \ln \left(\frac{\cosh \left(\frac{\beta \Omega_\Delta}{2} \right)}{\cosh \left(\frac{\beta \Omega}{2} \right)} \right). \quad (39)$$

5 Conclusions

Functional methods provide an efficient tool to compute the thermodynamics functions and critical properties of spin models. In the limit of zero temperature the results obtained by this means may be compared with the ones obtained directly from an exact or approximate quantum mechanical solution of the system involved.

In this paper we compute the free energy of the full Dicke model with dipole-dipole interaction at temperature β^{-1} and determine the critical temperature for the superradiant phase transition. To handle the non-Gaussian terms resulting from the presence of the dipole-dipole interaction we introduce a single auxiliary field which allow to compute an effective action in terms of two bosonic fields. This action showed to be manageable using the saddle point approximation and in this way we determine the corrections to the critical temperature of the full Dicke model computed in [17, 27]. This approach improves the approximation scheme of Ref. [37] where no correction was detected. For the full Dicke model with dipole-dipole interaction addressed in this paper the critical coupling associated with the quantum superradiant phase transition were obtained in Refs. [29, 30] following [3] by performing a Holstein Primakoff and then the diagonalization of the effective Hamiltonian so obtained. At zero temperature our result reproduces the critical coupling of the quantum phase transition discussed in Ref. [29, 30].

In Refs. [3, 29, 30] the relation of the entanglement of the collective modes of the Dicke models and the quantum phase transition was discussed. Since for finite temperature the thermal interaction usually induces decoherence in quantum systems it could be interesting to link this analysis with the thermodynamic behavior of the system. To this end, the excitation spectrum of the system should be computed and understood. This subject, as well as the effect of introducing disorder [39] in the dipole-dipole coupling are being investigated by the authors.

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