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Constructing “non-Kerrness” on compact domains

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Abstract

Given a compact domain of a 3-dimensional hypersurface on a vacuum spacetime, a scalar (the “non-Kerrness”) is constructed by solving a Dirichlet problem for a second order elliptic system. If such scalar vanishes, and a set of conditions are satisfied at a point, then the domain of dependence of the compact domain is isometric to a portion of a member of the Kerr family of solutions to the Einstein field equations. This construction is expected to be of relevance in the analysis of numerical simulations of black hole spacetimes.

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1 Introduction

The present article is concerned with the problem of measuring how different a given initial data set for the Einstein vacuum field equations is from a Kerr initial data set. In [1, 2, 4] this problem has been addressed by the construction of a geometric invariant —the *non-Kerrness*— on hypersurfaces with at least one asymptotic end. This setting, although convenient for theoretical discussions, is not ideal for numerical considerations where very often one needs to make use of bounded computational domains on an hypersurface. The purpose of this article is to provide a construction of non-Kerrness on bounded domains.

The construction of the non-Kerrness given in [1, 2, 4] is based on a very strong property of the Kerr spacetime: the existence of a *Killing-Yano tensor*. A Killing-Yano tensor is an antisymmetric, rank 2 tensor $Y_{\mu\nu}$ satisfying the equation

$$\nabla_{(\mu}Y_{\nu)\lambda} = 0.$$

Let $\zeta_\mu \equiv \epsilon_\mu{}^{\nu\lambda\rho}\nabla_\nu Y_{\lambda\rho}$ denote the codifferential of $Y_{\mu\nu}$. If $Y_{\mu\nu}$ is a Killing-Yano tensor, then ζ_μ satisfies the Killing vector equation. As discussed in [8], the theory of Killing-Yano tensors can be conveniently reformulated in terms of the existence of a valence 2 Killing spinor, $\kappa_{AB} = \kappa_{(AB)}$, satisfying the equation

$$\nabla_{A'(A}\kappa_{BC)} = 0. \tag{1}$$

The spinorial analogue of the codifferential ζ_μ is the spinor $\xi_{AA'} \equiv \nabla_{A'}{}^B\kappa_{AB}$. In general, if κ_{AB} satisfies the Killing spinor equation, then $\xi_{AA'}$ is a complex Killing vector. In the case of the Kerr spacetime the real and imaginary parts of this vector are proportional —and by multiplying

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with a complex constant, the imaginary part can be set to zero. In general, the existence of a Killing-Yano tensor is equivalent to existence of a Killing spinor κ_{AB} such that $\xi_{AA'}$ is real.

Killing spinors (or alternatively, Killing-Yano tensors) are useful in the characterisation of the Kerr spacetime as the existence of one of these objects severely restricts the algebraic type of the curvature of the spacetime. Furthermore, the implied existence of a real Killing vector allows to make contact with the theory of the Mars-Simon tensor —see [5, 6]. As a result of this analysis, it is possible to provide a purely local characterisation of the Kerr spacetime —see Theorem 1 in [6]. Alternatively, one can obtain a somewhat simpler characterisation if one combines local and global requirements: the existence of a stationary, asymptotically flat region with non-vanishing mass —see Theorem 2 in [6]. Precisely this result was used in the constructions of non-Kerrness on non-bounded 3-manifolds described in [1, 2, 4].

The construction of the non-Kerrness on bounded domains discussed in the present article makes use of the local spacetime characterisation of the Kerr spacetime given in Theorem 1 of [6] to show that if the non-Kerrness vanishes on some 3-dimensional bounded domain, then the initial data prescribed on that region is locally isometric to data for a Kerr spacetime. We expect that this result will be of utility to assess in a quantitative way how a given numerically constructed dynamical black hole spacetime evolves towards a stationary state described by the Kerr spacetime. In the process, it will be shown that the general theory of Killing spinor initial data sets used in [1, 2, 4] can be simplified.

Overview of the article

The content of this article is structured as follows: Section 2 provides a summary of key properties of spacetimes with Killing spinors. It also contains a reformulation in terms of spinors of a local characterisation of the Kerr spacetime by M. Mars. Finally, a brief discussion of the notion of Killing spinor candidates is provided. Section 3 provides a brief summary of the theory of the Killing spinor initial data equations which encode the existence of a Killing vector at the level of initial data. Section 4 gives a brief discussion of the notion of approximate Killing spinors, the approximate Killing spinor equations and the elliptic theory required to discuss the existence of solutions to this equation with Dirichlet boundary conditions. Section 5 provides a result regarding the realness of the Killing vector constructed from the Killing spinor, which will be required in our subsequent discussion. Section 6 provides our main result: a theorem which characterises Kerr initial data on a compact domain of a 3-dimensional manifold using the notion of approximate Killing spinors. Finally Section 7 provides some concluding remarks. There is an appendix (Appendix A) providing a proof of a theorem discussed in Section 3, which tells that one of the Killing spinor initial data equations can be omitted.

2 A local spacetime characterisation of the Kerr spacetime

Given a spacetime $(\mathcal{M}, g_{\nu\nu})$, let $C_{\mu\nu\lambda\rho}$ denote the Weyl tensor of the metric $g_{\mu\nu}$. Let $C_{AA'BB'CC'DD'}$ denote the spinorial counterpart of $C_{\mu\nu\lambda\rho}$. There exists a completely symmetric spinor Ψ_{ABCD} such that:

$$C_{AA'BB'CC'DD'} = \Psi_{ABCD}\bar{\epsilon}_{A'B'}\bar{\epsilon}_{C'D'} + \bar{\Psi}_{A'B'C'D'}\epsilon_{AB}\epsilon_{CD}.$$

We recall that the two classical invariants of the Weyl tensor are given by:

$$\mathcal{I} \equiv \tfrac{1}{2}\Psi_{ABCD}\Psi^{ABCD}, \qquad \mathcal{J} \equiv \tfrac{1}{6}\Psi_{ABCD}\Psi^{CDEF}\Psi_{EF}{}^{AB}.$$

2.1 Properties of spacetimes with Killing spinors

In what follows it is assumed one has a region $\mathcal{N}$ of the spacetime $(\mathcal{M}, g_{\mu\nu})$ where one has a solution κ_{AB} of the Killing spinor equation (1). It is then well known that the spacetime must be of Petrov type D, N or O —see e.g. [10]. In the sequel we will concentrate our attention to the case when $(\mathcal{M}, g_{\mu\nu})$ is of Petrov type D. In such case, there exist spinors α_A, β_A, $\alpha_Q\beta^Q = 1$, such that

$$\Psi_{ABCD} = \psi\alpha_{(A}\alpha_A\beta_C\beta_{D)}, \tag{2}$$

where

$$\psi \equiv -18\mathcal{J}/\mathcal{I}. \tag{3}$$

The valence 2 Killing spinor is then given by

$$\kappa_{AB} = \psi^{-1/3}\alpha_{(A}\beta_{B)}. \tag{4}$$

As in the introduction, let

$$\xi_{AA'} \equiv \nabla^Q{}_{A'}\kappa_{AQ}.$$

Then $\xi_{AA'}$ is (in general) a complex solution to Killing equation

$$\nabla_{AA'}\xi_{BB'} + \nabla_{BB'}\xi_{AA'} = 0.$$

If $\xi_{AA'}$ is real, we define the Killing form of $\xi_{AA'}$ by

$$F_{AA'BB'} \equiv \tfrac{1}{2}\left(\nabla_{AA'}\xi_{BB'} - \nabla_{BB'}\xi_{AA'}\right) = \nabla_{AA'}\xi_{BB'}.$$

Vacuum spacetimes admitting a Killing spinor such that $\xi_{AA'}$ is real will be said to belong to the *generalised Kerr-NUT class. In the rest of this section it is assumed that* $(\mathcal{M}, g_{\mu\nu})$ *is a generalised Kerr-NUT spacetime.*

As a consequence of the symmetries of $F_{AA'BB'}$, there exists a symmetric, valence 2 spinor ϕ_{AB} such that

$$F_{AA'BB'} = \phi_{AB}\bar{\epsilon}_{A'B'} + \bar{\phi}_{A'B'}\epsilon_{AB}, \qquad \phi_{AB} \equiv \tfrac{1}{2}F_{AQ'B}{}^{Q'}.$$

Using (4) one finds the following expressions for $\xi_{AA'}$, and ϕ_{AB} in terms of ψ and the principal spinors:

$$\xi_{AA'} = \tfrac{1}{2}\psi^{-4/3}\left(\alpha_A\beta^Q + \beta_A\alpha^Q\right)\nabla_{QA'}\psi,$$
$$\phi_{AB} = \tfrac{1}{4}\psi^{2/3}\alpha_{(A}\beta_{B)}.$$

For later use, we introduce the *norm of the Killing form*, the *norm of the Killing vector* and the *twist 1-form* via

$$\Phi \equiv \phi_{PQ}\phi^{PQ}, \qquad \lambda \equiv \xi_{AA'}\xi^{AA'}, \qquad \omega_{AA'} \equiv \epsilon_{AA'BB'CC'DD'}\xi^{BB'}\nabla^{CC'}\xi^{DD'},$$

where

$$\epsilon_{AA'BB'CC'DD'} \equiv \mathrm{i}\left(\epsilon_{AC}\epsilon_{BD}\bar{\epsilon}_{A'D'}\bar{\epsilon}_{B'C'} - \epsilon_{AD}\epsilon_{BC}\bar{\epsilon}_{A'C'}\bar{\epsilon}_{B'D'}\right)$$

is the spinorial counterpart of the completely antisymmetric volume form, $\epsilon_{\mu\nu\lambda\rho}$, of $g_{\mu\nu}$. Locally, $\omega_{AA'}$ is exact, so that there exists ω (the *twist potential*) such that $\omega_{AA'} = \nabla_{AA'}\omega$. Using λ and ω we define the *Ernst potential*, σ, by

$$\sigma \equiv \lambda + \mathrm{i}\omega.$$

Using expressions (2) and (4) one readily finds the following expressions for Φ, λ and $\omega_{AA'}$:

$$\Phi = -\tfrac{1}{32}\psi^{4/3}, \tag{5a}$$
$$\lambda = -\tfrac{1}{4}\psi^{-8/3}\nabla_{AA'}\psi\nabla^{AA'}\psi, \tag{5b}$$
$$\omega_{AA'} = \mathrm{Im}(4\phi_A{}^B\xi_{BA'}), \tag{5c}$$

In order to obtain an expression for the Ernst potential in terms of ψ, we notice the identities

$$\nabla_{AA'}(\psi^{1/3}) = \tfrac{16}{3}\phi_A{}^B\xi_{BA'}, \tag{6a}$$
$$\nabla_{AA'}\nabla^{AA'}\psi = \psi^2 + \tfrac{2}{3}\psi^{-1}\nabla_{AA'}\psi\nabla^{AA'}\psi, \tag{6b}$$
$$\nabla_{AA'}\lambda = \mathrm{Re}(4\phi_A{}^B\xi_{BA'}). \tag{6c}$$

Thus, one concludes that

$$\nabla_{AA'}\lambda + \mathrm{i}\omega_{AA'} = \tfrac{3}{4}\nabla_{AA'}\psi^{1/3}.$$

The latter can be integrated to give

$$\sigma - c = \tfrac{3}{4}\psi^{1/3}, \tag{7}$$

with c a complex constant. The real part of c is not arbitrary: using equations (6a) and (6c) one obtains that

$$\mathrm{Re}(c) = \lambda - \tfrac{3}{4}\mathrm{Re}(\psi^{1/3}). \tag{8}$$

2.2 A local characterisation of Kerr

The analysis of the *so called* Mars-Simon tensor presented in [5, 6] gives rise to a local characterisation of the Kerr spacetime among the class of spacetimes endowed with a Killing vector. This characterisation involves the Weyl tensor, the Killing form and the Ernst potential —see Theorem 1 in [6]. For the convenience of our subsequent analysis, here we present a slight generalisation of this result in the language of spinors.

Theorem 1 (Mars, 2000). *Let $(\mathcal{M}, g_{\mu\nu})$ be a smooth, vacuum spacetime admitting a Killing vector ξ^μ. Let $\mathcal{N} \subset \mathcal{M}$ be a non-empty open subset satisfying:*

(i) There is a point $p \in \mathcal{N}$ where $\Phi \neq 0$.

(ii) The Killing form and the Weyl tensor are related by

$$\Psi_{ABCD} = \varpi \phi_{(AB}\phi_{CD)},$$

where ϖ is a complex scalar function.

Then there exist two complex constants $\tilde{c}$ and k such that

$$\varpi = -\frac{12}{\tilde{c} - \sigma}, \qquad \Phi = -k(\tilde{c} - \sigma)^4, \qquad \text{on } \mathcal{N}.$$

If, in addition, $\mathrm{Re}(\tilde{c}) > 0$ and $k = \mathrm{Re}(k) > 0$ then $(\mathcal{N}, g_{\mu\nu})$ is locally isometric to a portion of the Kerr spacetime.

Remark. This result follows from —and is equivalent to— Theorem 1 in [6] by introducing a different normalisation in the Killing vector and exploiting the ambiguity in the definition of the Ernst potential[1].

2.3 Killing spinor candidates

The construction of non-Kerrness on a bounded domain requires the notion of a *Killing spinor candidate* introduced in [4]:

Definition 1. *Let $(\mathcal{M}, g_{\mu\nu})$ be a vacuum spacetime. Consider $\mathcal{N} \subset \mathcal{M}$ and on $\mathcal{N}$ a symmetric spinor ζ_{AB} satisfying*

$$\zeta_{AB} \neq 0, \quad \psi^{-1}\Psi_{PQRS}\zeta^{PQ}\zeta^{RS} - \tfrac{1}{6}\zeta_{PQ}\zeta^{PQ} \neq 0 \quad \text{on } \mathcal{N}.$$

The symmetric spinor given by

$$\breve{\kappa}_{AB} = \psi^{-1/3}\Xi^{-1/2}\left(\psi^{-1}\Psi_{ABPQ}\zeta^{PQ} - \tfrac{1}{6}\zeta_{AB}\right), \tag{9}$$

with

$$\Xi \equiv \psi^{-1}\Psi_{PQRS}\zeta^{PQ}\zeta^{RS} - \tfrac{1}{6}\zeta_{PQ}\zeta^{PQ},$$

will be called the ζ_{AB}-Killing spinor candidate on $\mathcal{N}$.The scalar ψ is obtained from the Weyl spinor Ψ_{ABCD} using formula (3)*.*

Formula (10) can be evaluated for any vacuum spacetime $(\mathcal{M}, g_{\mu\nu})$. The name Killing spinor candidate is justified by the following result also proved in [4]:

Proposition 2. *Let $(\mathcal{M}, g_{\mu\nu})$ be a vacuum spacetime. If on $\mathcal{N} \subset \mathcal{M}$, the spacetime is of Petrov type D and ζ_{AB} is a symmetric spinor satisfying*

$$\zeta_{AB} \neq 0, \quad \psi^{-1}\Psi_{PQRS}\zeta^{PQ}\zeta^{RS} - \tfrac{1}{6}\zeta_{PQ}\zeta^{PQ} \neq 0 \quad \text{on } \mathcal{N},$$

then

$$\kappa_{AB} = \psi^{-1/3}\Xi^{-1/2}\left(\psi^{-1}\Psi_{ABPQ}\zeta^{PQ} - \tfrac{1}{6}\zeta_{AB}\right) \tag{10}$$

is a Killing spinor on $\mathcal{N}$. The formula (10) *is independent of the choice of ζ_{AB}.*

[1] We thank M. Mars for pointing this out to us.

3 The Killing spinor initial data equations

Key for the construction of the non-Kerrness discussed in [1, 2, 4], is the idea of how to encode that the development of an initial data set $(\mathcal{S}, h_{ij}, K_{ij})$ admits a solution to the Killing spinor equation (1). This question can be addressed by means of the space-spinor decomposition of the Killing spinor equation (1). For a more detailed description see [2].

The space-spinor decomposition of equation (1) renders a set of 3 conditions intrinsic to the hypersurface $\mathcal{S}$:

$$\xi_{ABCD} = 0, \tag{11a}$$

$$\Psi_{(ABC}{}^{F}\kappa_{D)F} = 0, \tag{11b}$$

$$3\kappa_{(A}{}^{E}\nabla_{B}{}^{F}\Psi_{CD)EF} + \Psi_{(ABC}{}^{F}\xi_{D)F} = 0, \tag{11c}$$

where we have written

$$\xi_{ABCD} \equiv \nabla_{(AB}\kappa_{CD)}, \quad \xi_{AB} \equiv \frac{3}{2}\nabla_{(A}{}^{D}\kappa_{B)D}, \quad \xi \equiv \nabla^{PQ}\kappa_{PQ}, \tag{12}$$

and ∇_{AB} denotes the spinorial version of the Sen connection associated to the pair (h_{ij}, K_{ij}) of intrinsic metric and extrinsic curvature. It can be expressed in terms of the spinorial counterpart, D_{AB} of the Levi-Civita connection of the 3-metric h_{ij}, and the spinorial version, $K_{ABCD} = K_{(AB)(CD)} = K_{CDAB}$, of the second fundamental form K_{ij}. For example, given a valence 1 spinor π_A one has that

$$\nabla_{AB}\pi_C = D_{AB}\pi_C + \tfrac{1}{2}K_{ABC}{}^{Q}\pi_Q,$$

with the obvious generalisations to higher valence spinors. In equations (11b)-(11c), the spinor Ψ_{ABCD} denotes the restriction to the hypersurface $\mathcal{S}$ of the self-dual Weyl spinor. Crucially, the spinor Ψ_{ABCD} can be written entirely in terms of initial data quantities via the relations:

$$\Psi_{ABCD} = E_{ABCD} + \mathrm{i}B_{ABCD},$$

with

$$E_{ABCD} = -r_{(ABCD)} + \tfrac{1}{2}\Omega_{(AB}{}^{PQ}\Omega_{CD)PQ} - \tfrac{1}{6}\Omega_{ABCD}K,$$
$$B_{ABCD} = -\mathrm{i}\, D^{Q}{}_{(A}\Omega_{BCD)Q},$$

and where $\Omega_{ABCD} \equiv K_{(ABCD)}$, $K \equiv K_{PQ}{}^{PQ}$. Furthermore, the spinor r_{ABCD} is the Ricci tensor, r_{ij}, of the 3-metric h_{ij}.

In Appendix A it is shown that the second algebraic condition (11c) is, in fact, redundant and a consequence of the conditions (11a)-(11b). In particular it follows then that

Theorem 3. *Let equations* (11a)-(11b) *be satisfied for a symmetric spinor $\mathring{\kappa}_{AB}$ on an open set $\mathcal{U} \subset \mathcal{S}$. Then the Killing spinor equation* (1) *has a solution, κ_{AB}, on the future domain of dependence $\mathcal{D}^{+}(\mathcal{U})$.*

Remark. This means that the term I_2 in the invariants of [1, 2, 4] can be omitted.

4 Approximate Killing spinors

4.1 The approximate Killing spinor equation

The *spatial Killing spinor equation* (11a) can be regarded as a (complex) generalisation of the conformal Killing vector equation. As in the case of the conformal Killing equation, equation (11a) is clearly overdetermined. However, one can construct a generalisation of the equation which under suitable circumstances can always be expected to have a solution. One can do this by composing the operator in (11a) with its formal adjoint —see [1]. This procedure renders the equation

$$\mathbf{L}\kappa_{CD} \equiv \nabla^{AB}\nabla_{(AB}\kappa_{CD)} - \Omega^{ABF}{}_{(A}\nabla_{|DF|}\kappa_{B)C} - \Omega^{ABF}{}_{(A}\nabla_{B)F}\kappa_{CD} = 0, \tag{13}$$

which will be called the *approximate Killing spinor equation.* One has the following result proved in [2]:

Lemma 4. *The operator* $\mathbf{L}$ *defined by the left hand side of equation* (13) *is a formally self-adjoint elliptic operator.*

In order to discuss the solvability of equation on a bounded domain, $\mathcal{U} \subset \mathcal{S}$, (13) one has to supplement it with appropriate boundary conditions. On $\partial\mathcal{U}$ we will consider the homogeneous Dirichlet operator $\mathbf{B}$ given by

$$\mathbf{B}u(y) = u(y), \quad y \in \partial\mathcal{S}.$$

The combined operator $(\mathbf{L}, \mathbf{B})$ satisfies the so-called *Lopatinski-Shapiro compatibility conditions* —see [11] for detailed definitions and discussion. Thus, $(\mathbf{L}, \mathbf{B})$ is L-elliptic —see again [11], Theorem 10.7. Moreover, one has the following theorem —see also [7].

Theorem 5. *Let* $\mathbf{L}$ *denote a smooth second order homogeneous elliptic operator on* $\mathcal{U}$*. Furthermore, let* $\partial\mathcal{U}$ *be smooth and let* $\mathbf{B}$ *denote the Dirichlet boundary operator. Then for* $s \geq 2$ *the map*

$$(\mathbf{L}, \mathbf{B}) : H^s(\mathcal{U}) \rightarrow H^{s-2}(\mathcal{U}) \times H^{s-1/2}(\partial\mathcal{U})$$

is Fredholm. Furthermore, the boundary value problem

$$\mathbf{L}u(x) = f(x), \quad f \in H^0(\mathcal{U}), \quad x \in \mathcal{U},$$
$$u(y) = g(y), \quad g \in H^0(\partial\mathcal{U}), \quad y \in \partial\mathcal{U},$$

has a solution $u \in H^2(\mathcal{U})$ *if*

$$\int_{\mathcal{U}} f \cdot \nu d\mu = 0,$$

for all $\nu \in H^2(\mathcal{U})$ *such that*

$$\mathbf{L}^*\nu(x) = 0, \quad x \in \mathcal{U},$$
$$\nu(y) = 0, \quad y \in \partial\mathcal{U}.$$

Remark. If $\mathbf{L}$ has smooth coefficients and $\mathbf{L}u = 0$, then it follows from Weyl's Lemma —see e.g. [11]— that if a solution to the boundary value problem exists and the boundary data is smooth, then the solution must be, in fact, smooth —this is the so-called elliptic regularity.

In what follows let $n_{AB} = n_{(AB)}$ denote the spinorial counterpart of the inward pointing normal to $\partial\mathcal{U}$. As a consequence of our signature conventions one has that $n_{PQ}n^{PQ} = -1$. Theorem 5 will be used to establish the existence of solutions to the approximate Killing spinor equation (13) with Dirichlet boundary data given by the n_{AB}-Killing spinor candidate. In order to ensure that the Killing spinor candidate can be constructed on $\partial\mathcal{U}$, we define the set

$$\mathcal{Q} \equiv \{z \in \mathbb{C} \mid z = \Xi(p), \;\; p \in \partial\mathcal{U}\},$$

and make the assumption:

Assumption 1. *The initial data set* $(\mathcal{S}, h_{ij}, K_{ij})$ *and the compact set* $\mathcal{U}$ *are such that* Ξ *is a smooth function over* $\partial\mathcal{U}$ *satisfying*

(i) $0 \notin \mathcal{Q}$*;*

(ii) $\mathcal{Q}$ *does not encircle the point* $z = 0$*.*

As a consequence of this assumption one can choose a cut of the square root function on the complex plane such that $\Xi^{1/2}(p)$ is smooth for all $p \in \partial\mathcal{U}$.

One has the following result:

Proposition 6. *Let $(\mathcal{S}, h_{ij}, K_{ij})$ be an initial data set for the Einstein vacuum field equations. Furthermore, let $\mathcal{U} \subset \mathcal{S}$ be a compact subset with boundary $\partial\mathcal{S}$ satisfying Assumption 1. Then, there exists a unique smooth solution, κ_{AB}, to the approximate Killing spinor equation* (13) *with boundary value given by the n_{AB}-Killing spinor candidate given by equation* (9).

Proof. The proof of this result follows directly from the second part of Theorem 5. Notice that as the equation is is homogeneous, there is no potential obstruction to the existence of solutions and one does not need to verify the triviality of the Kernel of the adjoint operator as it is in the case with asymptotically Euclidean ends —see [1, 2, 4]. □

5 Reality of the Killing vector

As discussed in the introduction, the existence of a Killing spinor is not enough to single out the generalized Kerr-NUT family from the type D solutions. We also need that the Killing vector constructed from the Killing spinor is real. This section provides some tools to determine that.

5.1 Imaginary part of the Killing vector data

In what follows, let κ_{AB} solve the Killing spinor equation (1) in a spacetime domain $\mathcal{D}$, and let ξ and ξ_{AB} be defined as in (12). In this section we only study what happes in the domain $\mathcal{D}$. A computation using the suite `xAct` for `Mathematica` starting from equations (11a)-(11c) shows that

$$D_{AB}\mathrm{Im}(\xi^{AB}) = -\tfrac{1}{2}\mathrm{Im}(\xi)K, \tag{14a}$$

$$D_{(AB}\mathrm{Im}(\xi_{CD)}) = -\tfrac{1}{2}\mathrm{Im}(\xi)\Omega_{ABCD}. \tag{14b}$$

The equation (1) implies $\nabla\kappa_{AB} = -\frac{2}{3}\xi_{AB}$, where ∇ denotes the normal derivative $\tau^{AA'}\nabla_{AA'}$. Commuting derivatives and simplifying one obtains

$$\nabla\mathrm{Im}(\xi) = \mathrm{Im}(\xi^{AB})K_{AB}, \tag{15a}$$

$$\begin{aligned}\nabla\mathrm{Im}(\xi_{AB}) = {} & -\tfrac{1}{2}\mathrm{Im}(\xi)K_{AB} + \tfrac{1}{3}\mathrm{Im}(\xi_{AB})K + \Omega_{ABCD}\mathrm{Im}(\xi^{CD}) \\ & -D_{AB}\mathrm{Im}(\xi) - \mathrm{Im}(\xi_{(A}{}^{C})K_{B)C}. \end{aligned}\tag{15b}$$

Making a space spinor split of $\xi_{AA'} = \nabla^{B}{}_{A'}\kappa_{AB}$ and using equation (1), we find

$$\mathrm{Im}(\xi_{AA'}) = \tfrac{1}{2}\mathrm{Im}(\xi)\tau_{AA'} - \mathrm{Im}(\xi_{AB})\tau^{B}{}_{A'}.$$

After differentiating once more, making a further space spinor split, and using equation (14a) and (14b) we have:

Lemma 7. *Let κ_{AB} solve the Killing spinor equation* (1) *in a spacetime domain $\mathcal{D}$. Assume that*

$$\mathrm{Im}(\xi) = 0, \quad \mathrm{Im}(\xi_{AB}) = 0, \quad D_{AB}\mathrm{Im}(\xi) = 0, \quad D_{(A}{}^{C}\mathrm{Im}(\xi_{B)C}) = 0 \tag{16}$$

at a point $p \in \mathcal{D}$. Then $\mathrm{Im}(\xi_{AA'}) = 0$ and $\nabla_{AA'}\mathrm{Im}(\xi_{BB'}) = 0$ at p.

6 The non-Kerrness invariant

The approximate Killing spinor κ_{AB} obtained in Proposition 6 will now be used, in the spirit of [1], to construct a geometric invariant measuring the non-Kerrness of the initial data on the compact set $\mathcal{U}$. More precisely, we define

$$I \equiv \int_{\mathcal{U}} \nabla_{(AB}\kappa_{CD)}\widehat{\nabla^{AB}\kappa^{CD}}\mathrm{d}\mu + \int_{\mathcal{U}} \Psi_{(ABC}{}^{P}\kappa_{D)P}\widehat{\Psi^{ABCQ}\kappa^{D}{}_{Q}}\mathrm{d}\mu. \tag{17}$$

6.1 The main result

The main result of our analysis is the following theorem:

Theorem 8. *Let $(\mathcal{S}, h_{ij}, K_{ij})$ be an initial data set for the Einstein vacuum field equations, and let $\mathcal{U} \subset \mathcal{S}$ be a compact connected subset with boundary $\partial\mathcal{U}$ satisfying Assumption 1. Let I be as defined by equation (17) where κ_{AB} is given as the only solution to equation (13) with boundary behaviour given by the n_{AB}-Killing spinor candidate $\breve{\kappa}_{AB}$ where n_{AB} is the inward pointing normal to $\partial\mathcal{U}$. If:*

(i) $I = 0$*;*

(ii) there exists a point on $\mathcal{U}$ for which

$$\mathrm{Im}(\xi) = 0, \quad \mathrm{Im}(\xi_{AB}) = 0, \quad D_{AB}\mathrm{Im}(\xi) = 0, \quad D_{(A}{}^{C}\mathrm{Im}(\xi_{B)C}) = 0;$$

then the future domain of dependence, $D^+(\mathcal{U})$, of $\mathcal{U}$ is isometric to a subset of a generalised Kerr-NUT spacetime. If, in addition:

(iii) there exists a point on $\mathcal{U}$ for which $\Phi \neq 0$;

(iv) there exists a point on $\mathcal{U}$ for which

$$\lambda - \tfrac{3}{4}\mathrm{Re}(\psi^{1/3}) > 0, \tag{18}$$

then $D^+(\mathcal{U})$ is isometric to a portion of a Kerr spacetime.

Remark 1. If $D^+(\mathcal{U})$ is isometric to a portion of a Kerr spacetime, the conditions *(ii)*, *(iii)* and *(iv)* are satisfied on every point. Hence, the choice of which point to check the conditions in, is not important.

Remark 2. If $\mathcal{U}$ is not connected, the conditions *(ii)*, *(iii)* and *(iv)* needs to be checked for each connected component of $\mathcal{U}$.

Remark 3. The conditions *(iii)* and *(iv)* can be replaced by an asymptotic flatness condition.

Proof. If $I = 0$ then it follows from our smoothness assumptions that equations (11a)-(11b) are satisfied on $\mathcal{U}$. Hence, from Theorem 3 it follows that $D^+(\mathcal{U})$ will contain a Killing spinor κ_{AB}. Then $\xi_{AA'}$ is the spinor counterpart of a (possibly complex) Killing vector. Now, using assumption *(ii)* together with Lemma 7 gives $\mathrm{Im}(\xi_{AA'}) = 0$ and $\nabla_{AA'}\mathrm{Im}(\xi_{BB'}) = 0$ at a point. Using a standard result about Killing spinors (see Appendix C.3 in [9]), one concludes that $\mathrm{Im}(\xi) = \mathrm{Im}(\xi_{AB}) = 0$ everywhere on $D^+(\mathcal{U})$ so that $\xi_{AA'}$ is, in fact, real. Thus, $D^+(\mathcal{U})$ is isometric to a portion of a generalised Kerr-NUT spacetime.

As in the main text, let ϕ_{AB} denote the spinorial counterpart of the Killing form for of $\xi_{AA'}$. From the discussion in Subsection 2.1 one concludes that

$$\Psi_{ABCD} = \varpi\phi_{(AB}\phi_{CD)},$$

for some function ϖ. Now, if $\Phi \neq 0$ on $\mathcal{U}$, then using Theorem 1, one has that

$$\varpi = -\frac{12}{\tilde{c} - \sigma}, \qquad \Phi = -k(\tilde{c} - \sigma)^4,$$

for some (possibly complex) constants $\tilde{c}$ and k. Using formulae (7) and (5a), one can identify the constants c and $\tilde{c}$ and set $k = \frac{8}{81}$. Evaluating c at the point where (18) holds one obtains that $\mathrm{Re}(c) > 0$. Thus, the hypothesis of Theorem 1 hold and one concludes that $D^+(\mathcal{U})$ is isometric to a portion of the Kerr spacetime. □

7 Conclusions and discussion

In this paper we have devised a way to measure the deviation from Kerr initial data for bounded domains. The main result is presented in Theorem 8. In the previous papers [1, 2, 4], a similar result was obtained for cases where the computational domain reached spatial infinity. For such cases the asymptotic behaviour of the approximate Killing spinor could be specified in a way that helped us to exclude all other Petrov type D solutions. Therefore we could conclude that the data was Kerr data if and only if $I = 0$. As the present paper deals with bounded domains, we constructed the boundary data for the approximate Killing spinor from the curvature. The drawback is that this gives $I = 0$ for all type D solutions. Therefore, one requires conditions *(ii)*, *(iii)*, *(iv)* in Theorem 8 to single out the Kerr solution. An effort was put into formulating the conditions so they can be verified at a single arbitrarily chosen point of the computational domain. Furthermore, we have shown that a part of the invariant constructed in [1, 2, 4] can be omitted in the case of a bounded domain as well the unbounded case.

The results of this paper can be used to numerically evaluate how much any slice of a spacetime deviates from Kerr data. This gives a tool to quantify decay towards Kerr data for a numerically evolved spacetime. A project along these lines have been initiated.

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A Redundancy of the second algebraic condition

The purpose of the present appendix is to prove the assertion made in Theorem 3 that the second algebraic condition given by equation (11c) is a consequence of the conditions (11a) and (11b). As a consequence of this result, the conditions required on an initial data set to have a development with a valence 2 Killing spinor become completely analogue to those required to have a valence 1 Killing spinor —see e.g. [3].

The analysis in this appendix proceeds by discussing the various possible algebraic types that the spinor κ_{AB} can have. Our first result is the following:

Lemma 9. *Assume that the symmetric spinor* κ_{AB} *satisfies*

$$\kappa_{AB}\kappa^{AB} \neq 0, \quad \nabla_{(AB}\kappa_{CD)} = 0, \quad \Psi_{(ABC}{}^{F}\kappa_{D)F} = 0,$$

on an open subset $\mathcal{U} \subset \mathcal{S}$*. Then the algebraic condition* (11c) *is satisfied on* $\mathcal{U}$*.*

Proof. The condition $\kappa_{AB}\kappa^{AB} \neq 0$ allows us to choose a spin dyad (o_A, ι_A) and a scalar field $\varkappa$ such that $o_A\iota^A = 1$ and $\kappa_{AB} = e^{\varkappa} o_{(A}\iota_{B)}$. Similarly, the condition $\Psi_{(ABC}{}^{F}\kappa_{D)F} = 0$ implies that there is a scalar field ψ such that $\Psi_{ABCD} = \psi o_{(A}o_B\iota_C\iota_{D)}$.

In the next step we decompose the equation $\nabla_{(AB}\kappa_{CD)} = 0$ into its various components to obtain:

$$o^A o^B o^C \nabla_{AB} o_C = 0, \tag{19a}$$
$$o^A \iota^B o^C \nabla_{AB} o_C = -\tfrac{1}{2} o^A o^B \nabla_{AB}\varkappa, \tag{19b}$$
$$o^A o^B \iota^C \nabla_{AB}\iota_C - \iota^A\iota^B o^C \nabla_{AB} o_C = 2 o^A \iota^B \nabla_{AB}\varkappa, \tag{19c}$$
$$o^A \iota^B \iota^C \nabla_{AB}\iota_C = \tfrac{1}{2}\iota^A\iota^B\nabla_{AB}\varkappa, \tag{19d}$$
$$\iota^A\iota^B\iota^C\nabla_{AB}\iota_C = 0. \tag{19e}$$

These equations imply, in turn, that

$$
\begin{aligned}
e^{-\varkappa}\xi_{AB} = & -3o_A o_B o^C \iota^D \iota^F \nabla_{CD}\iota_F - 3\iota_A\iota_B o^C \iota^D o^F \nabla_{CD} o_F \\
& + \tfrac{3}{2} o_{(A}\iota_{B)}(o^C o^D \iota^F \nabla_{CD}\iota_F + \iota^C \iota^D o^F \nabla_{CD} o_F). \qquad (20)
\end{aligned}
$$

Now, it is well known that the spacetime Bianchi identity $\nabla^Q{}_{A'}\Psi_{ABCQ} = 0$ implies the constraint

$$\nabla^{CD}\Psi_{ABCD} = 0, \qquad (21)$$

on $\mathcal{S}$. Substituting $\Psi_{ABCD} = \psi o_{(A} o_B \iota_C \iota_{D)}$ and contracting with combinations of o^A and ι^A one finds that the content of (21) is given by

$$o^A o^B \nabla_{AB}\psi = 6\psi o^A \iota^B o^C \nabla_{AB} o_C, \qquad (22a)$$

$$o^B \iota^C \nabla_{BC}\psi = \tfrac{3}{2}\psi \iota^A \iota^B o^C \nabla_{AB} o_C - \tfrac{3}{2}\psi o^A o^B \iota^C \nabla_{AB}\iota_C, \qquad (22b)$$

$$\iota^A \iota^B \nabla_{AB}\psi = -6\psi o^A \iota^B \iota^C \nabla_{AB}\iota_C. \qquad (22c)$$

Using equation (20) and the Bianchi identities (22a)-(22c) we get

$$
\begin{aligned}
\Psi_{(ABC}{}^F \xi_{D)F} + 3\kappa_{(A}{}^F \nabla_B{}^H \Psi_{CD)FH} = & \tfrac{3}{4} e^{\varkappa}\psi \iota_A \iota_B \iota_C \iota_D o^M o^P o^Q \nabla_{PQ} o_M \\
& - \tfrac{3}{4} e^{\varkappa}\psi o_A o_B o_C o_D \iota^M \iota^P \iota^Q \nabla_{PQ}\iota_M.
\end{aligned}
$$

Finally using the information about the derivatives of the spin dyad contained in equations (19a)-(19e) one finds that we get that the second algebraic condition, equation (11c), is satisfied on $\mathcal{U}$. Notice that in this argument one could have had $\psi = 0$. □

Using similar methods as before, one obtains the following lemma:

Lemma 10. *Assume that the symmetric spinor κ_{AB} satisfies*

$$\kappa_{AB}\kappa^{AB} = 0, \quad \kappa_{AB}\hat{\kappa}^{AB} \neq 0, \quad \nabla_{(AB}\kappa_{CD)} = 0, \quad \Psi_{(ABC}{}^F \kappa_{D)F} = 0,$$

on an open subset $\mathcal{U} \subset \mathcal{S}$. Then the algebraic condition (11c) *is satisfied on $\mathcal{U}$.*

Proof. By assumption the κ_{AB} is algebraically special —that is, it has repeated principal spinors. Thus, there exists o_A such that $\kappa_{AB} = o_A o_B$. We then complete o_A to a normalised spinor dyad (o_A, ι_A). The equation $\nabla_{(AB}\kappa_{CD)} = 0$ is equivalent to

$$o^A o^B o^C \nabla_{(AB} o_{C)} = 0, \qquad (23a)$$

$$o^A o^B \iota^C \nabla_{(AB} o_{C)} = 0, \qquad (23b)$$

$$o^A \iota^B \iota^C \nabla_{(AB} o_{C)} = 0, \qquad (23c)$$

$$\iota^A \iota^B \iota^C \nabla_{(AB} o_{C)} = 0. \qquad (23d)$$

These equations imply, in turn, that

$$\xi_{AB} = -2 o_A o_B \iota^C \nabla_{CD} o^D + 2 o_{(A}\iota_{B)} o^C \nabla_{CD} o^D. \qquad (24)$$

The condition $\Psi_{(ABC}{}^F \kappa_{D)F} = 0$ implies that there is a scalar field ψ such that $\Psi_{ABCD} = \psi o_{(A} o_B o_C o_{D)}$. Using this together with (24) yields

$$
\begin{aligned}
\Psi_{(ABC}{}^F \xi_{D)F} + 3\kappa_{(A}{}^F \nabla_B{}^H \Psi_{CD)FH} = & -3 o_A o_B o_C o_D \psi o^P o^Q \iota^R \nabla_{(PQ} o_{R)} \\
& + 3 o_{(A} o_B o_C \iota_{D)} \psi o^P o^Q o^R \nabla_{(PQ} o_{R)}. \qquad (25)
\end{aligned}
$$

Finally using the relations (23a)-(23d) we get that the second algebraic condition, equation (11c), is satisfied on $\mathcal{U}$. □

With the aid of the previous two lemmas, one can provide a proof of Theorem 3 in the main text.

Proof. Let $\mathcal{U}_1$ be the set of all points in $\mathcal{S}$ where $\kappa_{AB}\kappa^{AB} \neq 0$ and $\mathcal{U}_2$ be the set of all points in $\mathcal{S}$ where $\kappa_{AB}\hat{\kappa}^{AB} \neq 0$. The scalar functions $\kappa_{AB}\kappa^{AB} : \mathcal{S} \to \mathbb{C}$ and $\kappa_{AB}\hat{\kappa}^{AB} : \mathcal{S} \to \mathbb{R}$ are continuous. Therefore, $\mathcal{U}_1$ and $\mathcal{U}_2$ are open sets. Now, let $\mathcal{V}_1$ and $\mathcal{V}_2$ denote, respectively, the interiors of $\mathcal{S} \setminus \mathcal{U}_1$ and $\mathcal{V}_1 \setminus \mathcal{U}_2$. On the open set $\mathcal{V}_1 \cap \mathcal{U}_2$ we have that $\kappa_{AB}\kappa^{AB} = 0$ and $\kappa_{AB}\hat{\kappa}^{AB} \neq 0$. Hence, by Lemma 10 the second algebraic condition, equation (11c), is satisfied on $\mathcal{V}_1 \cap \mathcal{U}_2$. Similarly, by Lemma 9 the condition (11c) is satisfied on $\mathcal{U}_1$. On the open set $\mathcal{V}_2$ we have that $\kappa_{AB} = 0$ and therefore equation (11c) is trivially satisfied on $\mathcal{V}_2$. Using the above sets, the 3-manifold $\mathcal{S}$ can be split as

$$\begin{aligned}\mathcal{S} &= \mathcal{U}_1 \cup \partial(\mathcal{S} \setminus \mathcal{U}_1) \cup \mathcal{V}_1, \\ &= \mathcal{U}_1 \cup \partial(\mathcal{S} \setminus \mathcal{U}_1) \cup (\mathcal{V}_1 \cap \mathcal{U}_2) \cup \partial(\mathcal{S} \setminus \mathcal{U}_2) \cup \mathcal{V}_2.\end{aligned}$$

As the sets $\partial(\mathcal{S} \setminus \mathcal{U}_1)$ and $\partial(\mathcal{S} \setminus \mathcal{U}_2)$ have measure zero, one has that the second algebraic condition, equation (11c), is satisfied almost everywhere on $\mathcal{S}$. The left hand side of equation (11c) is continuous and we can therefore conclude that (11c) is satisfied everywhere on $\mathcal{S}$. Finally, using Theorem 2 in [2] one obtains the existence of a valence-2 Killing spinor on $D^+(\mathcal{S})$. □

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