Constructing "non-Kerrness" on compact domains

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Given a compact domain of a three-dimensional hypersurface on a vacuum spacetime, a scalar (the "non-Kerrness") is constructed by solving a Dirichlet problem for a second order elliptic system. If such scalar vanishes, and a set of conditions are satisfied at a point, then the domain of dependence of the compact domain is locally isometric to a portion of a member of the Kerr family of solutions to the Einstein field equations. This construction is expected to be of relevance in the analysis of numerical simulations of black hole spacetimes. © *2012 American Institute of Physics*. [http://dx.doi.org/10.1063/1.3702569]

I. INTRODUCTION

The present article is concerned with the problem of measuring how different a given initial data set for the Einstein vacuum field equations is from a Kerr initial data set. In Refs. 1, 2, and 4, this problem has been addressed by the construction of a geometric invariant—the *non-Kerrness*—on hypersurfaces with at least one asymptotic end. This setting, although convenient for theoretical discussions, is not ideal for numerical considerations where very often one needs to make use of bounded computational domains on a hypersurface. The purpose of this article is to provide a construction of non-Kerrness on bounded domains.

The construction of the non-Kerrness given in Refs. 1, 2, and 4, is based on a very strong property of the Kerr spacetime: the existence of a *Killing-Yano tensor*. A Killing-Yano tensor is an antisymmetric, rank 2 tensor $Y_{\mu\nu}$ satisfying the equation

$$\nabla_{(\mu} Y_{\nu)\lambda} = 0.$$

Let $\zeta_\mu \equiv \epsilon_\mu{}^{\nu\lambda\rho}\nabla_\nu Y_{\lambda\rho}$ denote the codifferential of $Y_{\mu\nu}$. If $Y_{\mu\nu}$ is a Killing-Yano tensor, then ζ_μ satisfies the Killing vector equation. As discussed in Ref. 9, the theory of Killing-Yano tensors can be conveniently reformulated in terms of the existence of a valence 2 Killing spinor, $\kappa_{AB} = \kappa_{(AB)}$, satisfying the equation

$$\nabla_{A'(A}\kappa_{BC)} = 0. \tag{1}$$

The spinorial analogue of the codifferential ζ_μ is the spinor $\xi_{AA'} \equiv \nabla_{A'}{}^{B}\kappa_{AB}$. In general, if κ_{AB} satisfies the Killing spinor equation, then $\xi_{AA'}$ is a complex Killing vector. In the case of the Kerr spacetime, the real and imaginary parts of this vector are proportional—and by multiplying with a complex constant, the imaginary part can be set to zero. In general, the existence of a Killing-Yano tensor is equivalent to existence of a Killing spinor κ_{AB} such that $\xi_{AA'}$ is real.

Killing spinors (or alternatively, Killing-Yano tensors) are useful in the characterization of the Kerr spacetime as the existence of one of these objects severely restricts the algebraic type of the curvature of the spacetime. Furthermore, the implied existence of a real Killing vector allows to make contact with the theory of the Mars-Simon tensor—see Refs. 5 and 6. As a result of this analysis,

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it is possible to provide a purely local characterization of the Kerr spacetime—see Theorem 1 in Ref. 6. Alternatively, one can obtain a somewhat simpler characterization if one combines local and global requirements: the existence of a stationary, asymptotically flat region with non-vanishing mass—see Theorem 2 in Ref. 6. Precisely, this result was used in the constructions of non-Kerrness on non-bounded 3-manifolds described in Refs. 1, 2, and 4.

The construction of the non-Kerrness on bounded domains discussed in the present article makes use of the local spacetime characterization of the Kerr spacetime given in Theorem 1 of Ref. 6 to show that if the non-Kerrness vanishes on some three-dimensional bounded domain, then the initial data prescribed on that region is locally isometric to data for a Kerr spacetime. We expect that this result will be of utility to assess in a quantitative way how a given numerically constructed dynamical black hole spacetime evolves towards a stationary state described by the Kerr spacetime. In the process, it will be shown that the general theory of Killing spinor initial data sets used in Refs. 1, 2, and 4 can be simplified.

A. Overview of the article

The content of this article is structured as follows. Section II provides a summary of key properties of spacetimes with Killing spinors. It also contains a reformulation in terms of spinors of a local characterization of the Kerr spacetime by Mars. Finally, a brief discussion of the notion of Killing spinor candidates is provided. Section III provides a brief summary of the theory of the Killing spinor initial data equations which encode the existence of a Killing vector at the level of initial data. Section IV gives a brief discussion of the notion of approximate Killing spinors, the approximate Killing spinor equations, and the elliptic theory required to discuss the existence of solutions to this equation with Dirichlet boundary conditions. Section V provides a result regarding the realness of the Killing vector constructed from the Killing spinor, which will be required in our subsequent discussion. Section VI provides our main result: a theorem which characterizes Kerr initial data on a compact domain of a three-dimensional manifold using the notion of approximate Killing spinors. Finally, Sec. VII provides some concluding remarks. There is an appendix providing a proof of a theorem discussed in Sec. III, which tells that one of the Killing spinor initial data equations can be omitted.

B. Notation and conventions

All throughout, $(\mathcal{M}, g_{\mu\nu})$ will denote a smooth, orientable, and time orientable globally hyperbolic vacuum spacetime. Here, and in what follows, $\mu, \nu, \cdots$ denote abstract four-dimensional tensor indices. The metric $g_{\mu\nu}$ will be taken to have signature $(+, -, -, -)$. Let ∇_μ denote the Levi-Civita connection of $g_{\mu\nu}$. The sign of the Riemann tensor will be given by the equation

$$\nabla_\mu \nabla_\nu \xi_\zeta - \nabla_\nu \nabla_\mu \xi_\zeta = R_{\nu\mu\zeta}{}^{\eta} \xi_\eta.$$

Spinors will be used systematically. We follow the conventions of Ref. 8. In particular, $A, B, \ldots$ will denote abstract spinorial indices. Tensors and their spinorial counterparts are related by means of the solder form $\sigma_\mu{}^{AA'}$ satisfying $g_{\mu\nu} = \sigma_\mu^{AA'} \sigma_\nu^{BB'} \epsilon_{AB} \bar{\epsilon}_{A'B'}$, where ϵ_{AB} is the antisymmetric spinor and $\bar{\epsilon}_{A'B'}$ its complex conjugate copy. One has, for example, that $\xi_\mu = \sigma_\mu{}^{AA'} \xi_{AA'}$. Let $\nabla_{AA'}$ denote the spinorial counterpart of the spacetime connection ∇_μ.

II. A LOCAL SPACETIME CHARACTERIZATION OF THE KERR SPACETIME

Given a spacetime $(\mathcal{M}, g_{\nu\nu})$, let $C_{\mu\nu\lambda\rho}$ denote the Weyl tensor of the metric $g_{\mu\nu}$. Let $C_{AA'BB'CC'DD'}$ denote the spinorial counterpart of $C_{\mu\nu\lambda\rho}$. There exists a completely symmetric spinor Ψ_{ABCD} such that

$$C_{AA'BB'CC'DD'} = \Psi_{ABCD} \bar{\epsilon}_{A'B'} \bar{\epsilon}_{C'D'} + \bar{\Psi}_{A'B'C'D'} \epsilon_{AB} \epsilon_{CD}.$$

In terms of the spinor Ψ_{ABCD}, the Bianchi identity can be rewritten as

$$\nabla^{Q}{}_{A'}\Psi_{ABCQ} = 0. \tag{2}$$

We recall that the two classical invariants of the Weyl tensor are given by

$$\mathcal{I} \equiv \tfrac{1}{2}\Psi_{ABCD}\Psi^{ABCD},$$
$$\mathcal{J} \equiv \tfrac{1}{6}\Psi_{ABCD}\Psi^{CDEF}\Psi_{EF}{}^{AB}.$$

A. Properties of spacetimes with Killing spinors

In what follows it is assumed one has a region $\mathcal{N}$ of the spacetime $(\mathcal{M}, g_{\mu\nu})$ where one has a solution κ_{AB} of the Killing spinor equation, Eq. (1). It is then well known that the spacetime must be of Petrov type D, N, or O at every point where the Killing spinor exists—see, e.g., Ref. 11. In the sequel, we will concentrate our attention to the case when $(\mathcal{M}, g_{\mu\nu})$ is of Petrov type D. In such case, there exist spinors α_A, β_A, $\alpha_Q\beta^Q = 1$, such that

$$\Psi_{ABCD} = -\psi\alpha_{(A}\alpha_A\beta_C\beta_{D)}, \tag{3}$$

where

$$\psi \equiv 18\mathcal{J}/\mathcal{I}. \tag{4}$$

The sign convention used in this equation differs from the one used in Refs. 1, 2, and 4. The reason behind this choice is to avoid potential problems with the choice of branch of roots of complex quantities. The valence 2 Killing spinor is then given by

$$\kappa_{AB} = \psi^{-1/3}\alpha_{(A}\beta_{B)}, \tag{5}$$

where the branch with minimal absolute value of the complex argument is used. The conventions used gives a real and positive ψ for the Schwarzschild spacetime.

As in the introduction, let

$$\xi_{AA'} \equiv \nabla^{Q}{}_{A'}\kappa_{AQ}.$$

Then $\xi_{AA'}$ is (in general) a complex solution to Killing equation

$$\nabla_{AA'}\xi_{BB'} + \nabla_{BB'}\xi_{AA'} = 0.$$

If $\xi_{AA'}$ is real, we define the Killing form of $\xi_{AA'}$ by

$$F_{AA'BB'} \equiv \tfrac{1}{2}\left(\nabla_{AA'}\xi_{BB'} - \nabla_{BB'}\xi_{AA'}\right)$$
$$= \nabla_{AA'}\xi_{BB'}.$$

Vacuum spacetimes admitting a Killing spinor such that $\xi_{AA'}$ is real will be said to belong to the *generalized Kerr-NUT class*—see Refs. 1 and 2. *In the rest of this section, it is assumed that* $(\mathcal{M}, g_{\mu\nu})$ *is a generalized Kerr-NUT spacetime.*

As a consequence of the symmetries of $F_{AA'BB'}$, there exists a symmetric, valence 2 spinor ϕ_{AB} such that

$$F_{AA'BB'} = \phi_{AB}\bar{\epsilon}_{A'B'} + \bar{\phi}_{A'B'}\epsilon_{AB},$$
$$\phi_{AB} \equiv \tfrac{1}{2}F_{AQ'B}{}^{Q'}.$$

Using (5), one finds the following expressions for $\xi_{AA'}$, and ϕ_{AB} in terms of ψ and the principal spinors:

$$\xi_{AA'} = \tfrac{1}{2}\psi^{-4/3}\left(\alpha_A\beta^Q + \beta_A\alpha^Q\right)\nabla_{QA'}\psi,$$
$$\phi_{AB} = -\tfrac{3}{4}\Psi_{ABCD}\kappa^{CD} = -\tfrac{1}{4}\psi^{2/3}\alpha_{(A}\beta_{B)}.$$

The above expression for the spinor ϕ_{AB} is obtained using the Killing spinor equation and by commutation of covariant derivatives.

For later use, we introduce the *norm of the Killing form*, the *norm of the Killing vector* and the *twist 1-form* via

$$\Phi \equiv \phi_{PQ}\phi^{PQ}, \qquad \lambda \equiv \xi_{AA'}\xi^{AA'},$$
$$\omega_{AA'} \equiv \epsilon_{AA'BB'CC'DD'}\xi^{BB'}\nabla^{CC'}\xi^{DD'},$$

where

$$\epsilon_{AA'BB'CC'DD'} \equiv \mathrm{i}\,(\epsilon_{AC}\epsilon_{BD}\bar{\epsilon}_{A'D'}\bar{\epsilon}_{B'C'} - \epsilon_{AD}\epsilon_{BC}\bar{\epsilon}_{A'C'}\bar{\epsilon}_{B'D'})$$

is the spinorial counterpart of the completely antisymmetric volume form, $\epsilon_{\mu\nu\lambda\rho}$, of $g_{\mu\nu}$. Locally, $\omega_{AA'}$ is exact, so that there exists ω (the *twist potential*) such that $\omega_{AA'} = \nabla_{AA'}\omega$. Using λ and ω, we define the *Ernst potential*, σ, by

$$\sigma \equiv \lambda + \mathrm{i}\omega.$$

Using expressions (3) and (5), one readily finds the following expressions for Φ, λ and $\omega_{AA'}$:

$$\Phi = -\tfrac{1}{32}\psi^{4/3}, \tag{6a}$$

$$\lambda = -\tfrac{1}{4}\psi^{-8/3}\nabla_{AA'}\psi\nabla^{AA'}\psi, \tag{6b}$$

$$\omega_{AA'} = \mathrm{Im}(4\phi_A{}^B\xi_{BA'}), \tag{6c}$$

In order to obtain an expression for the Ernst potential in terms of ψ, we notice the identities

$$\nabla_{AA'}(\psi^{1/3}) = -\tfrac{16}{3}\phi_A{}^B\xi_{BA'}, \tag{7a}$$

$$\nabla_{AA'}\lambda = \mathrm{Re}(4\phi_A{}^B\xi_{BA'}). \tag{7b}$$

These identities follow from the Bianchi identity (2), the Killing spinor equation and commuting derivatives as necessary. One concludes that

$$\nabla_{AA'}\lambda + \mathrm{i}\omega_{AA'} = -\tfrac{3}{4}\nabla_{AA'}\psi^{1/3}.$$

The latter can be integrated to give

$$\sigma - c = -\tfrac{3}{4}\psi^{1/3}, \tag{8}$$

with c a complex constant. The real part of c is not arbitrary: using Eqs. (7a) and (7b), one obtains that

$$\mathrm{Re}(c) = \lambda + \tfrac{3}{4}\mathrm{Re}(\psi^{1/3}). \tag{9}$$

B. A local characterization of Kerr

The analysis of the *so-called* Mars-Simon tensor presented in Refs. 5 and 6 gives rise to a local characterization of the Kerr spacetime among the class of spacetimes endowed with a Killing vector. This characterization involves the Weyl tensor, the Killing form and the Ernst potential—see Theorem 1 in Ref. 6. For the convenience of our subsequent analysis, here we present a slight generalization of this result in the language of spinors.

Theorem 1 (Ref. 6): *Let $(\mathcal{M}, g_{\mu\nu})$ be a smooth, vacuum spacetime admitting a Killing vector ξ^μ. Let $\mathcal{N} \subset \mathcal{M}$ be a non-empty open subset satisfying*:

(i) *There is a point $p \in \mathcal{N}$ where $\Phi \neq 0$.*

(ii) *The Killing form and the Weyl tensor are related by*

$$\Psi_{ABCD} = \varpi \phi_{(AB}\phi_{CD)},$$

where ϖ is a complex scalar function.

Then there exist two complex constants $\tilde{c}$ and k such that

$$\varpi = -\frac{12}{\tilde{c} - \sigma}, \qquad \Phi = -k(\tilde{c} - \sigma)^4, \qquad \text{on } \mathcal{N}.$$

If, in addition, $Re(\tilde{c}) > 0$ and $k = Re(k) > 0$ then $(\mathcal{N}, g_{\mu\nu})$ is locally isometric to a portion of the Kerr spacetime.

Remark 1: This result follows from—and is equivalent to—Theorem 1 in Ref. 6 by introducing a different normalization in the Killing vector and exploiting the fact that ω is defined only up to an additive constant.[13]

Remark 2: As discussed in Ref. 6, it follows from the previous result that the Kerr spacetime is everywhere strictly of type D. In particular, this implies that $\psi \neq 0$.

C. Killing spinor candidates

The construction of non-Kerrness on a bounded domain requires the notion of a *Killing spinor candidate* introduced in Ref. 4:

Definition 2: Let $(\mathcal{M}, g_{\mu\nu})$ be a vacuum spacetime. Consider a point $p \in \mathcal{M}$ for which $\mathcal{I} \neq 0$, $\mathcal{J} \neq 0$, and a symmetric spinor ζ_{AB} satisfying at p,

$$\zeta_{AB} \neq 0, \quad \psi^{-1}\Psi_{PQRS}\zeta^{PQ}\zeta^{RS} + \tfrac{1}{6}\zeta_{PQ}\zeta^{PQ} \neq 0.$$

The symmetric spinor given by

$$\breve{\kappa}_{AB} = \psi^{-1/3}\Xi^{-1/2}\left(-\psi^{-1}\Psi_{ABPQ}\zeta^{PQ} - \tfrac{1}{6}\zeta_{AB}\right), \tag{10}$$

with

$$\Xi \equiv -\psi^{-1}\Psi_{PQRS}\zeta^{PQ}\zeta^{RS} - \tfrac{1}{6}\zeta_{PQ}\zeta^{PQ},$$

will be called the ζ_{AB}-Killing spinor candidate at p. The scalar ψ is obtained from the Weyl spinor Ψ_{ABCD} using formula (4).

Formula (10) can be evaluated for any vacuum spacetime $(\mathcal{M}, g_{\mu\nu})$ satisfying the explicit conditions in Definition 2, that is, it is not restricted to a special Petrov type. The name Killing spinor candidate is justified by the following result also proved in Ref. 4:

Proposition 3: Let $(\mathcal{M}, g_{\mu\nu})$ be a vacuum spacetime. If on $\mathcal{N} \subset \mathcal{M}$, the spacetime is of Petrov type D and ζ_{AB} is a symmetric spinor satisfying

$$\Xi = \psi^{-1}\Psi_{PQRS}\zeta^{PQ}\zeta^{RS} + \tfrac{1}{6}\zeta_{PQ}\zeta^{PQ} \neq 0,$$
$$\zeta_{AB} \neq 0 \quad \text{on } \mathcal{N},$$

and $\mathcal{N}$ contains no branch cuts of $\psi^{1/3}$ and $\Xi^{1/2}$, then

$$\kappa_{AB} = \psi^{-1/3}\Xi^{-1/2}\left(-\psi^{-1}\Psi_{ABPQ}\zeta^{PQ} - \tfrac{1}{6}\zeta_{AB}\right) \tag{11}$$

is a Killing spinor on $\mathcal{N}$. The formula (11) is independent of the choice of ζ_{AB}.

Remark. Different choices of branch cuts in $\psi^{1/3}$ and $\Xi^{1/2}$ only change the right hand side of (11) by a constant complex phase. The assumption on the no existence of branch cuts of $\psi^{1/3}$ and $\Xi^{1/2}$ is included to ensure the smooth existence of derivatives of the various fields—see also Assumption 7 below.

III. THE KILLING SPINOR INITIAL DATA EQUATIONS

Key for the construction of the non-Kerrness discussed in Refs. 1, 2, and 4, is the idea of how to encode that the development of an initial data set $(\mathcal{S}, h_{ij}, K_{ij})$ admits a solution to the Killing spinor equation, Eq. (1). This question can be addressed by means of the space-spinor decomposition of the Killing spinor equation, Eq. (1). For a more detailed description see Ref. 2.

In order to perform a space-spinor decomposition of Eq. (1), it is convenient to define the spinors

$$\xi_{ABCD} \equiv \nabla_{(AB}\kappa_{CD)}, \quad \xi_{AB} \equiv \tfrac{3}{2}\nabla_{(A}{}^{D}\kappa_{B)D}, \quad \xi \equiv \nabla^{PQ}\kappa_{PQ}, \tag{12}$$

where ∇_{AB} denotes the spinorial version of the Sen connection associated to the pair (h_{ij}, K_{ij}) of intrinsic metric and extrinsic curvature. It can be expressed in terms of the spinorial counterpart, D_{AB} of the Levi-Civita connection of the 3-metric h_{ij}, and the spinorial version, $K_{ABCD} = K_{(AB)(CD)} = K_{CDAB}$, of the second fundamental form K_{ij}. For example, given a valence 1 spinor π_A, one has that

$$\nabla_{AB}\pi_C = D_{AB}\pi_C + \tfrac{1}{2}K_{ABC}{}^{Q}\pi_Q,$$

with the obvious generalizations to higher valence spinors. For expressions involving the commutators, we refer to the paper.[2] The *Hermitian conjugate* of π_A is defined via

$$\hat{\pi}_A \equiv \tau_A{}^{E'}\bar{\pi}_{E'},$$

where $\tau^{AA'}$ is the normal to $\mathcal{S}$ with length $\sqrt{2}$. The Hermitian conjugate can be extended to higher valence symmetric spinors in the obvious way. It can be verified that $\xi_{ABCD}\hat{\xi}^{ABCD} \geq 0$.

Using the notation described in the previous paragraph, we find that the space-spinor decomposition of Eq. (1) renders a set of three conditions intrinsic to the hypersurface $\mathcal{S}$:

$$\xi_{ABCD} = 0, \tag{13a}$$

$$\Psi_{(ABC}{}^{F}\kappa_{D)F} = 0, \tag{13b}$$

$$3\kappa_{(A}{}^{E}\nabla_{B}{}^{F}\Psi_{CD)EF} + \Psi_{(ABC}{}^{F}\xi_{D)F} = 0, \tag{13c}$$

where the spinor Ψ_{ABCD} denotes, in a slight abuse of notation, the restriction to the hypersurface $\mathcal{S}$ of the self-dual Weyl spinor. For the ease of notation, a similar convention will be adopted for the restriction of other spacetime fields. Whether one is considering the field on spacetime or its restriction to $\mathcal{S}$ will always be clear from the context. Crucially, the spinor Ψ_{ABCD} in Eqs. (13b) and (13c) can be written entirely in terms of initial data quantities via the relations,

$$\Psi_{ABCD} = E_{ABCD} + \mathrm{i}B_{ABCD},$$

with

$$E_{ABCD} = -r_{(ABCD)} + \tfrac{1}{2}\Omega_{(AB}{}^{PQ}\Omega_{CD)PQ} - \tfrac{1}{6}\Omega_{ABCD}K,$$
$$B_{ABCD} = -\mathrm{i}\, D^{Q}{}_{(A}\Omega_{BCD)Q},$$

and where $\Omega_{ABCD} \equiv K_{(ABCD)}$, $K \equiv K_{PQ}{}^{PQ}$. Furthermore, the spinor r_{ABCD} is the Ricci tensor, r_{ij}, of the 3-metric h_{ij}.

In the Appendix, it is shown that the second algebraic condition (13c) is, in fact, redundant and a consequence of the conditions (13a) and (13b). In particular, it follows:

Theorem 4: *Let Eqs.* (13a) and (13b) *be satisfied for a symmetric spinor* $\breve{\kappa}_{AB}$ *on an open set* $\mathcal{U} \subset \mathcal{S}$. *Then the Killing spinor equation, Eq.* (1) *has a solution*, κ_{AB}, *on the future domain of dependence* $\mathcal{D}^+(\mathcal{U})$.

Remark. This means that the term I_2 in the invariants of Refs. 1, 2, and 4 can be omitted.

IV. APPROXIMATE KILLING SPINORS

A. The approximate Killing spinor equation

The *spatial Killing spinor equation, Eq.* (13a) can be regarded as a (complex) generalization of the conformal Killing vector equation. As in the case of the conformal Killing equation, Eq. (13a) is clearly overdetermined. However, one can construct a generalization of the equation which under suitable circumstances can always be expected to have a solution. One can do this by composing the operator in (13a) with its formal adjoint—see Ref. 1. This procedure renders the equation

$$\mathbf{L}\kappa_{CD} \equiv \nabla^{AB}\nabla_{(AB}\kappa_{CD)} - \Omega^{ABF}{}_{(A}\nabla_{|DF|}\kappa_{B)C} - \Omega^{ABF}{}_{(A}\nabla_{B)F}\kappa_{CD} = 0, \tag{14}$$

which will be called the *approximate Killing spinor equation*. One has the following result proved in Ref. 2:

Lemma 5: The operator $\mathbf{L}$ *defined by the left hand side of Eq.* (14) *is a formally self-adjoint elliptic operator*.

In order to discuss the solvability of Eq. (14) on a bounded domain, $\mathcal{U} \subset \mathcal{S}$, one has to supplement it with appropriate boundary conditions. On $\partial\mathcal{U}$, we will consider the homogeneous Dirichlet operator $\mathbf{B}$ given by

$$\mathbf{B}u(y) = u(y), \quad y \in \partial\mathcal{S}.$$

The combined operator $(\mathbf{L}, \mathbf{B})$ satisfies the so-called *Lopatinski-Shapiro compatibility conditions*—see Ref. 12 for detailed definitions and discussion. Thus, $(\mathbf{L}, \mathbf{B})$ is L-elliptic—see again, Theorem 10.7 of Ref. 12. Moreover, one has the following theorem—see also Ref. 7.

Theorem 6: *Let* $\mathbf{L}$ *denote a smooth second order homogeneous elliptic operator on* $\mathcal{U}$. *Furthermore, let* $\partial\mathcal{U}$ *be smooth and let* $\mathbf{B}$ *denote the Dirichlet boundary operator. Then for* $s \geq 2$ *the map*

$$(\mathbf{L}, \mathbf{B}) : H^s(\mathcal{U}) \to H^{s-2}(\mathcal{U}) \times H^{s-1/2}(\partial\mathcal{U})$$

is Fredholm. Furthermore, the boundary value problem

$$\mathbf{L}u(x) = f(x), \quad f \in H^0(\mathcal{U}), \quad x \in \mathcal{U},$$
$$u(y) = g(y), \quad g \in H^0(\partial\mathcal{U}), \quad y \in \partial\mathcal{U},$$

has a solution $u \in H^2(\mathcal{U})$ *if*

$$\int_{\mathcal{U}} f \cdot \nu \mathrm{d}\mu = 0,$$

for all $\nu \in H^2(\mathcal{U})$ *such that*

$$\mathbf{L}^*\nu(x) = 0, \quad x \in \mathcal{U},$$
$$\nu(y) = 0, \quad y \in \partial\mathcal{U}.$$

Remark 1. In the previous Theorem, the action of $\mathcal{B}$ on u is to be understood in the trace sense—see Ref. 12.

Remark 2: If $\mathbf{L}$ has smooth coefficients and $\mathbf{L}u = 0$, then it follows from Weyl's Lemma—see, e.g., Ref. 12—that if a solution to the boundary value problem exists and the boundary data is smooth, then the solution must be, in fact, smooth—this is the so-called elliptic regularity.

In what follows let $n_{AB} = n_{(AB)}$ denote the spinorial counterpart of the inward pointing normal to $\partial\mathcal{U}$. As a consequence of our signature conventions, one has that $n_{PQ}n^{PQ} = -1$. Theorem 6 will be used to establish the existence of solutions to the approximate Killing spinor equation, Eq. (14) with Dirichlet boundary data given by the n_{AB}-Killing spinor candidate. In order to ensure that the Killing spinor candidate can be constructed on $\partial\mathcal{U}$, we define the set

$$\mathcal{Q} \equiv \{z \in \mathbb{C} \mid z = \Xi(p), \quad p \in \partial\mathcal{U}\},$$

where we have chosen $\zeta_{AB} = n_{AB}$ in the function Ξ. We make the following assumption:

Assumption 7: The initial data set $(\mathcal{S}, h_{ij}, K_{ij})$ and the compact set $\mathcal{U}$ are such that $\mathcal{I} \neq 0$, $\mathcal{J} \neq 0$ on $\partial\mathcal{U}$ and that Ξ is a smooth function over $\partial\mathcal{U}$ satisfying

(i) $0 \notin \mathcal{Q}$,
(ii) $\mathcal{Q}$ does not encircle the point $z = 0$,

when we choose ζ_{AB} as the inward pointing normal to $\partial\mathcal{U}$.

Remarks: As a consequence of this assumption one can choose a cut of the square root function on the complex plane such that $\Xi^{1/2}(p)$ is smooth for all $p \in \partial\mathcal{U}$. Notice that the n_{AB}-Killing spinor candidate is only defined at $\partial\mathcal{U}$. The assumptions $\mathcal{I} \neq 0$, $\mathcal{J} \neq 0$ are justified on the basis that we are mainly interested in discussing configurations close to Kerr initial data—for which $\psi \neq 0$.

One has the following result:

Proposition 8: Let $(\mathcal{S}, h_{ij}, K_{ij})$ be an initial data set for the Einstein vacuum field equations. Furthermore, let $\mathcal{U} \subset \mathcal{S}$ be a compact subset with boundary $\partial\mathcal{S}$ satisfying Assumption 7. Then, there exists a unique smooth solution, κ_{AB}, to the approximate Killing spinor equation, Eq. (14) with boundary value given by the n_{AB}-Killing spinor candidate given pointwise by Eq. (10) on $\partial\mathcal{U}$.

Proof: The proof of this result follows directly from the second part of Theorem 6. Notice that as the equation is homogeneous, there is no potential obstruction to the existence of solutions and one does not need to verify the triviality of the Kernel of the adjoint operator as it is in the case with asymptotically Euclidean ends—see Refs. 1, 2, and 4. □

V. REALITY OF THE KILLING VECTOR

As discussed in the Introduction, the existence of a Killing spinor is not enough to single out the generalized Kerr-NUT family from the type D solutions. We also need that the Killing vector constructed from the Killing spinor is real. This section provides some tools to determine that.

A. Imaginary part of the Killing vector data

In what follows, let κ_{AB} solve the Killing spinor equation, Eq. (1) in a spacetime domain $\mathcal{D}$, and let ξ and ξ_{AB} be defined as in (12). In this section, we only study what happens in the domain $\mathcal{D}$. A computation using the suite `xAct` for MATHEMATICA starting from Eqs. (13a)–(13c) shows that

$$D_{AB}\mathrm{Im}(\xi^{AB}) = -\tfrac{1}{2}\mathrm{Im}(\xi)K, \tag{15a}$$

$$D_{(AB}\mathrm{Im}(\xi_{CD)}) = -\tfrac{1}{2}\mathrm{Im}(\xi)\Omega_{ABCD}. \tag{15b}$$

This can be seen by using Eqs. (18a) and (18b) in Ref. 2 and splitting into real and imaginary parts. Equation (1) implies $\nabla\kappa_{AB} = -\frac{2}{3}\xi_{AB}$, where ∇ denotes the normal derivative $\tau^{AA'}\nabla_{AA'}$. Commuting derivatives and simplifying one obtains

$$\nabla\mathrm{Im}(\xi) = \mathrm{Im}(\xi^{AB})K_{AB}, \tag{16a}$$

$$\nabla\mathrm{Im}(\xi_{AB}) = -\tfrac{1}{2}\mathrm{Im}(\xi)K_{AB} + \tfrac{1}{3}\mathrm{Im}(\xi_{AB})K + \Omega_{ABCD}\mathrm{Im}(\xi^{CD}) - D_{AB}\mathrm{Im}(\xi) - \mathrm{Im}(\xi_{(A}{}^{C})K_{B)C}, \tag{16b}$$

where K_{AB} is the acceleration vector. For more details about the derivation see Eqs. (32b) and (32c) in Ref. 2 and their derivations. Making a space spinor split of $\xi_{AA'} = \nabla^{B}{}_{A'}\kappa_{AB}$ and using Eq. (1), we find

$$\mathrm{Im}(\xi_{AA'}) = \tfrac{1}{2}\mathrm{Im}(\xi)\tau_{AA'} - \mathrm{Im}(\xi_{AB})\tau^{B}{}_{A'}.$$

After differentiating once more, making a further space spinor split, and using Eqs. (15a), (15b), (16a), and (16b), we have:

Lemma 9: Let κ_{AB} solve the Killing spinor equation, Eq. (1) in a spacetime domain $\mathcal{D}$. Assume that

$$Im(\xi) = 0, \quad Im(\xi_{AB}) = 0, \quad D_{AB}Im(\xi) = 0, \quad D_{(A}{}^{C}Im(\xi_{B)C}) = 0, \tag{17}$$

at a point $p \in \mathcal{D}$. Then $Im(\xi_{AA'}) = 0$ and $\nabla_{AA'}Im(\xi_{BB'}) = 0$ at p.

VI. THE NON-KERRNESS INVARIANT

The approximate Killing spinor κ_{AB} obtained in Proposition 8 will now be used, in the spirit of Ref. 1, to construct a geometric invariant measuring the non-Kerrness of the initial data on the compact set $\mathcal{U}$. More precisely, we define

$$I \equiv \int_{\mathcal{U}} \nabla_{(AB}\kappa_{CD)}\widehat{\nabla^{AB}\kappa^{CD}}\mathrm{d}\mu + \int_{\mathcal{U}} \Psi_{(ABC}{}^{P}\kappa_{D)P}\widehat{\Psi^{ABCQ}\kappa^{D}}{}_{Q}\mathrm{d}\mu. \tag{18}$$

A. The main result

The main result of our analysis is the following theorem:

Theorem 10: Let $(\mathcal{S}, h_{ij}, K_{ij})$ be an initial data set for the Einstein vacuum field equations, and let $\mathcal{U} \subset \mathcal{S}$ be a compact connected subset with boundary $\partial\mathcal{U}$ satisfying Assumption 7. Let I be as defined by Eq. (18) where κ_{AB} is given as the only solution to Eq. (14) with boundary behaviour given by the n_{AB}-Killing spinor candidate $\breve{\kappa}_{AB}$ where n_{AB} is the inward pointing normal to $\partial\mathcal{U}$. If

(i) $I = 0$,
(ii) *there exists a point on $\mathcal{U}$ for which*

$$\mathrm{Im}(\xi) = 0, \quad \mathrm{Im}(\xi_{AB}) = 0, \quad D_{AB}\mathrm{Im}(\xi) = 0, \quad D_{(A}{}^{C}\mathrm{Im}(\xi_{B)C}) = 0, \tag{19}$$

then the future domain of dependence, $D^{+}(\mathcal{U})$, of $\mathcal{U}$ is locally isometric to a subset of a generalized Kerr-NUT spacetime. If, in addition,
(iii) *there exists a point on $\mathcal{U}$ for which $\Phi \neq 0$,*
(iv) *there exists a point on $\mathcal{U}$ for which*

$$\lambda + \tfrac{3}{4}\mathrm{Re}(\psi^{1/3}) > 0, \tag{20}$$

then $D^{+}(\mathcal{U})$ is locally isometric to a portion of a Kerr spacetime. Conversely, on a compact subset $\mathcal{U} \subset \mathcal{S}$ of a Kerr initial data set, $(\mathcal{S}, h_{ij}, K_{ij})$, the properties (i), (ii), (iii), and (iv) are satisfied.

Remark 1: If $D^{+}(\mathcal{U})$ is locally isometric to a portion of a Kerr spacetime, the conditions *(ii), (iii),* and *(iv)* are satisfied on every point. Hence, the choice of which point to check the conditions in, is not important.

Remark 2: If $\mathcal{U}$ is not connected, the conditions (ii), (iii), and (iv) needs to be checked for each connected component of $\mathcal{U}$.

Remark 3: The conditions (iii) and (iv) can be replaced by an asymptotic flatness condition.

Proof: If $I = 0$, then it follows from our smoothness assumptions that Eqs. (13a) and (13b) are satisfied on $\mathcal{U}$. Hence, from Theorem 4, it follows that $D^+(\mathcal{U})$ will contain a Killing spinor κ_{AB}. Then $\xi_{AA'}$ is the spinor counterpart of a (possibly complex) Killing vector. Now, using assumption *(ii)* together with Lemma 9 gives $\text{Im}(\xi_{AA'}) = 0$ and $\nabla_{AA'}\text{Im}(\xi_{BB'}) = 0$ at a point. Using a standard result about Killing spinors (see Appendix C.3 in Ref. 10), one concludes that $\text{Im}(\xi) = \text{Im}(\xi_{AB}) = 0$ everywhere on $D^+(\mathcal{U})$ so that $\xi_{AA'}$ is, in fact, real. Thus, $D^+(\mathcal{U})$ is locally isometric to a portion of a generalized Kerr-NUT spacetime.

As in the main text, let ϕ_{AB} denote the spinorial counterpart of the Killing form for of $\xi_{AA'}$. From the discussion in Subsection II A, one concludes that

$$\Psi_{ABCD} = \varpi \phi_{(AB}\phi_{CD)},$$

for some function ϖ. Now, if $\Phi \neq 0$ on $\mathcal{U}$, then using Theorem 1, one has that

$$\varpi = -\frac{12}{\tilde{c} - \sigma}, \qquad \Phi = -k(\tilde{c} - \sigma)^4,$$

for some (possibly complex) constants $\tilde{c}$ and k. Using formulae (8) and (6a), one can identify the constants c and $\tilde{c}$ and set $k = \frac{8}{81}$. Evaluating c at the point where (20) holds, one obtains that $\text{Re}(c) > 0$. Thus, the hypothesis of Theorem 1 hold and one concludes that $D^+(\mathcal{U})$ is locally isometric to a portion of the Kerr spacetime.

Now, given a compact subset $\mathcal{U} \subset \mathcal{S}$ of a Kerr initial data set, $(\mathcal{S}, h_{ij}, K_{ij})$, one knows there exist a spinor κ_{AB} for which the spatial Killing spinor equation, Eq. (13a), and Eq. (13b) are satisfied. This spinor coincides at $\partial\mathcal{U}$ (up to an irrelevant constant numerical factor) with the n_{AB}-Killing spinor candidate. Thus, by uniqueness of the elliptic problem (14), the approximate Killing spinor obtained from solving the equation and κ_{AB} coincide (again, up to an irrelevant numerical factor) and one has $I = 0$ and (i) is satisfied. As κ_{AB} satisfies the spatial Killing spinor equations, it follows from the general theory of Ref. 2 that (ξ, ξ_{AB}) is a Killing vector initial data set. For Kerr this data corresponds to the real stationary Killing vector, thus (ii) is satisfied. Now, as $\psi \neq 0$ for the Kerr spacetime, one has from Eq. (6a) that $\Phi \neq 0$ and thus (iii) holds. Finally, an explicit computation with the Kerr spacetime shows that (20) holds for any point of the Kerr spacetime—hence one obtains (iv). □

VII. CONCLUSIONS AND DISCUSSION

In this paper, we have devised a way to measure the deviation from Kerr initial data for bounded domains. The main result is presented in Theorem 10. In the previous papers,[1,2,4] a similar result was obtained for cases where the computational domain reached spatial infinity. For such cases, the asymptotic behaviour of the approximate Killing spinor could be specified in a way that helped us to exclude all other Petrov type D solutions. Therefore, we could conclude that the data was Kerr data if and only if $I = 0$. As the present paper deals with bounded domains, we constructed the boundary data for the approximate Killing spinor from the curvature. The drawback is that this gives $I = 0$ for all type D solutions. Therefore, one requires conditions (ii), (iii), and (iv) in Theorem 10 to single out the Kerr solution. An effort was put into formulating the conditions so they can be verified at a single arbitrarily chosen point of the computational domain. Furthermore, we have shown that a part of the invariant constructed in Refs. 1, 2, and 4 can be omitted in the case of a bounded domain as well the unbounded case.

The results of this paper can be used to numerically evaluate how much any slice of a spacetime deviates from Kerr data. This gives a tool to quantify decay towards Kerr data for a numerically evolved spacetime. A project along these lines have been initiated.

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APPENDIX: REDUNDANCY OF THE SECOND ALGEBRAIC CONDITION

The purpose of the present appendix is to prove the assertion made in Theorem 4 that the second algebraic condition given by Eq. (13c) is a consequence of the conditions (13a) and (13b). As a consequence of this result, the conditions required on an initial data set to have a development with a valence 2 Killing spinor become completely analogue to those required to have a valence 1 Killing spinor—see, e.g., Ref. 3.

The analysis in this appendix proceeds by discussing the various possible algebraic types that the spinor κ_{AB} can have. Our first result is the following:

Lemma 11: Assume that the symmetric spinor κ_{AB} satisfies

$$\kappa_{AB}\kappa^{AB} \neq 0, \quad \nabla_{(AB}\kappa_{CD)} = 0, \quad \Psi_{(ABC}{}^F\kappa_{D)F} = 0,$$

on an open subset $\mathcal{U} \subset \mathcal{S}$. Then the algebraic condition (13c) is satisfied on $\mathcal{U}$.

Proof: The condition $\kappa_{AB}\kappa^{AB} \neq 0$ allows us to choose a spin dyad (o_A, ι_A) and a scalar field $\varkappa$ such that $o_A\iota^A = 1$ and $\kappa_{AB} = e^{\varkappa}o_{(A}\iota_{B)}$. Similarly, the condition $\Psi_{(ABC}{}^F\kappa_{D)F} = 0$ implies that there is a scalar field ψ such that $\Psi_{ABCD} = \psi o_{(A}o_B\iota_C\iota_{D)}$.

In the next step, we decompose the equation $\nabla_{(AB}\kappa_{CD)} = 0$ into its various components to obtain

$$o^Ao^Bo^C\nabla_{AB}o_C = 0, \tag{A1a}$$

$$o^A\iota^Bo^C\nabla_{AB}o_C = -\tfrac{1}{2}o^Ao^B\nabla_{AB}\varkappa, \tag{A1b}$$

$$o^Ao^B\iota^C\nabla_{AB}\iota_C - \iota^A\iota^Bo^C\nabla_{AB}o_C = 2o^A\iota^B\nabla_{AB}\varkappa, \tag{A1c}$$

$$o^A\iota^B\iota^C\nabla_{AB}\iota_C = \tfrac{1}{2}\iota^A\iota^B\nabla_{AB}\varkappa, \tag{A1d}$$

$$\iota^A\iota^B\iota^C\nabla_{AB}\iota_C = 0. \tag{A1e}$$

These equations imply, in turn, that

$$e^{-\varkappa}\xi_{AB} = -3o_Ao_Bo^C\iota^D\iota^F\nabla_{CD}\iota_F - 3\iota_A\iota_Bo^C\iota^Do^F\nabla_{CD}o_F + \tfrac{3}{2}o_{(A}\iota_{B)}(o^Co^D\iota^F\nabla_{CD}\iota_F + \iota^C\iota^Do^F\nabla_{CD}o_F). \tag{A2}$$

Now, it is well known that the spacetime Bianchi identity $\nabla^Q{}_{A'}\Psi_{ABCQ} = 0$ implies the constraint

$$\nabla^{CD}\Psi_{ABCD} = 0, \tag{A3}$$

on $\mathcal{S}$. Substituting $\Psi_{ABCD} = \psi o_{(A}o_B\iota_C\iota_{D)}$ and contracting with combinations of o^A and ι^A, one finds that the content of (A3) is given by

$$o^Ao^B\nabla_{AB}\psi = 6\psi o^A\iota^Bo^C\nabla_{AB}o_C, \tag{A4a}$$

$$o^B\iota^C\nabla_{BC}\psi = \tfrac{3}{2}\psi\iota^A\iota^Bo^C\nabla_{AB}o_C - \tfrac{3}{2}\psi o^Ao^B\iota^C\nabla_{AB}\iota_C, \tag{A4b}$$

$$\iota^A\iota^B\nabla_{AB}\psi = -6\psi o^A\iota^B\iota^C\nabla_{AB}\iota_C. \tag{A4c}$$

Using Eq. (A2) and the Bianchi identities (A4a)–(A4c), we get

$$\begin{aligned}\Psi_{(ABC}{}^{F}\xi_{D)F} + 3\kappa_{(A}{}^{F}\nabla_{B}{}^{H}\Psi_{CD)FH} &\\ = \tfrac{3}{4}e^{\varkappa}\psi\iota_A\iota_B\iota_C\iota_D o^M o^P o^Q \nabla_{PQ}o_M &\\ - \tfrac{3}{4}e^{\varkappa}\psi o_A o_B o_C o_D \iota^M\iota^P\iota^Q\nabla_{PQ}\iota_M .&\end{aligned}$$

Finally using the information about the derivatives of the spin dyad contained in Eqs. (A1a)–(A1e) one finds that we get that the second algebraic condition, Eq. (13c), is satisfied on $\mathcal{U}$. Notice that in this argument one could have had $\psi = 0$. □

Using similar methods as before, one obtains the following lemma:

Lemma 12: Assume that the symmetric spinor κ_{AB} satisfies

$$\kappa_{AB}\kappa^{AB} = 0, \quad \kappa_{AB}\hat{\kappa}^{AB} \neq 0, \quad \nabla_{(AB}\kappa_{CD)} = 0, \quad \Psi_{(ABC}{}^{F}\kappa_{D)F} = 0,$$

on an open subset $\mathcal{U} \subset \mathcal{S}$. Then the algebraic condition (13c) is satisfied on $\mathcal{U}$.

Proof: By assumption the κ_{AB} is algebraically special, that is, it has repeated principal spinors. Thus, there exists o_A such that $\kappa_{AB} = o_A o_B$. We then complete o_A to a normalized spinor dyad (o_A, ι_A). The equation $\nabla_{(AB}\kappa_{CD)} = 0$ is equivalent to

$$o^A o^B o^C \nabla_{(AB}o_{C)} = 0, \tag{A5a}$$

$$o^A o^B \iota^C \nabla_{(AB}o_{C)} = 0, \tag{A5b}$$

$$o^A \iota^B \iota^C \nabla_{(AB}o_{C)} = 0, \tag{A5c}$$

$$\iota^A \iota^B \iota^C \nabla_{(AB}o_{C)} = 0. \tag{A5d}$$

These equations imply, in turn, that

$$\xi_{AB} = -2 o_A o_B \iota^C \nabla_{CD} o^D + 2 o_{(A}\iota_{B)} o^C \nabla_{CD} o^D. \tag{A6}$$

The condition $\Psi_{(ABC}{}^{F}\kappa_{D)F} = 0$ implies that there is a scalar field ψ such that $\Psi_{ABCD} = \psi o_{(A}o_B o_C o_{D)}$. Using this together with (A6) yields

$$\begin{aligned}\Psi_{(ABC}{}^{F}\xi_{D)F} + 3\kappa_{(A}{}^{F}\nabla_{B}{}^{H}\Psi_{CD)FH} &\\ = -3 o_A o_B o_C o_D \psi o^P o^Q \iota^R \nabla_{(PQ}o_{R)} &\\ + 3 o_{(A}o_B o_C \iota_{D)}\psi o^P o^Q o^R \nabla_{(PQ}o_{R)}.&\end{aligned} \tag{A7}$$

Finally using the relations (A5a)–(A5d), we get that the second algebraic condition, Eq. (13c), is satisfied on $\mathcal{U}$. □

With the aid of the previous two lemmas, one can provide a proof of Theorem 4 in the main text.

Proof: Let $\mathcal{U}_1$ be the set of all points in $\mathcal{S}$ where $\kappa_{AB}\kappa^{AB} \neq 0$ and $\mathcal{U}_2$ be the set of all points in $\mathcal{S}$ where $\kappa_{AB}\hat{\kappa}^{AB} \neq 0$. The scalar functions $\kappa_{AB}\kappa^{AB} : \mathcal{S} \to \mathbb{C}$ and $\kappa_{AB}\hat{\kappa}^{AB} : \mathcal{S} \to \mathbb{R}$ are continuous. Therefore, $\mathcal{U}_1$ and $\mathcal{U}_2$ are open sets. Now, let $\mathcal{V}_1$ and $\mathcal{V}_2$ denote, respectively, the interiors of $\mathcal{S} \setminus \mathcal{U}_1$ and $\mathcal{V}_1 \setminus \mathcal{U}_2$. On the open set $\mathcal{V}_1 \cap \mathcal{U}_2$, we have that $\kappa_{AB}\kappa^{AB} = 0$ and $\kappa_{AB}\hat{\kappa}^{AB} \neq 0$. Hence, by Lemma 12, the second algebraic condition, Eq. (13c), is satisfied on $\mathcal{V}_1 \cap \mathcal{U}_2$. Similarly, by Lemma 11, the condition (13c) is satisfied on $\mathcal{U}_1$. On the open set $\mathcal{V}_2$, we have that $\kappa_{AB} = 0$ and therefore Eq. (13c) is trivially satisfied on $\mathcal{V}_2$. Using the above sets, the 3-manifold $\mathcal{S}$ can be split as

$$\mathrm{int}\mathcal{S} \subset \mathcal{U}_1 \cup (\mathcal{V}_1 \cap \mathcal{U}_2) \cup \mathcal{V}_2 \cup \partial\mathcal{U}_1 \cup \partial\mathcal{V}_2.$$

The left hand side of Eq. (13c) is continuous and vanishes on the open sets $\mathcal{U}_1$, $\mathcal{V}_1 \cap \mathcal{U}_2$, and $\mathcal{V}_2$. By continuity, it therefore also vanishes on the boundaries $\partial\mathcal{U}_1$ and $\partial\mathcal{V}_2$. We can therefore conclude that (13c) is satisfied everywhere on int $\mathcal{S}$. Again by continuity this extends to $\mathcal{S}$. Finally, using Theorem 2 in Ref. 2, one obtains the existence of a valence 2 Killing spinor on $D^+(\mathcal{S})$. $\square$

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