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Jumpstarting the all-loop S-matrix of planar $\mathcal{N} = 4$ super Yang-Mills

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Abstract: We derive a set of first-order differential equations obeyed by the S-matrix of planar maximally supersymmetric Yang-Mills theory. The equations, based on the Yangian symmetry of the theory, involve only finite and regulator-independent quantities and uniquely determine the all-loop S-matrix. When expanded in powers of the coupling they give derivatives of amplitudes as single integrals over lower-loop, higher-point amplitudes/Wilson loops. We outline a derivation for the equations using the Operator Product Expansion for Wilson loops. We apply them on a few examples at two- and three-loops, reproducing a recent result on the two-loop NMHV hexagon and fixing previously undermined coefficients in a recent Ansatz for the three-loop MHV hexagon. In addition, we consider amplitudes restricted to a two-dimensional subspace of Minkowski space and derive a particularly simple closed set of equations in that case.

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1 Introduction

$\mathcal{N} = 4$ supersymmetric Yang-Mills theory (SYM) is believed to be integrable in the planar limit [1, 2], cf. [3] for a recent review. This has made it possible to compute quantities of the theory, such as the spectrum of anomalous dimensions, at any value of the coupling [4–6]. On the other hand, remarkable structures in the S-matrix of the theory have been unraveled recently. Among them, a hidden, dual superconformal symmetry has been discovered both at strong [7] and weak coupling [8] for the S-matrix, which, together with the ordinary superconformal symmetry, generates an infinite-dimensional symmetry encoding the integrability of the theory, the so-called Yangian symmetry [9].

The dual superconformal symmetry can be understood as the symmetry of null polygonal Wilson loops in a dual spacetime. The bosonic Wilson loops are dual to maximally-helicity-violated (MHV) scattering amplitudes at strong [7] and weak coupling [11–13], and recently the duality has been generalized to arbitrary helicity (N^{k}MHV) amplitudes and supersymmetric Wilson loops [14, 15] (or with a closely related light-cone limit of correlation functions [16, 17]). Although generally tree amplitudes are Yangian invariant [18–21], the naive Yangian symmetry is broken for loop-level amplitudes/Wilson loops even if we consider finite quantities, such as the remainder and ratio functions [18, 22, 23]. In this paper, we will argue that the Yangian symmetry can be made exact for all-loop amplitudes/Wilson loops, which are in turn completely determined by the all-loop equations derived from the exact symmetry.

In order to discuss regulator-independent relations in a uniform way, it proves convenient to introduce the *BDS-subtracted* S-matrix element $R_{n,k}$,

$$A_{n,k} = A_n^{\rm BDS} \times R_{n,k} \tag{1.1}$$

where $A_{n,k}$ stands for the N^{k}MHV scattering amplitude and $A_n^{\rm BDS}$ for the exponentiated Ansatz proposed by Bern, Dixon and Smirnov (BDS) [24], including the MHV tree and coupling constant factor.

The BDS-subtracted S-matrix $R_{n,k}$ is infrared finite and regulator independent. It is invariant under the action of a chiral half of the dual superconformal symmetry as well as under dual conformal transformations. In this paper we propose a compact, all-loop equation for the action of other dual superconformal generators, denoted as $\bar{Q}$, in terms of a one-dimensional integral over the collinear limit of a higher-point amplitude:

$$\bar{Q}_a^A R_{n,k} = a \;\mathrm{Res}_{\epsilon=0} \int_{\tau=0}^{\tau=\infty} \left(d^{2|3}\mathcal{Z}_{n+1}\right)_a^A \left[R_{n+1,k+1} - R_{n,k} R^{\rm tree}_{n+1,1}\right] + \text{cyclic}, \tag{1.2}$$

where $a, A = 1, \ldots, 4$ are momentum-twistor indices and ϵ, τ parametrize $\mathcal{Z}_{n+1}$ in the collinear limit (τ being related to the longitudinal momentum fraction), Eq. (3.1). In this equation, $a = a(g^2)$ is one quarter of the cusp anomalous dimension

$$a := \frac{1}{4}\Gamma_{\rm cusp} = g^2 - \frac{\pi^2}{3} g^4 + \frac{11\pi^4}{45} g^6 + \ldots, \tag{1.3}$$

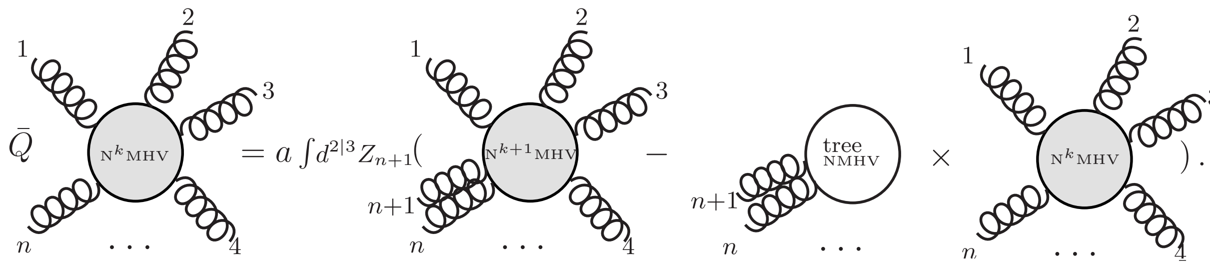


Figure 1. All-loop equation for planar $\mathcal{N}=4$ S-matrix.

known exactly at all values of the coupling $g^2 = \frac{g^2_{\rm YM} N_c}{16\pi^2}$ [4]. We expect the equation to be exact at any value of the coupling, but in this paper we will study it perturbatively with respect to a.

We find Eq. (1.2) natural and pleasing in many respects. First, it relates finite and regulator-independent quantities. Integrating out a particle with measure $(d^{2|3}\mathcal{Z})^A_a$ is virtually the simplest operation one could imagine, which carries the quantum numbers of $\bar{Q}$. The one-dimensional collinear integral over τ reflects the physical intuition that naive $\bar{Q}$ is violated because it causes asymptotic states to radiate collinearly. The presence of two terms on the right-hand side has a simple explanation: if the first term is viewed as the effect of $\bar{Q}$ on the amplitude, then the term with $R^{\rm tree}_{n+1,1}$ is due to the action on the $A^{\rm BDS}$ factor in Eq. (1.1). The proportionality to $\Gamma_{\rm cusp}$ of the second term is thus easy to understand, it being the constant of proportionality in the BDS Ansatz, while the structure itself is rigid: certain divergences which would violate conformal invariance cancel between the two terms. The fact that only $1 \to 2$ splitting appears to all loop orders, as opposed to $1 \to 3, 4, \ldots$ seems difficult to understand from the scattering amplitude viewpoint, and we can only derive it through the duality with Wilson loops.

The equation holds for generic configurations, that is, it neglects so-called distributional terms which are supported on singular configurations. These terms were used in [25] to determine tree amplitudes. By stripping off the MHV tree, which give rise to such terms, the tree amplitudes have been argued to be uniquely determined by requiring analytic properties such as the right collinear behavior, in addition to Yangian invariance [22]. In this spirit, we will assume that all the pertinent information is included by imposing in addition to Eq. (1.2) the correct collinear limits of BDS-subtracted amplitudes, which play the role of boundary conditions to Eq. (1.2).

Using the discrete parity symmetry of scattering amplitudes, we can derive an equivalent equation for the level-one generator

$$Q^{(1)a}_A R_{n,k} = a\ Z^a_n \lim_{\epsilon\to 0}\int_0^\infty \frac{d\tau (d\eta_{n+1})_A}{\tau}\left(R_{n+1,k} - \sum_{1\le j<i\le n-3} C_{n,i,j}\frac{\partial R_{n,k}}{\partial \chi_j}\right) + \text{cyclic}, \qquad (1.4)$$

where $C_{n,i,j}$ is given in Eq. (3.24). The level zero generators $\bar{Q}$ and Q together with $Q^{(1)}$

generate the full Yangian algebra.

It is significant that the right-hand sides of Eq. (1.2) and Eq. (1.4) take the form of linear operators acting on the BDS-subtracted S-matrix (viewing $R_{n,1}^{\rm tree}$ as a collection of constants, see (2.4)). This means that the right-hand side could be moved to the left, and the resulting operators interpreted as quantum-corrected $\bar{Q}$ and $Q^{(1)}$ which annihilate the S-matrix. In other words, these equations are not "anomaly equations" — their content is precisely that all Yangian anomalies can be removed by a simple redefinition of the generators. Symmetry generators which receive quantum corrections but nonetheless admit simple closed form expressions are not uncommon in integrable systems, see for instance [26]. This being said, we will continue to write these equations in the form of a naive (or bare) generator on the left, with the correction on the right-hand side, as this will prove most useful for our applications.

The form of (1.2) is very similar to that obtained by Sever and Vieira in the context of a proposed CSW-regularization of amplitudes [27]. The essential new features are the focus on the finite quantity R, which gives rise to the differenced form, and the advantage of working with integrated amplitude, which results in an all-loop relation with an overall proportionality to $\Gamma_{\rm cusp}$. Our formulas also reproduce the one-loop results of [28].

Our derivation of Eq. (1.2) will be based on the Operator Product Expansion (OPE) for null polygonal Wilson loops [29]. It will be supported by an explicit computation of the fermion dispersion relation to $\mathcal{O}(\Gamma_{\rm cusp}^2)$, finding agreement with [30]. We will also present strong explicit evidence for its all-loop validity, through two- and three-loop computations. In particular, we have reproduced the two-loop MHV [31] and NMHV hexagon [32], and obtained new results for NMHV heptagon and three-loop MHV hexagon, where we have fixed the two undetermined coefficients in the recent Ansatz for its symbol [33].

The paper is organized as follows. We start in section 2 with a short review on momentum twistors, BDS Ansatz and Yangian symmetry. In section 3, we explain how to use (1.2) to compute $\bar{Q}$, especially at one loop, how are MHV and NMHV amplitudes uniquely determined by (1.2), and how to derive (1.4) from it. We then employ the method to reproduce results for two-loop MHV amplitudes in section 4. We continue in section 5 to derive results for two-loop NMHV hexagon and heptagon, as well as three-loop MHV hexagon. In section 6, we outline a derivation of the equation. We also discuss two-dimensional kinematics in section 7. We finish with some conclusions and appendices containing techniques and details of some computations.

2 Momentum twistors, BDS Ansatz and Yangian symmetry

Since our discussion will center on the dual superconformal symmetry, it is advantageous to use momentum-twistor variables introduced by Hodges [34], which manifest the symmetry at least for tree amplitudes. The Wilson loop dual to n-point amplitude is formulated along a n-sided null polygon in a chiral superspace with coordinates (x,θ),

$$x_i^{\alpha\dot{\alpha}} - x_{i-1}^{\alpha\dot{\alpha}} = \lambda_i^{\alpha}\bar{\lambda}_i^{\dot{\alpha}}, \qquad \theta_i^{\alpha A} - \theta_{i-1}^{\alpha A} = \lambda_i^{\alpha}\eta_i^{A}, \tag{2.1}$$

for $i = 1, \ldots, n$, where $\alpha, \dot{\alpha}$ are SU(2) indices of spinors λ_i and their conjugates $\bar{\lambda}_i$ encoding the null momenta of n particles, and A is the SU(4) index of Grassmann variables η_i describing their helicity states. The momentum (super-) twistors are defined as

$$\mathcal{Z}_i = (Z_i^a, \chi_i^A) := (\lambda_i^\alpha, x_i^{\alpha\dot{\alpha}}\lambda_{i\alpha}, \theta_i^{\alpha A}\lambda_{i\alpha}). \tag{2.2}$$

We further define the totally antisymmetric contraction of four twistors, $\langle ijkl\rangle := \varepsilon_{abcd} Z_i^a Z_j^b Z_k^c Z_l^d$, and the basic R-invariant of five super-twistors,

$$[i\,j\,k\,l\,m] := \frac{\delta^{0|4}(\langle\langle i\,j\,k\,l\,m\rangle\rangle)}{\langle ijkl\rangle\langle jklm\rangle\langle klmi\rangle\langle lmij\rangle\langle mijk\rangle}, \tag{2.3}$$

where the argument of Grassmann delta function is $\langle\langle i\,j\,k\,l\,m\rangle\rangle^A := \chi_i^A\langle jklm\rangle$+cyclic. NMHV tree (divided by MHV tree), appearing in (1.2), is simply given by a sum of these R-invariants

$$R_{n,1}^{\text{tree}} = \sum_{1<i<j<n} [1\,i\,i{+}1\,j\,j{+}1]. \tag{2.4}$$

At loop level, the symmetry of amplitudes is broken by infrared divergences, which need to be regulated and subtracted for exact symmetry. Based on the known infrared behavior and the ABDK iterative relation [35], BDS have proposed an exponentiated Ansatz for all-loop MHV amplitudes in $D = 4 - 2\epsilon$ dimensions [24],

$$\frac{A_n^{\text{BDS}}}{A_{n,\text{MHV}}^{\text{tree}}} = 1 + \sum_{\ell=1}^{\infty} \tilde{g}^{2\ell} M_n^{(\ell)}(\epsilon) = \exp\left[\sum_{\ell=1}^{\infty} \tilde{g}^{2\ell}\left(f^{(\ell)}(\epsilon) M_n^{(1)}(\ell\epsilon) + C^{(\ell)} + E_n^{(\ell)}(\epsilon)\right)\right], \tag{2.5}$$

where $\tilde{g}^2 := 2g^2(4\pi e^{-\gamma})^\epsilon$ has been used as the parameter of loop expansion, $f^{(\ell)}(\epsilon) = \frac{1}{4}\Gamma_{\text{cusp}}^{(\ell)} + \mathcal{O}(\epsilon)$, $C^{(\ell)}$(non-vanishing for $\ell > 1$), are independent of kinematics or n, and $E_n^{(\ell)}$ vanish as $\epsilon \to 0$; by stripping off the MHV tree $A_{n,\text{MHV}}^{\text{tree}} = \frac{\delta^4(\sum_i \lambda_i\bar{\lambda}_i)\delta^{0|8}(\sum_i \lambda_i\eta_i)}{\langle 1\,2\rangle\langle 2\,3\rangle\ldots\langle n\,1\rangle}$, the one-loop amplitude $M_n^{(1)} := A_{n,\text{MHV}}^{(1)}/A_{n,\text{MHV}}^{\text{tree}}$ is given by

$$M_n^{(1)} = -\frac{1}{2\epsilon^2}\sum_{i=1}^{n}\left(-\frac{x_{i,i+2}^2}{\mu^2}\right)^{-\epsilon} + F_n^{(1)}(\epsilon), \tag{2.6}$$

where $F_n^{(1)}(\epsilon)$ is a sum of finite parts of the so-called two-mass easy box functions [24]. The BDS Ansatz is believed to be exact for $n = 4, 5$, in which case $R_{4,0} = R_{5,0} = R_{5,1}/R_{5,1}^{\text{tree}} = 1$, and R behaves simply under collinear limits, both k-preserving $R_{n,k} \to R_{n-1,k}$ and k-decreasing $\frac{\int d^4\chi_n R_{n,k}}{\int d^4\chi_n [n{-}2\,n{-}1\,n\,1\,2]} \to R_{n-1,k-1}$, as $\mathcal{Z}_n \to \mathcal{Z}_{n-1}$ (using a parametrization such as Eq. (3.1)).

As a consequence of Poincaré supersymmetry of scattering amplitudes, the BDS subtracted amplitude is invariant under a chiral half of dual superconformal symmetry, and is believed to be invariant under bosonic generators,

$$Q_A^a = \{Q_A^\alpha, \bar{S}_A^{\dot{\alpha}}\} := \sum_{i=1}^{n} Z_i^a \frac{\partial}{\partial\chi_i^A}, \qquad K_b^a := \sum_{i=1}^{n} Z_i^a \frac{\partial}{\partial Z_i^b}. \tag{2.7}$$

Naively the BDS-subtracted amplitude is not annihilated by generators in the other chiral half,

$$\bar{Q}^A_a = \{S^A_\alpha, \bar{Q}^A_{\dot\alpha}\} := \sum_{i=1}^n \chi_i^A \frac{\partial}{\partial Z_i^a}, \tag{2.8}$$

but as we can see from (1.2), the symmetry is restored by a quantum-corrected $\bar{Q}$. Note the correction is manifestly Q-invariant, thus the Q invariance and (1.2) imply the invariance under $K^a_b = \frac{1}{2}\{Q^a_A, \bar{Q}^A_b\}$. For Yangian symmetry, one needs at least one additional level-one generator, e.g. $Q^{(1)a}_A$ which contains the ordinary superconformal generator $s^\alpha_A := \sum_{i=1}^n \frac{\partial}{\partial \lambda_{i\alpha}} \frac{\partial}{\partial \eta_i^A}$,

$$Q^{(1)a}_A = \frac{1}{2}\left(\sum_{i<j} - \sum_{j<i}\right)\left(Z_i^a \frac{\partial}{\partial Z_i^b} Z_j^b \frac{\partial}{\partial \chi_j^A} - Z_i^a \frac{\partial}{\partial \chi_i^B} \chi_j^B \frac{\partial}{\partial \chi_j^A}\right). \tag{2.9}$$

Note that, although second order in derivatives, this is only first order in bosonic derivatives. Since s^α_A is the parity conjugate of $\bar{s}^A_{\dot\alpha}$, which is part of $\bar{Q}$, we will derive the Eq.(1.4) from the conjugate of Eq.(1.2).

3 The $\bar{Q}$ equation

In this section we elaborate on the evaluation of Eq. (1.2). It involves adding a particle in a collinear limit. In the case of edge n, we parameterize its (super-)twistor as

$$\mathcal{Z}_{n+1} = \mathcal{Z}_n - \epsilon \mathcal{Z}_{n-1} + C\epsilon\tau \mathcal{Z}_1 + C'\epsilon^2 \mathcal{Z}_2, \tag{3.1}$$

with $C = \frac{\langle n{-}1 n 23\rangle}{\langle n123\rangle}$ and $C' = \frac{\langle n{-}2\, n{-}1\, n1\rangle}{\langle n{-}2\, n{-}1\, 21\rangle}$. The collinear limit is $\epsilon \to 0$ and, physically, τ is related to the momentum fraction shared by particle $n{+}1$ in that limit. The most general collinear limit would require three parameters, and the third one, which we will not need, could be obtained by replacing C' with an order one function. The signs and normalization have been chosen such that, for real $\epsilon > 0$ and $\tau > 0$, Euclidean n-gons (configurations with positive cross-ratios) are approached by Euclidean $(n{+}1)$-gons.

The basic operation $\mathrm{res}_{\epsilon=0} \int (d^{2|3}\mathcal{Z}_{n+1})^A_a$ can be evaluated as follows. In our parametrization, the bosonic part of the measure is $(d^2 Z_{n+1})_a := (Z_{n+1} dZ_{n+1} dZ_{n+1})_a = C(n{-}1n1)_a \epsilon d\epsilon d\tau$, where only the dominant part at $\epsilon \to 0$ was kept. Thus

$$\mathrm{res}_{\epsilon=0} \int_{\tau=0}^{\tau=\infty} (d^{2|3}\mathcal{Z}_{n+1})^A_a = C(n{-}1n1)_a \; \mathrm{res}_{\epsilon=0} \int \epsilon d\epsilon \int_0^\infty d\tau (d^3\chi_{n+1})^A. \tag{3.2}$$

The notation $\mathrm{res}_{\epsilon=0}$ means to extract the coefficient of $\frac{d\epsilon}{\epsilon}$ in the $\epsilon \to 0$ limit. It is not trivial that this exists, but we will see that there are never singularities stronger than $\frac{d\epsilon}{\epsilon} \log^{\ell-1} \epsilon$ in $\bar{Q}$ of the ℓ-loop amplitude, and that the logarithms always go away after τ integration.

3.1 R-invariants

It is useful to illustrate the procedure on the simplest non-trivial object, NMHV R-invariants. If the R-invariant does not involve $\mathcal{Z}_{n+1}$, the Grassmann integral will produce zero. Furthermore, even if $\mathcal{Z}_{n+1}$ appears, a pole $1/\epsilon$ will be absent unless $\mathcal{Z}_n$ is also present. Thus the only R-invariants which give non-trivial results contain both $\mathcal{Z}_n$ and $\mathcal{Z}_{n+1}$.

Consider the invariant $[i\,j\,k\,n\,n{+}1]$ for i, j, k all distinct from $n{-}1$ and 1. After doing the χ_{n+1} integration one gets

$$\int (d^{2|3}\mathcal{Z}_{n+1})^A_a [i\,j\,k\,n\,n{+}1] = C(n{-}1n1)_a \int \frac{\epsilon d\epsilon d\tau \langle\langle i\,j\,k\,n\,n{+}1\rangle\rangle^A \langle ijkn\rangle^2}{\langle ijnn{+}1\rangle\langle jknn{+}1\rangle\langle kinn{+}1\rangle\langle ijkn{+}1\rangle}, \quad (3.3)$$

and plugging in the parametrization Eq. (3.1) and keeping the dominant term as $\epsilon \to 0$ gives

$$\begin{aligned}\mathrm{res}_{\epsilon=0} \int_{\tau=0}^{\tau=\infty} (d^{2|3}\mathcal{Z}_{n+1})^A_a [i\,j\,k\,n\,n{+}1] &= C(n{-}1n1)_a \mathrm{res}_{\epsilon=0} \frac{d\epsilon}{\epsilon} \int_0^\infty d\tau \frac{\langle\langle i\,j\,k\,n\,B\rangle\rangle^A \langle ijkn\rangle}{\langle ij\,nB\rangle\langle jknB\rangle\langle kinB\rangle} \\ &= C(n{-}1n1)_a \int_0^\infty d\tau \frac{\langle\langle i\,j\,k\,n\,B\rangle\rangle^A \langle ijkn\rangle}{\langle ijnB\rangle\langle jknB\rangle\langle kinB\rangle}, \quad (3.4)\end{aligned}$$

where $\mathcal{Z}_B := \mathcal{Z}_{n-1} - C\tau\mathcal{Z}_1$. We could perform the τ integral here, but it is advantageous not to do so and keep the τ-integrand untouched at this stage. This is because in later applications we will need this integral with additional dependence on τ inserted. However, we can simplify it a bit. It has three poles hence two linearly independent residues. Define the bitwistor $X = X(\tau) := n \wedge B$. Then the residue at $\langle ijX\rangle = 0$ gives

$$(n{-}1n1)_a \frac{\langle\langle i\,j\,k\,n\,[n{-}1\rangle\rangle^A \langle 1]ijn\rangle\langle ijkn\rangle}{\langle ijn1\rangle\langle jkn[n{-}1\rangle\langle 1]ijn\rangle\langle kin[n{-}1\rangle\langle 1]ijn\rangle} = (n{-}1n1)_a \frac{\langle\langle n{-}1\,n\,1\,i\,j\rangle\rangle^A}{\langle n{-}1n1i\rangle\langle n{-}1n1j\rangle}. \quad (3.5)$$

This can be rewritten using the nice identity

$$(n{-}1n1)_a \frac{\langle\langle n{-}1\,n\,1\,i\,j\rangle\rangle^A}{\langle n{-}1n1i\rangle\langle n{-}1n1j\rangle} = \bar{Q}^A_a \log \frac{\langle \bar{n}i\rangle}{\langle \bar{n}j\rangle},$$

where $(\bar{n}) := (n{-}1n1)$, so by adding the other contribution we have,

$$\mathrm{res}_{\epsilon=0} \int_{\tau=0}^{\tau=\infty} d^{2|3}\mathcal{Z}_{n+1}[i\,j\,k\,n\,n{+}1] = \int_0^\infty \left(d\log \frac{\langle Xij\rangle}{\langle Xjk\rangle} \bar{Q} \log \frac{\langle \bar{n}j\rangle}{\langle \bar{n}i\rangle} + d\log \frac{\langle Xjk\rangle}{\langle Xik\rangle} \bar{Q} \log \frac{\langle \bar{n}k\rangle}{\langle \bar{n}i\rangle} \right). \quad (3.6)$$

This is valid at the level of the τ-integrand, for $i, j, k \neq n{-}1, 1$. Other R-invariants are computed similarly, and we complete this subsection by giving the result of the $\mathrm{res}_{\epsilon=0} \int_{\tau=0}^{\tau=\infty} d^{2|3}\mathcal{Z}_{n+1}$ operation on these R-invariants, with $i, j \neq n{-}1, 1$,

$$\begin{aligned}[i\,j\,n{-}1\,n\,n{+}1] &\to \int d\log \frac{\langle Xij\rangle}{\langle Xn{-}2n{-}1\rangle} \bar{Q} \log \frac{\langle \bar{n}j\rangle}{\langle \bar{n}i\rangle}, \\ [i\,j\,n\,n{+}1\,1] &\to \int d\log \frac{\langle Xij\rangle}{\langle X12\rangle} \bar{Q} \log \frac{\langle \bar{n}j\rangle}{\langle \bar{n}i\rangle}, \\ [i\,n{-}1\,n\,n{+}1\,1] &\to \int d\log \frac{\langle Xn{-}2n{-}1\rangle}{\langle X12\rangle} \bar{Q} \log \frac{\langle \bar{n}2\rangle}{\langle \bar{n}i\rangle}. \quad (3.7)\end{aligned}$$

All other R-invariants giving zero. Note these expression all hold at the level of the τ-integrand.

3.2 $\bar{Q}$ of one-loop amplitudes

Armed with just this result, we are ready to evaluate the $\bar{Q}$ of any one-loop amplitude. The right-hand side of Eq. (1.2) reads

$$\bar{Q}R^{\text{1-loop}}_{n,k} = \text{res}_{\epsilon=0}\int_{\tau=0}^{\tau=\infty} d^{2|3}\mathcal{Z}_{n+1}\left(R^{\text{tree}}_{n+1,k+1} - R^{\text{tree}}_{n,k}R^{\text{tree}}_{n+1,1}\right) + \text{cyclic}. \tag{3.8}$$

Using the (P)BCFW formula for removing $\mathcal{Z}_{n+1}$ (associated to the shift $\mathcal{Z}_{n+1} \to \mathcal{Z}_{n+1} + z\mathcal{Z}_1$) [36], the parenthesis can be rewritten as

$$\sum_{i=2}^{n-2}[n\, n{+}1\, 1\, i\, i{+}1]\left(\sum_{k'=0}^{k} R^{\text{tree}}_{i+2,k'}(\widehat{n{+}1},1,\ldots,i,\hat{I}_i)R^{\text{tree}}_{n+1-i,k-k'}(\hat{I}_i,i{+}1,\ldots,n) - R^{\text{tree}}_{n,k}\right), \tag{3.9}$$

up to $\mathcal{Z}_{n+1}$-independent terms which do not contribute to the integral. The dependence on $\mathcal{Z}_{n+1}$ is in the R-invariant and in the shifted twistors, $\widehat{n{+}1} := (nn{+}1)\cap(1ii{+}1)$, $\hat{I}_i := (ii{+}1)\cap(nn{+}11)$. However, the parenthesis has a smooth collinear limit since no term depends simultaneously on both $\mathcal{Z}_n$ and $\mathcal{Z}_{n+1}$. Thus we can merely replace $\hat{\mathcal{Z}}_{n+1}$ by $\mathcal{Z}_n$ and $\hat{I}_i$ by its limit $I_i = (n{-}1n1)\cap(ii{+}1)$ (supersymmetrically, the fermions of I_i are taken from χ_i and χ_{i+1}). The $\mathcal{Z}_{n+1}$ dependence is limited to the R-invariant and using Eq. (3.7) the integral gives

$$\bar{Q}R^{\text{1-loop}}_{n,k} = \int_{\tau=0}^{\tau=\infty}\sum_{i=2}^{n-2} d\log\frac{\langle Xii{+}1\rangle}{\langle X12\rangle}\bar{Q}\log\frac{\langle\bar{n}i\rangle}{\langle\bar{n}i{+}1\rangle}(\text{parenthesis in Eq. (3.9)}) + \text{cyclic}. \tag{3.10}$$

In the parenthesis, $\widehat{n{+}1} \to n$, $\hat{I}_i \to I_i$, and nothing depends on τ.

We must show that this integral is convergent at its endpoints. Near $\tau = 0$, there is a pole due to $d\log\langle Xn{-}2n{-}1\rangle$ in the term $i = n{-}2$. However, the two terms in the parenthesis cancel in this case, so there is no problem. There is also a pole at $\tau = \infty$, due to $d\log\langle X12\rangle$ present in every term. It is nontrivial to see that it cancels out in the sum, but can be proved as follows.

Instead of using the (P)BCFW formula associated with shifting $\mathcal{Z}_{n+1}$, we could have used the BCFW formula associated with the shift $\mathcal{Z}_n \to \mathcal{Z}_n + z\mathcal{Z}_{n-1}$. Then we would have obtained the same formula but with the R-invariants replaced with $[i\, i{+}1\, n{-}1\, n\, n{+}1]$, and so $d\log\frac{\langle Xii{+}1\rangle}{\langle Xn{-}2n{-}1\rangle}$ in the integrand, but the parenthesis in the $\epsilon \to 0$ limit unchanged. This form would make convergence at $\tau = \infty$ manifest but not at $\tau = 0$. We conclude that overall convergence follows beautifully from the equality of the BCFW and (P)BCFW representations of tree amplitudes.

Given these cancelations, we can integrate Eq. (3.11) termwise by dropping the $d\log\langle X12\rangle$ factor and the $i = n{-}2$ term, obtaining simply

$$\begin{aligned}\bar{Q}R^{\text{1-loop}}_{n,k} = \sum_{i=2}^{n-3}&\log\frac{\langle n1ii{+}1\rangle}{\langle n{-}1nii{+}1\rangle}\bar{Q}\log\frac{\langle\bar{n}i\rangle}{\langle\bar{n}i{+}1\rangle}\\ &\times\left(\sum_{k'}R^{\text{tree}}(n,1,\ldots,i,I_i)R^{\text{tree}}(I_i,i{+}1,\ldots,n) - R^{\text{tree}}_{n,k}\right) + \text{cyclic},\end{aligned} \tag{3.11}$$

which agrees with the formula of [28] (there the product of tree amplitudes is interpreted in terms of unitarity cuts). The subtraction of $R_{n,k}^{\rm tree}$ arises because we are considering the BDS-subtracted amplitude, which is the one-loop N^{k}MHV ratio function in this case.

3.3 Uniqueness of $\bar{Q}$ solutions at MHV and NMHV

The $\bar{Q}$ equation is especially interesting because as we will see now, it fixes uniquely MHV *and* NMHV amplitudes (assuming that the right-hand side is known). This is not too difficult to see for MHV amplitudes using the momentum-twistor form of $\bar{Q}$, (2.8). Indeed, taking derivatives of the equation $\bar{Q}f(Z) = 0$ for any function of bosonic Z's, $f(Z)$, we have,

$$\frac{\partial}{\partial \chi_i^1}\bar{Q}_a^1 f(Z) = 0 \Rightarrow \frac{\partial}{\partial Z_i^a} f(Z) = 0. \tag{3.12}$$

This equation, for all particle labels i and twistor indices $a = 1\ldots 4$, implies that a bosonic function annihilated by $\bar{Q}$ is a constant. Thus the ambiguity of the $\bar{Q}$ equation is at most a constant, which can be fixed using the properties of the BDS-subtracted amplitudes in collinear limits.

For NMHV amplitudes, we have to work harder to restrict the kernel of $\bar{Q}$. A simple example which illustrates this at 5-points is $[12345]\log\frac{\langle 1234\rangle}{\langle 1235\rangle}$. This has vanishing $\bar{Q}$ because

$$[12345]\bar{Q}\log\frac{\langle 1234\rangle}{\langle 1235\rangle} = [12345]\frac{(123)\langle\langle 12345\rangle\rangle}{\langle 1234\rangle\langle 1235\rangle} \tag{3.13}$$

contains $\langle\langle 12345\rangle\rangle$ both explicitly and from the Grassmann delta function $\delta^{0|4}(\langle\langle 12345\rangle\rangle)$ in the R-invariant, hence vanishes. On the other hand, this expression is not acceptable because the argument of the logarithm is not conformal invariant (synonymous with little group invariance in what follows): it has non-vanishing weight with respect to 4 and 5. So it does not correspond to any real ambiguity. This turns out to be general: any NMHV expression with neutral little group and annihilated by both Q, $\bar{Q}$, is a sum of R-invariants with constant coefficients.

To prove this, we first note that by Q invariance alone, any NMHV expression can be written as

$$F = \sum_{2\leq j<k<l<m\leq n} [1\,j\,k\,l\,m] F_{j,k,l,m}(Z) \tag{3.14}$$

where the $\begin{pmatrix} n-1 \\ 4\end{pmatrix}$ $[1\,j\,k\,l\,m]$'s form a basis for all independent NMHV R-invariants at n-point [37]. Each $F_{j,k,l,m}(Z)$ is a conformal invariant function of the bosonic Z's.

To show that the $F_{j,k,l,m}(Z)$ must be constant, we pick $i \notin \{1,j,k,l,m\}$ and extract a specific component, $\chi_i^1\chi_j^1\chi_k^2\chi_l^3\chi_m^4$, of $Z_j^a\bar{Q}_a^1 F$. The only way χ_j^1 can arise is either from $\bar{Q}F_{j,k,l,m}$ or from a R-invariant, but since $Z_j^a\frac{\partial}{\partial Z_j^a}F = 0$, it can only arise from a R-invariant. Since only $[1\,j\,k\,l\,m]$ contains the prescribed components, we deduce that

$$Z_j^a\frac{\partial}{\partial Z_i^a}F_{j,k,l,m} = 0. \tag{3.15}$$

Repeating this with permutations of j, k, l, m shows that $F_{j,k,l,m}$ is independent of Z_i, and repeating for other i's shows that $F_{j,k,l,m}$ depends only on twistors $1, j, k, l, m$. But since there are no nontrivial little-group invariant functions of five twistors, $F_{j,k,l,m}$ must be a constant, QED.

All remaining constant ambiguities can be fixed by collinear limits. As mentioned, there are both k-preserving and k-decreasing collinear limits. It turns out that just four of the k-preserving limits suffice for any n. For instance, working in the same basis, the k-preserving collinear limit $\mathcal{Z}_1 \to \mathcal{Z}_2$ will fix all constants except those multiplying invariants of the form $[1\,2\ldots]$. Taking the limit $\mathcal{Z}_2 \to \mathcal{Z}_3$ will then fix the coefficient of all but those beginning with $[1\,2\,3\ldots]$, and so on.

These results open up the possibility of using the $\bar{Q}$ equation to compute nontrivial MHV and NMHV amplitudes. Given one-loop N^{k}MHV amplitudes as the seed for recursion, this will restrict the applications in this paper to two-loop MHV and NMHV and three-loop MHV. To go beyond NMHV it becomes necessary to use both the $\bar{Q}$ and $Q^{(1)}$ equations[1], or, equivalently, the $\bar{Q}$ equation and parity. Uniqueness then follows from a theorem proved in [38, 39]: *all Yangian invariants are combinations of compact contour integrals inside the Grassmannian* $G(k, n)$. We conclude that any N^{k}MHV expression annihilated by (naive) $Q, \bar{Q}, Q^{(1)}$ can only be a combinations of such invariants, multiplied by c-numbers, which we expect to be determined by collinear limits.

3.4 The one-loop NMHV hexagon

In the case $n = 6$, equation (3.11) evaluates more or less directly to

$$\bar{Q} R_{6,1}^{\text{1-loop}} = ((5) + (3)) \log u_3 \bar{Q} \log \frac{\langle 5612\rangle}{\langle 5613\rangle} + (1) \log u_3 \bar{Q} \log \frac{\langle 5613\rangle}{\langle 5614\rangle} + \text{cyclic}, \tag{3.16}$$

where (1) is the R-invariant [23456], (i) is obtained by a cyclic shift, and we have used that $R_{6,1}^{\text{tree}} = (1) + (3) + (5) = (2) + (4) + (6)$. The appearance of $\log u_3$ is easy to understand from the two terms in Eq. (3.11), because they correspond to two poles of the τ-integral, and so what multiplies the log has to be equal and opposite. We use the following cross-ratios

$$u_1 = \frac{\langle 1234\rangle\langle 4561\rangle}{\langle 1245\rangle\langle 3461\rangle}, \qquad u_2 = \frac{\langle 2345\rangle\langle 5612\rangle}{\langle 2356\rangle\langle 1245\rangle}, \qquad u_3 = \frac{\langle 3456\rangle\langle 6123\rangle}{\langle 3461\rangle\langle 2356\rangle}. \tag{3.17}$$

We have just shown that the information in Eq. (3.16) should suffice to determine $R_{6,1}$. Let us see how this works. The crucial step is to bring the right-hand side to a form where the argument of each $\bar{Q}$ is the logarithm of a conformal invariant cross-ratio; this form will be unique. This can be achieved by adding suitable combinations of zero in the form of equation (3.13), for which there is a systematic procedure.

[1] A simple counter-example to $\bar{Q}$ uniqueness is the invariant $\frac{\delta^{0|4}(\langle 1234\rangle\chi_5\chi_6+\text{cyclic})}{\langle 1234\rangle\cdots\langle 6123\rangle}$ which arises in the 6-point N^{2}MHV tree amplitude and depends on six twistors. Any conformal invariant cross-ratio of the six twistors multiplying it will be $\bar{Q}$-invariant.

The following procedure reduces this to a simple linear algebra problem. The first step is to remove the ambiguities in writing the R-invariants, by using the identity $(1)-(2)+(3)-(4)+(5)-(6)=0$ to remove (6). We can then use four distinct nontrivial representations of zero, $[(1)-(2)+(3)-(4)+(5)]$ times $\bar{Q}\frac{\langle 1234\rangle}{\langle 1235\rangle}$, $\bar{Q}\frac{\langle 1234\rangle}{\langle 1245\rangle}$, $\bar{Q}\frac{\langle 1234\rangle}{\langle 1345\rangle}$, $\bar{Q}\frac{\langle 1234\rangle}{\langle 2345\rangle}$, to remove e.g. the little group weight with respect to i of the coefficient of (i), for $i=1,\ldots 4$.

Actually, there is a final constraint: it is not trivial the little group weight with respect to 5 of the coefficient of (5) is also removed; but this is is the case. Then the coefficient of (i) has correct little group weight with respect to i for $i=1,\ldots,5$. The little-group weights with respect to other variables can then be removed using equation (3.13) with R-invariants $(1),\ldots,(5)$.

This procedure is simple to follow but not particularly illuminating, so we spare the reader the details, recording only the final result:

$$\bar{Q}R^{\text{1-loop}}_{6,1}=\left(R^{\text{tree}}_{6,1}\bar{Q}\log\frac{u_1u_2}{1-u_3}-((1)+(4))\bar{Q}\log u_2-((2)+(5))\bar{Q}\log u_1\right)\log u_3+\text{cyclic}. \tag{3.18}$$

This equation is equal to Eq. (3.16), but now the $\bar{Q}$ acts on conformal invariants. The upshot is that in this form we are allowed to directly integrate $\bar{Q}$:

$$R^{\text{1-loop}}_{6,1}=R^{\text{tree}}_{6,1}(\log u_2\log u_3+\text{Li}_2(1-u_3))-((1)+(4))\log u_2\log u_3+\text{cyclic}+C, \tag{3.19}$$

where C is an undetermined combination of R-invariants with c-number coefficients.

To fix C, we can consider collinear limits. For instance, the ratio function should vanish in the k-preserving limit $\mathcal{Z}_6\to\mathcal{Z}_5$, corresponding to $u_1\to 0$ and $u_2\to 1-u_3$. This limit probes the coefficient of (5) plus the coefficient of (6). In this limit, what we have in Eq. (3.19) goes to $\frac{\pi^2}{3}[12345]$. All other k-preserving limits go to the same number, allowing us to fix

$$C=-\frac{\pi^2}{3}R^{\text{tree}}_{6,1}. \tag{3.20}$$

This is the correct ratio function!

3.5 Derivation of equation (1.4) from equation (1.2)

This subsection lies a bit outside the main scope of this paper. As noted in the Introduction, the equation for $\bar{Q}$ together with parity symmetry of scattering amplitudes implies an equation for $Q^{(1)}$.

To derive it, the first step is to express Eq. (1.2) in the language of scattering amplitudes. We only need to do this for the two components of $\bar{Q}$ which coincide with the ordinary superconformal generators $\bar{s}^A_{\dot\alpha}$ [9]. Technically, we really only need to do this for the first term in the parenthesis of Eq. (1.2), and we can drop the explicit dependence on ϵ, reinstating it at the end. We find, after reinstating the MHV prefactor and changing variable $C\tau\to\frac{\langle n1\rangle}{\langle n-1n\rangle}x/(1-x)$,

$$\bar{s}^A_{\dot\alpha}A_{n,k}=\tilde\lambda_{n\dot\alpha}\lim_{\epsilon\to 0}\int_0^1 dx(d^3\chi)^A A_{n+1,k+1}(\ldots,\{\lambda_n,x\tilde\lambda_n,x\tilde\eta_n+\chi\},\{\lambda_n,(1-x)\tilde\lambda_n,(1-x)\tilde\eta_n-\chi\})+\ldots.$$

The variable x is the usual longitudinal momentum fraction. With the help of the *BCFW* computer package for the evaluation of tree amplitudes [40], we have verified that the integral gives the correct result acting on the NMHV 5,6,7 point tree amplitudes. The upshot is that in this form it is possible to immediately write down the parity-conjugate equation:[2]

$$s_A^\alpha A_{n,k} = \lambda_n^\alpha \lim_{\epsilon\to 0}\int_0^1 \frac{dx(d\chi)_A}{x(1-x)} A_{n+1,k}(\dots,\{\lambda_n, x\tilde\lambda_n, x\tilde\eta_n{+}\chi\},\{\lambda_n,(1{-}x)\tilde\lambda_n,(1{-}x)\tilde\eta_n{-}\chi\}){+}\dots,$$

where the denominator $1/x(1-x)$ comes from a little group transformation needed after interchanging λ and $\tilde\lambda$. The final step is to convert this equation back to momentum twistors:

$$Q_A^{(1)a} A_{n,k} = Z_n^a \lim_{\epsilon\to 0}\int_0^\infty \frac{d\tau}{\tau}(d\eta_{n+1})_A R_{n+1,k}(1,\dots,n{+}1) + \dots \tag{3.21}$$

where we have put back the ϵ dependence, $\mathcal{Z}_{n+1}(\epsilon,\tau)$ being again given by Eq. (3.1). Strictly speaking, s_A^α gives two out of the four twistor components of $Q_A^{(1)a}$. The remaining two components come for free, because the level-zero conformal symmetry of Wilson loops is unbroken acting on BDS-subtracted amplitudes.

This takes care of the first term in the parenthesis of Eq. (1.2). To deal with the second term we need the explicit form of acting $\bar Q$ on NMHV tree,

$$\mathrm{res}_{\epsilon=0} d^{2|3}\mathcal{Z}_{n+1}R^{\mathrm{tree}}_{n+1,1} = \sum_{i=2}^{n-3}\frac{\langle\langle \bar n\, i\, i{+}1\rangle\rangle}{\langle \bar n\, i\rangle\langle \bar n\, i{+}1\rangle} d\log\frac{\langle n\, n{+}1\, i\, i{+}1\rangle}{\langle n\, n{+}1\, n{-}2\, n{-}1\rangle}. \tag{3.22}$$

We take its parity conjugate using $\chi_i = \sum_{j=n-2}^{i}\langle ij\rangle\eta_j$ and then $\eta_j \to \frac{\partial}{\partial\eta_j}$. In terms of momentum twistors, the end result is

$$Q_A^{(1)a} R_{n,k} = a\; Z_n^a \lim_{\epsilon\to 0}\int_0^\infty \frac{d\tau}{\tau}\left((d\chi_{n+1})_A R_{n+1,k} - \sum_{1\le j<i\le n-3} C_{n,i,j}\frac{\partial R_{n,k}}{\partial \chi_j^A}\right) + \text{cyclic} \tag{3.23}$$

where

$$C_{n,i,j}(\tau) = \frac{\langle n\bar i\rangle\langle j\overline{i{+}1}\rangle - \langle n\overline{i{+}1}\rangle\langle j\bar i\rangle}{\langle n\bar i\rangle\langle n\overline{i{+}1}\rangle}\tau\frac{d}{d\tau}\log\frac{\langle nn{+}1ii{+}1\rangle}{\langle nn{+}1n{-}2n{-}1\rangle}, \tag{3.24}$$

which is the formula recorded in the Introduction.

Because this is a consequence of unbroken parity symmetry and the equation for $\bar Q$, this does not require separate verification. To ascertain that Eq. (3.23) contains no mistake, we have tested it on known expressions for 1-loop 6,7,8-point NMHV amplitudes.

[2] The easiest way to derive this equation is to consider the case where particle n is a positive-helicity gluon in the $\bar s_{\dot\alpha}^A$ equation. Then the χ integral gives eight terms on the right hand side, involving a minus-helicity fermion, or a scalar plus a plus-helicity fermion, with various R-symmetry assignments. Parity dictates that when n is a negative-helicity gluon, s_A^α should produce the eight parity conjugate terms. The correctness of the other cases follows by supersymmetry.

4 Two-loop MHV amplitudes

Armed with just the $\bar{Q}$ equation (1.2), expressions for one-loop NMHV ratio functions, and the $d^{2|3}\mathcal{Z}$ integral of R-invariants Eq. (3.7), we are now ready to analyze two-loop MHV amplitudes.

4.1 The square and the pentagon

In the cases $n = 4$ and $n = 5$, it is well-known that $\log R_{4,5} = 0$ [24]. Let us begin by reproducing this simple result starting from equation (1.2). For $n = 4$, this is essentially trivial for all loops provided $\log R_5 = 0$ at one lower loop order, so the equation reads

$$\bar{Q}R_4 = 0. \tag{4.1}$$

This implies that R_4 is a constant, which must be trivial by the boundary condition.

In the case $n = 5$, starting at two loops, the right-hand side is not so trivially zero. Rather, it involves the collinear limit of the one-loop six-point NMHV amplitude given in Eq. (3.19). Letting

$$\mathrm{res}_{\epsilon=0}\int_{\tau=0}^{\tau=\infty} d^{2|3}\mathcal{Z}_6 R_6^{(1)} = \bar{Q}\log\frac{\langle 4512\rangle}{\langle 4513\rangle}\times I, \tag{4.2}$$

we get that the R-invariants contribute to I as follows,

$$(1)\to d\log\frac{\tau}{\tau+1},\quad (2)\to d\log\tau,\quad (4)\to -d\log(\tau+1),\quad (3),(5),(6)\to 0. \tag{4.3}$$

In the collinear limit $u_1\to\epsilon^2$, $u_2\to\frac{1}{1+\tau}$, $u_3\to\frac{\tau}{1+\tau}$, allowing us to write $I = I_1 + I_2$ with

$$\begin{aligned} I_1 &= \int_0^\infty d\left(\log\frac{(1+\tau)^2}{\tau}\right)\log(1+\tau)\log(1+1/\tau), \\ I_2 &= \log\epsilon^2\times\int_0^\infty d\left(\log(1+\tau)\log(1+1/\tau)\right). \end{aligned} \tag{4.4}$$

We find that $I_1 = I_2 = 0$, confirming that $\bar{Q}R_5 = 0$ as expected. This is the first nontrivial hint that the equation is working beyond one-loop.

The reader might worry about the divergent prefactor $\log\epsilon^2$ in front of I_2. Shouldn't the $\epsilon\to 0$ limit entering our basic equation be well-defined? The answer is that the order of operations is important. The limit $\epsilon\to 0$ will always be well-defined provided the integration over τ is carried out first. If one were to take instead $\epsilon\to 0$ with fixed τ, one would find a divergence. This divergence has a simple explanation and is actually predicted by the Wilson loop OPE [29]. We will return to it in subsection 6.3 where we confirm the quantitative prediction for it, and also give the general argument for its cancelation after τ-integration.

4.2 The hexagon

For $n = 6$, we need the one-loop seven-point NMHV amplitude, which can be put in a compact form [41],

$$R_{7,1}^{(1)} = [1,\{2,3\},\{4,5,6\}]I_1 + [1,\{2,3\},\{4,5,6,7\}]I_1' + \text{cyclic}, \tag{4.5}$$

where

$$[i,\{i{+}1,...,j\},\{k,...,l\}]=\sum_{J=\{i+1,i+2\}}^{\{j-1,j\}}\sum_{K=\{k,k+1\}}^{\{l,k\}}[i,J,K], \tag{4.6}$$

$$\begin{aligned}
I_1 =&\ \mathrm{Li}_2(1-v_7v_3)+\mathrm{Li}_2(1-v_1)+\mathrm{Li}_2(1-v_3v_6)+\mathrm{Li}_2(1-v_6v_2)\\
&-\mathrm{Li}_2(1-v_1v_4)-\mathrm{Li}_2(1-v_6)-\mathrm{Li}_2(1-v_3)-\mathrm{Li}_2(1-v_5v_1)+\log v_7\log v_2,\\
I'_1 =&\ \mathrm{Li}_2(1-v_7)+\mathrm{Li}_2(1-v_6)+\mathrm{Li}_2(1-v_3)+\mathrm{Li}_2(1-v_5v_1)\\
&-\mathrm{Li}_2(1)-\mathrm{Li}_2(1-v_7v_3)-\mathrm{Li}_2(1-v_3v_6)+\log v_7\log v_6,
\end{aligned} \tag{4.7}$$

and I_i, I'_i are obtained by cyclic shifts for $i=2,...,7$. Here we need to define cross-ratios beyond six points

$$u_{i,j,k,l}=\frac{\langle i\,i{+}1\,j\,j{+}1\rangle\langle k\,k{+}1\,l\,l{+}1\rangle}{\langle i\,i{+}1\,k\,k{+}1\rangle\langle j\,j{+}1\,l\,l{+}1\rangle}, \tag{4.8}$$

and at seven points, a basis of cross-ratios can be chosen as $v_i := u_{i+1,i+3,i+4,i}$ for $i=1,...,7$. In the collinear limit $\mathcal{Z}_7\to\mathcal{Z}_6$, $v_4\to 0$, $v_3\to(1-v_2)/(1-v_2v_6)$ and $v_5\to(1-v_6)/(1-v_2v_6)$, thus the result depends on v_1,v_2,v_6,v_7.

The R-invariants appearing are not independent, and it is convenient to choose those containing the label 2 as a basis. Upon doing the integral over $d^{2|3}\mathcal{Z}_7$, only [12367], [12467], [23467], [23567] and [24567] contribute ([12567] does not contribute, because its coefficient has to vanish due to the k-decreasing collinear limit constraint), which produce, proceeding as in the five-point example,

$$\bar{Q}R^{(2)}_{6,0}=(I_{1,1}+I_{1,2})\bar{Q}\log\frac{\langle 5613\rangle}{\langle 5612\rangle}+(I_{2,1}+I_{2,2})\bar{Q}\log\frac{\langle 5614\rangle}{\langle 5612\rangle}+\text{cyclic}, \tag{4.9}$$

where

$$\begin{aligned}
I_{1,1} &= \int_0^\infty \begin{pmatrix} d\log(\frac{\tau}{\tau+u_3})\left(\log\frac{u_2(\tau+u_3)}{\tau}\log\frac{u_3(\tau+1)}{\tau+u_3}+\mathrm{Li}_2(1-u_3)-\mathrm{Li}_2(\frac{1-u_3}{\tau+u_3})\right)\\ +d\log(\tau+1)\left(\log u_1(\tau+1)\log\frac{\tau+u_3}{\tau+1}+\mathrm{Li}_2(1-u_3)-\mathrm{Li}_2(\frac{(1-u_3)\tau}{\tau+u_3})\right)\\ +d\log(\frac{\tau+u_3}{\tau+1})\left(\log\frac{u_2(\tau+u_3)}{u_3}\log\frac{\tau+1}{\tau+u_3}-\log(\tau+1)\log\frac{u_1(\tau+1)}{\tau+u_3}\right.\\ \left.+\mathrm{Li}_2(1-u_1)+\mathrm{Li}_2(1-u_2)+\log u_1\log u_2-\frac{\pi^2}{6}\right)\end{pmatrix},\\
I_{1,2} &= \log\epsilon^2\times\int_0^\infty d\left(\log\frac{u_3(\tau+1)}{\tau+u_3}\log(\frac{\tau}{\tau+u_3})+\log(\tau+1)\log\frac{\tau+u_3}{\tau+1}\right),
\end{aligned} \tag{4.10}$$

and

$$\begin{aligned}
I_{2,1} &= \int_0^\infty \begin{pmatrix} d\log\frac{\tau+u_3}{\tau}\left(\log\frac{u_2(\tau+u_3)}{\tau}\log\frac{u_3(\tau+1)}{\tau+u_3}-\log(\tau+1)\log\frac{\tau+1}{\tau}\right.\\ \left.+\mathrm{Li}_2(1-u_2)+\mathrm{Li}_2(1-u_3)-\mathrm{Li}_2(1-\frac{u_2}{\tau+1})-\mathrm{Li}_2(\frac{1-u_3}{\tau+1})\right)\\ +d\log(\tau+u_3)\left(\log\frac{\tau+u_3}{\tau}\log\frac{u_3}{\tau+u_3}+\mathrm{Li}_2(1-u_1)-\mathrm{Li}_2(1-\frac{u_1}{\tau+u_3})\right)\\ +d\log\frac{\tau+u_3}{\tau+v}\left(\mathrm{Li}_2(1-\frac{u_1\tau}{\tau+u_3})+\mathrm{Li}_2(1-\frac{u_2}{\tau+1})+\log\frac{u_1\tau}{\tau+u_3}\log\frac{u_2}{\tau+1}-\frac{\pi^2}{6}\right)\end{pmatrix},\\
I_{2,2} &= \log\epsilon^2\times\int_0^\infty d\left(\log\frac{\tau+u_3}{\tau}\log\frac{u_3}{\tau+u_3}\right).
\end{aligned} \tag{4.11}$$

The non-spacetime ratio $v = \frac{\langle 5624\rangle\langle 6123\rangle}{\langle 5623\rangle\langle 6124\rangle}$ is needed to produce a parity-odd part.

We emphasize that this comes directly out of the collinear limit of the heptagon. No manhandling has been applied, nor would have been necessary. We have, in the interest of this presentation, used standard dilogarithm identities to simplify the expression and hopefully make it more human-readable, but we have not used integration by part nor any manipulation which would affect the numerical value of the τ-integrand.

The divergent terms cancel upon integration: $I_{1,2} = I_{2,2} = 0$, just as in the pentagon example. This cancelation is of paramount importance to our approach, and after it is effected, we are left with two finite and manifestly conformal-invariant integrals $I_{1,1}$ and $I_{2,1}$. The mechanism for this cancelation is general and detailed in subsection 6.3.

The integrals produce trilogarithms. Computing them is not entirely trivial (for instance, Mathematica would not do them automatically), but obtaining their symbols is, following, for instance, the method of Appendix A. From the symbol it is not too difficult to obtain actual functions, and then fix beyond-the-symbol ambiguities using the differential computed in Appendix A. The resulting functions are quite simple

$$\begin{aligned} I_{1,1} &= \left(\frac{1}{3}\log^2 u_3 + \log u_1 \log u_2 + \sum_{i=1}^{3} \mathrm{Li}_2(1-u_i)\right)\log u_3 - 2\mathrm{Li}_3(1-1/u_3), \\ I_{2,1} &= -\frac{1}{2} I_6^{6D} + \mathrm{Li}_3(1-1/u_2) + \mathrm{Li}_3(1-1/u_3) - \mathrm{Li}_3(1-1/u_1) + \frac{1}{12}\log^3\frac{u_2 u_3}{u_1} \\ &\quad + \frac{1}{2}\log\frac{u_2 u_3}{u_1}\sum_{i=1}^{3}\mathrm{Li}_2(1-1/u_i), \end{aligned} \tag{4.12}$$

where I_6^{6D} is the six-dimensional massless hexagon integral [42, 43], reproduced in Appendix B alongside the definitions of $x^{\pm}$ and L_4^+ to be used shortly.

To complete the computation of $\bar{Q}R_6 \equiv dR_6$, we go back to Eq. (4.9) and add the other edges contribution via cyclic symmetry. For future reference, we record the simple result:

$$dR_{6,0}^{(2)} = I_6^{6D} d\log\frac{x^+}{x^-} + \left(I_{1,1} d\log\frac{1-u_3}{u_3} + \text{two cyclic}\right). \tag{4.13}$$

This agrees precisely with the differential of Goncharov, Spradlin, Vergu and Volovich's formula [31], derived from the results in [44],

$$\frac{1}{4}R_{6,0}^{(2)} = \sum_{i=1}^{3}\left(L_4^+(x^+u_i, x^-u_i) - \frac{1}{2}\mathrm{Li}_4(1-1/u_i)\right) - \frac{1}{8}\left(\sum_{i=1}^{3}\mathrm{Li}_2(1-1/u_i)\right)^2 + \frac{1}{24}J^4 + \frac{\pi^2}{12}J^2 + \frac{\pi^4}{72}. \tag{4.14}$$

Of course, in practice the step from Eq. (4.13) to Eq. (4.14) can be a very difficult one, and we do not wish to imply otherwise; we have simply gone the other way, taking the derivative of Eq. (4.14). Still, it is impressive how close to Eq. (4.14) the present formalism lands us, namely, on Eq. (4.13). Important qualitative features of the result, such as its finiteness, transcendental degree and conformal invariance, were manifest at every stage of the computation.

4.3 The differential of the n-gon

To obtain results for $n > 6$ two-loop MHV amplitudes, we need the $n+1 > 7$-point one-loop NMHV amplitudes. Since there are no qualitative differences between these $n+1 > 7$-point amplitudes and the seven-point one, the computation is similar in every respect. We have verified that the divergent terms integrate to zero for generic n, leaving a set of finite and manifestly conformal integrals, which are too lengthy to record here. However, we have explicitly obtained these integrals for $n = 6, 7, 8, 9$ (which is generic) using the present method, and we can compare this result with that given in [23] (specifically, equations (4.21) and (4.28) there). We find perfect agreement: numerically both one-dimensional integrals give the same to 30-digits precision on a few randomly generated Euclidean kinematic points, and symbolically, they give the same symbol. We recall that these integrals give degree-three transcendental functions characterizing the full differential of the amplitudes.

5 Two-loop NMHV and three-loop MHV amplitudes

5.1 The two-loop NMHV hexagon

Because one-loop N^2MHV amplitudes are known, there is no reason to stop at MHV level. (For one-loop amplitudes, we have used expressions based on the box-expansion and generalized unitarity expressed in momentum twistor space [21, 45–48].) The first step in our procedure to compute the NMHV hexagon is to take the collinear limit of the one-loop seven-point N^2MHV amplitude and extract the $d\epsilon/\epsilon$ term from the $d^{2|3}\mathcal{Z}$ integration. Just as in the previous cases, one obtains a one-dimensional integral over the variable τ, and after verifying that terms proportional to $\log\epsilon$ integrate to zero, one is left with a manifestly finite and conformal integral over polylogarithms of degree two. The integrals are not significantly more difficult than those appearing in the MHV case, and can be done similarly; we only record the result:

$$\begin{aligned}\bar{Q}R^{(2)}_{6,1} = {} & (6)\bar{Q}\log\frac{\langle\bar{6}2\rangle}{\langle\bar{6}4\rangle}f_1 + ((1)-(2)+(4)-(5))\bar{Q}\log\frac{\langle\bar{6}4\rangle}{\langle\bar{6}2\rangle}f_2 + ((2)-(4))\bar{Q}\log\frac{\langle\bar{6}4\rangle}{\langle\bar{6}2\rangle}f_3 \\ & + \left((6)\bar{Q}\log\frac{\langle\bar{6}2\rangle}{\langle\bar{6}3\rangle} + ((5)-(4))\bar{Q}\log\frac{\langle\bar{6}2\rangle}{\langle\bar{6}4\rangle}\right)f_4 + ((2)+(4))\bar{Q}\log\frac{\langle\bar{6}2\rangle}{\langle\bar{6}4\rangle}f_5 \\ & + (5)\bar{Q}\log\frac{\langle\bar{6}2\rangle}{\langle\bar{6}3\rangle}f_6 + (3)\bar{Q}\log\frac{\langle\bar{6}2\rangle}{\langle\bar{6}3\rangle}f_7, \end{aligned} \tag{5.1}$$

where $f_1, \ldots, f_7$ are degree-three transcendental functions reproduced in Appendix B.

The fact the result could be expanded over a basis of 7 linearly independent rational prefactors (of the form (R-invariant) times $\bar{Q}\log\frac{\langle\bar{n}i\rangle}{\langle\bar{n}j\rangle}$), times integrals with unit residues, follows from a general Grassmannian analysis. The key fact is that these prefactors all originate from seven-point N^2MHV leading singularities, which are combinations of the 15 independent residues in the $G(2,7)$ momentum twistor Grassmannian [20, 21]. In fact, we found that coming up with the full list of the 7 prefactors was the most nontrivial part in our derivation of the above equation. After this was known, the $\text{res}_{\epsilon=0}$ part of the $d^{2|3}\mathcal{Z}$ integration step could be easily automated on a computer.[3] There remained only the τ integration, which could be done automatically at the level of the symbol and with a bit of human input for the function.

After obtaining this equation, we are (already) essentially done. To complete this computation, we need to use cyclic symmetry to obtain the contribution of other edges, and plug the result into the exact same linear algebra problem as encountered in the one-loop example in subsection 3.4. Namely, starting from the 42 rational prefactors obtained from symmetrizing the above 7 over the 6 edges, we need to add "zero" in the form of equation (3.13) to make the argument of all $\bar{Q}$'s become cross-ratios. Just like at one-loop, we found exactly one potential obstruction, which vanished for the above f_i's, leaving 41 truly independent functions. We expect this counting to be the same at all higher loop orders. Expressing the amplitude in the form [49]

$$\begin{aligned} R_{6,1}^{(2)} = \frac{1}{2}\Big(&[(1)+(4)]V_3 + [(2)+(5)]V_1 + [(3)+(6)]V_2 \\ &+[(1)-(4)]\tilde{V}_3 + [(5)-(2)]\tilde{V}_1 + [(3)-(6)]\tilde{V}_2\Big), \end{aligned} \tag{5.2}$$

then the solution yields the differentials of each V's and $\tilde{V}$'s. The resulting formulas are reported in Appendix B, together with the definition of y variables.

From these differentials we can already check that the symbols of V and $\tilde{V}$ agree with those obtained recently by Dixon, Drummond and Henn [32], attached with their arXiv submission; they do. This is one first nontrivial check. But we are also interested in beyond-the-symbol information. We could in principle compute the differential of the results in [32] and compare with Appendix B, but we have contented ourselves with a numerical comparison.

To obtain numerical results we first need the value of the functions $V_1 \ldots V_3$ and $\tilde{V}_1 \ldots \tilde{V}_3$ at at least one point. Using the fact that V_1+V_2 and $\tilde{V}_3$ vanish in the $u_1 \to 0$ collinear limit, for instance, we could in principle evaluate these combinations at any point by integrating along a path connecting to this limit (choosing a path which remains in Euclidean kinematics). We would then use other paths to compute the other cyclically related combinations. However, we did not find this approach particularly convenient in practice. A more fruitful strategy is to first derive the amplitude at some other point away from a collinear limit. In fact, in the special case $u_1 = u_2 = u_3 = u$, it turns out that the differential simplifies dramatically

$$\begin{aligned} dV \;=\; & -I_6^{6D} d\log y + (2\text{Li}_3(1-u) + 4\text{Li}_3(1-1/u) + 5\log u \text{Li}_2(1-u) + \frac{4}{3}\log^3 u - \frac{4\pi^2}{3}\log u) d\log u \\ & -(6\text{Li}_3(1-u) + 6\text{Li}_3(1-1/u) + 6\log u \text{Li}_2(1-u) + 2\log^3 u - 2\pi^2 \log u) d\log(1-u), \\ d\tilde{V} \;=\; & 0. \end{aligned} \tag{5.3}$$

[3] We used a semi-numerical procedure, in which we evaluated numerically the $d^{2|3}\mathcal{Z}$ integral of the N^2MHV residues for a set of random integer-valued momentum twistors. We then used the analytic knowledge that the result should be an integral linear combination of 7 basic objects to promote the numerical result to an analytic one. Although semi-numerical, this procedure has no error bars and is rigorously exact.

where $V := V_1 = V_2 = V_3$ and $\tilde{V} := \tilde{V}_1 = \tilde{V}_2 = \tilde{V}_3 = 0$, allowing it to be integrated explicitly

$$\begin{aligned} V(u,u,u) &= -4L^+(x^+u, x^-u) - \frac{1}{18}J^4 - \frac{\pi^2}{9}J^2 + 2(\mathrm{Li}_4(u) + \frac{1}{6}\log^3 u \log(1-u)) \\ &\quad -6\mathrm{Li}_4(1-u) - 6\mathrm{Li}_4(1-1/u) + 4\mathrm{Li}_3(1-u)\log u - 5\mathrm{Li}_2(1-u)\mathrm{Li}_2(1-1/u) \\ &\quad + \frac{7}{24}\log^4 u - 2\pi^2\mathrm{Li}_2(1-u) - \frac{5\pi^2}{6}\log^2 u - 2\zeta(3)\log u + \frac{\pi^4}{10}. \end{aligned} \tag{5.4}$$

The first three terms are essentially as in $-1/3R^{(2)}_{6,0}$. To fix the constant, we have used numerical integration as explained in the previous paragraph, connecting these configurations to a collinear limit. We have computed the value of the constant at the three points $u = 1/3, 3/4$ and $5/6$; each point produced the same result. We then recognized this numerical result as $\pi^4/10$ and confirmed it to 40 digits. Because it is fully manifest from the formulation that the constant is a degree four transcendental number with order one rational coefficient, it does not seem necessary to supplement the numerics with an analytic computation.

The upshot of the formula is that it is very easy to deform any kinematical point to one on the line $u_1 = u_2 = u_3 = u$. Integrating the differential along such paths, we can evaluate efficiently the ratio function at any point. In particular, we have evaluated it on the kinematic point in [49]. Defining $V', \tilde{V}'$ by adding the one-loop shift $-\frac{\pi^2}{3}R^{(1)}_{6,1}$ and multiplying by $1/4$ to account for expanding in a as opposed to $2g^2$, we find on the kinematical point $(u_1, u_2, u_3) = (\frac{112}{85}, \frac{28}{17}, \frac{16}{5})$ (see Eq. (3.17))

$$V_1' = 12.6138748750304719319, \qquad \tilde{V}_1' = -0.121176561122269858950i \tag{5.5}$$

$$V_2' = 11.7057979933899946922, \qquad \tilde{V}_2' = 0.0306385302058078423O7i \tag{5.6}$$

$$V_3' = 14.4289552936316184920, \qquad \tilde{V}_3' = 0.0905380309164620166643i. \tag{5.7}$$

This was obtained by integrating along a simple linear path in cross-ratio space to the point $u = \frac{9}{4}$, but we have also tried a few other points and got the same result. The parity even objects V_i' agree precisely with the quantities called $V_i + R_6$ in [49] and the parity odd objects $\tilde{V}_i'$ agree within numerical accuracy with those given in [32]. After accounting for the same coupling constant shift, Eq. (5.4) can also be compared directly with Eq.(6.30) of [32]; we have compared the value at the two points given in Appendix D of [32] and found perfect agreement. Given that their symbols match, these numerical tests remove any doubt in our mind that the two expressions are equal.

5.2 The two-loop NMHV heptagon and the three-loop MHV hexagon

The NMHV heptagon can be attacked in an entirely similar way starting from the collinear limit of the known 1-loop N^{2}MHV octagon.

The first step is essentially kinematic and independent of loop order: one has to list all (rational) objects which can arise from taking residues of the $d^{2|3}\mathcal{Z}_7$ integral on octagon leading singularities. We found 42 linearly independent ones, all of the form (R-invariant) times $\bar{Q}\log\frac{\langle \bar{n}i\rangle}{\langle \bar{n}j\rangle}$, where i and j are momentum twistors or intersections of the momentum twistors entering the R-invariants. An example being $[23457]\bar{Q}\log\frac{\langle\bar{n}(23)\cap(457)\rangle}{\langle\bar{n}2\rangle\langle3457\rangle}$, but actually only three elements of the basis contained intersections. In general at ℓ-loop we expect to find the same 42 structures, each multiplying a pure transcendental integral over degree $2(\ell-1)$ functions. In the case at hands, over dilogarithms. Another purely kinematic step is the analog of the linear algebra problem encountered previously: out of the $7\times 42 = 294$ residues obtained by cyclic symmetry, one has to find all combinations which can be written $\bar{Q}\log$ of (conformal invariant object), possibly adding zero in the form of Eq. (3.13). We found 288 combinations, leaving 6 constraints on the integrals. These 288 combinations are independent of loop order, and, ultimately, determine what the last entry of the symbol can be at all loop orders.

We have not computed the resulting 42 integrals (each of which, manifestly, would give trilogarithms), but we have computed their symbol. This was essentially automatic using the method of Appendix A. Plugging the result into the solution of the linear algebra problem then gives the symbol of the amplitude. All entries of the symbol are either four-brackets or intersections of the type $\langle 12(\bar{4})\cap(\bar{6})\rangle$ or $\langle 23(745)\cap(671)\rangle$. We hope to analyze it further elsewhere.[4]

In this paper, our interest in the heptagon stems mostly from its connection with the three-loop MHV hexagon via the $\bar{Q}$ equation. In fact, as already familiar from our analysis of the two-loop MHV hexagon, in an appropriate basis out of the 15 independent R-invariants at 7-points only five survive $d^{2|3}\mathcal{Z}_7$ integration, namely, [12367], [12467], [23467], [23567] and [24567]. If we write $dR^{(3)}_{6,0} = d\log\frac{\langle 5613\rangle}{\langle 5612}I_1 + d\log\frac{\langle 5614\rangle}{\langle 5612\rangle} + \text{cyclic}$, it follows that we can write

$$\begin{aligned} I_1 &= \int_0^\infty \left(d\log(\tau+1)g_1 + d\log\frac{\tau+1}{\tau}g_4 + d\log\frac{\tau+1}{\tau+u_3}g_3\right) \\ I_2 &= \int_0^\infty \left(d\log(\tau+v)g_2 + d\log\frac{\tau+v}{\tau}g_5 + d\log\frac{\tau+u_3}{\tau+v}g_3\right) \end{aligned} \tag{5.8}$$

where $v = \frac{\langle 5624\rangle\langle 6123\rangle}{\langle 5623\rangle\langle 6124\rangle}$. The five functions g_i are pure degree-four transcendental functions determined by the collinear limit of the heptagon ratio function. On physical grounds (the τ integrals must converge), we know that $g_{1,2}$ must vanish at $\tau=\infty$ and $g_{4,5}$ must vanish at $\tau = 0$. Thus we can use integration by parts:

$$\begin{aligned} I_1 &= \int_0^\infty \left(\log(\tau+1)h_1 + \log\frac{\tau+1}{\tau}h_4 + \log\frac{\tau+1}{\tau+u_3}h_3\right) + g_3(0)\log u_3, \\ I_2 &= \int_0^\infty \left(\log(\tau+v)h_2 + \log\frac{\tau+v}{\tau}h_5 + \log\frac{\tau+u_3}{\tau+v}h_3\right) + g_3(0)\log\frac{v}{u_3}. \end{aligned} \tag{5.9}$$

where $h_i = -\frac{d}{d\tau}g_i$ are degree 3 functions. We see that only the collinear limit of the *differential* of the heptagon is needed, with the exception of $g_3(\tau=0)$, but one could argue that it is fixed by cyclic and parity symmetry of $dR^{(3)}_{6,0}$. We hope to use these equations in the future to study

[4]In its present unprocessed form, the result is too lengthy to be attached with this arXiv submission. It is available upon request to the authors.

the differential of the 3-loop MHV hexagon, beyond the symbol. In any event, our result for the symbol of the heptagon already gives the symbols of the g_i, which, after the nontrivial but entirely automated integration in Eq. (5.8), give the symbols of I_1 and I_2, which give, directly, the symbol of $R^{(3)}_{6,0}$. We now describe this result.

Recently, an Ansatz was constructed for the three-loop hexagon, based on reasonable physical assumptions about entries of its symbol [33] (most significantly, that they should all be products of momentum twistor four-brackets), on OPE constraints [29], and on requiring that that the last entry of the symbol should involve only brackets of the form $\langle i{-}1ii{+}1j\rangle$. This Ansatz contained many coefficients but, remarkably, in the end all but two could be determined by these authors. Recently Lipatov and collaborators, considering Regge limits using new results on the adjoint representation BFKL kernel, confirmed the value of a number of these coefficients [50].

There are three things we wish to add here. First, that all entries of the symbol should be four-brackets is manifest from our approach, since it follows from the symbol of the two-loop heptagon involving only momentum twistor intersections (together with the way symbols of integrals are built using e.g. the algorithm in Appendix A, and the fact that at six points all momentum twistor intersections become reducible to four-brackets). In turn, this property of the heptagon was essentially inherited from properties of the one-loop octagon in collinear limits. In this way it should also be possible to obtain general information about the symbol at $\ell \geq 4$ loops, although we will not do so here.

Second, the assumption about the last entry, conjectured in [23], can actually be derived from Eq. (1.2) and is therefore now proved. Indeed it follows from writing the NMHV heptagon in the form (R-invariants) times (pure transcendental functions), and using the general result for $d^{2|3}\mathcal{Z}$ on R-invariants, equation (3.7). The upshot is that these two assumptions made in [33] follow rigorously from Eq. (1.2), without doing any explicit computation. Note that although this form of the heptagon is not strictly proven to all loops, it is widely believed that it does hold [20], and assuming this then the statement about the last entry of the symbol for MHV amplitudes follows to all loops.

Third, the result of our first-principle computation of the three-loop hexagon symbol can be summarized very succinctly: the final two coefficients in [33] are $\alpha_1 = -\frac{3}{8}$ and $\alpha_2 = \frac{7}{32}$.

6 All-loop validity of the $\bar{Q}$ equation

In this section we would like to explain how we believe Eq. (1.2) could be proved, and show its consistency at any value of the coupling.

6.1 Outline of a derivation

Our proposed derivation of Eq. (1.2) starts from an expression in [15, 23] for the-right-hand side of $\bar{Q}$ in terms of insertion of a fermion operator on the edges of the chiral Wilson loop (defined on (x,θ) space):

$$\bar{Q}^{\dot{\alpha}A}\langle W_{n,k}\rangle \propto g^2 \oint dx_{\dot{\alpha}\alpha}\langle(\psi^A+F\theta^A+\ldots)^\alpha W_{n,k}\rangle. \tag{6.1}$$

This can be decomposed into a sum of n terms, one for each edge. Since each edge contribution is gauge invariant and meaningful, for the following discussion it will suffice to consider the contribution of edge n. For simplicity we will also assume that the (unbroken) Q supersymmetry has been used to set fermions χ_{n-1}, χ_n and χ_1 to zero. Then $\theta=0$ along that edge, and the formula reduces to the supersymmetry transformation law of a bosonic Wilson line in a suitable normalization.

In [23], the chiral Wilson loop with fermion insertion was computed in explicit examples using conventional Feynman diagram techniques. The new ingredient in the present paper is a simple yet powerful fact about the spectrum of excitations of the null Wilson loop: *the fermion insertion is the unique twist-one excitation with the quantum numbers of $\bar{Q}$.*

This is a powerful statement because it means that the right-hand side of Eq. (6.1) isn't really a new object. Rather, in the spirit of the Operator Product Expansion (OPE) of null polygonal Wilson loops [29], its expectation value can be extracted from any object having a nonzero overlap with it in the OPE limit. The rest of this derivation will thus be based on the analysis of [29]. The simplest possible object is the collinear limit of a $(n+1)-$point Wilson loop; this is depicted in Fig. 2. A good strategy to extract the piece with the right twist and quantum numbers in this limit is to write down the simplest operation with the quantum numbers of $\bar{Q}$, namely the $d^{2|3}\mathcal{Z}_{n+1}$ operation detailed in section 3.

To be more precise, the fermion insertion is part of a one-parameter family of insertions having bare twist one (at weak coupling), labeled by a position τ along the edge. In the quantum theory, operators in this family will renormalize among themselves. Thus we expect a relation of the form

$$\lim_{\epsilon\to 0} d^{2|3}\mathcal{Z}_{n+1}(\tau,\epsilon)\langle W_{n+1,k}(\tau,\epsilon)\rangle = \int_0^\infty d\tau'\,\tilde{F}(\tau,\tau',\epsilon)\langle\psi(\tau')W_{n,k}\rangle, \tag{6.2}$$

where the inserted twistor $\mathcal{Z}_{n+1}(\tau,\epsilon)$ is parametrized as in Eq. (3.1), and the right-hand side contains the Wilson loop with insertion we are interested in. Now we could instead consider the BDS-subtracted Wilson loop, and using the collinear limit properties of the BDS Ansatz we would find a similar equation with a slightly different F

$$\lim_{\epsilon\to 0} d^{2|3}\mathcal{Z}_{n+1}(\tau,\epsilon)R_{n+1,k}(\tau,\epsilon) = \frac{1}{A_n^{\rm BDS}}\int_0^\infty d\tau'\,F(\tau,\tau',\epsilon)\langle\psi(\tau')W_{n,k}\rangle. \tag{6.3}$$

The dependence on ϵ of the OPE coefficient $F(\tau,\tau',\epsilon)$ is governed, in the $\epsilon\to 0$ limit, by a renormalization group equation which we describe in subsection 6.3. The essential conclusion is that the total integral $\int_0^\infty d\tau$ does not renormalize, e.g., is ϵ-independent in the limit. Thus, for the total integral, which enters Eq. (6.1), $\tilde{F}\to\tilde{F}(a)$ depends only on the coupling, allowing us to write

$$\frac{1}{A_n^{\rm BDS}}\bar{Q}\langle W_{n,k}\rangle = \frac{g^2}{F(a)}\int \lim_{\epsilon\to 0}\int_{\tau=0}^{\tau=\infty} d^{2|3}\mathcal{Z}_{n+1}(\tau,\epsilon)R_{n+1,k}(\tau,\epsilon) + \text{cyclic}. \tag{6.4}$$

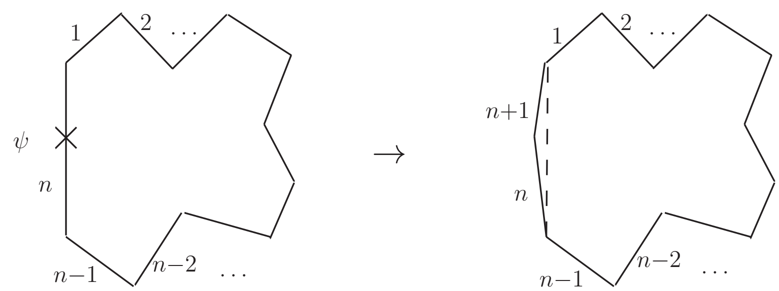


Figure 2. Fermion insertion on the Wilson loop versus kink insertion

An important subtlety at this point is that the integral $\int_0^\infty d\tau$ is singular due to endpoint divergences. As discussed in the next subsection, the integral is always at most single-logarithmic divergent, reflecting the behavior expected for $\bar{Q}$ of the log of an amplitude. What this means is that in a given ultraviolet regularization scheme the result will be well-defined, but it may depend on the scheme.[5]

A natural way to remove the scheme dependence is to divide by the BDS Ansatz, e.g. push the $1/A_n^{\rm BDS}$ factor on the left-hand side inside the $\bar{Q}$. Since the BDS Ansatz is 1-loop exact and proportional to a in the exponent, and since $\bar{Q}$ is first order in derivatives, this adds a term

$$\langle W_{n,k}\rangle \bar{Q}\frac{1}{A_n^{\rm BDS}} = -aR_{n,k}\int \lim_{\epsilon\to 0}\int_{\tau=0}^{\tau=\infty} d^{2|3}\mathcal{Z}_{n+1}(\tau,\epsilon)R^{\rm tree}_{n+1,1} + \text{cyclic}. \tag{6.5}$$

Adding the two equations Eq. (6.4) and Eq. (6.5) gives Eq. (1.2), up to the yet undermined function of the coupling $F(a)$. In the next subsection we will determine that $g^2/F(a) = a$, using the known fact that $\bar{Q}R_{n,k}$ must be finite and scheme-independent to all loops [24]. The point is that some cancelations are required to occur between the two terms.

There are various points in this derivation which may not be fully rigorous. For instance, we have assumed that a supersymmetric regularization of Wilson loop existed, but, as pointed out in [51], in the only regulator scheme which has been tried so far it may be necessary to add complicated counterterms to define the correct operator at the quantum level, changing the explicit form of the Wilson loop. This means that our derivation is not based on any explicitly known regulator. On the other hand, the explicit form of the operator was not really important for the derivation; we only really used simple physical properties about the excitation spectrum of the Wilson loop. Furthermore, in the end, everything is expressed in terms of finite and regulator-independent quantities. For these reasons, we believe that this derivation is quite robust.

Following the same steps for Wilson loops in theories with less supersymmetry, one would find that the fermion insertion ψ would no longer appear inside the chiral Wilson loop, so

[5]In the momentum space introduced in subsection 6.3, the anomalous dimension are $\mathcal{O}(p^2)$ while the endpoint divergences produce $1/p$ in the small p limit. This is why it is still safe to ignore entirely the anomalous dimensions in this discussion.

the right-hand side of Eq. (6.1) would be a genuine new object. So $\mathcal{N}=4$ SYM is special in being the only gauge theory in which all elementary fields circulate in a chiral superconnection. Still, one might be able to derive similar equations in other theories by enlarging the class of Wilson loops to be considered. From our viewpoint, the hallmark of integrability is not the $\bar{Q}$ equation itself, because as seen in subsection 3.3 it takes one only ever so far, but the existence of a similar equation for $Q^{(1)}$, which is known so far only for planar $\mathcal{N}=4$ SYM.

From the scattering amplitude perspective, we expect equations similar to Eq. (1.2) to be valid at one-loop in other theories as well (paralleling the results of [28]). On the other hand, as noted in Introduction, at higher loops it seems quite difficult, at least to the authors, to justify the absence of $1 \to 3, 4, \ldots$ splitting terms in theories that have no local Wilson loop dual.

6.2 Convergence of the τ integral

Let us consider the second term in Eq. (1.2), the BDS-subtraction term. Near $\tau = 0$ it looks like

$$-\, a R_{n,k} \bar{Q} \log \frac{\langle \bar{n} 2\rangle}{\langle \bar{n} n{-}2\rangle} \int_0 \frac{d\tau}{\tau}. \tag{6.6}$$

This is divergent, reflecting the infrared logarithms in the BDS Ansatz.

Poles in the τ-integrand originate from poles in scattering amplitudes, and this divergence can be traced to poles $1/\langle n{-}2 n{-}1 n n{+}1\rangle$ and $1/\langle n{-}1 n n{+}1 1\rangle$. Since the amplitude only has poles corresponding to physical channels, these are the only two possible poles which could contribute. Actually, the second pole blows up in the collinear limit regardless of τ (see Fig. 2). A constraint from the k-decreasing collinear limit implies that the coefficient of this pole is $R_{n,k}$, so this contribution cancels out between this term and a corresponding part in Eq. (6.6), even before we take $\tau \to 0$. The first pole, $1/\langle n{-}2 n{-}1 n n{+}1\rangle$, blows up when we take $\tau \to 0$ but this correspond to a soft limit of the amplitude, in which its coefficient also reduce to $R_{n,k}$. The analysis of divergences near $\tau = \infty$ is similar. We conclude that the finiteness of the τ-integrand, observed empirically in the previous sections, is a general fact which will remain true at any value of the coupling thanks to the nice collinear and soft limits of BDS-subtracted amplitudes.

Note that this is only true when the relative coefficient between the two terms is chosen as in Eq. (1.2). This is the reason why we believe that $g^2/F(a) = a = \frac{1}{4}\Gamma_{\text{cusp}}$ in Eq. (6.4) exactly in the coupling.

6.3 Absence of $\log \epsilon$ divergences

In the main text, we are interested in the zero-momentum component (total τ integral) of the difference between the two terms in Eq. (1.2). In this subsection, we will be be interested in the τ-dependence of just the fist term. In particular, we wish to understand the $\log \epsilon$ terms which arise in the $\epsilon \to 0$ limit at fixed τ.

From general field theory one might expect these logarithms to be related to the anomalous dimensions of local operators insertions on the Wilson loop. In the context of null polygonal Wilson loops this was formalized recently [29], and we refer the reader to this reference for more background. In the case at hand the key feature, just alluded to, is that the only insertions with the correct (bare) twist and quantum numbers are insertions of single fermions. The absence of multi-excitation states is a considerable simplification. This means that all pertinent operators are labeled by one a position along the edge, so we expect the renormalization group to act as convolution. Actually, the edge has a symmetry which is a combination of a longitudinal Lorentz boost and a dilatation leaving the position of the two cusps (and the orientation of neighboring segments) unchanged. In our variables this is generated by $\tau \to \alpha^{1/2}\tau$. It follows that the renormalization group equation is diagonalized in momentum space. The upshot is that we expect

$$\lim_{\epsilon\to 0} \frac{\epsilon}{d\epsilon} \int_{\tau=0}^{\tau=\infty} \int d^{2|3}\mathcal{Z}_{n+1} \tau^{\frac{ip}{2}} R^{(\ell)}_{n+1,1} = \log\epsilon \times (E(p)-1) + C(p) \tag{6.7}$$

where the so-called form factor $C(p)$ (which depends on helicity choices) is finite as $\epsilon \to 0$. The so-called dispersion relation $E(p)$ should match that of an elementary fermion excitation of the null edge (equivalent to excitations of the GKP string [52]), known exactly to all values of the coupling thanks to integrability [30] (see also Appendix B of [53]).

The cancelation of $\log\epsilon$ divergences at zero-momentum is very easy to understand from this formula: the energy $E(0) = 1$ is protected by Goldstone's theorem, the zero-momentum fermion being the Goldstone fermion for the breaking of supersymmetry caused by the Wilson loop background. The condition $E(0) = 1$ is also verified within the integrability framework [30]. This shows that the cancelations observed empirically in sections 4 and 5 are general and will hold exactly in the coupling.

6.4 The fermion dispersion relation

We now wish to check the prediction for $E(p)$ at finite p. Due to the physical origin of the divergences, it should suffice to check this for the collinear limit of six-point amplitudes, the dispersion relation being expected to be universal. We let

$$\lim_{\epsilon\to 0} \frac{\epsilon}{d\epsilon} d^2 Z_{n+1} \int d^{0|3}\chi_{n+1} R^{(\ell)}_{n+1,1} = \bar{Q} \log\frac{\langle 4512\rangle}{\langle 4513\rangle} \times \frac{d\tau}{\tau} \times I^{(\ell)}(\tau). \tag{6.8}$$

as in subsection 4.1, where ℓ denotes the loop order (e.g., the order in $a = \frac{1}{4}\Gamma_{\rm cusp}$), and $n = 5$.

Taking the collinear limit of the six-point tree amplitude (1) + (3) + (5) (the rules in subsection 4.1 can be useful here) gives

$$I^{(0)}(\tau) = \frac{1}{\tau+1}. \tag{6.9}$$

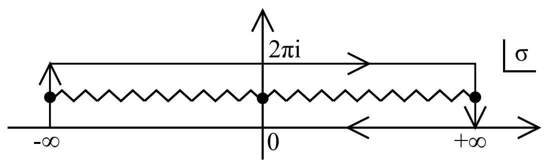
We will need the Fourier transform

$$I^{(0)}(p) = \int_{-\infty}^{\infty} \frac{d\sigma e^{\frac{ip}{2}\sigma}}{e^\sigma + 1} = \frac{\pi}{i\sinh\frac{\pi p}{2}}, \tag{6.10}$$

where $\tau = e^{\sigma}$. The specific form of this result is very useful, as it allows us to write the ratio

$$\frac{I^{(\ell)}(p)}{I^{(0)}(p)} = e^{\frac{\pi p}{2}} \frac{e^{-\pi p} - 1}{2\pi i} \int_{-\infty}^{\infty} d\sigma e^{\frac{ip}{2}\sigma} I^{(\ell)}(e^{\sigma}) = \frac{e^{\frac{\pi p}{2}}}{2\pi i} \oint_{\mathcal{C}} d\sigma e^{\frac{ip}{2}\sigma} I^{(\ell)}(e^{\sigma}), \tag{6.11}$$

where $\mathcal{C}$ is the rectangle contour:



This is valid for $-2 < \mathrm{Im}p < 0$, where the contributions from infinity can be neglected. Now, for any ℓ, $I^{(\ell)}(\tau)$ is an analytic function of τ with branch points at $\tau = 0, -1$, and ∞. This allows the contour to be deformed and expressed in terms of discontinuities on the horizontal line $\mathrm{Im}\ \sigma = \pi$:

$$\begin{aligned}
\frac{I^{(\ell)}(p)}{I^{(0)}(p)} &= \int_{-\infty}^{\infty} d\sigma e^{\frac{ip}{2}\sigma} \mathrm{Disc}\left[I^{(\ell)}(-e^{\sigma})\right] \\
&= \int_0^1 \frac{dx}{x} x^{\frac{ip}{2}} \mathrm{Disc}\left[I^{(\ell)}(-x)\right] + (p \to -p)
\end{aligned} \tag{6.12}$$

where $\mathrm{Disc}\left[I^{(\ell)}(-x)\right] := \frac{I^{(\ell)}(-x-i0) - I^{(\ell)}(-x+i0)}{2\pi i}$. It could be very interesting to interpret this discontinuity (in the cross-ratio regime $u_1 \to 0$, $u_2 = 1 - u_3$, $u_3 < 0$) as the imaginary part of the six-gluon amplitude in a physical Minkowski-signature regime.

To deal with such integrals, we found useful to think of $\frac{ip}{2}$ as a positive integer and use the language of harmonic sums. Recalling the 5-point τ integrand Eq. (4.4) and taking its discontinuity, we obtain

$$\begin{aligned}
\frac{I^{(1)}(p)}{I^{(0)}(p)} &= \left(M\left[\frac{2x}{(x-1)_+}\right] \log\epsilon + M\left[\frac{x+1}{(x-1)_+} \log(1-x)\right] - \frac{\pi^2}{6} \right) + (N \to -N) \\
&= \left(2S_1 \log\epsilon + \frac{S_1}{N} - S_1^2 - S_2 - \frac{\pi^2}{6} \right) + (N \to -N)
\end{aligned} \tag{6.13}$$

where $M[f] := \int_0^1 \frac{dx}{x} x^N f$ is the Mellin transform, $N := \frac{ip}{2}$, and the + prescription is the usual one, such that $\int_0^1 \frac{dx \log(1-x)^a}{(1-x)_+} = 0$ for $a \geq 0$. For integer N the harmonic sums are defined as $S_i = \sum_{n=1}^{N} \frac{1}{n^i}$ and $S_{i_1,i_2} = \sum_{n \geq n_1 \geq n_2} \frac{1}{n_1^{i_1} n_2^{i_2}}$, and elsewhere by analytic continuation [54]. The $-\pi^2/6$ term follows from a careful treatment of the x near 1 region, but can also be verified numerically quite unambiguously using the first form in Eq. (6.11).

At two-loops, we are interested only in the $\log\epsilon$-terms. Conveniently, these can be read off from our formula for the differential of the NMHV hexagon, in Appendix B, without doing any integration. The point is that the logarithms exclusively arise from terms proportional to $d\log u_1$. So, all we have to do, is take the expressions in Appendix B, drop all terms except those proportional to $d\log u_1$, and expand the degree three functions in powers of $\log u_1$ as $u_1 \to 0$. Then we use the simple rules $\log u_1 d\log u_1 \to 2\log^2\epsilon, d\log u_1 \to 2\log\epsilon$. We then have to take a discontinuity as a function of $\tau \to -x$. We obtain for the $\log^2\epsilon$ terms

$$\begin{aligned}\frac{1}{4}\frac{I^{(2)}(p)}{I^{(0)}(p)} &= \log^2\epsilon\left(M\left[\frac{(1{+}x)\log(1-x)-\frac{1}{2}\log x}{(1{-}x)_+}\right]+\frac{\pi^2}{12}\right)+(N\to -N)\\ &= \log^2\epsilon\left(S_1^2+\frac{1}{2}S_2-\frac{S_1}{N}+\frac{\pi^2}{6}\right)+(N\to -N),\end{aligned}\tag{6.14}$$

and for the single-logarithmic terms

$$\begin{aligned}\frac{1}{4}\frac{I^{(2)}(p)}{I^{(0)}(p)} &= \log\epsilon\left(M\left[\frac{x(\frac{\pi^2}{6}+\mathrm{Li}_2(x))+\frac{1}{2}(1+x)\log(1-x)(3\log(1-x)-\log x)}{(1-x)_+}\right]+\frac{1}{2}\zeta(3)\right)+(N\to -N)\\ &= \log\epsilon\left(-S_{1,2}-S_1S_2-S_1^3+\frac{S_2}{N}+\frac{3}{2}\frac{S_1^2}{N}-\frac{S_1}{2N^2}-\frac{\pi^2}{2}S_1-\frac{5}{2}\zeta(3)\right)+(N\to -N).\end{aligned}\tag{6.15}$$

The OPE prediction concerns the logarithm, so we need to add the combination

$$\begin{aligned}-\frac{1}{8}\left(I^{(1)}(p)\right)^2 &= +\log\epsilon\left(2S_{1,2}-S_3+S_1^3-\frac{S_2}{N}-\frac{3}{2}\frac{S_1^2}{N}+\frac{S_1}{N^2}+\frac{\pi^2}{2}S_1+3\zeta(3)\right)+(N\to -N)\\ &\quad -\log^2\epsilon\left(S_1^2+\frac{1}{2}S_2-\frac{S_1}{N}+\frac{\pi^2}{6}\right)+(N\to -N).\end{aligned}\tag{6.16}$$

This is the square of Eq. (6.13), although writing it as harmonic sums was not entirely trivial due to cross-terms between the $+N$ and $-N$ terms. We found that an efficient way to achieve this was to match the poles on the negative N axis. The $\log^2\epsilon^2$ terms cancel in the sum, and we obtain for the logarithm

$$\frac{1}{4}\left(\log I(p)\right)^{(2)} = \log\epsilon\left(S_{1,2}-S_3-S_1S_2+\frac{1}{2}\frac{S_1}{N^2}+\frac{1}{2}\zeta(3)\right)+(N\to -N)+\text{finite}.\tag{6.17}$$

This is to be compared with $(E(p)-1)\log\epsilon$ where for $E(p)$ we use the expansion to second order in Γ_{cusp} of the "large fermion" dispersion relation in Eq.(20) from [30]:

$$\begin{aligned}E(p)-1 &= \Gamma_{\text{cusp}}\left(\psi_+-\psi(1)\right)-\frac{\Gamma_{\text{cusp}}^2}{8}\left(\psi_+''+4\psi_-'(\psi_--\frac{1}{p})+6\zeta(3)\right)\\ &= \frac{1}{2}\Gamma_{\text{cusp}}S_1+\frac{1}{4}\Gamma_{\text{cusp}}^2(S_{1,2}-S_3-S_2S_1+\frac{S_1}{2N}+\frac{1}{2}\zeta(3))+(N\to -N)\end{aligned}$$

where $\psi_+ := \frac{1}{2}(\psi(1+\frac{ip}{2})+\psi(1-\frac{ip}{2}))$ and $\psi_- := \frac{i}{2}(\psi(1+\frac{ip}{2})-\psi(1-\frac{ip}{2}))$ in the first line, and on the second line we have converted the result to harmonic sums. Perhaps we should have said earlier, that the one-loop prediction $2S_1$ was matched by Eq. (6.13).

The perfect agreement confirms beautifully that our $d^{2|3}\mathcal{Z}$ integral is probing fermion excitations on the edges of the Wilson loop, as was expected from the OPE analysis. Second, and perhaps more importantly for us, it gives an independent confirmation to two-loop accuracy (besides the numerical check in subsection 5.1), that the prefactor in Eq. (1.2) has to be Γ_{cusp}.

7 Two-dimensional kinematics

7.1 Preliminaries

It can be useful to consider special kinematic configurations in which scattering amplitudes usually simplify. Following [57, 58] we now consider configurations of external momenta/edges of the Wilson loop which can be embedded inside a two-dimensional subspace of Minkowski space. This reduces the conformal group SU(2,2) to SL(2)×SL(2). Actually, we are interested in super-amplitudes, and as will become apparent soon it is very natural to consider a supersymmetric reduction to SU(2,2|4) to SL(2|2)×SL(2|2).

With no loss of generality we take the number of particles/edges, n, to be even. Even and odd labels are distinguished,

$$\mathcal{Z}_{2i-1} = (\lambda_i^{1+}, 0, \lambda_i^{2+}, 0, \chi_i^{1+}, 0, \chi_i^{2+}, 0), \qquad \mathcal{Z}_{2i} = (0, \lambda_i^{1-}, 0, \lambda_i^{2-}, 0, \chi_i^{1-}, 0, \tilde{\chi}_i^{2-}). \tag{7.1}$$

Four-brackets with two odd and two even labels factorize, $\langle 2i{-}1\, 2j{-}1\, 2k\, 2l\rangle := \langle ij\rangle[kl]$, and all others vanish. R-invariants which contain a generic reference twistor Z_*, two even and two odd labels will appear, and they factorize into odd and even parts,$[*\, i\, j\, k{-}1\, l{-}1] = (*\, i\, j)[*\, k{-}1\, l{-}1]$ where

$$(*\, i\, j) := \frac{\delta^{0|2}(\langle\langle *\, i\, j\rangle\rangle)}{\langle *\, i\rangle^+\langle i\, j\rangle^+\langle j\, *\rangle} \tag{7.2}$$

and similarly for the odd part $[*\, k\, l]$. These R-invariants satisfy a four-term identity $(a\, b\, c) - (a\, b\, d) + (a\, c\, d) - (b\, c\, d) = 0$. Finally, cross-ratios are also separated into even and odd sectors

$$u^+_{i,j} = \frac{\langle ij{+}1\rangle\langle i{+}1j\rangle}{\langle ij\rangle\langle i{+}1j{+}1\rangle}, \quad u^-_{i,j} = \frac{[ij{+}1][i{+}1j]}{[ij][i{+}1j{+}1]}. \tag{7.3}$$

In this notation, the NMHV tree amplitude (most easily extracted from the CSW form [59]) is

$$R^{\text{tree}}_{2n,1} = \frac{1}{2}\sum_{i,j}(*\, i\, j)\left([i\, j{-}1\, j] - [i{-}1\, j{-}1j]\right). \tag{7.4}$$

7.2 Collinear limits

The BDS-subtracted amplitudes behave simply under single-collinear limits. In two-dimensional kinematics, the natural limit instead collapses the length of an edge to zero, the so-called triple-soft-collinear limit. The behavior of amplitudes in this limit is well understood at both two-loops and at strong coupling [57, 58], let us recall the main conclusion. First, it is easy to see that nothing can diverge in this limit, just by dual conformal symmetry, to any loop order: for the hexagon the whole (BDS-subtracted) amplitude is just a constant since all cross-ratios are equal to 1. On the other hand, as is especially clear from the Wilson loop viewpoint where the limit has an OPE interpretation [29, 53], the limit involves only physics localized around one edge and so cannot depend on the number of points. So in general we expect a simple multiplicative relation $R_{2n} \to f(a)R_{2n-2}$ for some function of the coupling.

Actually, we will need both k-preserving or k-decreasing limits, which are not parity conjugate to each other in two-dimensional kinematics. This allows for two different constants. It will be useful to absorb them by a simple rescaling of the BDS-subtracted amplitudes:

$$R_{2n,k} := e^{(n-2)f_1(a)+kf_2(a)}\tilde{R}_{2n,k}. \tag{7.5}$$

By choosing f_1 and f_2 such that $\tilde{R}_{6,0} = 1$ and $\tilde{R}_{6,1} = R^{\text{tree}}_{6,1}$, then $\tilde{R}_{2n,k} \to \tilde{R}_{2n-2,k}$ in the k-preserving limit $\lambda^-_{n+1} \to \lambda^-_n$, and $(\int d^2\chi^+_n d^2\chi^-_n \tilde{R}_{2n,k})/(\int d^2\chi^+_n d^2\chi^-_n (n{-}1n1)[n{-}1n1]) \to \tilde{R}_{2n-2,k-1}$ in the k-decreasing case. It is known that

$$f_1(a) = -a^2\frac{\pi^4}{9} + \mathcal{O}(a^3), \quad f_2(a) = -a\frac{\pi^2}{3} + a^2\frac{7\pi^4}{30} + \mathcal{O}(a^3). \tag{7.6}$$

The strong coupling limits are also known: $f_1(a) = a\frac{2\pi}{3} + \mathcal{O}(1)$, $f_2(a) = 0 + \mathcal{O}(1)$, when $a \to \frac{\sqrt{g^2_{\text{YM}}N_c}}{8\pi} \to \infty$, as we extract from [56]. The result for f_1 at weak coupling has been known for a while [58], while the two-loop correction to f_2 follows from the value $\frac{8\pi^4}{45}A^{\text{tree}}$ of the two-loop value of $R_{6,1}$ in Eq. (5.4), also obtained recently by [32]. It would be nice to have a way to calculate these functions at all values of the coupling. Below we will discuss the functions $\tilde{R}_{2n,k}$ directly.

7.3 Two-dimensional Yangian equations

Eq. (1.2) involves a $(2n{+}1)$-gon which isn't very natural in two-dimensional kinematics. However, the key ingredient in its derivation was the fact that the collinear limit of the $(2n{+}1)$-gon had a nonzero overlap with the insertion of a fermion on the $2n$-gon. This suggests that we can get the same information out of the $(2n{+}2)$-gon. Namely, we write down a limit with the right quantum numbers. Take A and a to be even and add particles $2n{+}1$ and $2n{+}2$:

$$\begin{aligned}\bar{Q}^A_a\tilde{R}_{2n,k} \;&=\; a\int d^{1|2}\lambda^+_{n+1}\int d^{0|1}\lambda^-_{n+1}(\tilde{R}_{2n+2,k+1} - R^{\text{tree}}_{2n+2,1}\tilde{R}_{2n,k}) + \text{cyclic} \\ &:= a\;\lambda^-_{n\,a}\lim_{\lambda^-_{n+1}\to\lambda^-_n}\int_{\lambda^+_n}^{\lambda^+_1}\langle\lambda^+_{n+1}d\lambda^+_{n+1}\rangle\int d^2\chi^+_{n+1}(d\chi^-_{n+1})^A\,(\text{parenthesis}) + \text{cyclic}\end{aligned} \tag{7.7}$$

In principle, one might expect a nontrivial function of the coupling multiplying the first term, arising as an OPE coefficient similar to $g^2/F(a)$ above, but not in front of the second, it being associated with the BDS Ansatz. However that function would have to be independent of n, and arguing as follows it is possible to see that using $\tilde{R}$ this function can only be 1. The essential point is that some cancelations have to occur between the two terms.

On R-invariants not depending $n{+}1$, the integral gives zero, while in general

$$d^{1|2}\lambda^+_{n+1}d^{0|1}\lambda^-_{n+1}(n{+}1\,i\,j)[n{+}1\,k\,l] = \bar{Q}\log\frac{[nk]}{[nl]}d\log\frac{\langle\lambda^+_{n+1}\lambda_i\rangle}{\langle\lambda^+_{n+1}\lambda_j\rangle}. \tag{7.8}$$

The limit in Eq. (7.7) does not depend on how λ^-_{n+1} approaches λ^-_n, but for individual terms it may, thus it is useful to choose $\lambda^-_{n+1} = \lambda^-_n + \epsilon\lambda_1$ supersymmetrically. Then the previous equation is valid in all cases, with the substitution $[nn] \to [n1]$ when $k = n$ or $l = n$. The action on NMHV tree gives

$$d^{1|2}\lambda^+_{n+1} d^{0|1}\lambda^-_{n+1}\tilde{R}^{\text{tree}}_{2n+2,1} = \sum_{j=2}^{n} d\log\langle\lambda^+_{n+1}\lambda_j\rangle\bar{Q}\log\frac{[nj{-}1]}{[nj]}. \tag{7.9}$$

This agrees, for the generic term, with Eq.(3.13) of [57] for the BDS amplitude, but we see that the measure is logarithmically divergent at the endpoint $\lambda^+_{n+1} = \lambda^+_n$. On the other hand, thanks to the universal k-decreasing collinear limits of $\tilde{R}$, this divergence will cancel in the combination Eq. (7.7) as in subsection 6.2, showing that it is the correct combination.

Similarly, starting from Eq. (1.4) and noting that the essentially nontrivial term in it (the first term) can be interpreted as inserting a $\tilde{\psi}$ on edge $2n$, we expect that for even A and a

$$Q^{(1)a}_A\tilde{R}_{2n,k} = a\,\lambda^{a-}_n \lim_{\lambda^-_{n+1}\to\lambda^-_n}\int_0^\infty\frac{d\tau}{\tau}\left((d\chi^-_{n+1})_A R_{2n+2,k} - \sum_{1\le j<i\ne n-1} C'_{n,i,j}(\tau)\frac{\partial}{\partial\chi^{-A}}R_{2n,k}\right) + \text{cyclic}. \tag{7.10}$$

with $\lambda^+_{n+1}(\tau) = \lambda^+_n + \tau\lambda^+_1$, where the $C'_{n,i,j}$'s are some coefficients analogous to Eq. (3.24) but which we have not computed. Since these coefficients are expected to be universal, one way to obtain them would be to deduce them from $Q^{(1)}$ acting on the 1-loop NMHV amplitude given below.

Identical equations apply when both indices on $\bar{Q}$ and $Q^{(1)}$ are in the odd (plus) sector, if we take $\lambda^+_i \to \lambda^-_i, \lambda^-_i \to \lambda^+_{i+1}$.

7.4 From tree N^{2}MHV to one-loop NMHV to two-loop MHV

Let us now check whether it is indeed possible to compute the 2-loop amplitude in restricted kinematics, starting with tree amplitudes in *only* restricted kinematics. For the N^{2}MHV tree amplitude we start from the CSW representation [59], which has a simple 2d limit for the generic term, albeit the bondary terms are a bit complicated. After some massaging we managed to obtain the form:

$$\begin{aligned} R^{\text{tree}}_{2n,2} = {} & \frac{1}{2}\sum_{i<j<k<l<i}(*\,i\,j)(*\,k\,l)\left([i\,j{-}1\,j]-[i{-}1\,j{-}1j]\right)\left([k\,l{-}1\,l]-[k{-}1\,l{-}1l]\right) \\ & -\frac{1}{3}\sum_{i<j<l<i}(*\,i\,j)(*\,j\,l)\left(([l\,i{-}1\,i]-[j{-}1\,i{-}1\,i])([j\,l{-}1\,l]-[j{-}1\,l{-}1\,l])\right. \\ & \left.+([j\,i{-}1\,i]-[j{-}1\,i{-}1\,i])([j\,l{-}1\,l]-[i{-}1\,l{-}1\,l])\right). \end{aligned} \tag{7.11}$$

Computing the $\bar{Q}$ from edge n using Eq. (7.7) and Eq. (7.11) is a bit tedious, possibly because this form for the trees is not optimally simple. After patient bookkeeping we obtain the simple result

$$\bar{Q}\tilde{R}^{\text{1-loop}}_{2n,1} = \sum_{1\le j<k<l\le n}\log\frac{\langle 1k\rangle}{\langle nk\rangle}\bar{Q}\log\frac{[nk]}{[nk{-}1]}(jkl)([j\,l{-}1\,l]-[j{-}1\,l{-}1\,l]) + \text{cyclic}. \tag{7.12}$$

To integrate the $\bar{Q}$ we need to complete its arguments into cross-ratios. A successful strategy is to decompose the logarithms as a difference of two terms and collect the terms. Then for the generic term we immediately get cross-ratios, but we also get some boundary terms

$$\begin{aligned}\bar{Q}\tilde{R}^{\text{1-loop}}_{2n,1} = &\sum_{i<j<k<l<i} \log\langle ik\rangle(\bar{Q}\log u^-_{i-1,k-1})(jkl)([j\,l{-}1\,l]-[j{-}1\,l{-}1\,l]) - \sum_{i<j<k<i}\log\langle ik\rangle(ijk)\\ &\times\left(\bar{Q}\log\frac{[ik{-}1]}{[ik]}([i\,j{-}1\,j]-[i{-}1\,j{-}1\,j])+\bar{Q}\log\frac{[i{-}1k{-}1]}{[ik{-}1]}([k\,j{-}1\,j]-[k{-}1\,j{-}1\,j])\right).\end{aligned}$$

To complete the arguments of $\bar{Q}$ in the boundary terms one can in principle follow the systematic strategy of subsection 3.4. However in this case this is not necessary, as close inspection reveals that adding zero in the form $\bar{Q}\log\frac{[ij]}{[jk]}[i\,j\,k] = \bar{Q}\log\frac{[ij]}{[jk]}([k\,j{-}1\,j]-[i\,j{-}1\,j]+[i\,j{-}1\,k])$, antisymmetrized in $(i,i{-}1)$ and $(k,k{-}1)$, will do the trick. Thus, without much attempt at simplifying the result, we obtain the formula

$$\begin{aligned}\tilde{R}^{\text{1-loop}}_{2n,1} = &\sum_{i<j<k<l<i}(jkl)([j\,l{-}1\,l]-[j{-}1\,l{-}1\,l])\log\langle ik\rangle\log u^-_{i-1,k-1} - \sum_{i<j<k<i}\log\langle ik\rangle(ijk)\\ &\times\Big(\log u^-_{i,k-1,k,j}([i\,j{-}1\,j]-[i{-}1\,j{-}1\,j])+\log u^-_{k-1,i-1,i,j}([k\,j{-}1\,j]-[k{-}1\,j{-}1\,j])\\ &\quad+\Big(\log u^-_{i,j,j-1,k}[i\,j{-}1\,k]-(i\leftrightarrow i{-}1)-(j\leftrightarrow j{-}1)\Big)\Big).\end{aligned}$$

This is the general 1-loop NMHV amplitude in two-dimensional kinematics. We have checked this formula against the two-dimension limit of the box expansion for $2n = 8, 10, 12$.

To get the two-loop MHV amplitudes from this using the $\bar{Q}$ equation is much simpler, as it involves only the substitution $(*n{+}1j)[*n{+}1k] \to \log\frac{\langle 1j\rangle}{\langle nj\rangle}\log[nk]$ from Eq. (7.8); we obtain

$$\begin{aligned}R^{\text{2-loop}}_{2n,0} = &-\sum_{i<j<k<l<i}\log\langle ik\rangle\log\langle jl\rangle\log u^-_{i-1,k-1}\log u^-_{j-1,l-1} - 2\sum_{i<j<k<i}\log\langle ij\rangle\log\langle jk\rangle\\ &\times(\log u^-_{j-1,k-1}\log u^-_{k-1,i,i-1,j}+\log u^-_{i-1,j-1}\log u^-_{j-1,k,k-1,i}+\log u^-_{i-1,k-1}\log u^-_{l,j,j-1,k-1})\\ &-\sum_{i<j<i}\log^2\langle ij\rangle\log u^-_{i-1,j-1}\log(1-u^-_{i-1,j-1}).\end{aligned}\tag{7.13}$$

where in the parenthesis the second and third terms are permutations of the first one. We have verified that this result is equivalent to the forms given in [53, 57], albeit its form is not immediately as elegant.

Something unusual has to be said. We found that obtaining a reasonably intelligible form of the N^{2}MHV tree amplitude in 2D kinematics was by far the most time-consuming part of this computation. Obtaining the NMHV 1-loop result from it, involved some patient bookkeeping, but no special difficulty. Finally, obtaining the two-loop MHV formula was by far the simplest step. While we hope that simpler expressions for the tree and one-loop amplitudes will be obtained in the future, our main objective here was to confirm that it was possible to obtain these results using only input from two-dimensional amplitudes.

8 Conclusion

We have proposed an all-loop equation for the $\bar{Q}$ symmetry acting on BDS-subtracted planar S-matrix of $\mathcal{N}=4$ SYM, as well as its parity-conjugate for $Q^{(1)}$ symmetry, which, interpreted as quantum-corrected symmetry generators, amounts to exact Yangian symmetry. In principle, these equations can be used to determine all-loop S-matrix uniquely, at at any value of the coupling. As a perturbative study, we have applied the $\bar{Q}$ equation to reproduce results for MHV n-gon, NMHV hexagon at two loops, and fix the undetermined coefficients in an Ansatz for three-loop MHV hexagon. The equations relate dual Wilson loops with fermion insertions along edges to higher-point ones in the collinear limit, and we have outlined a derivation in the spirit of OPE. In particular, we have reproduced the fermion dispersion relations to the second order of Γ_{cusp}, as a strong consistency check for the proposal.

It should be possible to extend our results in the more loops/more legs directions, with a manageable amount of time. For instance, higher-point 3-loop MHV amplitudes could be analyzed. Using parity the N^{2}MHV heptagon is equivalent to the NMHV heptagon (whose symbol we have been able to compute at two loops, though not its function), and from it the three-loop NMHV hexagon could be obtained. The two-loop N^{2}MHV octagon should be uniquely determined using parity symmetry together with the $\bar{Q}$ equation; from this one could obtain all three-loop heptagons and 4-loop hexagons. With, perhaps, the help of a big enough computer, we foresee no essential difficulty in obtaining the symbol of these objects. Without doing detailed computations, it might also be possible to make general qualitative statements about these objects.

In addition to two- and three-loop results, we have also obtained certain all-loop predictions for NMHV hexagon and heptagon, namely what prefactors can appear which multiply degree-$(2\ell-1)$ functions for the $\bar{Q}$ of ℓ-loop amplitudes, equivalent to what can appear in the last entry of the symbol after integrating the $\bar{Q}$. Based on classifications of residues in the Grassmannian $G(2,n)$, it should be straightforward to generalize such predictions to NMHV n-gon, in analog of that of MHV n-gon. More importantly, by considering the $Q^{(1)}$ equation, we hope to carry such analysis for N^{k}MHV cases, which would provide useful information on the all-loop structure of general amplitudes.

Corresponding functions, as opposed to symbols, could also be obtained provided one is able to carry out the one-dimensional integrals which appear, but this might require some new ideas. Clearly it would be important to better understand results which are already known, for instance the differential of the general MHV n-gon [23] and the three-loop MHV hexagon. Indeed the undetermined coefficients in the Ansatz of [33] at three-loops are now completely fixed, and it would be fascinating to see the corresponding function.

It would also be fascinating to study our equations at strong coupling, paralleling the the strong coupling application of OPE. It is consistent with all which is presently known at strong coupling, in that it express $\bar{Q}\log R_{n,0}$ as $\sqrt{\lambda}$ times a ratio function, which is known to be of order 1. However, to test it in a nontrivial way, one would have to obtain the ratio function, which corresponds to a one-loop computation in the string theory.

Amplitudes restricted to a two-dimensional subspace of four-dimensions seem especially promising. With a suitable supersymmetric reduction preserving a $SL(2\|2)\times SL(2\|2)$ symmetry, we have seen that these amplitudes form a closed set under the Yangian equations. They are uniquely determined by equations (7.7) and (7.10), without any reference to the underlying four-dimensional theory. There is the intriguing possibility, that the main complexities of loop amplitudes in these kinematics might already be present in some way inside the tree amplitudes. With some effort, one should be able to derive the recent three-loop MHV octagon [60], from a nice form of tree N^3MHV tetradecagon, and it might be possible to get some closed-form result for ℓ-loop N^kMHV $2n$-gons, starting from tree N^{k+l}MHV $2(n+l)$-gons (the combinatorics are worked out in [40]). Finally, we speculate that these equations might put planar S-matrix (at least in two-dimensional kinematics) in the same position as the anomalous dimension was put by Bethe equations a decade ago.

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A Taking differentials of one-dimensional integrals

Consider an integral of the form

$$\int_0^\infty d\log(x+a)F(x,u_i), \tag{A.1}$$

or, more generally, a linear combination of such integrals such that the total integrand converges both at zero and infinity.

Then its differential is the sum of the following terms:

- A term
$$-F(x{=}0,u_i)d\log a \tag{A.2}$$
- For each term of the form $G_j(x,u_i)d\log(x+x_j)$ in the differential of F, a term
$$+(d\log(a-x_j))\int_0^\infty (d\log\frac{x+a}{x+x_j})G_j(x,u_i) \tag{A.3}$$
- For each term of the form $H_j(x,u_j)d\log f$ in the differential of F, where f is independent of x, a term
$$+(d\log f)\int_0^\infty d\log(x-a)G_0(x,u_i). \tag{A.4}$$

The proof is left to the reader; it is more or less integration by parts. Sometimes it may happen that some $x_j=0$, in which case some intermediate expressions will be ill-defined. This can be dealt with efficiently by moving the boundary to $x=\epsilon$, the ϵ-dependence then canceling at the end provided the integral is convergent.

This algorithm for computing derivatives can be used to efficiently (and automatically) compute the symbol of any one-dimensional integral; the result depends only on the symbol of F, not on its functional representative.

B Special functions for MHV and NMHV hexagons

The six-dimensional massless hexagon integral is [42, 43]:

$$\begin{aligned}
I_6^{6D} &= -\frac{1}{3}J^3-\frac{\pi^2}{3}J+2\sum_{i=1}^{3}L^-(x^+u_i,x^-u_i),\\
J &= \sum_{i=1}^{3}(\ell_1(x^+u_i)-\ell_1(x^-u_i)),\\
x^+ &= \frac{\langle 1245\rangle\langle 3461\rangle\langle 2356\rangle}{\langle 1234\rangle\langle 3456\rangle\langle 5612\rangle},\quad x^- = \frac{\langle 1245\rangle\langle 3461\rangle\langle 2356\rangle}{\langle 6123\rangle\langle 4561\rangle\langle 2345\rangle}.
\end{aligned}\tag{B.1}$$

This expression is valid for all Euclidean kinematics, u_1, u_2, u_3 real and positive. There appear the special combinations introduced in [31]

$$\begin{aligned}
L_n^+(x^+, x^-) &= \frac{\log(x^+x^-)^n}{(2n)!!} + \sum_{m=0}^{n-1} \frac{(-1)^m}{(2m)!!} \log(x^+x^-)^m \left(\ell_{n-m}(x^+) + \ell_{n-m}(x^-)\right), \\
L_n^-(x^+, x^-) &= \sum_{m=0}^{n-1} \frac{(-1)^m}{(2m)!!} \log(x^+x^-)^m \left(\ell_{n-m}(x^+) - \ell_{n-m}(x^-)\right), \qquad \text{(B.2)}
\end{aligned}$$

where

$$\ell_n(x) = \frac{1}{2}(\mathrm{Li}_n(x) - (-1)^n \mathrm{Li}_n(1/x)). \qquad \text{(B.3)}$$

Regarding these special functions, it is important to note that in our applications the arguments x^+ and x^- are the two roots of a quadratic equation with real coefficients. The functions L^+ and $L^-/(x^+ - x^-)$ are always real, regardless of whether the roots are real or complex, and remain analytic in the transition region. When the two roots are complex, the standard branch of the polylogarithms is to be used (the one defined on the complex plane minus the line $[1, \infty)$). When the two roots become real, one is instructed to add opposite infinitesimal quantities $x^+ \to x^+ + i\epsilon$, $x^- \to x^- - i\epsilon$; the result will not depend on the sign of ϵ.

The collinear limit of the N^{2}MHV two-loop amplitude gives rise to following seven integrals related to the two-loop NMHV 6-point amplitudes:

$$\begin{aligned}
\bar{Q}R_6^{\mathrm{NMHV}} &= (6)\bar{Q}\log\frac{\langle\bar{6}2\rangle}{\langle\bar{6}4\rangle} f_1 + ((1)-(2)+(4)-(5))\bar{Q}\log\frac{\langle\bar{6}4\rangle}{\langle\bar{6}2\rangle} f_2 + ((2)-(4))\bar{Q}\log\frac{\langle\bar{6}4\rangle}{\langle\bar{6}2\rangle} f_3 \\
&+ \left((6)\bar{Q}\log\frac{\langle\bar{6}2\rangle}{\langle\bar{6}3\rangle} + ((5)-(4))\bar{Q}\log\frac{\langle\bar{6}2\rangle}{\langle\bar{6}4\rangle}\right) f_4 + ((2)+(4))\bar{Q}\log\frac{\langle\bar{6}2\rangle}{\langle\bar{6}4\rangle} f_5 \\
&+ (5)\bar{Q}\log\frac{\langle\bar{6}2\rangle}{\langle\bar{6}3\rangle} f_6 + (3)\bar{Q}\log\frac{\langle\bar{6}2\rangle}{\langle\bar{6}3\rangle} f_7, \qquad \text{(B.4)}
\end{aligned}$$

where (1) notates the R-invariant [23456] and

$$\begin{aligned}
f_1 &= \frac{1}{2}I_6^{6D} + \mathrm{Li}_3(1-u_2) + \mathrm{Li}_3(1-1/u_2) - \mathrm{Li}_3(1-u_1) - \mathrm{Li}_3(1-1/u_1) + \frac{\pi^2}{3}\log u_3 \\
&+ \frac{1}{2}(\mathrm{Li}_2(1-u_1) + \mathrm{Li}_2(1-u_2) + \mathrm{Li}_2(1-u_3))\log\frac{u_1}{u_2u_3} + \frac{\log^3 u_1}{6} - \frac{\log^3 u_2}{6} \\
&- \left(\mathrm{Li}_2(1-u_2) + \frac{1}{2}\mathrm{Li}_2(1-u_3)\right)\log u_3 - \frac{3}{4}\log u_1 \log^2 u_3 - \frac{1}{4}\log u_2 \log^2 u_3, \\
f_2 &= \mathrm{Li}_3(1-u_3) - \frac{1}{2}\left(\mathrm{Li}_2(1-u_3) + \frac{1}{2}\log u_3 \log u_1u_2 - \log u_1 \log u_2\right)\log u_3, \\
f_3 &= -\mathrm{Li}_3(1-1/u_3) + \frac{1}{6}\log^3 u_3 + \frac{\pi^2}{6}\log u_3, \\
f_4 &= \left(\mathrm{Li}_2(1-u_1) + \mathrm{Li}_2(1-u_2) + \mathrm{Li}_2(1-u_3) + \log u_1 \log u_3 - \frac{\pi^2}{3}\right)\log u_3, \\
f_5 &= \left(\mathrm{Li}_2(1-u_3) - \frac{\pi^2}{6}\right)\log\frac{u_1}{u_2}, \quad f_6 = \log^2 u_3 \log\frac{u_2}{u_1}, \quad f_7 = \log u_2 \log u_3 \log\frac{u_3}{u_1}. \qquad \text{(B.5)}
\end{aligned}$$

Expressing the six-gluon NMHV amplitude as

$$
\begin{aligned}
R_{6,1} \;=\; & \frac{1}{2}\Big([(1)+(4)]V_3+[(2)+(5)]V_1+[(3)+(6)]V_2 \\
& +[(1)-(4)]\tilde{V}_3+[(5)-(2)]\tilde{V}_1+[(3)-(6)]\tilde{V}_2\Big)\,,
\end{aligned} \tag{B.6}
$$

we obtain from this, as explained in the main text, the differential

$$
\begin{aligned}
dV_3 \;=\; & -\frac{1}{2}I_6^{6D}d\log\frac{y_2}{y_3}+(dV_3)_1 d\log\frac{u_1}{(1-u_2)(1-u_3)}+(dV_3)_2 d\log\frac{u_2}{1-u_2}+(dV_3)_3 d\log\frac{u_3}{1-u_3} \\
& +(dV_3)_4 d\log\frac{1-u_1}{u_2u_3}
\end{aligned}
$$

where

$$
y_1=\frac{\langle 1235\rangle\langle 2346\rangle\langle 1456\rangle}{\langle 1234\rangle\langle 2456\rangle\langle 1356\rangle},\quad y_2=\frac{\langle 2345\rangle\langle 1356\rangle\langle 1246\rangle}{\langle 1345\rangle\langle 2346\rangle\langle 1256\rangle},\quad y_3=\frac{\langle 1345\rangle\langle 2456\rangle\langle 1236\rangle}{\langle 1235\rangle\langle 3456\rangle\langle 1246\rangle},
$$

$$
\begin{aligned}
(dV_3)_1 \;=\; & 2\mathrm{Li}_3(1-u_2)+\mathrm{Li}_3(1-1/u_2)+2\mathrm{Li}_3(1-u_3)+\mathrm{Li}_3(1-1/u_3)-\frac{1}{6}\log^3(u_2u_3) \\
& +\mathrm{Li}_2(1-u_1)\log u_2u_3+\mathrm{Li}_2(1-u_2)\log\frac{u_3^2}{u_1}+\mathrm{Li}_2(1-u_3)\log\frac{u_2^2}{u_1} \\
& +\frac{1}{2}\log u_1\left(\log^2 u_2u_3+2\log u_2\log u_3\right)+\frac{\pi^2}{3}\log\frac{u_1}{u_2^2u_3^2}, \\
(dV_3)_2 \;=\; & \mathrm{Li}_3(1-u_1)+\mathrm{Li}_3(1-1/u_1)+\mathrm{Li}_3(1-u_2)+2\mathrm{Li}_3(1-1/u_2)-3\mathrm{Li}_3(1-u_3)-2\mathrm{Li}_3(1-1/u_3) \\
& +\frac{1}{2}\mathrm{Li}_2(1-u_1)\log\frac{u_2}{u_1u_3^3}+\frac{1}{2}\mathrm{Li}_2(1-u_2)\log\frac{u_1}{u_2u_3^3}+\frac{1}{2}\mathrm{Li}_2(1-u_3)\log\frac{u_1^3u_3}{u_2^3}+\frac{\pi^2}{3}\log u_2u_3 \\
& +\frac{1}{3}\log^3 u_1u_3-\frac{1}{3}\log^3 u_2-\frac{1}{2}\log u_1u_2\log^2 u_1u_3+\frac{1}{2}\log u_1\log u_3\log\frac{u_2^3}{u_3^2}-\frac{\pi^2}{2}\log\frac{u_1u_2}{u_3}, \\
(dV_3)_3 \;=\; & (dV)_2 \text{ with } u_2\leftrightarrow u_3, \\
(dV_3)_4 \;=\; & -\left(\mathrm{Li}_2(1-u_1)+\mathrm{Li}_2(1-u_2)+\mathrm{Li}_2(1-u_3)+\log u_1\log u_2u_3-\log u_2\log u_3-\frac{\pi^2}{3}\right)\log u_1
\end{aligned}
$$

and

$$
\begin{aligned}
d\tilde{V}_3 \;=\; & \frac{1}{2}I_6^{6D}d\log\frac{v(1-w)}{(1-v)w}+(d\tilde{V}_3)_1 d\log y_1+(d\tilde{V}_3)_2 d\log y_2y_3+(d\tilde{V}_3)_3 d\log\frac{y_2}{y_3}, \\
(d\tilde{V}_3)_1 \;=\; & \mathrm{Li}_3(1-1/u_2)-\mathrm{Li}_3(1-1/u_3)+\left(\mathrm{Li}_2(1-u_1)-\frac{\pi^2}{3}\right)\log\frac{u_2}{u_3}-\frac{1}{6}\log^3\frac{u_2}{u_3} \\
(d\tilde{V}_3)_2 \;=\; & -\mathrm{Li}_3(1-u_2)+\mathrm{Li}_3(1-u_3)+\frac{1}{2}\mathrm{Li}_2(1-u_2)\log\frac{u_1u_2}{u_3}-\frac{1}{2}\mathrm{Li}_2(1-u_3)\log\frac{u_1u_3}{u_2} \\
& +\frac{1}{2}\left(\mathrm{Li}_2(1-u_1)+\frac{1}{2}\log u_1\log u_2u_3-\frac{\pi^2}{3}\right)\log\frac{u_2}{u_3}, \\
(d\tilde{V}_3)_3 \;=\; & \mathrm{Li}_3(1-u_1)+\mathrm{Li}_3(1-1/u_1)-\left(\frac{1}{2}\mathrm{Li}_2(1-u_1)+\frac{1}{4}\log^2\frac{u_2}{u_3}+\frac{1}{6}\log^2 u_1+\frac{\pi^2}{6}\right)\log u_1.
\end{aligned}
$$

These differentials are integrable, e.g., $d^2=0$.

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