# Improved Breakdown Criterion for Einstein Vacuum Equations in CMC Gauge

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**Abstract**

Let $\mathcal{M}_* = \bigcup_{t\in[t_0,t_*)} \Sigma_t$ be a part of vacuum globally hyperbolic space-time $(\mathbf{M}, \mathbf{g})$, foliated by constant mean curvature hypersurfaces $\Sigma_t$ with $t_0 < t_* < 0$. We improve the existing breakdown criteria for Einstein vacuum equations by showing that the foliation can be extended beyond $t_*$ provided the second fundamental form $k$ and the lapse function $n$ satisfy the weaker condition

$$\int_{t_0}^{t_*} (\|k\|_{L^\infty(\Sigma_t)} + \|\nabla \log n\|_{L^\infty(\Sigma_t)})dt < \infty.$$

The proof of this result relies on the second main result of the paper, which gives a uniform lower bound on the null radius of injectivity. © 2011 Wiley Periodicals, Inc.

## 1 Introduction

Let $(\mathbf{M}, \mathbf{g})$ be a (3+1)–dimensional vacuum globally hyperbolic space-time, i.e., $\mathbf{g}$ is a Lorentz metric of signature $(-,+,+,+)$ satisfying the Einstein vacuum equations

$$\mathbf{Ric}(\mathbf{g}) = 0,$$

and every causal curve intersects a Cauchy surface at precisely one point. If $(\mathbf{M}, \mathbf{g})$ has a compact, constant mean curvature (CMC) Cauchy surface $\Sigma_0$ with mean curvature $t_0 < 0$, then there exists a foliation of a neighborhood of $\Sigma_0$ by compact CMC surfaces, and the mean curvature varies monotonically from slice to slice. The CMC conjecture states that there is a foliation in $\mathbf{M}$ of CMC Cauchy surfaces with mean curvatures taking on all allowable values; i.e., the mean curvatures take all values in $(-\infty, 0)$ if $\Sigma_0$ is of Yamabe type $-1$ or 0, while the mean curvatures take on all values in $(-\infty, \infty)$ if $\Sigma_0$ is of Yamabe type $+1$. Some progress has been made [3]; the CMC conjecture, however, remains open. An important step to attack the CMC conjecture is to provide a reasonable breakdown criterion to detect what may happen when the CMC foliation cannot be extended.

In order to set up the framework, in this paper we assume that $\mathcal{M}_*$ is a part of the space-time $(\mathbf{M}, \mathbf{g})$ foliated by CMC hypersurfaces $\Sigma_t$ with mean curvature $t$ satisfying $t_0 \le t < t_*$ for some $t_0 < t_* < 0$. We shall refer to $\Sigma_0 := \Sigma_{t_0}$ as the

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initial slice. Thus, $\mathcal{M}_* = \bigcup_{t\in[t_0,t_*)} \Sigma_t$ with $t_* < 0$ and there is a time function $t$ defined on $\mathcal{M}_*$, monotonically increasing toward the future, such that each $\Sigma_t$ is a level hypersurface of $t$ with the lapse function $n$ and the second fundamental form $k$ defined by

$$n := (-\mathbf{g}(\mathbf{D}t, \mathbf{D}t))^{-1/2} \quad \text{and} \quad k(X, Y) := -\mathbf{g}(\mathbf{D}_X\mathbf{T}, Y),$$

where $\mathbf{T}$ denotes the future directed unit normal to $\Sigma_t$, $\mathbf{D}$ denotes the space-time covariant differentiation associated with $\mathbf{g}$, and $X$ and $Y$ are vector fields tangent to $\Sigma_t$. Let $g$ be the induced Riemannian metric on $\Sigma_t$, and let $\nabla$ be the corresponding covariant differentiation. For any coordinate chart $\mathcal{O} \subset \Sigma_0$ with coordinates $x = (x^1, x^2, x^3)$, let $x^0 = t, x^1, x^2, x^3$ be the transported coordinates on $[t_0, t_*) \times \mathcal{O}$ obtained by following the integral curves of $\mathbf{T}$. Under these coordinates the metric $\mathbf{g}$ takes the form

$$\mathbf{g} = -n^2\, dt^2 + g_{ij}\, dx^i\, dx^j. \tag{1.1}$$

Moreover, relative to these coordinates there hold the evolution equations

$$\partial_t g_{ij} = -2nk_{ij}, \tag{1.2}$$

$$\partial_t k_{ij} = -\nabla_i\nabla_j n + n(R_{ij} + \operatorname{Tr} k k_{ij} - 2k_{ia}k^a_j), \tag{1.3}$$

and the constraint equations

$$R - |k|^2 + (\operatorname{Tr} k)^2 = 0, \tag{1.4}$$

$$\nabla^j k_{ji} - \nabla_i \operatorname{Tr} k = 0, \tag{1.5}$$

on each $\Sigma_t$, where $R_{ij}$ and $R$ denote the Ricci curvature and the scalar curvature of the induced metric $g$ on $\Sigma_t$, and $\operatorname{Tr} k$ denotes the trace of $k$, i.e., $\operatorname{Tr} k = g^{ij}k_{ij}$. Since $\operatorname{Tr} k = t$ on $\Sigma_t$, it follows from the above equations that

$$\operatorname{div} k = 0 \tag{1.6}$$

and

$$-\Delta n + |k|^2 n = 1 \tag{1.7}$$

on each $\Sigma_t$.

The first important breakdown criterion was given by Anderson [2], who showed that if

$$\sup_{t\in[t_0,t_*)} \|\mathbf{R}\|_{L^\infty(\Sigma_t)} = \Lambda_0 < \infty \tag{1.8}$$

for all $t_* < 0$, then the CMC foliation exists for all values in $[t_0, 0)$, where $\mathbf{R}$ denotes the Riemannian curvature tensor of the space-time $(\mathbf{M}, \mathbf{g})$.

Recently Klainerman and Rodnianski [12] provided a new breakdown criterion which shows that the CMC foliation can be extended beyond any value $t_* < 0$ for which

$$\sup_{t\in[t_0,t_*)} (\|k\|_{L^\infty(\Sigma_t)} + \|\nabla \log n\|_{L^\infty(\Sigma_t)}) = \Lambda_0 < \infty. \tag{1.9}$$

This condition refers only to the second fundamental form $k$ and the lapse function $n$, which requires one degree less of differentiability, in contrast to the breakdown criterion of Anderson. Moreover, (1.8) implies (1.9) by purely elliptic estimates. Therefore, the result in [12] is a significant improvement. The argument in [12] relies heavily on the tools from the theory of hyperbolic equations. The analogous result has been extended to nonvacuum space-time in [13].

If we consider the Einstein equation expressed relative to the wave coordinates, by energy estimates one can see that the breakdown does not occur unless

$$\int_{t_0}^{t_*} \|\partial \mathbf{g}\|_{L^\infty}\, dt = \infty. \tag{1.10}$$

This condition, however, is not geometric since it depends on the choice of a full coordinate system. Observe that the components of the second fundamental form $k$ and $\nabla n$ can be viewed as part of the components of $\partial \mathbf{g}$. It is natural to ask if we have an integral form of breakdown criterion involving $k$ and $n$ only. The first main result of the present paper confirms this and provides a geometric counterpart of (1.10), which can be viewed as an improved version of the breakdown criterion of Klainerman and Rodnianski.

THEOREM 1.1 (Main Theorem I[1]). *Let $(\mathcal{M}_*, \mathbf{g})$ be a globally hyperbolic development of $\Sigma_0$ foliated by the CMC level hypersurfaces of a time function $t < 0$. Then the space-time together with the foliation $\Sigma_t$ can be extended beyond any value $t_* < 0$ for which*

$$\int_{t_0}^{t_*} (\|k\|_{L^\infty(\Sigma_t)} + \|\nabla \log n\|_{L^\infty(\Sigma_t)})dt = \mathcal{K}_0 < \infty. \tag{1.11}$$

*More precisely, the CMC foliation of the space-time can be extended to $[t_0, t_* + \delta_*)$ for some $0 < \delta_* \le -t_*$ depending only on $\mathcal{K}_0$, $|\Sigma_0|$, $t_0$, and $t_*$ and suitable norms of the initial data.*

We fix the convention for the deformation tensor of $\mathbf{T}$, expressed relative to an orthonormal frame $\{e_0 = \mathbf{T}, e_1, e_2, e_3\}$, as

$$\pi_{\alpha\beta} = -\mathbf{g}(\mathbf{D}_{e_\alpha}\mathbf{T}, e_\beta), \quad \alpha, \beta = 0, 1, 2, 3.$$

It is easy to check that

$$\pi_{00} = 0, \quad \pi_{0i} = -\nabla_i \log n, \quad \pi_{i0} = 0, \quad \pi_{ij} = k_{ij}, \quad i, j = 1, 2, 3. \tag{1.12}$$

Consequently, condition (1.11) can be formulated as

$$\|\pi\|_{L_t^1 L_x^\infty(\mathcal{M}_*)} := \int_{t_0}^{t_*} \|\pi\|_{L^\infty(\Sigma_t)} dt = \mathcal{K}_0 < \infty. \tag{A1}$$

To see the difficulties posed by the weaker condition (1.11), let us review the mechanism in the proof of [12]. In order to continue the foliation, according to the

[1] Our method applies equally well to the case where the $\Sigma_t$ are asymptotically flat and maximal, i.e., $\mathrm{Tr}\, k = 0$, and can also be extended to Einstein space-time with matters.

local existence theorem given in [5, theorem 10.2.1], one must establish a global uniform bound for the curvature tensor $\mathbf{R}$ and $L^2$-bounds for its first two covariant derivatives. Since $(\mathbf{M}, \mathbf{g})$ is a vacuum space-time, by virtue of the Bianchi identity, $\mathbf{R}$ verifies a wave equation of the form

$$\Box_{\mathbf{g}} \mathbf{R} = \mathbf{R} \star \mathbf{R}, \tag{1.13}$$

where $\Box$ denotes the covariant wave operator $\Box = \mathbf{D}^\alpha \mathbf{D}_\alpha$. Based on higher energy estimates, it is standard to show that the $L^2$-bounds for $\mathbf{DR}$ and $\mathbf{D}^2\mathbf{R}$ can be bounded in terms of the $L^\infty$-norm of $\mathbf{R}$. Thus, the derivation of the $L^\infty$-bound of $\mathbf{R}$ is a crucial step. In order to achieve this goal, Klainerman and Rodnianski [10] succeeded in representing $\mathbf{R}(p)$, for each $p \in \mathcal{M}_*$, by a Kirchhoff-Sobolev formula of the form

$$\mathbf{R}(p) = -\int_{\mathcal{N}^-(p,\tau)} \mathbf{A} \cdot (\mathbf{R} \star \mathbf{R}) + \text{other terms}$$

where $\mathbf{A}$ is a 4-covariant tensor defined as a solution of a transport equation along $\mathcal{N}^-(p,\tau)$ with appropriate initial data at the vertex $p$, and $\mathcal{N}^-(p,\tau)$ denotes the portion of the null boundary $\mathcal{N}^-(p)$ in the time interval $[t(p)-\tau, t(p)]$. The past null cone $\mathcal{N}^-(p)$ is in general an achronal Lipschitz hypersurface ruled by the set of past null geodesics from $p$. In order to derive all necessary estimates, one must show that $\mathcal{N}^-(p)$ remains a smooth hypersurface in the time slab $[t(p)-\tau, t(p))$ for some universal constant $\tau > 0$. Therefore, it is necessary to provide a uniform lower bound for the past null radius of injectivity at all $p \in \mathcal{M}_*$.

Let us recall briefly the definition of the past null radius of injectivity at $p$; one may consult [11] for more details. We parametrize the set of past null vectors in $T_p\mathbf{M}$ in terms of $\omega \in \mathbb{S}^2$, the standard sphere in $\mathbb{R}^3$. Then, for each $\omega \in \mathbb{S}^2$, let $l_\omega$ be the null vector in $T_p\mathbf{M}$ normalized with respect to the future, unit, timelike vector $\mathbf{T}_p$ by

$$\mathbf{g}(l_\omega, \mathbf{T}_p) = 1,$$

and let $\Gamma_\omega(s)$ be the past null geodesic with initial data $\Gamma_\omega(0) = p$ and $\frac{d}{ds}\Gamma_\omega(0) = l_\omega$. We define the null vector field $L$ on $\mathcal{N}^-(p)$ by

$$L(\Gamma_\omega(s)) = \frac{d}{ds}\Gamma_\omega(s)$$

which may only be smooth almost everywhere on $\mathcal{N}^-(p)$ and can be multivalued on a set of exceptional points. We can choose the parameter $s$ with $s(p) = 0$ so that

$$\mathbf{D}_L L = 0 \quad \text{and} \quad L(s) = 1.$$

This $s$ is called the affine parameter.

The past null radius of injectivity $i_*(p)$ at $p$ is then defined to be the supremum over all the values $s_0 > 0$ for which the exponential map

$$\mathfrak{g}_p : (s, \omega) \to \Gamma_\omega(s)$$

is a global diffeomorphism from $(0, s_0) \times \mathbb{S}^2$ to its image in $\mathcal{N}^-(p)$. It is known that $i_*(p) > 0$ for each $p$, $\mathcal{N}^-(p)$ is smooth within the null radius of injectivity, and

$$i_*(p) = \min\{s_*(p), l_*(p)\},$$

where $s_*(p)$, the past null radius of conjugacy at $p$, is defined to be the supremum over all values $s_0 > 0$ such that the exponential map $\mathfrak{g}_p$ is a local diffeomorphism from $(0, s_0) \times \mathbb{S}^2$ to its image in $\mathcal{N}^-(p)$, and $l_*(p)$, the past cut locus radius at $p$, is defined to be the smallest value of $s_0$ for which there exist two distinct null geodesics $\Gamma_1$ and $\Gamma_2$ from $p$ with $\Gamma_1(s_0) = \Gamma_2(s_0)$.

For the CMC foliation, it is convenient to introduce the past null radius of injectivity $i_*(p, t)$ at each $p$ with respect to the global time function $t$. We define $i_*(p, t)$ to be the supremum over all the values $\tau > 0$ for which the exponential map

$$\mathcal{G}_p : (t, \omega) \to \Gamma_\omega(s(t)) \tag{1.14}$$

is a global diffeomorphism from $(t(p) - \tau, t(p)) \times \mathbb{S}^2$ to its image in $\mathcal{N}^-(p)$. We remark that $s$ is a function not only depending on $t$ but also on $\omega$; we suppress $\omega$ just for convenience. It is known that

$$i_*(p, t) = \min\{s_*(p, t), l_*(p, t)\},$$

where $s_*(p, t)$ is defined to be the supremum over all values $\tau > 0$ such that the map $\mathcal{G}_p$ is a local diffeomorphism from $(t(p) - \tau, t(p)) \times \mathbb{S}^2$ to its image, and $l_*(p, t)$ is defined to be the smallest value of $\tau > 0$ for which there exist two distinct null geodesics $\Gamma_1(s(t))$ and $\Gamma_2(s(t))$ from $p$ that intersect at a point with $t = t(p) - \tau$.

In [11] Klainerman and Rodnianski provided a uniform lower bound on the null radius of injectivity under the assumption (1.9). In order to complete the proof of Theorem 1.1, we provide a uniform lower bound on the null radius of injectivity under the weaker condition (1.11), which is contained in the second main result of the present paper.

THEOREM 1.2 (Main Theorem II). *Assume that $\mathcal{M}_*$ is a globally hyperbolic development of $\Sigma_0$ satisfying condition* (1.11). *Then for all $p \in \mathcal{M}_*$ there holds*

$$i_*(p, t) > \min\{\delta_*, t(p) - t_0\}, \tag{1.15}$$

*where $\delta_* > 0$ is a universal constant.*[2]

In order to prove this result, it is useful to review the essential steps in the work of Klainerman and Rodnianski in [11]. The first step is to show that

$$s_*(p, t) > \min\{l_*(p, t), \delta_*\} \tag{1.16}$$

[2] A universal constant always means a constant depending only on $Q_0$, $\mathcal{K}_0$, $|\Sigma_0|$, $t_*$, and the number $I_0 > 0$ such that $I_0^{-1} \le (g_{ij}) \le I_0$ on the initial slice $\Sigma_0$, where $Q_0$ denotes the Bel-Robinson energy on the initial slice $\Sigma_0$, which will be defined in Section 2. Throughout this paper $C$ always denotes a universal constant.

for some universal constant $\delta_* > 0$. This can be achieved by showing that

$$\sup_{\mathcal{N}^-(p,\tau)} \left| \operatorname{tr}\chi - \frac{2}{s(t)} \right| \le C \tag{1.17}$$

with $\tau := \min\{l_*(p,t), \delta_*\}$, where $\chi$ is the null second fundamental form $\chi_{AB} = \mathbf{g}(\mathbf{D}_A L, e_B)$ of the two-dimensional spacelike surface $S_t := \mathcal{N}^-(p) \cap \Sigma_t$ with $(e_A)_{A=1,2}$ being a frame field tangent to $S_t$. The analogue has been carried out in [7, 8, 9, 14] for geodesic foliations under the boundedness assumption of the curvature flux. In order to adapt those arguments to prove (1.17) for the time foliations, one needs to show that $t(p) - t$ and $s$ are comparable and the geodesic curvature flux (see [11]) is bounded, both of which rely on the relation

$$|a - 1| \le \frac{1}{2} \quad \text{on } \mathcal{N}^-(p,\tau), \tag{1.18}$$

where $a$, the null lapse function, is defined by $a^{-1} := \mathbf{g}(\mathbf{T}, L)$ with $a(p) = 1$. Note that along a null geodesic

$$\frac{dt}{ds} = -(an)^{-1}, \quad \frac{da}{ds} = \nu, \quad \nu := k_{NN} - \nabla_N \log n,$$

where $N$ is the unit inward normal of $S_t$ in $\Sigma_t$. If (1.9) is satisfied, one can see that (1.18) holds for $t(p) - \delta_* \le t \le t(p)$ for some universal $\delta_* > 0$, and consequently $s$ and $t(p) - t$ are comparable. However, under the weaker condition (1.11) only, it is highly nontrivial to obtain (1.18). We observe that (1.18) can be achieved by establishing

$$\|\nu\|^2_{L^\infty_\omega L^2_t(\mathcal{N}^-(p,\tau))} := \sup_{\omega\in\mathbb{S}^2} \int_{\Gamma_\omega} an|\nu|^2\, dt \le C \tag{1.19}$$

where $\Gamma_\omega$ is the portion of a past null geodesic that initiates from $p$ and is contained in $\mathcal{N}^-(p,\tau)$ for some universal constant $\delta_* > 0$.

How to obtain such an estimate on $\nu$ is the first difficulty we encounter. Under the assumption (1.11) only, it relies crucially on the following two ingredients:[3]

(1) there holds for $\nabla\!\!\!/\,\nu$ the decomposition

$$\nabla\!\!\!/\,\nu = \nabla_L P + Q \tag{1.20}$$

with $P$ and $Q$ appropriate $S_t$ tangent tensors;

(2) there holds

$$\|\nabla\!\!\!/(\nu, P)\|_{L^2(\mathcal{N}^-(p,\tau))} + \|\nabla_L(\nu, P)\|_{L^2(\mathcal{N}^-(p,\tau))} \le C. \tag{1.21}$$

As one of the important observations in our work, the decomposition of the form (1.20) is derived in [15, 16]. How to obtain the estimate for $\nu$ in (1.21) still poses a substantial difficulty due to the weaker assumption. The estimate for $\nabla_N \log n$ of

[3] $\nabla\!\!\!/$ denotes the connection with respect to the induced metric $\gamma$ on $S_t$.

the form (1.21) can be obtained by elliptic estimates and the trace inequality. By an elliptic estimate, in view of

$$\operatorname{div} k = 0, \quad \operatorname{curl} k = H, \tag{1.22}$$

where $H$ denotes the magnetic part of $\mathbf{R}$, we can only derive $\|k\|_{H^1_x(\Sigma)} \le C$, which, by the classic trace theorem, loses a half derivative if restricted to the null cone. However, (1.21) requires the $L^2$ control of one derivative of $k_{NN}$ on null cones. Hence we must adopt a novel approach, which significantly surpasses the one via an elliptic estimate and the trace inequality. This motivates the application of the tensorial wave equation for $k$, which symbolically is given by

$$\Box k = k \cdot \mathrm{Ric} + n^{-2}\nabla^2 \dot{n} + \pi \cdot \nabla k - n^{-3}\dot{n}\nabla^2 n + \pi\cdot\pi\cdot\pi + k\cdot\nabla^2 n - n^{-1}k. \tag{1.23}$$

We then prove by the energy method that the $k$-flux satisfies

$$\|\nabla\!\!\!/ k\|_{L^2(\mathcal{N}^-(p,\tau))} + \|\nabla_L k\|_{L^2(\mathcal{N}^-(p,\tau))} \le C, \tag{1.24}$$

which schematically gives the desired control on $k_{NN}$.

The treatment for $P$ and $Q$ in (1.20) has to be coupled with the proof of a series of estimates for the Ricci coefficients on the null hypersurface $\mathcal{N}^-(p,\tau)$ including (1.17) by a delicate bootstrap argument. Hence, under condition (1.11) only, (1.17), (1.18), and (1.19) should be proved simultaneously. The proof, though close to the spirit of the works [7, 8, 9, 14], is very involved and entails new observations on the delicate structures of Ricci coefficients. We refer the reader to [15, 16] for full details.

The next step is to find a good system of local space-time coordinates under which $\mathbf{g}$ is comparable to the Minkowski metric. More precisely, for a sufficiently small constant $\epsilon > 0$, one needs to show that there exists a constant $\delta_* > 0$, depending only on $\epsilon$ and some universal constants, for which each geodesic ball $B_{\delta_*}(p)$ with $p \in \Sigma_t$ admits local coordinates $x = (x^1, x^2, x^3)$ such that under the corresponding transport coordinates $x^0 = t, x^1, x^2, x^3$ the metric $\mathbf{g}$ has the expression (1.1) with

$$|n - n(p)| \le \epsilon \quad \text{and} \quad |g_{ij} - \delta_{ij}| \le \epsilon \tag{1.25}$$

on $B_{\delta_*}(p) \times [t(p) - \delta_*, t(p)]$. The existence of such local coordinates together with (1.17) will enable us to show that $\mathcal{N}^-(p,\delta_*)$ is close to the flat cone and consequently $l_*(p,t) \ge \delta_*$.

The part on $n$ in (1.25) can be established by elliptic estimates on $n$ and $\partial_t n$. The derivation of the result for $g$ under the weaker condition (1.11), however, presents one of the core difficulties, which invokes new methods and a second application of (1.23).

By the Bel-Robinson energy bound $\mathcal{Q}(t) \le C$ and a result of Anderson [1], one can control the lower bound of the harmonic radius on $\Sigma_t$ such that with the coordinates $x = (x^1, x^2, x^3)$ on $B_{\delta_*}(p) \subset \Sigma_t$,

$$|g_{ij}(x, t(p)) - \delta_{ij}| \le \frac{1}{2}\epsilon.$$

The challenge is to control the time evolution of $g$. Using (1.2), one has[4]

$$|g_{ij}(x,t(p)) - g_{ij}(x,t)| \lesssim \int_t^{t(p)} |k(x,t')|dt'. \tag{1.26}$$

If (1.9) holds, or more generally, if

$$\int_t^{t(p)} \|k(t')\|^q_{L^\infty(\Sigma_{t'})} dt' \le \Lambda_0 < \infty \tag{1.27}$$

for some $q > 1$, then with $\delta_*$ sufficiently small

$$|g_{ij}(x,t(p)) - g_{ij}(x,t)| \le \Lambda_0^{1/q}(t(p)-t)^{1-1/q} < \frac{1}{2}\epsilon. \tag{1.28}$$

Without a uniform positive lower bound on the null radius of injectivity, deriving (1.27) only under assumption (1.11) is essentially to attack the $L^2$ curvature conjecture, which is still an open and extremely hard problem. Under assumption (1.11), our strategy is to establish directly that

$$\sup_{x\in\Sigma} \int_t^{t(p)} |k(x,t')|^2\, dt' \le C. \tag{1.29}$$

This together with (1.26) gives

$$\begin{aligned} |g_{ij}(x,t(p)) - g_{ij}(x,t)| &\lesssim \left(\int_t^{t(p)} |k(x,t')|^2\, dt'\right)^{1/2} (t(p)-t)^{1/2} \\ &\lesssim (t(p)-t)^{1/2}, \end{aligned}$$

which implies $|g_{ij}(x,t(p)) - g_{ij}(x,t)| < \frac{1}{2}\epsilon$ as long as $\delta_*$ is appropriately chosen.

The major part of the present paper is therefore to establish (1.29) under the weaker condition (1.11). To this end, we will use the Kirchhoff parametrix to represent $k$ as

$$-4\pi n(p) k(p) \cdot J = \int_{\mathcal{N}^-(p,\tau)} \Box k \cdot \mathbf{A} + \text{other terms}$$

for any $\delta < i_*(p,t)$, where $J$ is any 2-covariant tensor at $p$ tangent to $\Sigma_{t(p)}$ and $\mathbf{A}$ is the $\Sigma$-tangent tensor defined by

$$\mathbf{D}_L \mathbf{A}_{ij} + \frac{1}{2}\operatorname{tr}\chi \mathbf{A}_{ij} = 0 \ \text{ on } \mathcal{N}^-(p,\tau), \qquad \lim_{t\to t(p)} (t(p)-t)\mathbf{A}_{ij} = J.$$

It can be shown that $\|r\mathbf{A}\|_{L^\infty(\mathcal{N}^-(p,\tau))} \lesssim 1$ together with other estimates on $\mathbf{A}$, where $r = \sqrt{(4\pi)^{-1}|S_t|}$ and $|S_t|$ denotes the area of $S_t$. Thus

$$n(p)|k(p)| \lesssim \int_{\mathcal{N}^-(p,\tau)} r^{-1}|\Box k| + \text{other terms}.$$

[4] We use $\Phi_1 \lesssim \Phi_2$ to mean that $\Phi_1 \le C\Phi_2$ for some universal constant $C$.

Next we let $p$ move along an integral curve $\Phi(t)$ of $\mathbf{T}$ to get the representations of $k$ at all points on this curve. Then we can reduce the proof of (1.29) to showing

$$\int_{t(p)-\tau}^{t(p)} \left| \int_{\mathcal{N}^-(\Phi(t),t-t(p)+\tau)} r^{-1}|\Box k| + \cdots \right|^2 dt \lesssim 1. \tag{1.30}$$

In view of (1.23), we have to employ various estimates of $k$ and $n$ on the null cones, which will be established by delicate analysis. We emphasize that due to the severe loss of derivatives arising from the restriction from space-time to null cones, under the assumption of (1.11) only, the Kirchhoff parametrix is not powerful enough to establish the Strichartz estimate in (1.27). One of the key innovations of our approach lies in using (1.30) to prove (1.29), which is sufficient for the purpose of controlling the evolution of metrics. As seen in (1.30), integrating $n(p)^2|k(p)|^2$ with $p$ moving along the time axis leads to an integral over $\bigcup_{t\in(t(p)-\tau,t(p))} \mathcal{N}^-(\Phi(t), t-t(p)+\tau)$, which tackles the difficulty coming from restriction and enables us to obtain the sharp estimate in (1.29).

This paper is organized as follows. In Section 2 we collect some preliminary results that will be used frequently. In Section 3 we establish various elliptic estimates on the lapse function $n$; in particular, we show that $n$ can be bounded from below and above by positive universal constants. In Section 4 we provide the sketch of the proof of Theorem 1.2. We describe how to use the bootstrap argument to establish (1.17) and other related estimates on the null cones. We then show how to use estimate (1.29) to obtain a good system of local space-time coordinates, which is crucial for completing the proof of Theorem 1.2. The proof of (1.29) occupies the next five sections. In Section 5 we derive a tensorial wave equation for $k$ and in Section 6 we provide the estimate for the so-called $k$-flux, which will be defined later. In Section 7 we provide some trace estimates on the surfaces $S_t$. We then use these results in Section 8 to establish various estimates for $k$, $n$, and $\chi$ on the null cones. In Section 9 we adapt the Kirchhoff-Sobolev formula in [10] to represent the second fundamental form $k$, through which we give the proof of (1.29) under condition (1.11) and thus complete the proof of Theorem 1.2. Finally, in Section 10 we complete the proof of Theorem 1.1.

## 2 Preliminaries

For the lapse function $n$, by using the elliptic equation $-\Delta n + |k|^2 n = 1$ and the maximum principle it easily follows that

$$\frac{1}{\|k\|_{L^\infty(\Sigma_t)}} \le n \le \frac{3}{t^2} \quad \text{on } \Sigma_t. \tag{2.1}$$

In the next section, by virtue of condition (A1) on $k$, we will show that $n$ in fact can be bounded from below by a positive constant uniformly for all $t \in [t_0, t_*)$. Thus $C^{-1} \le n \le C$ on $\mathcal{M}_I$ for some universal constant $C > 0$.

For each slice $\Sigma_t$, we use $|\Sigma_t|$ to denote its volume. Then, by using $\partial_t g_{ij} = -2nk_{ij}$ and $\operatorname{Tr} k = t$ on $\Sigma_t$ we have

$$\frac{d}{dt}(|t|^3|\Sigma_t|) = \int_{\Sigma_t} t^2(t^2 n - 3) d\mu_g \le 0.$$

This implies that $|t|^3|\Sigma_t|$ is decreasing with respect to $t$. Consequently,

$$|\Sigma_t| \le \frac{|t_0|^3}{|t|^3}|\Sigma_{t_0}| \le \frac{|t_0|^3}{|t_*|^3}|\Sigma_{t_0}| \quad \forall t_0 \le t \le t_*. \tag{2.2}$$

## 2.1 Bel-Robinson Energy

We start with a brief review of Bel-Robinson energy; one may consult [5] for more details. Associated to the Weyl tensor $\mathbf{R}$, the Bel-Robinson tensor is the full symmetric, traceless tensor defined by

$$\mathbf{Q}[\mathbf{R}]_{\alpha\beta\gamma\delta} = \mathbf{R}_{\alpha\lambda\gamma\mu}\mathbf{R}_{\beta}{}^{\lambda}{}_{\delta}{}^{\mu} + {}^\star\mathbf{R}_{\alpha\lambda\gamma\mu}{}^\star\mathbf{R}_{\beta}{}^{\lambda}{}_{\delta}{}^{\mu}, \tag{2.3}$$

where ${}^\star\mathbf{R}$ denotes the Hodge dual of $\mathbf{R}$. On each leaf $\Sigma_t$ we introduce the Bel-Robinson energy

$$\mathcal{Q}(t) := \int_{\Sigma_t} \mathbf{Q}[\mathbf{R}](\mathbf{T}, \mathbf{T}, \mathbf{T}, \mathbf{T}) d\mu_{\Sigma_t}.$$

Since $\mathbf{R}_{\alpha\beta} = 0$, an integration by parts shows for $t_0 \le t < t_*$ that

$$\mathcal{Q}(t) = \mathcal{Q}(t_0) - 3\int_{t_0}^{t}\int_{\Sigma_{t'}} n\mathbf{Q}[\mathbf{R}]_{\alpha\beta 00}\pi^{\alpha\beta}\, d\mu_{\Sigma_{t'}}\, dt'.$$

Let $E$ and $H$ denote the electric and magnetic parts of the curvature tensor $\mathbf{R}$ defined by

$$E(X, Y) = \mathbf{g}(\mathbf{R}(X, \mathbf{T})\mathbf{T}, Y), \quad H(X, Y) = \mathbf{g}({}^\star\mathbf{R}(X, \mathbf{T})\mathbf{T}, Y). \tag{2.4}$$

It is well-known that $E$ and $H$ are traceless symmetric 2-tensors tangent to $\Sigma_t$ with

$$|\mathbf{R}|^2 = |E|^2 + |H|^2, \quad |\mathbf{Q}[\mathbf{R}]| \le 4(|E|^2 + |H|^2)$$

and

$$\mathbf{Q}(\mathbf{T}, \mathbf{T}, \mathbf{T}, \mathbf{T}) = |E|^2 + |H|^2.$$

Therefore

$$\mathcal{Q}(t) \le \mathcal{Q}(t_0) + 12\int_{t_0}^{t} \|n\pi\|_{L^\infty(\Sigma_{t'})}\mathcal{Q}(t')dt'.$$

By the Gronwall inequality it follows that

$$\mathcal{Q}(t) \le \mathcal{Q}(t_0)\exp\left(12\int_{t_0}^{t} \|n\pi\|_{L^\infty(\Sigma_{t'})}dt'\right)$$

for all $t \in [t_0, t_*)$. Therefore, in view of condition (A1), we obtain the uniform boundedness of the Bel-Robinson energy.

LEMMA 2.1. *Under condition* (A1)*, there exists a constant $C$ depending only on $\mathcal{K}_0$ and $t_*$ such that*

$$\mathcal{Q}(t) \le C Q_0^2$$

*for all $t \in [t_0, t_*)$, where $Q_0^2 := \mathcal{Q}(t_0)$.*

Consequently, we have the following:

LEMMA 2.2. *Let condition* (A1) *hold. Then on any CMC leaf $\Sigma_t \subset \mathcal{M}_*$ there holds*

$$\int_{\Sigma_t} \left( |\nabla k|^2 + \frac{1}{4}|k|^4 \right) + \int_{\Sigma_t} |\mathrm{Ric}|^2 \lesssim Q_0^2. \tag{2.5}$$

PROOF. The inequality on $k$ follows from [12, prop. 8.4] and Lemma 2.1. The inequality on Ric then follows from the identity $R_{ij} - k_{ia}k^{aj} + \mathrm{Tr}\, k\, k_{ij} = E_{ij}$. □

## 2.2 Harmonic Coordinates

For any coordinate chart $\mathcal{O} \subset \Sigma_0$ with local coordinates $x = (x^1, x^2, x^3)$, we denote by $x^0 = t, x^1, x^2, x^3$ the transported coordinates on $[t_0, t_*) \times \mathcal{O}$ obtained by transporting along the integral curves of $\mathbf{T}$. The following is an immediate consequence of (A1) and (1.2).

PROPOSITION 2.3. *Let assumption* (A1) *hold. There exists a positive constant $C_0$ depending only on $\mathcal{K}_0$ such that, relative to the induced transported coordinates $x^0 = t$, $x^1$, $x^2$, $x^3$ in $[t_0, t_*) \times \mathcal{O}$ we have*

$$C_0^{-1}|\xi|^2 \le g_{ij}(t, x)\xi^i \xi^j \le C_0|\xi|^2. \tag{2.6}$$

PROOF. This is [12, prop. 2.4], which was stated under the stronger condition (1.9); the proof, however, requires only the weaker assumption (A1). □

Using Proposition 2.3, one can derive a uniform lower bound on the volume radius for all the slices $\Sigma_t$; see [11, prop. 4.4]. In view of $\|\mathrm{Ric}\|_{L^2(\Sigma_t)} \le C$ in Lemma 2.2 and (2.2) on $|\Sigma_t|$, we may apply [1, theorem 3.5] to obtain the following results on the existence of harmonic coordinates.

PROPOSITION 2.4. *Let assumption* (A1) *hold. For any $\epsilon > 0$, there exists $r_0 > 0$ depending on $\epsilon$, $Q_0$, $\mathcal{K}_0$, $|\Sigma_0|$, and $t_*$ such that every geodesic ball $B_r(p) \subset \Sigma_t$ with $r \le r_0$ admits a system of harmonic coordinates $x = (x^1, x^2, x^3)$ under which*

$$(1 + \epsilon)^{-1}\delta_{ij} \le g_{ij} \le (1 + \epsilon)\delta_{ij}, \tag{2.7}$$

$$r \int_{B_r(p)} |\partial^2 g_{ij}|^2 \, d\mu_g \le \epsilon. \tag{2.8}$$

We will not use the full strength of this result. The crucial part in our applications is the existence of local coordinates $x = (x^1, x^2, x^3)$ on each $B_{r_0}(p) \subset \Sigma_t$ satisfying (2.7) with $r_0 > 0$ depending only on $\epsilon$, $Q_0$, $\mathcal{K}_0$, $|\Sigma_0|$, and $t_*$.

## 2.3 Sobolev-Type Inequalities

We will give several Sobolev-type inequalities under assumption (A1). These inequalities are useful in establishing various estimates.

LEMMA 2.5. *Let assumption* (A1) *hold on* $\mathcal{M}_*$. *Then for any smooth tensor field* $F$ *on* $\Sigma_t \subset \mathcal{M}_*$ *and any* $2 \le p \le 6$ *there holds*

$$\|F\|_{L^p(\Sigma_t)} \le C\big(\|\nabla F\|_{L^2(\Sigma_t)}^{(3/2)-(3/p)} \|F\|_{L^2(\Sigma_t)}^{(3/p)-(1/2)} + \|F\|_{L^2(\Sigma_t)}\big), \tag{2.9}$$

*where* $C$ *is a constant depending only on* $\mathcal{K}_0$ *and* $p$.

PROOF. This is [12, cor. 2.7]. □

The following inequality is useful in deriving $L^\infty$-bounds of certain quantities.

LEMMA 2.6. *Let assumption* (A1) *hold on* $\mathcal{M}_*$. *Then for any smooth tensor field* $F$ *on* $\Sigma_t \subset \mathcal{M}_*$ *and* $3 < p \le 6$ *there holds*

$$\begin{aligned}\|F\|_{L^\infty(\Sigma_t)} \le C\big(&\|\nabla^2 F\|_{L^2(\Sigma_t)}^{(3/2)-(3/p)} \|\nabla F\|_{L^2(\Sigma_t)}^{(3/p)-(1/2)} \\ &+ \|\nabla F\|_{L^2(\Sigma_t)} + \|F\|_{L^2(\Sigma_t)}\big),\end{aligned}$$

*where* $C$ *is a constant depending only on* $\mathcal{K}_0$ *and* $p$.

PROOF. By using a partition of unity, the Sobolev embedding $W^{1,p}(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$ with $p > 3$, and (2.6) in Proposition 2.3, it is easy to derive for any scalar function $f$ on $\Sigma_t$ that

$$\|f\|_{L^\infty(\Sigma_t)} \le C(\|\nabla f\|_{L^p(\Sigma_t)} + \|f\|_{L^p(\Sigma_t)}).$$

Now we take $f = |F|^2$ in the above inequality. It yields

$$\begin{aligned}\|F\|_{L^\infty(\Sigma_t)}^2 &\le C(\|\nabla |F|^2\|_{L^p(\Sigma_t)} + \||F|^2\|_{L^p(\Sigma_t)}) \\ &\le C(\|\nabla F\|_{L^p(\Sigma_t)} + \|F\|_{L^p(\Sigma_t)})\|F\|_{L^\infty(\Sigma_t)}.\end{aligned}$$

This implies for $p > 3$ that

$$\|F\|_{L^\infty(\Sigma_t)} \le C(\|\nabla F\|_{L^p(\Sigma_t)} + \|F\|_{L^p(\Sigma_t)}).$$

The desired inequality then follows from Lemma 2.5. □

# 3 Elliptic Estimates for the Lapse Function *n*

In this section, we establish a series of elliptic estimates on the lapse function $n$ together with $n^{-1}$ and $\dot{n} := \partial_t n$ under assumption (A1). These results will be repeatedly used in later sections. Throughout this paper we will use $C$ to denote a universal constant.

### 3.1 Estimates on *n*

PROPOSITION 3.1. *On each $\Sigma_t \subset \mathcal{M}_*$ there holds*

$$\|\nabla^2 n\|_{L^2(\Sigma_t)} + \|\nabla n\|_{L^2(\Sigma_t)} \le C.$$

PROOF. By multiplying equation (1.7) by $n$ and integrating over $\Sigma_t$, we obtain $\int_{\Sigma_t}(|\nabla n|^2 + |k|^2 n^2) = \int_{\Sigma_t} n$. In view of (2.1) and (2.2), this gives $\|\nabla n\|_{L^2} \le C$.

In order to obtain the bound on $\|\nabla^2 n\|_{L^2(\Sigma_t)}$, we use the Böchner identity

$$\int_{\Sigma_t} |\nabla^2 n|^2 = \int_{\Sigma_t} (|\Delta n|^2 - R_{ij}\nabla^i n \nabla^j n).$$

It then follows from equation (1.7), Lemma 2.2, and the Hölder inequality that

$$\|\nabla^2 n\|_{L^2} \lesssim \|k\|_{L^4}^2 + |\Sigma_t|^{1/2} + \|\mathrm{Ric}\|_{L^2}^{1/2}\|\nabla n\|_{L^4} \lesssim 1 + \|\nabla n\|_{L^4}.$$

With the help of Lemma 2.2, we have

$$\|\nabla^2 n\|_{L^2} \lesssim 1 + \|\nabla^2 n\|_{L^2}^{3/4}\|\nabla n\|_{L^2}^{1/4} + \|\nabla n\|_{L^2},$$

which together with the bound on $\|\nabla n\|_{L^2}$ implies $\|\nabla^2 n\|_{L^2} \le C$. □

In order to derive further estimates, we need the following inequality:

LEMMA 3.2. *For any* 1*-form $F$ on $\Sigma_t \subset \mathcal{M}_*$ there holds*

$$\|\nabla^2 F\|_{L^2(\Sigma_t)} \le C(\|\Delta F\|_{L^2(\Sigma_t)} + \|\nabla F\|_{L^2(\Sigma_t)} + \|F\|_{L^2(\Sigma_t)}). \tag{3.1}$$

PROOF. It is well-known that for any 1-form $F$ on $\Sigma_t$ there holds the Böchner identity

$$\begin{aligned}\int_{\Sigma_t} |\Delta F|^2 &= \int_{\Sigma_t} |\nabla^2 F|^2 - \frac{1}{2}\int_{\Sigma_t} R_{diac}R_{miac}F_d F_m \\ &\quad + \int_{\Sigma_t} R_{ad}\nabla_d F_i \nabla_a F_i - \int_{\Sigma_t} R_{idac}\nabla_c F_d \nabla_a F_i.\end{aligned} \tag{3.2}$$

Since $\Sigma_t$ is three-dimensional, the Riemannian curvature tensor is completely determined by its Ricci curvature. Thus, we may use (3.2), the Hölder inequality, Lemma 2.2, Lemma 2.5, and Lemma 2.6 to obtain the estimate

$$\begin{aligned}\|\nabla^2 F\|_{L^2} &\lesssim \|\Delta F\|_{L^2} + \|\mathrm{Ric}\|_{L^2}^{1/2}\|\nabla F\|_{L^4} + \|F\|_{L^\infty}\|Ric\|_{L^2} \\ &\lesssim \|\Delta F\|_{L^2} + \big(\|\nabla^2 F\|_{L^2}^{3/4}\|\nabla F\|_{L^2}^{1/4} + \|\nabla F\|_{L^2}\big).\end{aligned}$$

With the help of Young's inequality, inequality (3.1) follows immediately. □

PROPOSITION 3.3. *On each $\Sigma_t \subset \mathcal{M}_*$ there hold*

$$\|\nabla^3 n\|_{L^2(\Sigma_t)} \le C(\|\nabla n\|_{H^1(\Sigma_t)} + \|k\|_{L^\infty(\Sigma_t)}), \tag{3.3}$$

$$\|\nabla n\|_{L^\infty(\Sigma_t)} \le C\big(\|\nabla n\|_{H^1(\Sigma_t)} + \|k\|_{L^\infty(\Sigma_t)}^{(3/2)-(3/p)}\|\nabla^2 n\|_{L^2(\Sigma_t)}^{(3/p)-(1/2)}\big), \tag{3.4}$$

*where* $3 < p \le 6$.

PROOF. A simple application of Lemma 3.2 to $F = \nabla n$ gives

$$\|\nabla^3 n\|_{L^2} \lesssim \|\Delta\nabla n\|_{L^2} + \|\nabla^2 n\|_{L^2} + \|\nabla n\|_{L^2}. \tag{3.5}$$

By (1.7) and the commutation formula $\Delta\nabla_i n = \nabla_i \Delta n + R_{ij}\nabla_j n$, we can estimate

$$\|\Delta\nabla n\|_{L^2} \lesssim \|k\|_{L^6}^2 \|\nabla n\|_{L^6} + \|k\|_{L^\infty}\|\nabla k\|_{L^2} + \|\mathrm{Ric}\|_{L^2}\|\nabla n\|_{L^\infty}.$$

Plugging this into (3.5) and using Lemma 2.2 and Lemma 2.5 gives

$$\|\nabla^3 n\|_{L^2} \lesssim \|\nabla n\|_{L^\infty} + \|\nabla n\|_{H^1} + \|k\|_{L^\infty}.$$

Using Lemma 2.6 for the term $\|\nabla n\|_{L^\infty}$ with $p = 4$, we then obtain

$$\|\nabla^3 n\|_{L^2} \lesssim \|\nabla^3 n\|_{L^2}^{3/4}\|\nabla^2 n\|_{L^2}^{1/4} + \|\nabla n\|_{H^1} + \|k\|_{L^\infty}.$$

This implies (3.3). Inequality (3.4) follows from (3.3) and Lemma 2.6. □

By integrating (3.3) and (3.4) in time, in view of (A1) and Proposition 3.1 we obtain the following mixed norm estimates.

PROPOSITION 3.4. *Let $1 \le b < 2$. Then there hold*

$$\|\nabla^3 n\|_{L_t^1 L_x^2(\mathcal{M}_*)} \le C \quad \textit{and} \quad \|\nabla n\|_{L_t^b L_x^\infty(\mathcal{M}_*)} \le C.$$

## 3.2 Estimates on $n^{-1}$

We now show that $n$ is bounded below by a positive constant uniformly for all $t_0 \le t < t_*$. We achieve this by establishing the following estimates.

PROPOSITION 3.5. *On each $\Sigma_t \subset \mathcal{M}_*$ there hold*

$$\|\nabla^2(n^{-1})\|_{L^2(\Sigma_t)} + \|n^{-1}\|_{L^\infty(\Sigma_t)} \le C.$$

PROOF. We first have from the Bochner identity that

$$\|\nabla^2(n^{-1})\|_{L^2}^2 \le \|\Delta(n^{-1})\|_{L^2}^2 + \|\mathrm{Ric}\|_{L^2}\|\nabla(n^{-1})\|_{L^4}^2. \tag{3.6}$$

From (1.7) we can derive $\Delta(n^{-1}) = 2n^{-3}|\nabla n|^2 + n^{-2} - |k|^2 n^{-1}$. Consequently, it follows from the Hölder inequality that

$$\|\Delta(n^{-1})\|_{L^2} \lesssim \|n^{-1}\nabla n\|_{L^4}\|\nabla(n^{-1})\|_{L^4} + \|k\|_{L^6}^2\|n^{-1}\|_{L^6} + \|n^{-1}\|_{L^4}^2.$$

Combining this inequality with (3.6) and using the Sobolev embedding $H^1(\Sigma) \hookrightarrow L^p(\Sigma)$ with $2 \le p \le 6$, which is a consequence of Lemma 2.5, we obtain

$$\begin{aligned}\|\nabla^2(n^{-1})\|_{L^2} \lesssim\ & \|n^{-1}\nabla n\|_{L^4}\|\nabla(n^{-1})\|_{L^4} \\ & + (\|n^{-1}\|_{H^1} + \|k\|_{L^6}^2)\|n^{-1}\|_{H^1} \\ & + \|\mathrm{Ric}\|_{L^2}^{1/2}\|\nabla(n^{-1})\|_{L^4}\end{aligned} \tag{3.7}$$

We need to estimate $\|n^{-1}\nabla n\|_{L^4}$. To this end, we multiply equation (1.7) by $n^{-l}$ for some positive integer $l$ and then integrate by parts over $\Sigma_t$ to obtain

$$\int_{\Sigma_t} (l n^{-l-1}|\nabla n|^2 + n^{-l}) = \int_{\Sigma_t} n^{-l+1}|k|^2. \tag{3.8}$$

By using (3.8) with $l = 7$ we obtain

$$\|n^{-1}\nabla n\|_{L^4} \le \Big(\int_{\Sigma_t} n^{-8}|\nabla n|^2\Big)^{1/8}\Big(\int_{\Sigma_t} |\nabla n|^6\Big)^{1/8} \lesssim \|k\|_{L^4}^{1/4}\,\|n^{-2}\|_{L^6}^{3/8}\,\|\nabla n\|_{L^6}^{3/4}.$$

In view of Lemma 2.5 and Proposition 3.1 we have $\|\nabla n\|_{L^6} \le C$. In view of Lemma 2.5 and (3.8) with $l = 5$ we also have

$$\|n^{-2}\|_{L^6} \lesssim \|n^{-2}\|_{H^1} \lesssim \Big(\int_{\Sigma_t} n^{-4}|k|^2\Big)^{1/2} + \|n^{-1}\|_{L^4}^2 \lesssim (1 + \|k\|_{L^6})\|n^{-1}\|_{H^1}^2.$$

Therefore

$$\|n^{-1}\nabla n\|_{L^4} \lesssim \big(1 + \|k\|_{L^6}^{3/8}\big)\|k\|_{L^4}^{1/4}\|n^{-1}\|_{H^1}^{3/4}.$$

Combining this inequality with (3.7) and using Lemma 2.2 to bound $\|k\|_{L^4}$, $\|k\|_{L^6}$, and $\|\mathrm{Ric}\|_{L^2}$ yields

$$\|\nabla^2(n^{-1})\|_{L^2} \lesssim \|n^{-1}\|_{H^1}^{3/4}\,\|\nabla(n^{-1})\|_{L^4} + (\|n^{-1}\|_{H^1} + 1)\|n^{-1}\|_{H^1} + \|\nabla(n^{-1})\|_{L^4}.$$

Applying Lemma 2.5 to the term $\|\nabla(n^{-1})\|_{L^4}$ and then using Young's inequality, we obtain

$$\|\nabla^2(n^{-1})\|_{L^2} \lesssim \|n^{-1}\|_{H^1}^4 + \|n^{-1}\|_{H^1}. \tag{3.9}$$

In order to estimate $\|n^{-1}\|_{H^1}$, we use (3.8) with $l = 3$ to obtain $\|\nabla(n^{-1})\|_{L^2} \lesssim \|k\|_{L^4}\|n^{-1}\|_{L^4}$. Applying Lemma 2.5 to $\|n^{-1}\|_{L^4}$ and using Young's inequality we derive

$$\|\nabla(n^{-1})\|_{L^2} \lesssim (\|k\|_{L^4} + \|k\|_{L^4}^4)\|n^{-1}\|_{L^2} \lesssim \|n^{-1}\|_{L^2}. \tag{3.10}$$

The combination of (3.9) and (3.10) gives

$$\|\nabla^2(n^{-1})\|_{L^2} + \|\nabla(n^{-1})\|_{L^2} \lesssim \|n^{-1}\|_{L^2}^4 + \|n^{-1}\|_{L^2}.$$

Note that (3.8) with $l = 2$ gives $\|n^{-1}\|_{L^2}^2 \le \|k\|_{L^4}^2\|n^{-1}\|_{L^2}$, which implies $\|n^{-1}\|_{L^2} \le \|k\|_{L^4}^2 \le C$. Consequently, $\|n^{-1}\|_{H^2} \le C$. With the help of Lemma 2.6 the estimate $\|n^{-1}\|_{L^\infty} \le C$ follows immediately. □

### 3.3 Estimates on $\dot{n} := \partial_t n$

With the help of (1.2), (1.3), (1.6), and (1.7) and the fact $\mathrm{Tr}\,k = t$, we derive that

$$\begin{aligned}\Delta\dot{n} = &-4nk^{ij}\nabla_i\nabla_j n + |k|^2\dot{n} - 2k_i^a\nabla^i n\nabla_a n + \mathrm{Tr}\,k|\nabla n|^2\\ &+ 2nR_{ij}k^{ij} + 2n|k|^2\,\mathrm{Tr}\,k.\end{aligned} \tag{3.11}$$

Now we multiply equation (3.11) by $\dot{n}$ and integrate over $\Sigma_t$. By using the boundedness of $n$ and the Hölder inequality we obtain

$$\int_{\Sigma_t} (|\nabla \dot{n}|^2 + |k|^2 |\dot{n}|^2) \le \big(\|\nabla^2 n\|_{L^2} + \|\nabla n\|_{L^4}^2 + \|\mathrm{Ric}\|_{L^2}\big)\|k\|_{L^4}\,\|\dot{n}\|_{L^4} + \|k\|_{L^6}^3\,\|\dot{n}\|_{L^2}.$$

In view of Lemma 2.2 and Proposition 3.1 we have

$$\int_{\Sigma_t} (|\nabla \dot{n}|^2 + |k|^2 |\dot{n}|^2) \lesssim \|\dot{n}\|_{L^4} + \|\dot{n}\|_{L^2} \lesssim \|\nabla \dot{n}\|_{L^2} + \|\dot{n}\|_{L^2}.$$

Recall that $|k|^2 = |\widehat{k}|^2 + t^2/3$ and $|t| \ge |t_*| > 0$. Thus

$$\|\nabla \dot{n}\|_{L^2}^2 + \|\dot{n}\|_{L^2}^2 \lesssim \|\nabla \dot{n}\|_{L^2} + \|\dot{n}\|_{L^2}.$$

We therefore obtain the following:

LEMMA 3.6. *On each $\Sigma_t \subset \mathcal{M}_*$, there holds*

$$\|\nabla \dot{n}\|_{L^2(\Sigma_t)} + \|\dot{n}\|_{L^2(\Sigma_t)} \le C. \tag{3.12}$$

Now we are ready to give some mixed-norm estimates on $\dot{n}$.

PROPOSITION 3.7. *Let $1 \le b < 2$. Then there hold*

$$\|\nabla^2 \dot{n}\|_{L_t^1 L_x^2(\mathcal{M}_*)} \le C \quad \text{and} \quad \|\dot{n}\|_{L_t^b L_x^\infty(\mathcal{M}_*)} \le C.$$

PROOF. In view of (A1), it suffices to establish on each $\Sigma_t$ the inequalities

$$\|\nabla^2 \dot{n}\|_{L^2(\Sigma_t)} \lesssim \|k\|_{L^\infty(\Sigma_t)} + 1, \tag{3.13}$$

$$\|\dot{n}\|_{L^\infty(\Sigma_t)} \lesssim \|k\|_{L^\infty(\Sigma_t)}^{3/2-3/p} + 1, \tag{3.14}$$

for any $3 < p \le 6$.

By the Böchner identity and the fact $\|\mathrm{Ric}\|_{L^2} \le C$, we have

$$\|\nabla^2 \dot{n}\|_{L^2}^2 \le \|\Delta \dot{n}\|_{L^2}^2 + \|\nabla \dot{n}\|_{L^4}^2.$$

Applying Lemma 2.5 to $\|\nabla \dot{n}\|_{L^4}$ and using Young's inequality and (3.12), it follows that

$$\|\nabla^2 \dot{n}\|_{L^2} \lesssim \|\Delta \dot{n}\|_{L^2} + \|\dot{n}\|_{L^2} \lesssim \|\Delta \dot{n}\|_{L^2} + 1. \tag{3.15}$$

By virtue of the estimates in Lemma 2.2, Proposition 3.1, and (3.12), it follows from (3.11) that $\|\Delta \dot{n}\|_{L^2} \lesssim \|k\|_{L^\infty} + 1$. Thus $\|\nabla^2 \dot{n}\|_{L^2} \lesssim \|k\|_{L^\infty} + 1$, which is exactly (3.13). Inequality (3.14) follows from Lemma 2.6, (3.13), and (3.12). □

## 4 Null Radius of Injectivity: Proof of Main Theorem II

In this section we sketch the proof of Theorem 1.2. The complete proof is rather involved and requires a delicate bootstrap argument. For any $t_0 < t_1 < t_*$ we consider the slab $\mathcal{M}_I = \bigcup_{t \in I} \Sigma_t$ with $I = [t_0, t_1]$. We set, for each $p \in \mathcal{M}_I$,

$$\widetilde{i}_*(p,t) = \begin{cases} +\infty & \text{if } i_*(p,t) > t(p) - t_0, \\ i_*(p,t) & \text{otherwise,} \end{cases}$$

and define

$$i_* := \min\{\widetilde{i}_*(p,t) : p \in \mathcal{M}_I\}. \tag{4.1}$$

Due to the compactness of $\mathcal{M}_I$, we have $i_* > 0$. In order to complete the proof of Theorem 1.2, it suffices to show that $i_* > \delta_*$ for some universal constant $\delta_* > 0$.

We will use the following result concerning the lower bound on the null radius of injectivity of a globally hyperbolic space-time, which has essentially been proved in [11].

THEOREM 4.1. *Let $C^{-1} \le n \le C$ on $\mathcal{M}_I$ for some constant $C > 0$. Then there exists a small constant $\epsilon > 0$ depending only on $C$ such that if, for some constant $\delta_* > 0$, the three conditions stated below hold for all $p \in \mathcal{M}_I$, then there holds $i_* > \delta_*$; i.e., the null radius of injectivity satisfies*

$$i_*(p,t) > \min\{\delta_*, t(p) - t_0\}$$

*for all $p \in \mathcal{M}_I$. Those conditions are*:

(C1) *the null radius of conjugacy satisfies*

$$s_*(p,t) > \min\{i_*, \delta_*\};$$

(C2) *for each $t$ satisfying*

$$0 \le t(p) - t \le \min\{i_*, \delta_*\},$$

*the metric $\gamma_t$ on $\mathbb{S}^2$, obtained by restricting the metric $g$ on $\Sigma_t$ to $S_t := \mathcal{N}^-(p) \cap \Sigma_t$ and then pulling it back to $\mathbb{S}^2$ by the exponential map $\mathcal{G}(t, \cdot)$, satisfies*

$$|\gamma_t(X,X) - \overset{\circ}{\gamma}(X,X)| < \epsilon \overset{\circ}{\gamma}(X,X) \quad \forall X \in T\mathbb{S}^2,$$

*where $\overset{\circ}{\gamma}$ is the standard metric on $\mathbb{S}^2$;*

(C3) *on $\mathcal{U}_p := I_p \times B_{\delta_*}(p)$ with $I_p := [t(p) - \min\{i_*, \delta_*\}, t(p)]$ and $B_{\delta_*}(p) \subset \Sigma_{t(p)}$ a geodesic ball, there is a system of coordinates $x^\alpha$ with $x^0 = t$ relative to which the metric $\mathbf{g}$ is close to the Minkowski metric $\mathbf{m}_{\alpha\beta} = -n(p)dt^2 + \delta_{ij}\, dx^i\, dx^j$ in the sense that*

$$|n - n(p)| + |g_{ij} - \delta_{ij}| < \epsilon \quad \text{on } \mathcal{U}_p.$$

Theorem 4.1 provides a general framework to estimate the null radius of injectivity from below. Under condition (1.9), in [11] Klainerman and Rodnianski showed that conditions (C1)–(C3) hold with a universal constant $\delta_*>0$; thus they derived a universal lower bound on the null radius of injectivity.

In the following we will describe how to verify conditions (C1)–(C3) under assumption (A1). To this end, for each $p\in\mathcal{M}_I$ consider the past null cone $\mathcal{N}^-(p)$, let $s$ be its affine parameter, and let $S_t=\mathcal{N}^-(p)\cap\Sigma_t$. Then $S_t$ is diffeomorphic to $\mathbb{S}^2$ for each $t$ satisfying $t(p)-i_*(p,t)<t<t(p)$. Let $\gamma$ be the restriction of $\mathbf{g}$ to $S_t$, and let $|S_t|$ be the corresponding area. The radius of $S_t$ is defined to be

$$r:=\sqrt{(4\pi)^{-1}|S_t|}, \tag{4.2}$$

which is a function of $t$ only.

On $\mathcal{N}^-(p,\tau)\setminus\{p\}$ with $\tau<i_*(p,t)$ we can define a conjugate null vector $\underline{L}$ with $\mathbf{g}(L,\underline{L})=-2$ and such that $\underline{L}$ is orthogonal to the leaves $S_t$. In addition, we can choose $(e_A)_{A=1,2}$ tangent to $S_t$ such that $(e_A)_{A=1,2}$, $e_3=\underline{L}$, and $e_4=L$ form a null frame. The null second fundamental forms $\chi$ and $\underline{\chi}$, the torsion $\zeta$, and the Ricci coefficient $\underline{\zeta}$ of the foliation $S_t$ are then defined as follows:

$$\chi_{AB}=\mathbf{g}(\mathbf{D}_A L,e_B),\qquad \underline{\chi}_{AB}=\mathbf{g}(\mathbf{D}_A\underline{L},e_B),$$
$$\zeta_A=\frac{1}{2}\,\mathbf{g}(\mathbf{D}_A L,\underline{L}),\qquad \underline{\zeta}_A=\frac{1}{2}\,\mathbf{g}(e_A,\mathbf{D}_L\underline{L}).$$

In addition, we define $\operatorname{tr}\chi=\gamma^{AB}\chi_{AB}$ and $\widehat{\chi}_{AB}=\chi_{AB}-\frac{1}{2}\operatorname{tr}\chi\gamma_{AB}$. We can define $\operatorname{tr}\underline{\chi}$ and $\widehat{\underline{\chi}}$ similarly.

We introduce the null lapse function

$$a^{-1}:=\mathbf{g}(L,\mathbf{T}).$$

Then $a>0$ and $a(p)=1$. It is easy to see that

$$L=-a^{-1}(\mathbf{T}+N),\quad \underline{L}=-a(\mathbf{T}-N),$$

where $N$ denotes the unit inward normal to $S_t$ in $\Sigma_t$. We also introduce the function

$$\nu:=-n^{-1}\nabla_N n+k_{NN},$$

which is relevant to the estimate on $a$.

For any $S_t$-tangent tensor field $F$ we define the norm $\|F\|_{L_\omega^\infty L_t^2(\mathcal{N}^-(p,\tau))}$ by

$$\|F\|^2_{L_\omega^\infty L_t^2(\mathcal{N}^-(p,\tau))}:=\sup_{\omega\in\mathbb{S}^2}\int_{t(p)-\tau}^{t(p)}|F|^2na\,dt:=\sup_{\omega\in\mathbb{S}^2}\int_{\Gamma_\omega}|F|^2na\,dt,$$

where $\Gamma_\omega$ denotes the portion of a past null geodesic from $p$ contained in $\mathcal{N}^-(p,\tau)$.

The following result is sufficient to prove conditions (C1)–(C3) in Theorem 4.1.

THEOREM 4.2. *Let assumption* (A1) *hold. Then there exist universal constants* $\delta_* > 0$ *and* $C_* > 0$ *such that for any* $p \in \mathcal{M}_I$ *there holds*

$$\int_{t(p)-\tau}^{t(p)} |k(\Phi(t))|^2 \, dt \le C_* \tag{4.3}$$

*with* $\Phi$ *the integral curve of* $\mathbf{T}$ *through* $p$, *and*

$$|a-1| \le \frac{1}{2}, \quad \left|\operatorname{tr}\chi - \frac{2}{s}\right| \le C_*, \quad \|\hat{\chi}\|^2_{L^\infty_\omega L^2_t(\mathcal{N}^-(p,\tau))} \le C_*, \tag{4.4}$$

*on any null cones* $\mathcal{N}^-(p,\tau)$, *where* $\tau := \min\{i_*, \delta_*\}$.

In fact, the estimate on $\operatorname{tr}\chi$ in (4.4) implies condition (C1); see [4, 6]. Next we will show that the estimates in (4.4) imply condition (C2). To see this, recall that $\frac{ds}{dt} = -na$ and $\frac{d}{ds}\gamma_{AB} = 2\chi_{AB}$. Then

$$\frac{d}{dt}(s^{-2}\gamma_{AB}) = -na(-2s^{-3}\gamma_{AB} + 2s^{-2}\chi_{AB}).$$

Let $X \in T\mathbb{S}^2$ be any vector field. We integrate the above equation along any null geodesic and note $\lim_{t\to t(p)^-} s(t)^{-2}\gamma(t) = \mathring{\gamma}$ (see [14]); it follows that

$$|s(t)^{-2}\gamma(X,X) - \mathring{\gamma}(X,X)| \le \int_t^{t(p)} \left(2|\hat{\chi}| + \left|\operatorname{tr}\chi - \frac{2}{s(t')}\right|\right) s(t')^{-2}\gamma(X,X) na \, dt'.$$

Let $\Theta := 2|\hat{\chi}| + |\operatorname{tr}\chi - 2/s|$. We then have

$$|s(t)^{-2}\gamma(X,X) - \mathring{\gamma}(X,X)| \le \int_t^{t(p)} \Theta |s(t')^{-2}\gamma(X,X) - \mathring{\gamma}(X,X)| na \, dt' + \mathring{\gamma}(X,X) \int_t^{t(p)} \Theta(t') na \, dt'.$$

Therefore, it follows from the Gronwall inequality that

$$|s(t)^{-2}\gamma(X,X) - \mathring{\gamma}(X,X)| \le \mathring{\gamma}(X,X) \int_t^{t(p)} \Theta na \, dt' \exp\left(\int_t^{t(p)} \Theta(t') na \, dt'\right).$$

Since $0 < n \le 3/t_*^2$, estimate (4.4) in Theorem 4.2 implies

$$\int_t^{t(p)} \Theta na \, dt' \le C\big((t(p)-t)^{1/2} + (t(p)-t)\big) \le C(t(p)-t)^{1/2}$$

and consequently

$$|s^{-2}\gamma(X,X) - \mathring{\gamma}(X,X)| \le C(t(p)-t)^{1/2}\mathring{\gamma}(X,X) \tag{4.5}$$

for all $t(p) - \min\{i_*, \delta_*\} \le t < t(p)$, where $C$ is a universal constant. Condition (C2) is thus verified.

The verification of condition (C3), using estimate (4.3), is given in the following result:

LEMMA 4.3. *Let assumption* (A1) *hold. For any* $\epsilon > 0$, *there exists a constant* $\delta_* > 0$ *depending only on* $\mathcal{Q}_0$, $\mathcal{K}_0$, $t_*$, *and* $\epsilon$ *such that for every point* $p \in \mathcal{M}_I$ *there exists on* $\mathcal{U}_p := I_p \times B_{\delta_*}(p)$ *with* $I_p = [t(p) - \min\{i_*, \delta_*\}, t(p)]$ *a system of transported coordinates* $t, x = (x^1, x^2, x^3)$ *relative to which* $\mathbf{g}$ *is close to the Minkowski metric* $\mathbf{m}(p) = -n(p)^2\, dt^2 + \delta_{ij}\, dx^i\, dx^j$ *in the sense that*

$$|g_{ij} - \delta_{ij}| < \epsilon \quad \text{and} \quad |n - n(p)| < \epsilon. \tag{4.6}$$

PROOF. It follows from Proposition 2.4 that there exists a constant $\delta_0 > 0$ depending only $\mathcal{K}_0$, $\mathcal{Q}_0$, $t_*$, and $\epsilon$ such that every geodesic ball $B_{\delta_0}(p) \subset \Sigma_{t(p)}$ admits a system of harmonic coordinates $x = (x^1, x^2, x^3)$ under which

$$(1 + \epsilon/2)^{-1}\delta_{ij} \le g_{ij} \le (1 + \epsilon/2)\delta_{ij}. \tag{4.7}$$

Under the transported coordinates $t, x = (x^1, x^2, x^3)$, let $p = (t(p), 0)$ and let $q = (t, x)$ be an arbitrary point in $I_p \times B_{\delta_*}(p)$ with $I_p = [t(p) - \min\{i_*, \delta_*\}, t(p)]$, where $0 < \delta_* \le \delta_0$ is a constant to be determined. By using the equation $\partial_t g_{ij} = -2nk_{ij}$ we have

$$|g_{ij}(t, x) - g_{ij}(t(p), x)| = \left| \int_t^{t(p)} \partial_t g_{ij}(t', x) dt' \right| \le 2 \int_t^{t(p)} n|k| dt'.$$

Using the bound $0 < n \le 3/t_*$, the Hölder inequality, and estimate (4.3) in Theorem 4.2, it follows for some universal constant $C_1 > 0$ that

$$|g_{ij}(t, x) - g_{ij}(t(p), x)| \le C_1 (t(p) - t)^{1/2} \le C_1 \delta_*^{1/2}.$$

In view of (4.7), we thus obtain

$$\begin{aligned} |g_{ij}(t, x) - \delta_{ij}| &\le |g_{ij}(t, x) - g_{ij}(t(p), x)| + |g_{ij}(t(p), x) - \delta_{ij}| \\ &\le C_1 \delta_*^{1/2} + \frac{\epsilon}{2}, \end{aligned} \tag{4.8}$$

which gives the first inequality in (4.6) by letting $C_1 \delta_*^{1/2} < \epsilon/2$.

Next we prove the second inequality in (4.6). From Proposition 3.7 we have

$$\begin{aligned} |n(t, x) - n(t(p), x)| &\le \int_t^{t(p)} |\dot{n}(t', x)| dt' \\ &\le (t(p) - t)^{1/4} \|\dot{n}\|_{L_t^{4/3} L_x^\infty} \le C_2 \delta_*^{1/4}, \end{aligned}$$

while by employing Morrey's estimate, Lemma 2.5, and Proposition 3.1, we have

$$\begin{aligned}|n(t(p),x)-n(t(p),0)| &\le C_2\delta_*^{1/4}\|\nabla n\|_{L^4(\Sigma_{t(p)})}\\ &\le C_2\delta_*^{1/4}\big(\|\nabla^2 n\|_{L^2}^{3/4}\|\nabla n\|_{L^2}^{1/4}+\|\nabla n\|_{L_x^2}\big)\\ &\le C_2\delta_*^{1/4},\end{aligned}$$

where $C_2 > 0$ is a universal constant. Therefore $|n(t,x)-n(p)| \le 2C_2\delta_*^{1/4}$, which implies the second inequality in (4.3) by further letting $2C_2\delta_*^{1/4} < \epsilon$. □

The proof of Theorem 4.2 is based on a delicate bootstrap argument. We first fix some notation and terminology. Related to the deformation tensor $\pi_{\alpha\beta}$ of $\mathbf{T}$, we introduce the $\Sigma_t$-tangent tensor $h_\alpha^\mu h_\beta^\nu \pi_{\mu\nu}$, where $h_\alpha^\beta = \delta_\alpha^\beta + \mathbf{T}_\alpha \mathbf{T}^\beta$ denotes the projection tensor. It is easy to see that $k_{ij} = h_i^\mu h_j^\nu \pi_{\mu\nu}$, and thus this tensor is an extension of $k$. We will denote it by the same notation $k$, i.e.,

$$k_{\alpha\beta} = h_\alpha^\mu h_\beta^\nu \pi_{\mu\nu}. \tag{4.9}$$

Note that $k_{0\alpha} = k_{\alpha 0} = 0$.

Corresponding to the null vector $L$, let $\nabla_L k$ be the $\Sigma_t$-tangent tensor defined by $\nabla_L k_{ij} := h_i^\alpha h_j^\beta \mathbf{D}_L k_{\alpha\beta}$ and let

$$|\nabla_L k|^2 = g^{ii'} g^{jj'} \nabla_L k_{ij} \nabla_L k_{i'j'}.$$

We also introduce $\nabla\!\!\!/ k$ by $\nabla\!\!\!/_A k_{ij} := \nabla_A k_{ij}$ and set

$$|\nabla\!\!\!/ k|^2 = \gamma^{AB} g^{ii'} g^{jj'} \nabla_A k_{ij} \nabla_B k_{i'j'}.$$

Corresponding to the second fundamental form $k$, then, for each $p \in \mathcal{M}_I$, we introduce on the null cone $\mathcal{N}^-(p,\tau)$ the $k$-flux

$$\mathcal{F}[k](p,\tau) = \int_{\mathcal{N}^-(p,\tau)} (|\nabla\!\!\!/ k|^2 + |\nabla_L k|^2), \tag{4.10}$$

where, for each function $f$ and $\tau < i_*(p,t)$,

$$\int_{\mathcal{N}^-(p,\tau)} f := \int_{t(p)-\tau}^{t(p)} \int_{S_t} f n a \, d\mu_\gamma \, dt.$$

Corresponding to the time foliation, we recall the null components of the Riemannian curvature tensor $\mathbf{R}$ as follows:

$$\begin{aligned}\alpha_{AB} &= \mathbf{R}(L,e_A,L,e_B), & \underline{\alpha}_{AB} &= \mathbf{R}(\underline{L},e_A,\underline{L},e_B),\\ \beta_A &= \frac{1}{2}\mathbf{R}(e_A,L,\underline{L},L), & \underline{\beta}_A &= \frac{1}{2}\mathbf{R}(e_A,\underline{L},\underline{L},L),\\ \rho &= \frac{1}{4}\mathbf{R}(\underline{L},L,\underline{L},L), & \sigma &= \frac{1}{4}{}^\star\mathbf{R}(\underline{L},L,\underline{L},L),\end{aligned} \tag{4.11}$$

The corresponding curvature flux $\mathcal{R}(p,\tau)$ on the null cone $\mathcal{N}^-(p,\tau)$ is given by

$$\mathcal{R}(p,\tau) = \int_{t(p)-\tau}^{t(p)} \int_{S_t} (|\alpha|^2 + |\beta|^2 + |\rho|^2 + |\sigma|^2 + |\underline{\beta}|^2) n a \, d\mu_\gamma \, dt.$$

The following result says that once the null lapse $a$ is well controlled, then the $k$-flux and the curvature flux can be bounded by a universal constant.

THEOREM 4.4. *Let condition* (A1) *hold. Then there exists a universal constant* $C_* \ge 1$ *such that for all* $p \in \mathcal{M}_I$ *if* $|a-1| \le \frac{1}{2}$ *on* $\mathcal{N}^-(p,\tau)$ *for some* $0 < \tau \le i_*$, *then there holds*

$$\mathcal{R}(p,\tau) + \mathcal{F}[k](p,\tau) \le C_*.$$

We will prove Theorem 4.4 in Section 6. This result requires $\frac{1}{2} \le a \le \frac{3}{2}$ on $\mathcal{N}^-(p,\tau)$, which is obvious for small $\tau > 0$ since $a(p) = 1$. In order for the above result to be applicable, we must show that there is a universal constant $\delta_* > 0$ such that the same bound on $a$ holds with $\tau := \min\{i_*, \delta_*\}$, and so does the same bound on $\mathcal{R}(p,\tau) + \mathcal{F}[k](p,\tau)$. We will use a bootstrap argument to achieve this together with various estimates on $\operatorname{tr}\chi$, $\widehat{\chi}$, and $\nu$. That is, we will make the following bootstrap assumptions:

$$|a-1| \le \frac{1}{2}, \tag{BA1}$$

$$\left|\operatorname{tr}\chi - \frac{2}{s}\right| \le \mathcal{E}_0, \tag{BA2}$$

$$\|\widehat{\chi}\|^2_{L^\infty_\omega L^2_t(\mathcal{N}^-(p,\tau))} \le \mathcal{E}_0, \tag{BA3}$$

$$\|\nu\|^2_{L^\infty_\omega L^2_t(\mathcal{N}^-(p,\tau))} \le \mathcal{E}_0, \tag{BA4}$$

on the null cone $\mathcal{N}^-(p,\tau)$ for all $p \in \mathcal{M}_I$, where $0 < \tau \le i_*$ and $\mathcal{E}_0 \ge 1$ are two numbers satisfying $\mathcal{E}_0 \tau \le 1$. Due to the continuity of the quantities involved and the compactness of $\mathcal{M}_I$, the bootstrap assumptions (BA1)–(BA4) hold automatically for sufficiently small $\tau > 0$. Our goal is to show that we can choose universal constants $\mathcal{E}_0 \ge 1$ and $\delta_* > 0$ such that (BA1)–(BA4) hold with $\tau = \min\{i_*, \delta_*\}$. We will achieve this by showing that the estimates in (BA1)–(BA4) can be improved.

We will first derive various intermediate consequences of the bootstrap assumptions. In particular, we will derive the estimate on the important quantity $\mathcal{N}_1[\not{\pi}]$, which is defined as follows. For any $S_t$ tangent tensor field $F$ defined on the null cone $\mathcal{N}^-(p,\tau)$, the Sobolev norm $\mathcal{N}_1[F](p,\tau)$ is defined by

$$\begin{aligned}\mathcal{N}_1[F](p,\tau) := \|r^{-1}F\|_{L^2(\mathcal{N}^-(p,\tau))} + \|\nabla_L F\|_{L^2(\mathcal{N}^-(p,\tau))} \\ + \|\not{\nabla} F\|_{L^2(\mathcal{N}^-(p,\tau))}.\end{aligned} \tag{4.12}$$

Now we introduce $\not{\pi}$, related to the deformation tensor $\pi$ of $\mathbf{T}$ whose components, under transported coordinates, are given in (1.12). We set $\lambda = -\operatorname{Tr} k/3 = -t/3$

and let $\widehat{k}$ be the traceless part of $k$. We decompose $\widehat{k}$ on each $S_t$ by introducing components

$$\eta_{AB} = \widehat{k}_{AB}, \quad \epsilon_A = \widehat{k}_{AN}, \quad \delta = \widehat{k}_{NN}, \tag{4.13}$$

where $(e_A)_{A=1,2}$ is an orthonormal frame on $S_t$ and $N$ is the inward unit normal of $S_t$ in $\Sigma_t$. Let $\widehat{\eta}_{AB}$ be the traceless part of $\eta$. Since $\delta^{AB}\eta_{AB} = -\delta$, we have $\widehat{\eta}_{AB} = \eta_{AB} + \frac{1}{2}\delta_{AB}\delta$. We denote by $\widehat{\not{k}}$, $\not\nabla\widehat{\not{k}}$ and $\not{\pi}_0$ the collections

$$\widehat{\not{k}} = (\delta, \epsilon, \widehat{\eta}), \quad \not\nabla\widehat{\not{k}} = (\not\nabla\delta, \not\nabla\epsilon, \not\nabla\widehat{\eta}), \quad \not{\pi}_0 = (\not\nabla \log n, \nabla_N \log n),$$

respectively. We define $\not{\pi}$ to be the collection

$$\not{\pi} = (\widehat{\not{k}}, \not{\pi}_0, \lambda). \tag{4.14}$$

We then define $\mathcal{N}_1[\not{\pi}](p,\tau)$ according to (4.12) with $F$ replaced by $\not{\pi}$.

With the help of the bound on $k$-flux given in Theorem 4.4 and various estimates on the lapse $n$ given in Section 3, we will show that $\mathcal{N}_1[\not{\pi}](p,\tau)$ can be bounded in a suitable way under (A1) and the bootstrap assumptions.

THEOREM 4.5. *Let* (A1) *hold. Then there exists a universal constant $C$ such that under the bootstrap assumptions* (BA1)–(BA3) *with $\mathcal{E}_0\tau \le 1$ there holds*

$$\mathcal{N}_1[\not{\pi}](p,\tau) \le C \tag{4.15}$$

*for all $p \in \mathcal{M}_I$.*

We will prove Theorem 4.5 in Section 8. From Theorem 4.4 and Theorem 4.5 it follows that

$$\mathcal{R}(p,\tau) + \mathcal{N}_1[\not{\pi}](p,\tau) \le C_0, \tag{4.16}$$

where $C_0 \ge 1$ is a universal constant.

With the help of (4.16), we can establish the following result, which enables us to improve the estimates in the bootstrap assumptions.

THEOREM 4.6. *There exist two universal constants $\delta_0 > 0$ and $C_1 \ge 1$ such that, under the bootstrap assumptions* (BA1)–(BA4) *with $\mathcal{E}_0\tau \le 1$, if $\tau < \min\{i_*, \delta_0\}$ then there hold*

$$|a-1| \le C_1\tau^{1/2}, \tag{4.17}$$

$$\left|\operatorname{tr}\chi - \frac{2}{s}\right| \le C_1, \tag{4.18}$$

$$\|\widehat{\chi}\|^2_{L^\infty_\omega L^2_t(\mathcal{N}^-(p,\tau))} \le C_1, \tag{4.19}$$

$$\|\nu\|^2_{L^\infty_\omega L^2_t(\mathcal{N}^-(p,\tau))} \le C_1 \tag{4.20}$$

*on the null cones $\mathcal{N}^-(p,\tau)$ for all $p \in \mathcal{M}_I$.*

The significance of Theorem 4.6 lies in that it allows us to choose $\mathcal{E}_0 \geq 1$ and $\delta_* > 0$ universal such that (BA1)–(BA4) hold on $\mathcal{N}^-(p,\tau)$ with $\tau = \min\{i_*, \delta_*\}$. To see this, we choose $\mathcal{E}_0$ and $\delta_*$ in the way that

$$\mathcal{E}_0 := 2C_1 \quad \text{and} \quad \delta_* = \min\{(4C_1)^{-2}, \delta_0\}. \tag{4.21}$$

With such $\mathcal{E}_0$ and $\delta_*$, estimates (4.17)–(4.20) imply that estimates (BA1)–(BA4) can be improved as

$$|a-1| \leq \frac{1}{4}, \qquad \left|\operatorname{tr}\chi - \frac{2}{s}\right| \leq \frac{1}{2}\mathcal{E}_0,$$

$$\|\hat{\chi}\|^2_{L^\infty_\omega L^2_t(\mathcal{N}^-(p,\tau))} \leq \frac{1}{2}\mathcal{E}_0, \quad \|\nu\|^2_{L^\infty_\omega L^2_t(\mathcal{N}^-(p,\tau))} \leq \frac{1}{2}\mathcal{E}_0$$

on $\mathcal{N}^-(p,\tau)$ if $\tau \leq \min\{i_*, \delta_*\}$. By repeated use of Theorem 4.4, Theorem 4.5, and Theorem 4.6, the bootstrap principle implies that the estimates in the bootstrap assumptions (BA1)–(BA4) hold with $\tau = \min\{i_*, \delta_*\}$, where $\mathcal{E}_0$ and $\delta_*$ are determined by (4.21), which are positive universal constants. Consequently, we obtain (4.4) in Theorem 4.2.

We remark that results analogous to Theorem 4.6 have been proved in [7, 14] for the geodesic foliations where only the bound of the curvature flux is used. In time foliations, however, the proof of Theorem 4.6 relies not only on the curvature flux but also on $\mathcal{N}_1[\not{\pi}]$.

Assuming (4.20), the following simple argument shows how to derive (4.17) from (BA1). Recall that $a^{-1} = \mathbf{g}(L, \mathbf{T})$ and $L = -a^{-1}(N + \mathbf{T})$. We have

$$\frac{d}{ds}a^{-1} = \mathbf{g}(L, \mathbf{D}_L\mathbf{T}) = a^{-2}\mathbf{g}(N, \mathbf{D}_\mathbf{T}\mathbf{T}) + a^{-2}\mathbf{g}(N, \mathbf{D}_N\mathbf{T}).$$

Recall also that $\mathbf{D}_\mathbf{T}\mathbf{T} = n^{-1}\nabla n$ and $k_{NN} = -\langle N, \mathbf{D}_N\mathbf{T}\rangle$; we obtain $\frac{d}{ds}a^{-1} = -a^{-2}(\pi_{0N} + k_{NN})$. Consequently,

$$L(a) = \frac{d}{ds}a = \pi_{0N} + k_{NN}. \tag{4.22}$$

Since $\frac{ds}{dt} = -na$, we have $\frac{d}{dt}a = -na(\pi_{0N} + k_{NN})$. Integrating the above equation along null geodesics initiating from $p$ and using $a(p) = 1$ yields

$$a - 1 = \int_t^{t(p)} (\pi_{0N} + k_{NN})na\, dt' = \int_t^{t(p)} \nu na\, dt'.$$

From (BA1) and (4.20) it then follows that $|a-1| \leq C(t(p)-t)^{1/2} \leq C\tau^{1/2}$ for all $t(p) - \tau \leq t \leq t(p)$.

The derivation of (4.18)–(4.20), however, is highly nontrivial. The complete proof is contained in [15, 16], where other related estimates for Ricci coefficients are proved simultaneously.

In order to complete the proof of Theorem 4.2, it remains to prove (4.3), which is restated in the following result:

THEOREM 4.7. *Assume that condition* (A1) *holds. Then there exist universal constants* $\delta_* > 0$ *and* $C > 0$ *such that*

$$\int_{t(p)-\min\{i_*,\delta_*\}}^{t(p)} |k(\Phi(t))|^2 n\, dt \le C$$

*for all* $p \in \mathcal{M}_I$*, where* $\Phi$ *denotes the integral curve of* $\mathbf{T}$ *through* $p$*.*

The proof of Theorem 4.7 forms the core part of the present paper. It is based on the formula of $\Box k$ given in Section 5 and a Kirchhoff-Sobolev representation for $k$ given in Section 9 together with various estimates on null cones derived in Section 8.

## 5 Tensorial Wave Equation for the Second Fundamental Form

In this section we will derive the formula for $\Box k$, where $k$ is defined in (4.9), whose projection to $\Sigma_t$ is exactly the second fundamental form.

PROPOSITION 5.1. *The tensor* $k$ *defined by* (4.9) *satisfies the tensorial wave equation*

$$\begin{aligned}\Box k_{ij} &= -n^{-3}\dot{n}\nabla_i\nabla_j n + n^{-2}\nabla_i\nabla_j\dot{n} + 2\pi_{0a}(\nabla^a k_{ij} - \nabla_i k_j^a - \nabla_j k_i^a)\\ &\quad - 2\operatorname{Tr} k R_{ij} - R k_{ij} + R \operatorname{Tr} k g_{ij}\\ &\quad + 2(k_i^a R_{aj} + k_j^a R_{ai}) - 2R_{ab}k^{ab} g_{ij}\\ &\quad + n^{-1}(2k_i^a\nabla_a\nabla_j n + 2k_j^a\nabla_a\nabla_i n - \Delta n k_{ij} - \operatorname{Tr} k \nabla_i\nabla_j n)\\ &\quad + 2k_{ia}k^{ab}k_{bj} - \pi_{0a}\pi_0^a k_{ij} - n^{-1}k_{ij}.\end{aligned} \tag{5.1}$$

PROOF. We first recall that

$$\Box k_{ij} = -\mathbf{D}_0\mathbf{D}_0 k_{ij} + g^{pq}\mathbf{D}_p\mathbf{D}_q k_{ij}.$$

By using $k_{0\alpha} = k_{\alpha 0} = 0$ and $\mathbf{D}_i e_j = \nabla_i e_j - k_{ij}\mathbf{T}$, we can obtain through a straightforward calculation that

$$g^{pq}\mathbf{D}_p\mathbf{D}_q k_{ij} = \triangle k_{ij} + \operatorname{Tr} k \mathbf{D}_0 k_{ij} + 2k_{ia}k^{ab}k_{bj}.$$

By using $\mathbf{D}_\mathbf{T}\mathbf{T} = n^{-1}\nabla^i n e_i = -\pi_0^i e_i$ and $k_{0\alpha} = k_{\alpha 0} = 0$, we can obtain

$$\begin{aligned}\mathbf{D}_0\mathbf{D}_0 k_{ij} &= e_0(\mathbf{D}_0 k_{ij}) + k_i^a\mathbf{D}_0 k_{aj} + k_j^a\mathbf{D}_0 k_{ia} + \pi_{0a}\nabla^a k_{ij}\\ &\quad + \pi_{0i}\mathbf{D}_0 k_{0j} + \pi_{0j}\mathbf{D}_0 k_{i0}.\end{aligned}$$

It is easy to see $\mathbf{D}_0 k_{0j} = \pi_{0a}k_j^a$. From equation (1.3) it also follows that

$$\mathbf{D}_0 k_{ij} = e_0(k_{ij}) + 2k_{ia}k_j^a = -n^{-1}\nabla_i\nabla_j n + R_{ij} + \operatorname{Tr} k\, k_{ij}. \tag{5.2}$$

Consequently,

$$\begin{aligned}\mathbf{D}_0\mathbf{D}_0 k_{ij} &= e_0(\mathbf{D}_0 k_{ij}) + \pi_{0a}\nabla^a k_{ij} - n^{-1}(k_i^a\nabla_a\nabla_j n + k_j^a\nabla_a\nabla_i n)\\ &\quad + (k_i^a R_{aj} + k_j^a R_{ai}) + 2\operatorname{Tr} k\, k_{ia}k_j^a + \pi_{0i}\pi_{0a}k_j^a + \pi_{0j}\pi_{0a}k_i^a.\end{aligned}$$

Therefore

$$\begin{aligned} \Box k_{ij} &= -e_0(\mathbf{D}_0 k_{ij}) - \pi_{0a}\nabla^a k_{ij} - \pi_{0i}\pi_{0a}k_j^a - \pi_{0j}\pi_{0a}k_i^a \\ &\quad + n^{-1}(k_i^a \nabla_a \nabla_j n + k_j^a \nabla_a \nabla_i n) - (k_i^a R_{aj} + k_j^a R_{ai}) \\ &\quad - 2\,\mathrm{Tr}\, k k_{ia} k_j^a + \triangle k_{ij} + \mathrm{Tr}\, k \mathbf{D}_0 k_{ij} + 2k_{ia}k^{ab}k_{bj}. \end{aligned} \tag{5.3}$$

We need to compute $e_0(\mathbf{D}_0 k_{ij})$. It follows from (5.2) and $\mathrm{Tr}\, k = t$ that

$$\begin{aligned} e_0(\mathbf{D}_0 k_{ij}) &= n^{-3}\dot{n}\nabla_i\nabla_j n - n^{-2}\partial_t(\nabla_i\nabla_j n) + n^{-1}\partial_t R_{ij} \\ &\quad + n^{-1}k_{ij} + \mathrm{Tr}\, k\mathbf{D}_0 k_{ij} - 2\,\mathrm{Tr}\, k k_{ia} k_j^a. \end{aligned} \tag{5.4}$$

In order to compute $\partial_t(\nabla_i\nabla_j n)$ and $\partial_t R_{ij}$, let $\Gamma_{ij}^a$ denote the Christoffel symbol of $\Sigma_t$. Then it follows from the equation $\partial_t g_{ij} = -2nk_{ij}$ that

$$\dot{\Gamma}_{ij}^a = -n(\nabla_i k_j^a + \nabla_j k_i^a - \nabla^a k_{ij}) - \nabla_i n k_j^a - \nabla_j n k_i^a + \nabla^a n k_{ij}.$$

Using $\mathrm{div}\, k = 0$ and $\mathrm{Tr}\, k = t$, this in particular implies $\dot{\Gamma}_{aj}^a = -\mathrm{Tr}\, k\nabla_j n$. Therefore, noting that $\partial_t(\nabla_i\nabla_j n) = \nabla_i\nabla_j\dot{n} - \dot{\Gamma}_{ij}^a\nabla_a n$, we can obtain

$$\begin{aligned} \partial_t(\nabla_i\nabla_j n) &= \nabla_i\nabla_j\dot{n} + n\nabla_a n(\nabla_i k_j^a + \nabla_j k_i^a - \nabla^a k_{ij}) \\ &\quad + (\nabla_i n k_j^a + \nabla_j n k_i^a)\nabla_a n - |\nabla n|^2 k_{ij}. \end{aligned} \tag{5.5}$$

Noting also that $\partial_t R_{ij} = \nabla_a\dot{\Gamma}_{ij}^a - \nabla_i\dot{\Gamma}_{aj}^a$ and $\mathrm{div}\, k = 0$, we have

$$\begin{aligned} \partial_t R_{ij} &= \nabla_a n(2\nabla^a k_{ij} - \nabla_i k_j^a - \nabla_j k_i^a) - n(\nabla_a\nabla_i k_j^a + \nabla_a\nabla_j k_i^a - \triangle k_{ij}) \\ &\quad + \Delta n k_{ij} - (\nabla_a\nabla_i n\cdot k_j^a + \nabla_a\nabla_j n\cdot k_i^a) + \mathrm{Tr}\, k\, \nabla_i\nabla_j n. \end{aligned}$$

With the help of the commutation formula

$$\nabla_a\nabla_i k_j^a = [\nabla_a, \nabla_i]k_j^a = R_j{}^a{}_{bi}k_a^b + R_{ai}k_j^a$$

and the curvature decomposition formula

$$R_j{}^a{}_{bi} = g_{jb}R_i^a + R_{jb}\delta_i^a - R_{ij}\delta_b^a - R_b^a g_{ji} - \frac{1}{2}(g_{jb}\delta_i^a - g_{ij}\delta_b^a)R,$$

we obtain

$$\begin{aligned} \nabla_a\nabla_i k_j^a &= 2R_{ia}k_j^a + R_{ja}k_i^a - \mathrm{Tr}\, kR_{ij} - R_{ab}k^{ab}g_{ij} \\ &\quad - \frac{1}{2}Rk_{ij} + \frac{1}{2}R\,\mathrm{Tr}\, k g_{ij}. \end{aligned}$$

Consequently,

$$\begin{aligned} \partial_t R_{ij} &= \nabla_a n(2\nabla^a k_{ij} - \nabla_i k_j^a - \nabla_j k_i^a) - (\nabla_a\nabla_i n k_j^a + \nabla_a\nabla_j n k_i^a) \\ &\quad + n\triangle k_{ij} + \triangle n k_{ij} - 3n(R_{ia}k_j^a + R_{ja}k_i^a) + 2n\,\mathrm{Tr}\, kR_{ij} \\ &\quad + 2nR_{ab}k^{ab}g_{ij} + nRk_{ij} - nR\,\mathrm{Tr}\, k g_{ij} + \mathrm{Tr}\, k\nabla_i\nabla_j n. \end{aligned} \tag{5.6}$$

Plugging (5.5) and (5.6) into (5.4) and using $\pi_{0i} = -n^{-1}\nabla_i n$ yields

$$\begin{aligned} e_0(\mathbf{D}_0 k_{ij}) = {} & n^{-3}\dot{n}\nabla_i\nabla_j n - n^{-2}\nabla_i\nabla_j\dot{n} - \pi_{0a}(3\nabla^a k_{ij} - 2\nabla_i k_j^a - 2\nabla_j k_i^a) \\ & - \pi_{0i}\pi_{0a}k_j^a - \pi_{0j}\pi_{0a}k_i^a + \pi_{0a}\pi_0^a k_{ij} \\ & - n^{-1}(\nabla_a\nabla_i n k_j^a + \nabla_a\nabla_j n k_i^a - \operatorname{Tr} k \nabla_i\nabla_j n) + \triangle k_{ij} \\ & + n^{-1}\triangle n k_{ij} - 3(R_{ia}k_j^a + R_{ja}k_i^a) + 2\operatorname{Tr} k R_{ij} + 2R_{ab}k^{ab}g_{ij} \\ & + Rk_{ij} - R\operatorname{Tr} k g_{ij} + n^{-1}k_{ij} + \operatorname{Tr} k \mathbf{D}_0 k_{ij} - 2\operatorname{Tr} k k_i^a k_{aj}. \end{aligned}$$

Plugging the above equation into (5.3) gives the desired equation. □

## 6 Proof of Theorem 4.4

In this section we will complete the proof of Theorem 4.4; i.e., we will show that if $|a-1| \le \frac{1}{2}$ on $\mathcal{N}^-(p,\tau)$ for some $0 < \tau \le i_*$, then

$$\mathcal{R}(p,\tau) + \mathcal{F}[k](p,\tau) \le C_*,$$

where $C_*$ is a universal constant.

We will use the following result (see [5, lemma 8.1.1]):

LEMMA 6.1. *Let $P$ be a vector field defined on the domain $\mathcal{J}^-(p,\tau)$. Then*

$$\int_{\mathcal{N}^-(p,\tau)} \mathbf{g}(P,L) = \int_{\mathcal{J}^-(p,\tau)} \mathbf{D}^\mu P_\mu - \int_{\Sigma_{t(p)-\tau}\cap\mathcal{J}^-(p)} \mathbf{g}(P,\mathbf{T})d\mu_g,$$

*where $\mathcal{J}^-(p)$ denotes the causal past of $p$, $\mathcal{J}^-(p,\tau)$ denotes the portion of $\mathcal{J}^-(p)$ in the slab $[t(p)-\tau, t(p)]$, and*

$$\int_{\mathcal{J}^-(p,\tau)} f = \int_{t(p)-\tau}^{t(p)} dt\left(\int_{\Sigma_t\cap\mathcal{J}^-(p)} f n\, d\mu_g\right).$$

We first show the boundedness of the curvature flux $\mathcal{R}(p,\tau)$. We introduce $P_\mu = Q[\mathbf{R}]_{\mu\beta\gamma\delta}\mathbf{T}^\beta\mathbf{T}^\gamma\mathbf{T}^\delta$ with the Bel-Robinson tensor $\mathbf{Q}[\mathbf{R}]$ defined in Section 2. We may apply Lemma 6.1 to obtain

$$\int_{\mathcal{N}^-(p,\tau)} \mathbf{g}(P,L) = \int_{\mathcal{J}^-(p,\tau)} \mathbf{D}^\mu P_\mu - \int_{\Sigma_{t(p)-\tau}\cap\mathcal{J}^-(p)} \mathbf{Q}[\mathbf{R}](\mathbf{T},\mathbf{T},\mathbf{T},\mathbf{T})d\mu_g.$$

Since $\mathbf{R}_{\alpha\beta} = 0$, a direct calculation shows $\mathbf{D}^\mu P_\mu = -3\pi^{\alpha\beta}\mathbf{Q}[\mathbf{R}]_{\alpha\beta\gamma\delta}\mathbf{T}^\gamma\mathbf{T}^\delta$. With the help of (A1) and Lemma 2.1, the above identity implies

$$\left|\int_{\mathcal{N}^-(p,\tau)} \mathbf{g}(P,L)\right| \lesssim C. \tag{6.1}$$

Note that $\mathbf{g}(P, L) = \mathbf{Q}[\mathbf{R}](\mathbf{T}, \mathbf{T}, \mathbf{T}, L)$ and $\mathbf{T} = -\frac{1}{2}(aL + a^{-1}\underline{L})$. Since $|a-1| \le \frac{1}{2}$ on $\mathcal{N}^-(p,\tau)$, it follows from [5, lemma 7.3.1] that $-\mathbf{g}(P, L)$ is equivalent to

$$|\alpha|^2 + |\beta|^2 + |\underline{\beta}|^2 + |\rho|^2 + |\sigma|^2.$$

Thus, there holds, for some universal constant $C > 0$,

$$C^{-1}\mathcal{R}(p,\tau) \le \left| \int_{\mathcal{N}^-(p,\tau)} \mathbf{g}(P, L) \right| \le C\mathcal{R}(p,\tau).$$

By (6.1), we conclude that $\mathcal{R}(p,\tau) \le C_*$ for some universal constant $C_*$.

Next we will show the boundedness of the $k$-flux $\mathcal{F}[k](p,\tau)$. With the help of the projection tensor

$$h^{\alpha\beta} = \mathbf{g}^{\alpha\beta} + \mathbf{T}^\alpha \mathbf{T}^\beta,$$

for any tensor field $U_{\alpha_1\alpha_2\cdots\alpha_m}$ in $T\mathcal{M}$, we define $|U|$ as follows:

$$|U|^2 = h^{IJ} U_I U_J = h^{\alpha_1\beta_1} \cdots h^{\alpha_m\beta_m} U_{\alpha_1\alpha_2\cdots\alpha_m} U_{\beta_1\beta_2\cdots\beta_m},$$
$$h^{IJ} = h^{\alpha_1\beta_1} \cdots h^{\alpha_m\beta_m}, \quad U_I = U_{\alpha_1\alpha_2\cdots\alpha_m}, \quad U_J = U_{\beta_1\beta_2\cdots\beta_m}.$$

For any $\Sigma_t$-tangent tensor field $U$ in $\mathcal{M}_I$, we define the energy momentum tensor $Q[U]_{\alpha\beta}$ associated with the covariant wave operator acting on tensors as follows:

$$Q[U]_{\alpha\beta} := h^{IJ} \mathbf{D}_\alpha U_I \mathbf{D}_\beta U_J - \frac{1}{2} \mathbf{g}_{\alpha\beta} h^{IJ} \mathbf{g}^{\mu\nu} \mathbf{D}_\mu U_I \mathbf{D}_\nu U_J.$$

We have

$$\begin{aligned}\mathbf{D}^\beta Q[U]_{\alpha\beta} &= h^{IJ} \mathbf{D}_\alpha U_I \Box U_J + h^{IJ} (\mathbf{D}_\beta \mathbf{D}_\alpha U_I - \mathbf{D}_\alpha \mathbf{D}_\beta U_I) \mathbf{D}^\beta U_J \\ &\quad + \mathbf{D}^\beta h^{IJ} \left( \mathbf{D}_\alpha U_I \mathbf{D}_\beta U_J - \frac{1}{2} \mathbf{g}_{\alpha\beta} \mathbf{g}^{\mu\nu} \mathbf{D}_\mu U_I \mathbf{D}_\nu U_J \right)\end{aligned}$$

It is easy to see that the last term can be written symbolically as $\pi \cdot \mathbf{D}U \cdot \mathbf{D}U$.

We apply the above equation to $U = k$. Since $h^{0\alpha} = 0$ and $h^{ij} = g^{ij}$, we have

$$\begin{aligned}\mathbf{D}^\beta (Q[k]_{\alpha\beta} \mathbf{T}^\alpha) &= \mathbf{D}^\beta \mathbf{T}^\alpha Q[k]_{\alpha\beta} + \mathbf{D}^\beta Q[k]_{0\beta} \\ &= -k^{ij} Q[k]_{ij} - \pi^{0j} Q[k]_{0j} + \mathbf{D}_0 k^{ij} \Box k_{ij} \\ &\quad + [\mathbf{D}_a, \mathbf{D}_0] k_{ij} \nabla^a k^{ij} + \pi \cdot \mathbf{D}k \cdot \mathbf{D}k.\end{aligned} \tag{6.2}$$

In view of the commutation formula

$$[\mathbf{D}_m, \mathbf{D}_0] k_{ij} = \mathbf{R}_i{}^b{}_{m0} k_{bj} + \mathbf{R}_j{}^b{}_{m0} k_{ib} = -\epsilon_{ib}^s H_{sm} k_j^b - \epsilon_{jb}^s H_{sm} k_i^b,$$

we derive symbolically

$$\begin{aligned}\mathbf{D}^\beta (Q[k]_{\alpha\beta} \mathbf{T}^\alpha) &= -k^{ij} Q[k]_{ij} - \pi^{0j} Q[k]_{0j} + \mathbf{D}_0 k^{ij} \Box k_{ij} \\ &\quad + H \cdot k \cdot \nabla k + \pi \cdot \mathbf{D}k \cdot \mathbf{D}k.\end{aligned}$$

From the definition of $Q[k]$, it is easy to see that

$$Q[k]_{00} = \frac{1}{2}(|\mathbf{D}_0 k|^2 + |\nabla k|^2), \tag{6.3}$$

$$Q[k]_{0j} = \mathbf{D}_0 k_{pq} \nabla_j k^{pq}, \tag{6.4}$$

$$Q[k]_{ij} = \nabla_i k_{pq} \nabla_j k^{pq} - \frac{1}{2} g_{ij}(-|\mathbf{D}_0 k|^2 + |\nabla k|^2). \tag{6.5}$$

Therefore

$$\begin{aligned} \mathbf{D}^\beta(Q[k]_{\alpha\beta}\mathbf{T}^\alpha) = \frac{1}{2}\operatorname{Tr} k(-|\mathbf{D}_0 k|^2 + |\nabla k|^2) + k \cdot \nabla k \cdot \nabla k \\ + \mathbf{D}_0 k \cdot \Box k + H \cdot k \cdot \nabla k + \pi \cdot \mathbf{D} k \cdot \mathbf{D} k. \end{aligned} \tag{6.6}$$

We now apply Lemma 6.1 to $P^\beta := \mathbf{T}^\alpha Q[k]_\alpha^\beta$ and obtain

$$\int_{\mathcal{N}^-(p,\tau)} Q[k](\mathbf{T}, L) + \int_{\Sigma_{t(p)-\tau} \cap \mathcal{J}^-(p)} Q[k]_{00} = \int_{\mathcal{J}^-(p,\tau)} \mathbf{D}^\beta(Q[k]_{\alpha\beta}\mathbf{T}^\alpha). \tag{6.7}$$

For the null pair $L$ and $\underline{L}$, it is easy to see that

$$Q[k](L, L) = |\nabla_L k|^2, \quad Q[k](\underline{L}, L) = |\nabla\!\!\!/ k|^2.$$

Since $\mathbf{T} = -\frac{1}{2}(aL + a^{-1}\underline{L})$, we have

$$\begin{aligned} Q[k](\mathbf{T}, L) &= -\frac{1}{2}(aQ[k](L, L) + a^{-1}Q[k](\underline{L}, L)) \\ &= -\frac{1}{2}(a|\nabla_L k|^2 + a^{-1}|\nabla\!\!\!/ k|^2). \end{aligned}$$

Since $|a - 1| \le \frac{1}{2}$, the $k$-flux defined in (4.10) satisfies the inequality

$$-\int_{\mathcal{N}^-(p,\tau)} Q[k](\mathbf{T}, L) \le \mathcal{F}[k](p, \tau) \le -4 \int_{\mathcal{N}^-(p,\tau)} Q[k](\mathbf{T}, L).$$

Thus we derive from (6.7) and (6.3) that

$$\begin{aligned} \mathcal{F}[k](p, \tau) \le 4\left| \int_{\mathcal{J}^-(p,\tau)} \mathbf{D}^\beta(Q[k]_{\alpha\beta}\mathbf{T}^\alpha)\right| \\ + 2 \int_{\Sigma_{t(p)-\tau} \cap \mathcal{J}^-(p)} (|\mathbf{D}_0 k|^2 + |\nabla k|^2). \end{aligned} \tag{6.8}$$

In view of (5.2), Lemma 2.2, Proposition 3.1, and Proposition 3.5, we have

$$\begin{aligned} \int_{\Sigma_t} (|\mathbf{D}_0 k|^2 + |\nabla k|^2) \lesssim \\ \|\nabla^2 n\|^2_{L^2(\Sigma_t)} + \|\mathrm{Ric}\|^2_{L^2(\Sigma_t)} + \|k\|^4_{L^4(\Sigma_t)} + \|\nabla k\|^2_{L^2(\Sigma_t)} \le C. \end{aligned} \tag{6.9}$$

Moreover, in view of (6.6), (A1), Lemma 2.2, and the above inequality we have

$$
\begin{aligned}
&\Big| \int_{\mathcal{J}^-(p,\tau)} \mathbf{D}^\beta (Q[k]_{\alpha\beta}\mathbf{T}^\alpha)\Big| \\
&\lesssim \int_{t(p)-\tau}^{t(p)} \|\mathbf{D}_0 k\|_{L^2(\Sigma_{t'})} \|\Box k\|_{L^2(\Sigma_{t'})} dt' \\
&\quad + \int_{t(p)-\tau}^{t(p)} \|\pi\|_{L^\infty(\Sigma_{t'})} \big(\|\mathbf{D}_0 k\|^2_{L^2(\Sigma_{t'})} + \|\nabla k\|^2_{L^2(\Sigma_{t'})}\big) dt' \\
&\quad + \int_{t(p)-\tau}^{t(p)} \|k\|_{L^\infty(\Sigma_{t'})} \|H\|_{L^2(\Sigma_{t'})} \|\nabla k\|_{L^2(\Sigma_{t'})} dt' \\
&\lesssim \int_{t(p)-\tau}^{t(p)} \|\Box k\|_{L^2(\Sigma_{t'})} dt' + \int_{t(p)-\tau}^{t(p)} \|\pi\|_{L^\infty(\Sigma_{t'})} dt' \\
&\le C + C\int_{t(p)-\tau}^{t(p)} \|\Box k\|_{L^2(\Sigma_{t'})} dt'.
\end{aligned}
$$

Therefore

$$
\mathcal{F}[k](p,\tau) \le C + C\int_{t(p)-\tau}^{t(p)} \|\Box k\|_{L^2(\Sigma_{t'})} dt'. \tag{6.10}
$$

Recall the formula for $\Box k$ given in Proposition 5.1, which symbolically can be written as

$$
\begin{aligned}
\Box k &= -n^{-3}\dot{n}\nabla^2 n + n^{-2}\nabla^2\dot{n} + \pi\cdot\pi\cdot\pi + k\cdot\nabla^2 n \\
&\quad + k\cdot \mathrm{Ric} + \pi\cdot\nabla k - n^{-1}k.
\end{aligned}
$$

Since $C^{-1} \le n \le C$, we obtain

$$
\begin{aligned}
\|\Box k\|_{L_t^1 L_x^2} &\lesssim \|\dot{n}\|_{L_t^1 L_x^\infty} \|\nabla^2 n\|_{L_t^\infty L_x^2} + \|\nabla^2\dot{n}\|_{L_t^1 L_x^2} + \|\pi\|_{L_t^1 L_x^\infty}\|\pi\|^2_{L_t^\infty L_x^4} \\
&\quad + \|k\|_{L_t^1 L_x^\infty} \|\nabla^2 n\|_{L_t^\infty L_x^2} + \|k\|_{L_t^1 L_x^\infty} \|\mathrm{Ric}\|_{L_t^\infty L_x^2} \\
&\quad + \|k\|_{L_t^1 L_x^2} + \|\pi\|_{L_t^1 L_x^\infty}\|\nabla k\|_{L_t^\infty L_x^2}.
\end{aligned}
$$

In view of assumption (A1), Lemma 2.2, Proposition 3.1, Proposition 3.7, and (6.9), it follows that

$$
\|\Box k\|_{L_t^1 L_x^2} \le C(1 + \|\pi\|_{L_t^1 L_x^\infty} + \tau) \le C.
$$

Combining the above inequality with (6.10) completes the proof of Theorem 4.4.

## 7 Trace Estimates

For a point $p \in \mathcal{M}_I$, let $s$ be the affine parameter on the null cone $\mathcal{N}^-(p)$, and let $r$ be the radius of $S_t := \mathcal{N}^-(p) \cap \Sigma_t$, which is defined by (4.2). On each $S_t$

we introduce the ratio of area elements

$$v_t(\omega) = \frac{\sqrt{|\gamma|}}{\sqrt{|\mathring{\gamma}|}}, \quad \omega \in \mathbb{S}^2. \tag{7.1}$$

We will first show that all the quantities $s$, $r$, $v_t^{1/2}$, and $t(p)-t$ are comparable under the bootstrap assumptions (BA1)–(BA3). Here we say two quantities $\varphi$ and $\psi$ are comparable in the sense that $C^{-1}\psi \le \varphi \le C\psi$ for some universal constant $C > 0$.

LEMMA 7.1. *Under the bootstrap assumptions* (BA1)–(BA3)*, the four quantities* $s(t)$*,* $r(t)$*,* $v_t^{1/2}$*, and* $t(p) - t$ *are comparable on the null cone* $\mathcal{N}^-(p,\tau)$ *with* $\tau \le \min\{i_*, \delta_*\}$*, where* $\delta_* > 0$ *is a universal constant.*

PROOF. The comparability of $s$ and $t(p) - t$ follows from the relation $\frac{ds}{dt} = -na$ and the bootstrap assumption (BA1). Similar to the derivation of (4.5), we have under the bootstrap assumptions (BA1)–(BA3) that

$$|\gamma - s(t)^2\mathring{\gamma}| \le \frac{1}{2}s(t)^2\mathring{\gamma} \tag{7.2}$$

for all $t(p) - \min\{i_*, \tau, \delta_*\} \le t < t(p)$, where $\delta_*$ is a universal constant. This implies immediately that $\frac{1}{2}s(t)^2 \le v_t \le \frac{3}{2}s(t)^2$. Consequently, $v_t$ and $t(p) - t$ are comparable. Thus for the area $|S_t|$ of $S_t$ there holds

$$C^{-1}(t(p)-t)^2 \le |S_t| \le C(t(p)-t)^2$$

for some universal constant $C$. This together with the definition of $r$ gives the comparability of $r$ and $t(p) - t$. □

### 7.1 Optical Function

In this section we give a brief review of the construction of optical functions; one may see [5] for more information.

For any point $p \in \mathcal{M}_I$, let $J^-(p)$ be the causal past, and let $\mathcal{N}^-(p)$ and $\mathcal{I}^-(p)$ denote, respectively, the null boundary and the interior. For each $0 < \tau < i_*$ with $i_*$ defined by (4.1), let $\mathcal{J}^-(p,\tau)$, $\mathcal{N}^-(p,\tau)$, and $\mathcal{I}^-(p,\tau)$ denote the portions of $\mathcal{J}^-(p)$, $\mathcal{N}^-(p)$, and $\mathcal{I}^-(p)$ in the time slab $[t(p)-\tau, t(p)]$, respectively. Let $\Phi$ be the integral curve of $\mathbf{T}$ through $p$ with $\Phi(t(p)) = p$. According to the definition of $i_*$, all the null cones $\mathcal{N}^-(\Phi(t), \tau + t - t(p))$, with $t(p) - \tau \le t \le t(p)$ and $\tau < i_*$, are disjoint, and their union forms $\mathcal{N}^-(p,\tau)$. We now define $u$ to be the function, constant on each $\mathcal{N}^-(\Phi(t), t + \tau - t(p))$, such that

$$u(\Phi(t)) = \int_{t_0}^{t} n(\Phi(t'))dt'.$$

Such $u$, which will be called an optical function, is a well-defined smooth function on $\mathcal{J}^-(p,\tau)$ and satisfies the eikonal equation

$$\mathbf{g}^{\alpha\beta}\partial_\alpha u \partial_\beta u = 0.$$

It is clear that the level sets $C_u$ of $u$ are the incoming null cones in the time slab $[t(p)-\tau, t(p)]$ with vertices on $\Phi$, and $\mathbf{T}(u) = 1$ on $\Phi$. Moreover, the null geodesic vector $L$ defined before can be written as $L = \mathbf{g}^{\alpha\beta}\partial_\beta u \partial_\alpha$.

For each $t \in [t(p)-\tau, t(p)]$, we define $u_M(t)$ and $u_m(t)$, respectively, to be the largest and smallest values of $u$ for which the part of the cone $C_u$ that lies in the future of $\Sigma_t$ is contained in $\mathcal{J}^-(p)$, i.e.,

$$u_M(t) = u(p) \quad \text{and} \quad u_m(t) = u(\Phi(t)).$$

For each $u(\Phi(t(p)-\tau)) \le u \le u(p)$, we also define $t_M(u)$ and $t_m(u)$ to be the largest and smallest value of $t$ for which $\Sigma_t$ intersects $C_u$, respectively. It is clear that $t_M(u)$ is the value of $t$ at the vertex of $C_u$ and $t_m(u) = t(p)-\tau$. Note that both $u_M$ and $t_m$ are independent of $t$.

We set

$$S_{t,u} := C_u \cap \Sigma_t,$$

which is a smooth surface for each $t(p)-\tau \le t \le t(p)$ and $u_M \le u < u_m(t)$. The corresponding radius function is defined as

$$r(t,u) := \sqrt{(4\pi)^{-1}|S_{t,u}|},$$

where $|S_{t,u}|$ denotes the area of $S_{t,u}$ with respect to the metric $\gamma$.

The following result follows readily from Lemma 7.1 and the definition of $u$.

PROPOSITION 7.2. *Under the bootstrap assumptions* (BA1)–(BA3) *on* $\mathcal{N}^-(p,\tau)$ *for all* $p \in \mathcal{M}_I$, *there hold*

$$C^{-1} \le \frac{t_M(u)-t}{r(t,u)} \le C \quad \textit{and} \quad C^{-1} \le \frac{u-u_m(t)}{r(t,u)} \le C$$

*for all* $t(p)-\min\{i_*, \tau, \delta_*\} < t < t(p)$, *where* $C$ *and* $\delta_*$ *are two positive universal constants.*

In view of the above notation, it is clear that

$$\mathcal{N}^-(p,\tau) = \bigcup_{t\in[t(p)-\tau, t(p)]} S_{t,u_M}.$$

Let $\mathrm{Int}(S_{t,u_M})$ be the interior of $S_{t,u_M}$ in $\Sigma_t$; then

$$\mathrm{Int}(S_{t,u_M}) = \bigcup_{u\in[u_M, u_m(t)]} S_{t,u} \quad \text{and} \quad \mathcal{J}^-(p,\tau) = \bigcup_{t\in[t(p)-\tau, t(p)]} \mathrm{Int}(S_{t,u_M}).$$

The following result can be found in [5], which is crucial in deriving trace estimates.

LEMMA 7.3. *For any scalar* $f$ *satisfying* $\lim_{u\to u_m(t)} \int_{S_{t,u}} f\, d\mu_\gamma = 0$, *there holds*

$$\int_{S_{t,u_M}} f\, d\mu_\gamma = -\int_{u_m(t)}^{u_M} \int_{S_{t,u}} (\nabla_N f + \mathrm{tr}\,\theta f) a\, d\mu_{\gamma_u}\, du,$$

*where $N$ denotes the unit inward normal to $S_{t,u}$ in $\Sigma_t$, and $\theta$ denotes the corresponding second fundamental form.*

## 7.2 Trace Estimates

We will rely on the following trace inequality:

LEMMA 7.4. *Under the bootstrap assumptions* (BA1)–(BA3) *on $\mathcal{N}^-(p,\tau)$ with $\mathcal{E}_0\tau \le 1$, for any $\Sigma_t$-tangent tensor field $F$ there holds*

$$\|r^{-1/2}F\|_{L^2(S_t)} \lesssim \|\nabla F\|_{L^2(\Sigma_t)} + \|F\|_{L^2(\Sigma_t)},$$

*where $S_t := \mathcal{N}^-(p,\tau)\cap\Sigma_t$ and $r := \sqrt{(4\pi)^{-1}|S_t|}$.*

In Section 4 we have verified condition (C2). Therefore Lemma 7.4 can be proved by the standard procedure. Using Lemma 7.4, we can derive the following result.

PROPOSITION 7.5. *Let the bootstrap assumptions* (BA1)–(BA3) *hold on $\mathcal{N}^-(p,\tau)$ with $\mathcal{E}_0\tau \le 1$. Then for any $\Sigma_t$-tangent tensor field $F$ there hold*

$$\|F\|^2_{L^2(S_t)} \lesssim \|F\|_{H^1(\Sigma_t)}\|F\|_{L^2(\Sigma_t)}, \tag{7.3}$$

$$\|F\|_{L^4(S_t)} \lesssim \|F\|_{H^1(\Sigma_t)}, \tag{7.4}$$

*for all $t(p)-\tau \le t < t(p)$.*

PROOF. Let $\phi(u)$ be a smooth cutoff function satisfying $\phi(u_M)=1$, $0\le\phi\le 1$, and $\operatorname{supp}(\phi)\subset[\frac{u_m+u_M}{2}, u_M]$. It then follows from Lemma 7.3 that

$$\|F\|^2_{L^2(S_t)} = -\int_{\operatorname{Int}(S_t)} (\nabla_N|\phi F|^2 + \operatorname{tr}\theta|\phi F|^2)a\,d\mu_\gamma\,du' = \mathrm{I}_1 + \mathrm{I}_2, \tag{7.5}$$

where

$$\mathrm{I}_1 = -2\int_{\operatorname{Int}(S_t)} (\phi^2 F\cdot\nabla_N F + \phi\nabla_N\phi|F|^2)a\,d\mu_\gamma\,du',$$

$$\mathrm{I}_2 = -\int_{\operatorname{Int}(S_t)} \operatorname{tr}\theta|\phi F|^2 a\,d\mu_\gamma\,du'.$$

Since the bootstrap assumption (BA1) implies $\frac12 \le a \le \frac32$, it is easy to see that

$$\left|\int_{\operatorname{Int}(S_t)} \phi^2 F\cdot\nabla_N F a\,d\mu_\gamma\,du'\right| \lesssim \|\nabla_N F\|_{L^2(\Sigma_t)}\|F\|_{L^2(\Sigma_t)}$$

and

$$\left|\int_{\operatorname{Int}(S_t)} \phi\nabla_N\phi|F|^2 a\,d\mu_\gamma\,du'\right| \lesssim \frac{1}{u_M-u_m}\int_{\frac{u_m+u_M}{2}}^{u_M}\int_{S_{t,u'}} |F|^2\,d\mu_\gamma\,du'.$$

It follows from Lemma 7.4 that

$$\int_{S_{t,u'}} |F|^2 \, d\mu_\gamma \lesssim \|r^{-1/2} F\|_{L^2(S_{t,u'})} \|F\|_{L^2(S_{t,u'})} r^{1/2}$$
$$\lesssim \|F\|_{H^1(\Sigma_t)} \|F\|_{L^2(S_{t,u'})} r^{1/2},$$

where $r := r(t,u')$. From Proposition 7.2 it follows that $r(t,u') \lesssim u' - u_m$. Thus

$$\left| \int_{\mathrm{Int}(S_t)} \phi \nabla_N \phi |F|^2 a \, d\mu_\gamma \, du' \right|$$
$$\lesssim \frac{1}{u_M - u_m} \|F\|_{H^1(\Sigma_t)} \|F\|_{L^2(\Sigma_t)} \left( \int_{\frac{u_m+u_M}{2}}^{u_M} (u' - u_m) du' \right)^{1/2}$$
$$\lesssim \|F\|_{H^1(\Sigma_t)} \|F\|_{L^2(\Sigma_t)}.$$

We therefore obtain

$$|\mathrm{I}_1| \lesssim \|F\|_{H^1(\Sigma_t)} \|F\|_{L^2(\Sigma_t)}.$$

In order to estimate the term $I_2$, we recall that $\mathrm{tr}\,\theta = -a\,\mathrm{tr}\,\chi + \delta^{AB} k_{AB}$. Since the bootstrap assumption (BA2) implies $|\mathrm{tr}\,\chi - 2/s| \le \mathcal{E}_0$ on each $S_{t,u'}$ and Proposition 7.2 implies that $s$, $t(p) - t$, and $r$ are comparable, we have

$$|\mathrm{I}_2| \lesssim (\mathcal{E}_0 \tau + 1) \int_{u_m}^{u_M} \int_{S_{t,u'}} r^{-1} |\phi F|^2 \, d\mu_\gamma \, du' + \int_{u_m}^{u_M} \int_{S_{t,u'}} |k| |\phi F|^2 \, d\mu_\gamma \, du'$$
$$\lesssim \int_{u_m}^{u_M} \int_{S_{t,u'}} r^{-1} |\phi F|^2 \, d\mu_\gamma \, du' + \|k\|_{L^3(\Sigma_t)} \|F\|^2_{L^3(\Sigma_t)}.$$

Recall that $\|k\|_{L^3(\Sigma_t)} \le C$ from Lemma 2.2 and apply Lemma 2.5 to $\|F\|^2_{L^3(\Sigma_t)}$; we obtain

$$|\mathrm{I}_2| \lesssim \|F\|_{H^1(\Sigma_t)} \|F\|_{L^2(\Sigma_t)} + \int_{\frac{u_m+u_M}{2}}^{u_M} \int_{S_{t,u'}} r^{-1} |F|^2 \, d\mu_\gamma \, du'.$$

Now we use Lemma 7.4 again and note that Proposition 7.2 implies $r(t,u')^{-1} \lesssim (u' - u_m)^{-1}$; we have

$$\int_{\frac{u_m+u_M}{2}}^{u_M} \int_{S_{t,u'}} r^{-1} |F|^2 \, d\mu_\gamma \, du'$$
$$\lesssim \|F\|_{H^1(\Sigma_t)} \|F\|_{L^2(\Sigma_t)} \left( \int_{\frac{u_m+u_M}{2}}^{u_M} (u' - u_m)^{-1} \, du' \right)^{1/2}$$
$$\lesssim \|F\|_{H^1(\Sigma_t)} \|F\|_{L^2(\Sigma_t)}.$$

Therefore

$$|\mathrm{I}_2| \lesssim \|F\|_{H^1(\Sigma_t)} \|F\|_{L^2(\Sigma_t)}.$$

The proof of (7.3) is complete.

Applying (7.5) with $|F|$ replaced by $|F|^2$ and using Sobolev embedding, we can obtain (7.4) in a similar fashion. □

As a consequence, we obtain the following:

PROPOSITION 7.6. *Let the bootstrap assumptions* (BA1)–(BA3) *hold on* $\mathcal{N}^-(p,\tau)$ *with* $\mathcal{E}_0\tau \le 1$. *Let* $S_t := \mathcal{N}^-(p,\tau) \cap \Sigma_t$ *and let* $r$ *be defined by* (4.2). *Let* $\pi_0$ *denote the tensor* $-\nabla \log n$.

(i) *Let* $\underline{\pi}$ *denote either* $k$ , $\pi_0$, *or* $\mathbf{D}_0 \log n$; *then for* $t(p) - \tau \le t \le t(p)$

$$\|\underline{\pi}\|_{L^4(S_t)} \le C, \tag{7.6}$$

$$\|r^{-1/2}\underline{\pi}\|_{L^2(S_t)} \le C. \tag{7.7}$$

(ii) *Let* $F$ *denote either* $n^{-1}\nabla^2 n$ *or* $n^{-2}\nabla \dot{n}$; *then*

$$\|F\|_{L^2(\mathcal{N}^-(p,\tau))} \le C. \tag{7.8}$$

(iii) *For* $\pi_0$, *there holds*

$$\|\nabla_L \pi_0\|_{L^2(\mathcal{N}^-(p,\tau))} + \|\mathbf{D}_0\pi_0\|_{L^2(\mathcal{N}^-(p,\tau))} + \|\nabla\pi_0\|_{L^2(\mathcal{N}^-(p,\tau))} \le C. \tag{7.9}$$

PROOF.

(i) It follows that $\|\underline{\pi}\|_{H^1(\Sigma_t)} \le C$ from Lemma 2.2, Proposition 3.1, and Lemma 3.6. Thus (7.6) follows from (7.4) in Proposition 7.5 and (7.7) follows from Lemma 7.4.

(ii) For $F = (n^{-1}\nabla^2 n, n^{-2}\nabla\dot{n})$ it follows from Proposition 3.1, Proposition 3.4, Lemma 3.6, and Proposition 3.7 that

$$\|\nabla F\|_{L_t^1 L_x^2(\mathcal{M}_*)} \le C \quad \text{and} \quad \|F\|_{L_t^\infty L_x^2(\mathcal{M}_*)} \le C.$$

Applying (7.3) to $F$ yields

$$\|F\|^2_{L^2(\mathcal{N}^-(p,\tau))} \lesssim \|F\|_{L_t^1 H_x^1(\mathcal{M}_*)} \|F\|_{L_t^\infty L_x^2(\mathcal{M}_*)} \lesssim C.$$

(iii) By straightforward calculation, symbolically we have

$$\mathbf{D}_0\pi_0 = -n^{-2}\nabla\dot{n} + \underline{\pi}\cdot\pi_0, \quad \nabla\pi_0 = -n^{-1}\nabla^2 n + \underline{\pi}\cdot\pi_0,$$
$$\nabla_L\pi_0 = a^{-1}n^{-2}\nabla\dot{n} - a^{-1}\nabla\pi_0 - a^{-1}\underline{\pi}\cdot\pi_0.$$

Therefore, (7.9) follows immediately from (7.6) and (7.8).

□

# 8 Estimates on the Null Cones

## 8.1 Structure Equations on the Null Cones

In Section 4 we introduced the null pair $L, \underline{L}$ on the null cone $\mathcal{N}^-(p,\tau)$ and defined the null second fundamental forms $\chi, \underline{\chi}$ and the Ricci coefficients $\zeta, \underline{\zeta}$. We also introduced in (4.11) the null components $\alpha$, $\beta$, $\rho$, and $\sigma$ of the curvature tensor $\mathbf{R}$. There hold on null cones the following structure equations:

$$\frac{d \operatorname{tr} \chi}{ds} + \frac{1}{2}(\operatorname{tr} \chi)^2 = -|\widehat{\chi}|^2, \tag{8.1}$$

$$\frac{d \widehat{\chi}_{AB}}{ds} + \operatorname{tr} \chi \widehat{\chi}_{AB} = \alpha_{AB}, \tag{8.2}$$

$$\frac{d}{ds} \zeta_A = -\chi_{AB}\zeta_B + \chi_{AB}\underline{\zeta}_B - \beta_A, \tag{8.3}$$

$$\frac{d}{ds} \operatorname{tr} \underline{\chi} + \frac{1}{2} \operatorname{tr} \chi \operatorname{tr} \underline{\chi} = 2 \operatorname{div} \underline{\zeta} - \widehat{\chi} \cdot \underline{\widehat{\chi}} + 2|\underline{\zeta}|^2 + 2\rho. \tag{8.4}$$

Moreover, $\zeta$ satisfies the following Hodge system:

$$\operatorname{div} \zeta = -\mu - \rho + \frac{1}{2} \widehat{\chi} \cdot \underline{\widehat{\chi}} - |\zeta|^2 - \frac{1}{2} a\delta \operatorname{tr} \chi - a\lambda \operatorname{tr} \chi, \tag{8.5}$$

$$\operatorname{curl} \zeta = \sigma - \frac{1}{2} \widehat{\chi} \wedge \underline{\widehat{\chi}}, \tag{8.6}$$

where $\mu$ and $\underline{\mu}$ are the mass aspect functions defined by

$$\mu = -\frac{1}{2} \mathbf{D}_3 \operatorname{tr} \chi + \frac{a^2}{4}(\operatorname{tr} \chi)^2 - \omega \operatorname{tr} \chi, \tag{8.7}$$

$$\underline{\mu} = \mathbf{D}_4 \operatorname{tr} \underline{\chi} + \frac{1}{2} \operatorname{tr} \chi \cdot \operatorname{tr} \underline{\chi}, \tag{8.8}$$

$$\omega = \frac{1}{2}(\mathbf{D}_3 \log a + ak_{NN} - a\pi_{0N}). \tag{8.9}$$

These equations can be found in [5, pp. 351–360], where more structure equations have been derived.

### 8.2 Proof of Theorem 4.5

The main purpose of this subsection is to prove Theorem 4.5 concerning the boundedness of $\mathcal{N}_1[\not{\pi}]$ under the bootstrap assumptions (BA1)–(BA3) on $\mathcal{N}^-(p,\tau)$ with $0 < \tau \le i_*$ and $\mathcal{E}_0\tau \le 1$ for any $p \in \mathcal{M}_I$, where $\not{\pi}$ is defined by (4.14) and the Sobolev norm $\mathcal{N}_1[F]$ for any $S_t$-tangent tensor field $F$ is defined by (4.12). We can restate Theorem 4.5 in the following form:

PROPOSITION 8.1. *Let* $\not{\pi}$ *be the* $S_t$*-tangent tensor field defined in* (4.14)*, and let* $\overline{\pi} := (k, -\nabla \log n)$*. Then, under the bootstrap assumptions* (BA1)–(BA4) *with* $\mathcal{E}_0\tau \le 1$*, there hold*

$$\|r^{-1}\overline{\pi}\|_{L^2(\mathcal{N}^-(p,\tau))} \le C, \tag{8.10}$$

$$\|\nabla\!\!\!/ \not{\pi}\|_{L^2(\mathcal{N}^-(p,\tau))} \le C, \tag{8.11}$$

$$\|\nabla\!\!\!/_L \not{\pi}\|_{L^2(\mathcal{N}^-(p,\tau))} \le C. \tag{8.12}$$

We have obtained in Theorem 4.4 and (7.9) that

$$\|\nabla\!\!\!/ \bar{\pi}\|_{L^2(C_u)} + \|\nabla_L \bar{\pi}\|_{L^2(C_u)} \le C. \tag{8.13}$$

Let $N$ be the unit inward normal to $S_t$ in $\Sigma_t$, and let $\theta$ be the second fundamental form of $S_t$, i.e., $\theta_{AB} = g(\nabla_A N, e_B)$. Then there hold

$$\nabla_A N = \theta_{AB} e_B, \quad \nabla_B e_A = \nabla\!\!\!/_B e_A - \theta_{AB} N.$$

This enables us to derive symbolically that

$$\nabla\!\!\!/ \pi\!\!\!/ = \nabla\!\!\!/ \bar{\pi} + \operatorname{tr}\theta \cdot \pi\!\!\!/ + \widehat{\theta} \cdot \pi\!\!\!/. \tag{8.14}$$

Recall also that $\mathbf{D}_L L = 0$, $\mathbf{D}_L \underline{L} = 2\underline{\zeta}_A e_A$, and $\mathbf{D}_L e_A = \nabla\!\!\!/_4 e_A + \underline{\zeta}_A e_4$. We have, in view of $\frac{dt}{ds} = -(an)^{-1}$, that

$$\nabla\!\!\!/_L \pi\!\!\!/ = \nabla_L \bar{\pi} + \pi\!\!\!/ \cdot \underline{\zeta} + (an)^{-1}. \tag{8.15}$$

In order to show Proposition 8.1, we need three auxiliary lemmas. We will use the following norms for $\Sigma_t$-tangent tensor fields $F$ on null cones $\mathcal{N}^-(p,\tau)$:

$$\|F\|^q_{L^q_x L^\infty_t(\mathcal{N}^-(p,\tau))} := \int_{\mathbb{S}^2} \sup_{t\in\Gamma_\omega} (v_t |F|^q_g) d\mu_{\mathbb{S}^2},$$

$$\|F\|^q_{L^q_\omega L^\infty_t(\mathcal{N}^-(p,\tau))} := \int_{\mathbb{S}^2} \sup_{t\in\Gamma_\omega} |F|^q_g \, d\mu_{\mathbb{S}^2}.$$

where $v_t$ is defined by (7.1), and $\Gamma_\omega$, $\omega \in \mathbb{S}^2$, denotes the portion of an incoming null geodesic initiating from $p$ in the time slab $[t(p)-\tau, t(p)]$. In the following argument we will suppress $\mathcal{N}^-(p,\tau)$ in these norms for simplicity.

LEMMA 8.2. *For any $S_t$-tangent tensor field $F$, there hold the estimates*

$$\|r^{-1/2} F\|_{L^2_x L^\infty_t} + \|F\|_{L^4_x L^\infty_t} \lesssim \mathcal{N}_1[F], \tag{8.16}$$

$$\|F\|^2_{L^4_x L^\infty_t} \lesssim (\|\nabla\!\!\!/_L F\|_{L^2} + \|r^{-1} F\|_{L^2}) \|F\|_{L^\infty_\omega L^2_t}. \tag{8.17}$$

PROOF. We refer to [7, 14] for the proof of (8.16). In the following we will prove (8.17). Let $v_t$ be defined by (7.1). We first integrate along any past null geodesic initiating from $p$ to get

$$v_t |F|^4 = \lim_{t\to t(p)} (v_t |F|^4) - \int_t^{t(p)} \frac{d}{dt'} (v_{t'} |F|^4) dt'. \tag{8.18}$$

For the estimate of the first term on the right of (8.18), we proceed as follows: Let $\varphi$ be a smooth cutoff function defined on $[t(p)-\tau, t(p)]$ satisfying $0 \le \varphi \le 1$, $\varphi(t(p)) = 1$, and $\operatorname{supp}\varphi \subset [t(p)-\tau/2, t(p)]$. Then

$$\lim_{t\to t(p)} v_t |F|^4 = \int_{t(p)-\tau}^{t(p)} \left( \frac{d}{dt}(v_t |F|^4)\varphi^4 + 4 v_t |F|^4 \varphi^3 \frac{d}{dt}\varphi \right) dt. \tag{8.19}$$

Since $|\frac{d}{dt}\varphi| \lesssim (t(p)-t)^{-1}$, we have from Lemma 7.1 that $|\frac{d}{dt}\varphi| v_t^{\frac{1}{2}} \lesssim 1$. Using $0 \le \varphi \le 1$, it then follows from (8.18) and (8.19) that

$$\|F\|^4_{L_x^4 L_t^\infty} = \int_{\mathbb{S}^2} \sup_{t(p)-\tau \le t \le t(p)} (v_t |F|^4) \lesssim \mathrm{I} + \mathrm{II}, \tag{8.20}$$

where

$$\mathrm{I} = \int_{\mathbb{S}^2} \int_{t(p)-\tau}^{t(p)} \left| \frac{d}{dt}(v_t |F|^4) \right|, \quad \mathrm{II} = \int_{\mathbb{S}^2} \int_{t(p)-\tau}^{t(p)} v_t^{1/2} |F|^4.$$

Since

$$\frac{d}{dt}(v_t|F|^4) = -na(\mathrm{tr}\,\chi v_t |F|^4 + 4 v_t |F|^2 \nabla\!\!\!/_L F \cdot F),$$

we have

$$\begin{aligned} \mathrm{I} &\lesssim \big(\|v_t^{1/2} \nabla\!\!\!/_L F\|_{L^2_\omega L^2_t} + \|\mathrm{tr}\,\chi v_t^{1/2} F\|_{L^2_\omega L^2_t}\big) \|F\|_{L^\infty_\omega L^2_t} \|v_t^{1/2} |F|^2\|_{L^2_\omega L^\infty_t} \\ &\lesssim (\|\nabla\!\!\!/_L F\|_{L^2} + \|\mathrm{tr}\,\chi F\|_{L^2}) \|F\|_{L^\infty_\omega L^2_t} \|F\|^2_{L^4_x L^\infty_t}. \end{aligned}$$

By the bootstrap assumption (BA2) and Lemma 7.1 we have

$$\begin{aligned} \|\mathrm{tr}\,\chi F\|_{L^2} &\lesssim \left\| \mathrm{tr}\,\chi - \frac{2}{s} \right\|_{L^\infty} \tau \|r^{-1} F\|_{L^2} + \|r^{-1} F\|_{L^2} \\ &\lesssim (\mathcal{E}_0 \tau + 1) \|r^{-1} F\|_{L^2} \lesssim \|r^{-1} F\|_{L^2}. \end{aligned}$$

Therefore

$$\mathrm{I} \lesssim (\|\nabla\!\!\!/_L F\|_{L^2} + \|r^{-1} F\|_{L^2}) \|F\|_{L^\infty_\omega L^2_t} \|F\|^2_{L^4_x L^\infty_t}.$$

It is easy to see that

$$\begin{aligned} |\mathrm{II}| &\lesssim \|F\|_{L^2_\omega L^2_t} \|F\|_{L^\infty_\omega L^2_t} \|v_t^{1/2} |F|^2\|_{L^2_\omega L^\infty_t} \\ &\lesssim \|r^{-1} F\|_{L^2} \|F\|_{L^\infty_\omega L^2_t} \|F\|^2_{L^4_x L^\infty_t}. \end{aligned}$$

Combining the estimates for I and II with (8.20) gives (8.17). □

LEMMA 8.3. *For any $S_t$-tangent tensor field $F$ satisfying*

$$\nabla\!\!\!/_L F + \frac{m}{2} \mathrm{tr}\,\chi F = G \cdot F + H \tag{8.21}$$

*with $m \ge 1$ an integer and $G$ a tensor field of suitable type, if $\lim_{t \to t(p)} r(t)^m F = 0$ and $\sup_{\omega \in \mathbb{S}^2} \int_{t(p)-\tau}^{t(p)} na|G|^2\, dt \le \Delta_0^2$, the following estimates hold*:

$$\|F\|_{L^2_\omega L^2_t} \lesssim e^{C \Delta_0 \tau^{1/2}} \|H\|_{L^2}, \tag{8.22}$$

$$\|r^{1/2} F\|_{L^2_\omega L^\infty_t} \lesssim e^{C \Delta_0 \tau^{1/2}} \|H\|_{L^2}. \tag{8.23}$$

PROOF. Because $\frac{d}{dt} v_t = -na \operatorname{tr} \chi v_t$, along any past null geodesic initiating from $p$ we have

$$\frac{d}{dt}(v_t^m |F|^2) = -2nav_t^m \langle H + F \cdot G, F \rangle.$$

With the help of the $\lim_{t \to t(p)} r^m |F| = 0$, it follows for $t(p) - \tau \leq t \leq t(p)$ that

$$\begin{aligned} v_t^m |F|^2 &= 2 \int_t^{t(p)} nav_{t'}^m \langle H + F \cdot G, F \rangle \\ &\leq 2 \int_t^{t(p)} nav_{t'}^m (|F||H| + |F|^2 |G|). \end{aligned}$$

By a simple argument we can derive

$$v_t^{m/2} |F| \leq \exp\left( \int_t^{t(p)} |G| na \right) \int_t^{t(p)} nav_{t'}^{m/2} |H| \exp\left( - \int_{t'}^{t(p)} na|G| \right) dt'.$$

In view of $\sup_{\omega \in \mathbb{S}^2} \int_{t(p)-\tau}^{t(p)} na|G|^2 \, dt \leq \Delta_0^2$, we have

$$\exp\left( \int_t^{t(p)} na|G| \right) \leq e^{C\Delta_0 \tau^{1/2}}.$$

Thus by using Lemma 7.1 and $m \geq 1$, we have

$$\begin{aligned} |F| &\leq e^{C\Delta_0 \tau^{1/2}} v_t^{-m/2} \int_t^{t(p)} v_{t'}^{m/2} |H| na \, dt' \\ &\lesssim e^{C\Delta_0 \tau^{1/2}} (t(p) - t)^{-1} \int_t^{t(p)} r|H| dt'. \end{aligned} \tag{8.24}$$

To derive (8.22), we integrate the above inequality along a null geodesic initiating from vertex $p$. By the Hardy-Littlewood inequality we obtain

$$\|F\|_{L_t^2} \lesssim e^{C\Delta_0 \tau^{1/2}} \left\| \frac{1}{t(p) - t} \int_t^{t(p)} r|H| \right\|_{L_t^2} \lesssim e^{C\Delta_0 \tau^{1/2}} \|rH\|_{L_t^2}. \tag{8.25}$$

Integrating (8.25) with respect to the angular variable $\omega \in \mathbb{S}^2$ yields (8.22).

Next we multiply (8.24) by $r^{1/2}$ to obtain

$$\sup_{t(p)-\tau \leq t \leq t(p)} r^{1/2} |F| \lesssim e^{C\Delta_0 \tau^{1/2}} \|rH\|_{L_t^2},$$

which, by taking the $L_\omega^2$-norm, gives (8.23). □

LEMMA 8.4. *For $\widehat{\chi}$ there hold the estimates*

$$\|r^{-1} \widehat{\chi}\|_{L^2} + \|r^{1/2} \widehat{\chi}\|_{L_\omega^2 L_t^\infty} + \|\nabla\!\!\!/_L \widehat{\chi}\|_{L^2} \leq C, \tag{8.26}$$

$$\|\widehat{\chi}\|_{L_x^4 L_t^\infty} \leq C \mathcal{E}_0^{1/4}. \tag{8.27}$$

PROOF. We will use the transport equation (8.2), i.e.,

$$\nabla\!\!\!/_L \widehat{\chi} + \operatorname{tr}\chi\widehat{\chi} = \alpha. \tag{8.28}$$

Recall that $r\widehat{\chi} \to 0$ as $t \to t(p)$; see [14]. Recall also that $\|\alpha\|_{L^2} \le C$; see Theorem 4.4. It then follows from Lemma 8.3 that

$$\|r^{1/2}\widehat{\chi}\|_{L^2_\omega L^\infty_t} + \|\widehat{\chi}\|_{L^2_\omega L^2_t} \le C.$$

Next we use (8.28) again to estimate $\|\nabla\!\!\!/_L \widehat{\chi}\|_{L^2}$. In view of the bootstrap assumption (BA2) and the comparability of $r$, $s$, and $t(p) - t$ given in Lemma 7.1, we have

$$\|\operatorname{tr}\chi\widehat{\chi}\|_{L^2} \lesssim \left\|\operatorname{tr}\chi - \frac{2}{s}\right\|_{L^\infty} \|r\widehat{\chi}\|_{L^2_t L^2_\omega} + \|r^{-1}\widehat{\chi}\|_{L^2} \le C.$$

Thus, from (8.28) it follows that $\|\nabla\!\!\!/_L \widehat{\chi}\|_{L^2} \lesssim \|\operatorname{tr}\chi\widehat{\chi}\|_{L^2} + \|\alpha\|_{L^2} \le C$. We therefore complete the proof of (8.26).

By making use of (8.17) and (8.26) together with the bootstrap assumption (BA3), we obtain

$$\|\widehat{\chi}\|_{L^4_x L^\infty_t} \lesssim (\|\nabla\!\!\!/_L \widehat{\chi}\|_{L^2} + \|r^{-1}\widehat{\chi}\|_{L^2})^{1/2} \|\widehat{\chi}\|^{1/2}_{L^\infty_\omega L^2_t} \le C\mathcal{E}_0^{1/4},$$

which gives (8.27). □

Now we are ready to complete the proof of Proposition 8.1.

PROOF OF PROPOSITION 8.1. We first prove (8.10). Let $|\overline{\pi}| := |\overline{\pi}|_g$. It is easy to check that

$$\nabla\!\!\!/_L(s^{-1}|\overline{\pi}|^2) + \operatorname{tr}\chi s^{-1}|\overline{\pi}|^2 = s^{-1}\left(\operatorname{tr}\chi - \frac{2}{s}\right)|\overline{\pi}|^2 + s^{-2}|\overline{\pi}|_g^2 + 2s^{-1}\nabla_L\overline{\pi}\cdot\overline{\pi}.$$

We integrate the above equation along the null cone $\mathcal{N}^-(p,\tau)$. By Lemma 7.1, it is easy to see that $\int_{S_t} s^{-1}|\overline{\pi}|^2 \to 0$ as $t \to t(p)$. Thus, by integration by parts we obtain

$$\int_{S_{t(p)-\tau}} s^{-1}|\overline{\pi}|^2 = \int_{\mathcal{N}^-(p,\tau)} \left(s^{-2}|\overline{\pi}|^2 + s^{-1}\left(\operatorname{tr}\chi - \frac{2}{s}\right)|\overline{\pi}|^2 + 2s^{-1}\nabla_L\overline{\pi}\cdot\overline{\pi}\right) na\, d\mu_\gamma\, dt.$$

By Lemma 7.1 and (7.7) in Proposition 7.6 we have

$$\left|\int_{S_{t(p)-\tau}} s^{-1}|\overline{\pi}|^2\right| \lesssim \|r^{-1/2}\overline{\pi}\|^2_{L^2(S_{t(p)-\tau})} \le C.$$

By (BA2), Lemma 7.1, and (7.7),

$$\left|\int_{\mathcal{N}^-(p,\tau)} nas^{-1}\left(\operatorname{tr}\chi - \frac{2}{s}\right)|\overline{\pi}|^2\, d\mu_\gamma\, dt\right| \le C\mathcal{E}_0\tau \le C.$$

By (8.13) we have

$$\left|\int_{\mathcal{N}^-(p,\tau)} s^{-1}\nabla_L\overline{\pi}\cdot\overline{\pi} n a\, d\mu_\gamma\, dt\right| \lesssim \|\nabla_L\overline{\pi}\|_{L^2}\|s^{-1}\overline{\pi}\|_{L^2} \le C\|s^{-1}\overline{\pi}\|_{L^2}.$$

Therefore

$$\|s^{-1}\overline{\pi}\|_{L^2}^2 \le C + C\|s^{-1}\overline{\pi}\|_{L^2}, \tag{8.29}$$

which implies $\|s^{-1}\overline{\pi}\|_{L^2} \le C$. Consequently, in view of Lemma 7.1, (8.10) follows. As a byproduct, we have from (BA2) and Lemma 7.1 that

$$\|\mathrm{tr}\,\chi\overline{\pi}\|_{L^2} \lesssim \|s^{-1}\overline{\pi}\|_{L^2} + \left\|\mathrm{tr}\,\chi - \frac{2}{s}\right\|_{L^\infty}\tau\|s^{-1}\overline{\pi}\|_{L^2} \le C(1+\mathcal{E}_0\tau) \le C. \tag{8.30}$$

Next we will show (8.11) by using equation (8.14). Using $\theta_{AB} = -a\chi_{AB} + k_{AB}$, we have from (7.6) and (8.27) that

$$\|\widehat{\theta}\cdot\not{\pi}\|_{L^2} \lesssim \|\not{\pi}\|_{L^4}(\|k\|_{L^4} + \|\widehat{\chi}\|_{L^4}) \le C(\mathcal{E}_0^{1/4}+1)\tau^{1/2} \le C.$$

Since $\mathrm{tr}\,\theta = -a\,\mathrm{tr}\,\chi + \delta^{AB}k_{AB}$, we have from (7.6) and (8.30) that

$$\|\mathrm{tr}\,\theta\not{\pi}\|_{L_t^2L_x^2} \lesssim \|k\|_{L^4}\|\not{\pi}\|_{L^4} + \|\mathrm{tr}\,\chi\not{\pi}\|_{L^2} \le C.$$

Consequently, in view of (8.13) and (8.14), (8.11) follows immediately.

Using $\underline{\zeta} = \not{\nabla}\log n - \epsilon$ and (8.15), we can derive (8.12) easily from (8.13) and (7.6). □

## 8.3 Estimates for Ricci Coefficients

LEMMA 8.5. *For the Ricci coefficient $\zeta$ and the null lapse $a$ there hold*

$$\|r^{1/2}\zeta\|_{L_\omega^2L_t^\infty} + \|r^{-1}\zeta\|_{L^2} + \|\not{\nabla}_L\zeta\|_{L^2} \le C, \tag{8.31}$$

$$\|r^{1/2}\not{\nabla}\log a\|_{L_\omega^2L_t^\infty} + \|r^{-1}\not{\nabla}\log a\|_{L^2} + \|\not{\nabla}_L\not{\nabla}\log a\|_{L^2} \le C. \tag{8.32}$$

PROOF. From the transport equation (8.3) we have

$$\not{\nabla}_L\zeta + \frac{1}{2}\mathrm{tr}\,\chi\cdot\zeta = -\widehat{\chi}\cdot\zeta + \chi\cdot\underline{\zeta} - \beta. \tag{8.33}$$

Since (BA3) implies $\|\widehat{\chi}\|_{L_\omega^\infty L_t^2} \le \mathcal{E}_0^{1/2}$ with $\mathcal{E}_0\tau \le 1$, it follows from Lemma 8.3 and the relation $\chi = \widehat{\chi} + \frac{1}{2}\mathrm{tr}\,\chi\gamma$ that

$$\|r^{1/2}\zeta\|_{L_\omega^2L_t^\infty} + \|r^{-1}\zeta\|_{L^2} \lesssim \|\beta\|_{L^2} + \|\widehat{\chi}\cdot\underline{\zeta}\|_{L^2} + \|\mathrm{tr}\,\chi\cdot\underline{\zeta}\|_{L^2}$$

From Theorem 4.4 we have $\|\beta\|_{L^2} \le C$. Recall that $\underline{\zeta} = \not{\nabla}\log n - \epsilon$, which is a combination of terms in $\not{\pi}$. By (8.30) we have $\|\mathrm{tr}\,\chi\,\underline{\zeta}\|_{L^2} \le C$. Therefore

$$\|r^{1/2}\zeta\|_{L_\omega^2L_t^\infty} + \|r^{-1}\zeta\|_{L^2} \le C(\mathcal{E}_0\tau+1) + \|\widehat{\chi}\cdot\underline{\zeta}\|_{L^2}.$$

In view of (7.6) in Proposition 7.6, (8.27) in Lemma 8.4, and $\mathcal{E}_0\tau \le 1$, we have

$$\|r^{1/2}\zeta\|_{L_\omega^2L_t^\infty} + \|r^{-1}\zeta\|_{L^2} \le C + \tau^{1/2}\|\widehat{\chi}\|_{L_x^4L_t^\infty}\|\underline{\zeta}\|_{L_t^\infty L_x^4} \le C.$$

Consequently, it follows from (8.33), (BA2), and (BA3) that $\|\nabla\!\!\!/_L \zeta\|_{L^2} \le C$. We thus obtain (8.31).

In order to show (8.32), we use the relation $\zeta = \nabla\!\!\!/ \log a + \epsilon$. By Proposition 8.1,

$$\|r^{1/2}\epsilon\|_{L^2_\omega L^\infty_t} + \|\epsilon\|_{L^2_\omega L^2_t} + \|\nabla\!\!\!/_L \epsilon\|_{L^2} \le C.$$

Thus, the estimates for $\nabla\!\!\!/ \log a$ follow. □

LEMMA 8.6. *For the* $\underline{\mu}$ *defined by* (8.8) *there holds* $\|\underline{\mu}\|_{L^2} \le C$ *on* $\mathcal{N}^-(p,\tau)$.

PROOF. Recall that by (8.4), $\underline{\mu} = 2\,\mathrm{div}\,\underline{\zeta} - \hat{\chi}\cdot\underline{\hat{\chi}} + 2|\underline{\zeta}|^2 + 2\rho$. We have from Theorem 4.4, Proposition 7.6, and Theorem 4.5 that

$$\|\underline{\mu}\|_{L^2} \lesssim \|\nabla\!\!\!/ \underline{\zeta}\|_{L^2} + \|\underline{\zeta}\|^2_{L^4} + \|\rho\|_{L^2} + \|\hat{\chi}\cdot\underline{\hat{\chi}}\|_{L^2} \lesssim C + \|\hat{\chi}\cdot\underline{\hat{\chi}}\|_{L^2}.$$

Recall also that $\underline{\chi}_{AB} = -a^2\chi_{AB} + 2ak_{AB}$; we have from (8.27) and Proposition 7.6 that $\|\underline{\mu}\|_{L^2} \lesssim C + \|\hat{\chi}\|_{L^4}(\|\hat{\chi}\|_{L^4} + \|k\|_{L^4}) \le C$. □

Using $a\chi_{AB} = -\theta_{AB} + k_{AB}$ again, we can summarize the estimates obtained so far in this section as follows:

PROPOSITION 8.7. *There exist universal constants* $\delta_0 > 0$ *and* $C_* > 0$ *such that, under the bootstrap assumptions* (BA1)–(BA3) *with* $\mathcal{E}_0\tau \le 1$, *if* $\tau < \min\{i_*, \delta_0\}$, *then there hold*

$$\|r^{-1/2}\underline{\pi}\|_{L^2(S_{t,u})} \le C, \tag{8.34}$$

$$\|\underline{\pi}\|_{L^4(S_{t,u})} \le C, \tag{8.35}$$

$$\mathcal{N}_1[\pi\!\!\!/](p,\tau) \le C, \tag{8.36}$$

$$\|n^{-1}\nabla^2 n, n^{-2}\nabla\dot{n}\|_{L^2} \le C, \tag{8.37}$$

$$\|r^{1/2}(\hat{\chi}, \overline{\pi}, \zeta, \nabla\!\!\!/ \log a, \widehat{\theta})\|_{L^2_\omega L^\infty_t} \le C, \tag{8.38}$$

$$\|(\hat{\chi}, \overline{\pi}, \zeta, \nabla\!\!\!/ \log a, \widehat{\theta})\|_{L^2_t L^2_\omega} \le C, \tag{8.39}$$

$$\|\nabla\!\!\!/_L(\hat{\chi}, \zeta, \nabla\!\!\!/ \log a, \widehat{\theta})\|_{L^2} \le C, \tag{8.40}$$

*where* $\underline{\pi} = (n^{-1}\partial_t \log n, \overline{\pi})$.

The above estimates provide the intermediate steps toward the proof of Theorem 4.6. The complete proof, however, requires more estimates on $\hat{\chi}$, $\zeta$, and $\underline{\zeta}$ as follows. Since the arguments are rather lengthy, we will report them in [15, 16].

PROPOSITION 8.8. *There exist universal constants* $\delta_0 > 0$ *and* $C_* > 0$ *such that, under the bootstrap assumptions* (BA1)–(BA4) *with* $\mathcal{E}_0\tau \le 1$, *if* $\tau < \min\{i_*, \delta_0\}$, *then there hold*

$$\left\|\mathrm{tr}\,\chi - \frac{2}{s}\right\|_{L^\infty} \le C_*, \tag{8.41}$$

$$\|\hat{\chi}\|_{L^\infty_\omega L^2_t} + \|\zeta\|_{L^\infty_\omega L^2_t} \le C_*, \tag{8.42}$$

$$\|\nu\|_{L^\infty_\omega L^2_t} + \|\underline{\zeta}\|_{L^\infty_\omega L^2_t} \le C_*, \tag{8.43}$$

$$\mathcal{N}_1[\widehat{\chi}, \zeta, \nabla\!\!\!/ \log a, \widehat{\theta}](p,\tau) \le C_*, \tag{8.44}$$

$$\|r^{1/2}(\nabla\!\!\!/ \operatorname{tr}\chi, \mu)\|_{L^2_x L^\infty_t} + \|(\nabla\!\!\!/ \operatorname{tr}\chi, \mu)\|_{L^2} \le C_*, \tag{8.45}$$

*on the null cone $\mathcal{N}^-(p,\tau)$ for all $p \in \mathcal{M}_I$.*

The estimates in Proposition 8.7 and Proposition 8.8 give Theorem 4.6. Thus, we may use a bootstrap argument, as explained in Section 4, to conclude that all the estimates in the above two propositions hold on the null cones $\mathcal{N}^-(p,\tau)$ for all $p \in \mathcal{M}_I$ with $\tau = \min\{i_*, \delta_*\}$ for some universal constant $\delta_* > 0$.

We conclude this section with an application to estimate $\|\underline{\pi}\|_{L^2_u L^2_\omega(\mathrm{Int}(S_{t,u}))}$, where, for any $\Sigma$-tangent tensor $F$,

$$\|F\|^2_{L^2_u L^2_\omega(\mathrm{Int}\, S_{t,u})} = \int_{u_m}^{u} \int_{S_{t,u'}} r'^{-2} |F|^2_g a \, d\mu_\gamma \, du'$$

with $r' = r(t,u')$.

PROPOSITION 8.9. *For $\underline{\pi} = (n^{-1}\partial_t \log n, \overline{\pi})$, there holds $\|\underline{\pi}\|_{L^2_u L^2_\omega(\mathrm{Int}(S_{t,u}))} \le C$.*

PROOF. It is convenient to introduce the new null pair $L' := \mathbf{T} + N$, $\underline{L}' := \mathbf{T} - N$. Let $\chi'$, $\underline{\chi}'$, $\zeta'$, and $\underline{\zeta}'$ denote the Ricci coefficients corresponding to the null frame $(e_A)_{A=1,2}$, $e_3' = \underline{L}'$, $e_4' = L'$. Since $L = -a^{-1}L'$ and $\underline{L} = -a\underline{L}'$, it is easy to see

$$\chi = -a^{-1}\chi', \quad \underline{\chi} = -a\underline{\chi}', \quad \zeta = \zeta', \quad \underline{\zeta} = \underline{\zeta}'.$$

From (8.1), (8.7), (8.9), and (8.5), we can derive

$$\nabla\!\!\!/_N \operatorname{tr}\chi' + \frac{1}{2}(\operatorname{tr}\chi')^2 = -\frac{1}{2}\delta \operatorname{tr}\chi' + 2\lambda \operatorname{tr}\chi' - \widehat{\chi}'(\widehat{\chi}' + \widehat{\eta}) - (\operatorname{div}\zeta + |\zeta|^2 + \rho), \tag{8.46}$$

which, multiplied by $|\underline{\pi}| := |\underline{\pi}|_g$, implies

$$\nabla_N(\operatorname{tr}\chi' |\underline{\pi}|_g{}^2) + \operatorname{tr}\theta(\operatorname{tr}\chi'|\underline{\pi}|^2_g) - \frac{1}{2}|\operatorname{tr}\chi'\underline{\pi}|^2_g = \\ \left\{-\frac{3}{2}\delta \operatorname{tr}\chi' - \widehat{\chi}'(\widehat{\chi}' + \widehat{\eta}) - (\operatorname{div}\zeta + |\zeta|^2 + \rho)\right\}|\underline{\pi}|^2 + 2\operatorname{tr}\chi'\nabla_N\underline{\pi}\cdot\underline{\pi}.$$

In view of Lemma 7.3, integrating the above equation over $\mathrm{Int}(S_{t,u})$ gives

$$\begin{aligned}
&\frac{1}{2}\int_{u_m}^{u}\int_{S_{t,u'}} (\operatorname{tr}\chi')^2|\underline{\pi}|^2 a\, d\mu_\gamma\, du' \\
&\quad = -\int_{S_{t,u}} \operatorname{tr}\chi'|\underline{\pi}|^2 + \int_{u_m}^{u}\int_{S_{t,u'}} (-2\nabla_N\underline{\pi}\cdot\operatorname{tr}\chi'\underline{\pi} + \rho|\underline{\pi}|^2)a\, d\mu_\gamma\, du' \\
&\qquad + \int_{u_m}^{u}\int_{S_{t,u'}} \left(\frac{3}{2}\delta\operatorname{tr}\chi' + |\zeta|^2 + \widehat{\chi}'(\widehat{\chi}' + \widehat{\eta})\right)|\underline{\pi}|^2 a\, d\mu_\gamma\, du' \\
&\qquad + \int_{u_m}^{u}\int_{S_{t,u'}} -\zeta\cdot\nabla\!\!\!/(|\underline{\pi}|^2 a) d\mu_\gamma\, du'.
\end{aligned} \tag{8.47}$$

By (BA2), Lemma 7.1, and (7.7),

$$\left|\int_{S_{t,u}} \operatorname{tr}\chi'|\underline{\pi}|^2\, d\mu_\gamma\right| \lesssim \|r^{-1/2}\underline{\pi}\|^2_{L^2(S_{t,u})} \le C.$$

By Lemma 2.2, Proposition 3.1, and (3.12),

$$\begin{aligned}
\left|\int_{u_m}^{u}\int_{S_{t,u'}} \nabla_N\underline{\pi}\cdot\operatorname{tr}\chi'\underline{\pi} a\, d\mu_\gamma\, du'\right| &\lesssim \|\nabla_N\underline{\pi}\|_{L^2(\Sigma_t)}\,\|\operatorname{tr}\chi'\underline{\pi}\|_{L^2(\Sigma_t)} \\
&\le C\,\|\operatorname{tr}\chi'\underline{\pi}\|_{L^2(\Sigma_t)}
\end{aligned}$$

and

$$\begin{aligned}
\left|\int_{u_m}^{u'}\int_{S_{t,u}} \frac{3}{2}\delta\operatorname{tr}\chi'|\underline{\pi}|^2 a\, d\mu_\gamma\, du'\right| &\lesssim \|k\|_{L^6(\Sigma_t)}\,\|\underline{\pi}\|^2_{L^6(\Sigma_t)} + \|\underline{\pi}\|^3_{L^3(\mathrm{Int}(S_{t,u}))} \\
&\lesssim (\|\nabla k\|_{L^2(\Sigma_t)} + \|\underline{\pi}\|_{H^1(\Sigma_t)})\|\underline{\pi}\|^2_{H^1(\Sigma_t)} \\
&\le C.
\end{aligned}$$

By Lemma 2.1 and (7.6),

$$\left|\int_{u_m}^{u}\int_{S_{t,u'}} \rho|\underline{\pi}|^2 a\, d\mu_\gamma\, du'\right| \lesssim \|\rho\|_{L^2(\Sigma_t)}\|\underline{\pi}\|^2_{L^4(\mathrm{Int}\, S_{t,u})} \le C(u-u_m)^{1/2}.$$

Since $\zeta_A = \nabla\!\!\!/_A \log a + \epsilon_A$, we have

$$\begin{aligned}
&\left|\int_{u_m}^{u}\int_{S_{t,u'}} \zeta\nabla\!\!\!/(a|\underline{\pi}|^2) d\mu_\gamma\, du'\right| \\
&\quad = \left|\int_{u_m}^{u}\int_{S_{t,u'}} (\nabla\!\!\!/\log a|\underline{\pi}|^2\zeta + \nabla\!\!\!/|\underline{\pi}|^2\zeta)a\, d\mu_\gamma\, du'\right| n \le
\end{aligned}$$

$$\lesssim \|\nabla\!\!\!/ \underline{\pi}\|_{L^2(\mathrm{Int}(S_{t,u}))} \sup_{u_m \le u' \le u} (\|\underline{\pi}\|_{L^4(S_{t,u'})}\|\zeta\|_{L^4(S_{t,u'})})(u-u_m)^{1/2}$$
$$+ \int_{u_m}^{u}\int_{S_{t,u'}} (|\zeta|^2|\underline{\pi}|^2 + |\zeta||\underline{\pi}|^3) a \, d\mu_\gamma \, du'.$$

In view of Lemma 2.2 and Propositions 3.1 and (3.12), we derive

$$\|\nabla\!\!\!/ \underline{\pi}\|_{L^2(\mathrm{Int}(S_{t,u}))} \le \|\nabla \underline{\pi}\|_{L^2(\Sigma_t)} \le C,$$

while in view of (8.44), (8.16), and (7.6) we have

$$\sup_{u_m \le u' \le u} \|\zeta\|_{L^4(S_{t,u'})} \le C, \qquad \sup_{u_m \le u' \le u} \|\underline{\pi}\|_{L^4(S_{t,u'})} \le C.$$

Consequently,

$$\int_{u_m}^{u}\int_{S_{t,u'}} (|\zeta|^2|\underline{\pi}|^2 + |\zeta||\underline{\pi}|^3) a \, d\mu_\gamma \, du'$$
$$\lesssim \sup_{u_m \le u' \le u} \big(\|\zeta\|^2_{L^4(S_{t,u'})}\|\underline{\pi}\|^2_{L^4(S_{t,u'})}\big)(u-u_m)$$
$$+ \sup_{u_m \le u' \le u} \big(\|\zeta\|_{L^4(S_{t,u'})}\|\underline{\pi}\|^3_{L^4(S_{t,u'})}\big)(u-u_m)$$
$$\le C(u-u_m).$$

Therefore, we obtain

$$\left|\int_{u_m}^{u}\int_{S_{t,u'}} \zeta \nabla\!\!\!/ (a|\underline{\pi}|_g^2) d\mu_\gamma \, du'\right| \le C(1+(u-u_m)^{1/2})(u-u_m)^{1/2}.$$

In view of (8.44), (8.16), and (7.6), by a similar argument we obtain

$$\left|\int_{u_m}^{u}\int_{S_{t,u'}} (|\zeta|^2 + \hat{\chi}'(\hat{\chi}' + \hat{\eta}))|\underline{\pi}|^2 a \, d\mu_\gamma \, du'\right| \lesssim$$
$$\int_{u_m}^{u}\int_{S_{t,u'}} (|\underline{\pi}|^2(|\hat{\chi}|^2 + |\zeta|^2) + |\hat{\chi}| \cdot |\underline{\pi}|^3) d\mu_\gamma \, du' \le C(u-u_m).$$

Combining all the above estimates with (8.47), using $\chi' = -a\chi$ and (BA1), and noting $u - u_m \lesssim \tau \lesssim 1$ yields

$$\|\mathrm{tr}\, \chi \underline{\pi}\|^2_{L^2(\mathrm{Int}(S_{t,u}))} \le C + C\|\mathrm{tr}\, \chi \underline{\pi}\|_{L^2(\mathrm{Int}(S_{t,u}))},$$

which implies $\|\mathrm{tr}\chi\underline{\pi}\|_{L^2(\mathrm{Int}(S_{t,u}))} \le C$. This together with (BA2) implies the desired inequality. □

## 9 Proof of Theorem 4.7

In this section we will complete the proof of Theorem 4.7. For any $p \in \mathcal{M}_I$, let $\Phi(t)$ be the integral curve of $\mathbf{T}$ through $p$ with $\Phi(t(p)) = p$. For each $p_t := \Phi(t)$, we will represent $k(p_t)$ in terms of a Kirchhoff-Sobolev formula over a past null cone with vertex $p_t$. We then use the estimates established in the previous sections to obtain $\int_{t(p)-\tau}^{t(p)} |k(\Phi(t))|^2 n\, dt \le C$ for some universal constant $C$.

### 9.1 Derivation of the Kirchhoff Parametrix

We first revisit the formulation of the Kirchhoff parametrix in [10]. We define $\mathbf{A}$ to be a $\Sigma_t$-tangent 2-tensor satisfying

$$(9.1)\quad (\mathbf{D}_L \mathbf{A})_{ij} + \frac{1}{2} \operatorname{tr} \chi \mathbf{A}_{ij} = 0 \ \text{ on } \mathcal{N}^-(p,\tau), \qquad \lim_{t \to t(p)^-} (t(p) - t)\mathbf{A}_{ij} = J_{ij},$$

where $J \in T_p \Sigma_{t(p)}$ and $|J|_g = 1$. This $\mathbf{A}$ is similar to the one defined in [12] but with the modification that $\mathbf{A}$ is $\Sigma_t$-tangent. Since we have obtained in Propositions 8.7 and 8.8 the estimates on

$$\left\| \operatorname{tr} \chi - \frac{2}{s} \right\|_{L^\infty}, \quad \|\nabla\!\!\!/ \operatorname{tr} \chi\|_{L^2}, \quad \|r^{1/2} \nabla\!\!\!/ \operatorname{tr} \chi\|_{L_x^2 L_t^\infty},$$
$$\|r^{-1}(\zeta + \underline{\zeta})\|_{L^2}, \quad \|\hat{\chi}, \nu, \underline{\zeta}\|_{L_\omega^\infty L_t^2}, \quad \mathcal{R}(p,\tau),$$

on the null cone $\mathcal{N}^-(p,\tau)$, we may adapt the proof in [12] to obtain the following estimates on $\mathbf{A}$.

PROPOSITION 9.1. *For the tensor* $\mathbf{A}$ *defined by* (9.1) *there hold*

$$(9.2)\quad \|\nabla\!\!\!/ \mathbf{A}\|_{L^2(\mathcal{N}^-(p,\tau))} + \|r^{1/2} \nabla\!\!\!/ \mathbf{A}\|_{L_x^2 L_t^\infty(\mathcal{N}^-(p,\tau))} + \|r\mathbf{A}\|_{L^\infty(\mathcal{N}^-(p,\tau))} \le C,$$

*where* $C$ *is a universal constant.*

Now we derive the Kirchhoff-Sobolev formula for any $\Sigma_t$-tangent 2-tensor $\Psi_I$, $I = \{i, j\}$; see [10, 13]. According to the definition of $\Box \Psi_I$, we have under the null frame $(e_A)_{A=1,2}$, $e_3 = \underline{L}$, $e_4 = L$, that

$$\Box \Psi_I = -\frac{1}{2} \mathbf{D}_{43} \Psi_I - \frac{1}{2} \mathbf{D}_{34} \Psi_I + \delta^{AB} \mathbf{D}_{AB} \Psi_I.$$

Recall that $\mathbf{D}_L L = 0$, $\mathbf{D}_L \underline{L} = 2\underline{\zeta}_A e_A$, and $\mathbf{D}_B e_A = \nabla\!\!\!/_B e_A + \frac{1}{2} \chi_{AB} e_3 + \frac{1}{2} \underline{\chi}_{AB} e_4$. We can obtain

$$\mathbf{D}_{43} \Psi_I = \mathbf{D}_4 (\mathbf{D}_3 \Psi)_I - 2\underline{\zeta}^A \mathbf{D}_A \Psi_I,$$
$$\delta^{AB} \mathbf{D}_{AB} \Psi_I = \delta^{AB} \nabla\!\!\!/_A \nabla\!\!\!/_B \Psi_I - \frac{1}{2} \operatorname{tr} \underline{\chi} \mathbf{D}_4 \Psi_I - \frac{1}{2} \operatorname{tr} \chi \mathbf{D}_3 \Psi_I.$$

Noting the commutation formula $\mathbf{D}_{34}\Psi_I - \mathbf{D}_{43}\Psi_I = \mathbf{R}_i{}^\alpha{}_{34}\Psi_{\alpha j} + \mathbf{R}_j{}^\alpha{}_{34}\Psi_{i\alpha}$, we obtain

$$\Box\Psi_I = -\mathbf{D}_4(\mathbf{D}_3\Psi)_I + 2\underline{\zeta}^A\mathbf{D}_A\Psi_I - \frac{1}{2}\operatorname{tr}\underline{\chi}\mathbf{D}_4\Psi_I - \frac{1}{2}\operatorname{tr}\chi\mathbf{D}_3\Psi_I$$
$$+ \delta^{AB}\nabla\!\!\!/_A\nabla\!\!\!/_B\Psi_I - \frac{1}{2}\mathbf{R}_i{}^\alpha{}_{34}\Psi_{\alpha j} - \frac{1}{2}\mathbf{R}_j{}^\alpha{}_{34}\Psi_{i\alpha}.$$

We multiply the above equation by $\mathbf{A}_I$ and integrate over $\mathcal{N}^-(p,\tau)$ to obtain

$$\begin{aligned}\int_{\mathcal{N}^-(p,\tau)} \Box\Psi_I\mathbf{A}^I = {}& \Xi_1 + \Xi_2 \\ & + \int_{\mathcal{N}^-(p,\tau)} (2\underline{\zeta}^A\mathbf{D}_A\Psi_I\cdot\mathbf{A}^I + \delta^{AB}\nabla\!\!\!/_A\nabla\!\!\!/_B\Psi_I\cdot\mathbf{A}^I) \\ & - \frac{1}{2}\int_{\mathcal{N}^-(p,\tau)} (\mathbf{R}_i{}^\alpha{}_{34}\Psi_{\alpha j} + \mathbf{R}_j{}^\alpha{}_{34}\Psi_{i\alpha})\mathbf{A}^{ij}.\end{aligned} \tag{9.3}$$

where

$$\Xi_1 = \int_{\mathcal{N}^-(p,\tau)} \left(-\mathbf{D}_4(\mathbf{D}_3\Psi)_I\cdot\mathbf{A}^I - \frac{1}{2}\operatorname{tr}\chi\mathbf{D}_3\Psi_I\cdot\mathbf{A}^I\right),$$
$$\Xi_2 = -\frac{1}{2}\int_{\mathcal{N}^-(p,\tau)} \operatorname{tr}\underline{\chi}\mathbf{D}_4\Psi_I\cdot\mathbf{A}^I.$$

For $\Xi_1$, integration by parts gives

$$\begin{aligned}\Xi_1 = {}& -\int_{S_{t(p)-\tau}} \mathbf{D}_3\Psi_I\cdot\mathbf{A}^I + \lim_{t\to t(p)}\int_{S_t}\mathbf{D}_3\Psi_I\cdot\mathbf{A}^I \\ & + \int_{\mathcal{N}^-(p,\tau)}\left(\mathbf{D}_4\mathbf{A}^I + \frac{1}{2}\operatorname{tr}\chi\mathbf{A}^I\right)\cdot\mathbf{D}_3\Psi_I.\end{aligned}$$

Since $\lim_{t\to t(p)}(t(p)-t)^2\mathbf{A} = 0$, we have in view of (9.1) that

$$\Xi_1 = -\int_{S_{t(p)-\tau}} \mathbf{D}_3\Psi_I\cdot\mathbf{A}^I + \int_{\mathcal{N}^-(p,\tau)}\Omega_1(\Psi),$$

where $\Omega_1(\Psi) = \mathbf{D}_4\mathbf{A}^{0i}\cdot\mathbf{D}_3\Psi_{0i} + \mathbf{D}_4\mathbf{A}^{i0}\cdot\mathbf{D}_3\Psi_{i0}$.

For $\Xi_2$, in view of (9.1) and the fact that $\Psi$ is $\Sigma_t$-tangent, we first have

$$\operatorname{tr}\underline{\chi}\mathbf{D}_4\Psi_I\cdot\mathbf{A}^I = \mathbf{D}_4(\Psi_I\cdot\mathbf{A}^I\operatorname{tr}\underline{\chi}) + \frac{1}{2}\operatorname{tr}\chi\operatorname{tr}\underline{\chi}\mathbf{A}^I\cdot\Psi_I - \mathbf{D}_4\operatorname{tr}\underline{\chi}\cdot\Psi_I\cdot\mathbf{A}^I;$$

thus integration by parts yields

$$\Xi_2 = \int_{\mathcal{N}^-(p,\tau)} \frac{1}{2}\underline{\mu}\mathbf{A}^I \cdot \Psi_I - \frac{1}{2}\Big(\int_{S_{t(p)-\tau}} \Psi_I \cdot \mathbf{A}^I \operatorname{tr}\underline{\chi} - \lim_{t\to t(p)} \int_{S_t} \Psi_I \cdot \mathbf{A}^I \operatorname{tr}\underline{\chi}\Big),$$

where $\underline{\mu}$ is defined in (8.8).

In view of $\operatorname{tr}\underline{\chi} = -a^2 \operatorname{tr}\chi + 2a\delta^{AB}k_{AB}$ and $a(p) = 1$, we have

$$\lim_{t\to t(p)} \frac{1}{2}\int_{S_t} \Psi_I \cdot \mathbf{A}^I \operatorname{tr}\underline{\chi} = -4\pi n(p)\langle \Psi, J\rangle.$$

Hence

$$\Xi_2 = \int_{\mathcal{N}^-(p,\tau)} \frac{1}{2}\underline{\mu}\mathbf{A}^I \cdot \Psi_I - \frac{1}{2}\int_{S_{t(p)-\tau}} \Psi_I \cdot \mathbf{A}^I \operatorname{tr}\underline{\chi} - 4\pi n(p)\langle \Psi, J\rangle.$$

Therefore we derive

$$\begin{aligned} 4\pi n(p)\langle \Psi, J\rangle = &\int_{\mathcal{N}^-(p,\tau)} \Big(-\Box \Psi_I \cdot \mathbf{A}^I + \frac{1}{2}\underline{\mu}\Psi_I \cdot \mathbf{A}^I + \Omega_1(\Psi)\Big) \\ &- \int_{S_{t(p)-\tau}} \Big(\mathbf{D}_3\Psi_I \cdot \mathbf{A}^I + \frac{1}{2}\operatorname{tr}\underline{\chi}\Psi_I \cdot \mathbf{A}^I\Big) \\ &+ \int_{\mathcal{N}^-(p,\tau)} (2\underline{\zeta}^B \mathbf{D}_B \Psi_I \cdot \mathbf{A}^I - \nabla\!\!\!/_B \Psi_I \cdot \nabla\!\!\!/^B \mathbf{A}^I) \\ &- \frac{1}{2}\int_{\mathcal{N}^-(p,\tau)} (\mathbf{R}_i{}^{\alpha}{}_{34}\Psi_{\alpha j} + \mathbf{R}_j{}^{\alpha}{}_{34}\Psi_{i\alpha})\mathbf{A}^{ij}. \end{aligned} \tag{9.4}$$

We apply (9.4) to the tensor field $\Psi = k$ and obtain the following:

THEOREM 9.2. *Let $p \in \mathcal{M}_I$, let $\Phi(t)$ be the integral curve of $\mathbf{T}$ through $p$ with $\Phi(t(p)) = p$, and let $p_t = \Phi(t)$. Let $\mathbf{A}$ be a $\Sigma_t$-tangent 2-tensor on $\mathcal{J}^-(p,\tau)$ satisfying* (9.1) *on each null cone $C_u := \mathcal{N}^-(p_t, t - t(p) + \tau)$, where $u = u(t) = \int_{t_0}^t n|_\Phi dt$ for $t_m := t(p) - \tau \le t \le t(p)$. Then there holds*

$$\begin{aligned} 4\pi n(p_t)\langle k(p_t), J\rangle = &\, I(p_t) + J(p_t) + K(p_t) + L(p_t) \\ &+ \mathfrak{E}(p_t) + \int_{C_u} \Omega_1(k), \end{aligned} \tag{9.5}$$

*where* $\Omega_1(k) = \mathbf{D}_4\mathbf{A}^{0i}\cdot\mathbf{D}_3 k_{0i} + \mathbf{D}_4\mathbf{A}^{i0}\cdot\mathbf{D}_3 k_{i0}$ *and*

$$I(p_t) = -\int_{C_u} \mathbf{A}\cdot\Box k, \qquad J(p_t) = -\frac{1}{2}\int_{C_u} \mathbf{A}\cdot\mathbf{R}(\cdot,\cdot,\underline{L},L)\cdot k,$$

$$K(p_t) = \frac{1}{2}\int_{C_u} \underline{\mu}\mathbf{A}\cdot k, \qquad L(p_t) = \int_{C_u} (-\nabla\!\!\!/^B\mathbf{A}\cdot\nabla\!\!\!/_B k + 2\underline{\zeta}^B\cdot\nabla\!\!\!/_B k\cdot\mathbf{A}),$$

$$\mathfrak{E}(p_t) = -\int_{S_{t_m,u}} \left(\mathbf{D}_3 k\cdot\mathbf{A} + \frac{1}{2}\operatorname{tr}\underline{\chi} k\cdot\mathbf{A}\right).$$

## 9.2 Main Estimates

In the following we will use the representation formula given in Theorem 9.2 to show that

$$\int_{t(p)-\tau}^{t(p)} |k(p_t)|^2 n\, dt \le C$$

for some universal constant $C$. We proceed as follows.

• *Estimate on* $I(p_t)$. We use the expression of $\Box k$ given in Proposition 5.1, which symbolically can be written as

$$\begin{aligned}\Box k &= -n^{-3}\dot{n}\nabla^2 n + n^{-2}\nabla^2\dot{n} + \pi\cdot\pi\cdot\pi + k\cdot\nabla^2 n\\ &\quad + k\cdot\text{Ric} + \pi\cdot\nabla k - n^{-1}k.\end{aligned}$$

It then follows from Proposition 9.1 that

$$\begin{aligned}|I(p_t)| &\lesssim \int_{C_u} r^{-1}\big(|\dot{n}\nabla^2 n| + |\nabla^2\dot{n}| + |\pi|^3 + |k||\nabla^2 n| + |k||\text{Ric}|\\ &\qquad + |\pi||\nabla k| + |k|\big)\\ &\lesssim \|\nabla^2 n\|_{L^2(C_u)}\,\|r^{-1}\dot{n}\|_{L^2(C_u)} + \|r^{-1}\nabla^2\dot{n}\|_{L^1(C_u)} + \int_{C_u} r^{-1}|\pi|^3\\ &\quad + \|r^{-1}k\|_{L^2(C_u)}\,\|\nabla^2 n\|_{L^2(C_u)} + \|\text{Ric}\|_{L^2(C_u)}\,\|r^{-1}k\|_{L^2(C_u)}\\ &\quad + \|r^{-1}\pi\|_{L^2(C_u)}\,\|\nabla k\|_{L^2(C_u)} + \|r^{-1}k\|_{L^1(C_u)}.\end{aligned}$$

Therefore, with the help of Proposition 7.6 and Proposition 8.1, we have

$$\begin{aligned}|I(p_t)| &\lesssim \|r^{-1}\dot{n}\|_{L^2(C_u)} + \|r^{-1}\nabla^2\dot{n}\|_{L^1(C_u)} + \int_{C_u} r^{-1}|\pi|^3\\ &\quad + \|\text{Ric}\|_{L^2(C_u)} + \|\nabla k\|_{L^2(C_u)} + C.\end{aligned}$$

Now we consider $\int_{t_m}^{t(p)} |I(p_t)|^2\, dt$. Using $\frac{du}{dt} = n$ and Proposition 8.9 we have

$$\begin{aligned}\int_{t_m}^{t(p)} \|r^{-1}\dot{n}\|^2_{L^2(C_{u(t)})} n\, dt &= \int_{u(t_m)}^{u(t(p))} \|r^{-1}\dot{n}\|^2_{L^2(C_u)}\, du\\ &= \int_{u(t_m)}^{u(t(p))} \int_{t_m}^{t_M(u)} \int\limits_{S_{t',u}} r^{-2}|\dot{n}|^2 na\, d\mu_\gamma\, dt'\, du\\ &= \int_{t_m}^{t(p)} \int_{u(t')}^{u(t(p))} \int\limits_{S_{t',u}} r^{-2}|\dot{n}|^2 na\, d\mu_\gamma\, du\, dt'\\ &\lesssim \int_{t_m}^{t(p)} \|r^{-1}\dot{n}\|^2_{L^2(Int(S_{t',u(t(p))}))}\, dt' \le C\tau.\end{aligned}$$

By a similar argument, we have from Lemma 2.2 that

$$\int_{t_m}^{t(p)} \big(\|\mathrm{Ric}\|^2_{L^2(C_u)} + \|\nabla k\|^2_{L^2(C_u)}\big) n\, dt \le C\tau.$$

Therefore

$$\begin{aligned}&\int_{t_m}^{t(p)} |I(p_t)|^2 n\, dt \lesssim\\ &\qquad C\tau + \int_{t_m}^{t(p)} \|r^{-1}\nabla^2\dot{n}\|^2_{L^1(C_u)} n\, dt + \int_{t_m}^{t(p)} \Big(\int\limits_{C_u} r^{-1}|\pi|^3\Big)^2 n\, dt.\end{aligned}$$

By using the Minkowski inequality and Proposition 3.7 we have

$$\begin{aligned}&\left(\int_{t_m}^{t(p)} \|r^{-1}\nabla^2\dot{n}\|^2_{L^1(C_u)} n\, dt\right)^{1/2}\\ &= \left(\int_{u(t_m)}^{u(t(p))} \left(\int_{t_m}^{t_M(u)} r^{-1}\|an\nabla^2\dot{n}\|_{L^1(S_{t',u})} dt'\right)^2 du\right)^{1/2}\\ &\le \int_{t_m}^{t(p)} \left(\int_{u(t')}^{u(t(p))} r^{-2}\|an\nabla^2\dot{n}\|^2_{L^1(S_{t',u})}\, du\right)^{1/2} dt'\\ &\lesssim \int_{t_m}^{t(p)} \|\nabla^2\dot{n}\|_{L^2(\mathrm{Int}(S_{t',u(t(p))}))} dt' \le C.\end{aligned}$$

Finally, we have from Proposition 7.6 and (8.10) that

$$\int\limits_{C_u} r^{-1}|\pi|^3 \lesssim \int_{t_m}^{t_M(u)} \|r^{-1}\pi\|_{L^2(S_{t',u})}\, \|\pi\|^2_{L^4(S_{t',u})} dt' \le C(t_M(u) - t_m)^{1/2}.$$

Thus, by Lemma 7.1 we obtain

$$\int_{t_m}^{t(p)} \left( \int_{C_u} r^{-1} |\pi|^3 \right)^2 n \, dt \le C\tau^2.$$

Combining the above estimates we therefore obtain

$$\int_{t_m}^{t(p)} |I(p_t)|^2 n \, dt \le C + C\tau^2 \lesssim C.$$

• *Estimate on* $J(p_t)$. It follows from Proposition 9.1, Theorem 4.4, and Proposition 8.1 that

$$|J(p_t)| \lesssim \|r\mathbf{A}\|_{L^\infty(C_u)} \|r^{-1}k\|_{L^2(C_u)} \mathcal{R}(p_t, \tau + t - t(p)) \le C.$$

Thus

$$\int_{t_m}^{t(p)} |J(p_t)|^2 n \, dt \le C(t(p) - t_m) \le C\tau \le C.$$

• *Estimate on* $K(p_t)$. It follows from Proposition 9.1 and Proposition 8.1 that

$$|K(p_t)| \le \|r\mathbf{A}\|_{L^\infty(C_u)} \|r^{-1}\overline{\pi}\|_{L^2(C_u)} \|\underline{\mu}\|_{L^2(C_u)} \lesssim \|\underline{\mu}\|_{L^2(C_u)}.$$

From Lemma 8.6 we then obtain $|K(p_t)| \le C$. Therefore

$$\int_{t_m}^{t(p)} |K(p_t)|^2 n \, dt \le C(t(p) - t_m) \le C\tau \le C.$$

• *Estimate on* $L(p_t)$. It follows from the Hölder inequality that

$$|L(p_t)| \lesssim \|\nabla\!\!\!/ \mathbf{A}\|_{L^2(C_u)} \|\nabla\!\!\!/ k\|_{L^2(C_u)} + \|r\mathbf{A}\|_{L^\infty(C_u)} \|r^{-1}\underline{\zeta}\|_{L^2(C_u)} \|\nabla\!\!\!/ k\|_{L^2(C_u)}.$$

Therefore, we obtain from Proposition 9.1, Theorem 4.4, and Proposition 8.1 that $|L(p_t)| \le C$, which gives

$$\int_{t_m}^{t(p)} |L(p_t)|^2 n \, dt \le C(t(p) - t_m) \le C\tau \le C.$$

• *Estimate on* $\mathfrak{E}(p_t)$. We first have from Proposition 9.1 that

$$|\mathfrak{E}(p_t)| \lesssim r^{-1} \|\mathbf{D}_3 k\|_{L^1(S_{t_m,u})} + r^{-1} \|\mathrm{tr}\, \underline{\chi} k\|_{L^1(S_{t_m,u})}.$$

Using the definition of $r$ we then obtain

$$|\mathfrak{E}(p_t)| \lesssim \|\mathbf{D}_3 k\|_{L^2(S_{t_m,u})} + r^{-1} \|\mathrm{tr}\, \underline{\chi} k\|_{L^1(S_{t_m,u})}.$$

Since $\operatorname{tr}\underline{\chi} = -a^2 \operatorname{tr}\chi + 2a\delta^{AB}k_{AB}$, we have, with the help of (BA1) and (BA2), that

$$\begin{aligned}\|\operatorname{tr}\underline{\chi}k\|_{L^1(S_{t_m,u})} &\lesssim \left\|\operatorname{tr}\chi - \frac{2}{s}\right\|_{L^\infty(C_u)} \|k\|_{L^1(S_{t_m,u})} + r^{-1}\|k\|_{L^1(S_{t_m,u})} \\ &\quad + \|k\|^2_{L^2(S_{t_m,u})} \\ &\lesssim r^{-1}\|k\|_{L^1(S_{t_m,u})} + \|k\|^2_{L^2(S_{t_m,u})} \\ &\lesssim \|k\|_{L^2(S_{t_m,u})} + r\|k\|^2_{L^4(S_{t_m,u})}.\end{aligned}$$

Consequently,

$$|\mathfrak{E}(p_t)| \lesssim \|\mathbf{D}_3 k\|_{L^2(S_{t_m,u})} + r^{-1}\|k\|_{L^2(S_{t_m,u})} + \|k\|^2_{L^4(S_{t_m,u})}.$$

Therefore, using $\frac{du}{dt} = n$, we have

$$\begin{aligned}\int_{t_m}^{t(p)} |\mathfrak{E}(p_t)|^2\, dt &\lesssim \int_{u(t_m)}^{u(t(p))} |\mathfrak{E}(p_t)|^2\, du \\ &\lesssim \|\mathbf{D}_3 k\|^2_{L^2(\Sigma_{t_m})} + \|r^{-1}k\|^2_{L^2(\operatorname{Int}(S_{t_m,u}))} + \|k\|^4_{L^4(\Sigma_{t_m})}\end{aligned}$$

It follows from Lemma 2.2 and Proposition 8.9 that

$$\int_{t_m}^{t(p)} |\mathfrak{E}(p_t)|^2\, dt \lesssim \|\mathbf{D}_3 k\|^2_{L^2(\Sigma_{t_m})} + C.$$

Recall that $\underline{L} = -a(\mathbf{T} - N)$. So $\mathbf{D}_3 k = -a(\mathbf{D}_0 k - \nabla_N k)$. Recall also that $\mathbf{D}_0 k = -n^{-1}\nabla^2 n + \operatorname{Ric} + k \operatorname{Tr} k$. Thus

$$\begin{aligned}\|\mathbf{D}_3 k\|_{L^2(\Sigma_{t_m})} &\lesssim \|\nabla^2 n\|_{L^2(\Sigma_{t_m})} + \|\operatorname{Ric}\|_{L^2(\Sigma_{t_m})} \\ &\quad + \|k\|^2_{L^4(\Sigma_{t_m})} + \|\nabla k\|_{L^2(\Sigma_{t_m})}.\end{aligned}$$

It follows from Lemma 2.2 and Proposition 3.1 that $\|\mathbf{D}_3 k\|_{L^2(\Sigma_{t_m})} \le C$. Therefore

$$\int_{t_m}^{t(p)} |\mathfrak{E}(p_t)|^2 n\, dt \le C.$$

• *Estimate on* $\int_{C_u} \Omega_1(k)$. By straightforward calculation we have $\Omega_1(k) = \mathbf{A}\cdot\overline{\pi}\cdot\overline{\pi}\cdot\overline{\pi}$. It follows from Proposition 9.1 that

$$|\Omega_1(k)| \lesssim \int_{C_u} r^{-1}|\overline{\pi}|^3.$$

Therefore, one can use the similar argument in the estimate of $I(p_t)$ to get

$$\int_{t_m}^{t(p)} |\Omega_1(k)|^2 n\, dt \lesssim \int_{t_m}^{t(p)} \left|\int_{C_u} r^{-1}|\overline{\pi}|^3\right|^2 n\, dt \le C\tau^2 \le C.$$

## 10 Proof of Main Theorem I

In this section, based on Theorem 1.2, we will follow the idea in [12] to give the proof of Theorem 1.1. According to the local existence theorem given in [12, prop. 6.1] (see also [5, theorem 10.2.1]), it suffices to show that the quantity

$$\mathcal{R}_* := \|\mathrm{Ric}\|_{H^2(\Sigma_t)} + \|k\|_{H^3(\Sigma_t)} \tag{10.1}$$

on each slice $\Sigma_t$ with $t_0 \le t < t_*$ is uniformly bounded.

Since $(\mathbf{M}, \mathbf{g})$ is a vacuum space-time, by virtue of the Bianchi identity, $\mathbf{R}$ satisfies a wave equation of the form $\Box \mathbf{R} = \mathbf{R} \star \mathbf{R}$. Based on higher-energy estimates it is standard to show that

$$\|\mathbf{DR}(t)\|_{L^2}^2 \lesssim \|\mathbf{DR}(t_1)\|_{L^2}^2 + \int_{t_1}^{t} \|\mathbf{R}(t')\|_{L^\infty}^2 \, dt' \tag{10.2}$$

and

$$\|\mathbf{D}^2\mathbf{R}(t)\|_{L^2}^2 \lesssim \|\mathbf{D}^2\mathbf{R}(t_1)\|_{L^2}^2 + \int_{t_1}^{t} \|\mathbf{DR}(t')\|_{L^2}^2 \|\mathbf{R}(t')\|_{L^\infty}^2 \, dt' \tag{10.3}$$

for all $t_0 \le t_1 \le t < t_*$. The derivation has been given in [12] under assumption (1.9); the argument, however, depends only on condition (A1).

Thus, the derivation of the $L^\infty$-bound of $\mathbf{R}$ is a crucial step. As in [10] one can represent $\mathbf{R}(p)$, for each $p \in \mathcal{M}_*$, by a Kirchhoff-Sobolev formula over the null cone $\mathcal{N}^-(p,\tau)$, where $\tau > 0$ is a universal constant such that $i_*(p,t) \ge \tau$ whose existence is guaranteed by Theorem 1.2. One can then follow the delicate argument in [12] to derive that

$$\begin{aligned} \|\mathbf{R}(t)\|_{L^\infty} &\lesssim \\ &\tau^{-1} \sup_{t' \in [t-2\tau, t-\tau/2]} \left(\|\mathbf{R}(t')\|_{L^2} + \|\mathbf{DR}(t')\|_{L^2} + \|\mathbf{D}^2\mathbf{R}(t')\|_{L^2}\right). \end{aligned} \tag{10.4}$$

The derivation of (10.4) requires the estimates on

$$\mathcal{R}(p,\tau), \quad \left\|\mathrm{tr}\,\chi - \frac{2}{s}\right\|_{L^\infty(\mathcal{N}^-(p,\tau))}, \quad \|\hat{\chi}, \nu, \zeta, \underline{\zeta}\|_{L^\infty_\omega L^2_t(\mathcal{N}^-(p,\tau))},$$
$$\|\mu, \nabla\!\!\!/\, \mathrm{tr}\,\chi\|_{L^2(\mathcal{N}^-(p,\tau))}, \quad \|r^{1/2} \nabla\!\!\!/\, \mathrm{tr}\,\chi\|_{L^2_x L^\infty_t(\mathcal{N}^-(p,\tau))},$$
$$\|r^{-1}(\zeta + \underline{\zeta})\|_{L^2(\mathcal{N}^-(p,\tau))},$$

which are provided by Proposition 8.7 and Proposition 8.8 under condition (A1). Combining estimates (10.2)–(10.4) gives

$$\|\mathbf{R}(t)\|_{H^2} \lesssim \tau^{-1} \sup_{t' \in [t-\tau, t-\tau/2]} \|\mathbf{R}(t')\|_{H^2}.$$

Iterating this estimate as many times as needed, in steps of size $\tau/2$, yields

$$\sup_{t \in [t_0, t_*)} \|\mathbf{R}(t)\|_{H^2} \le C, \tag{10.5}$$

where $C$ is a positive constant depending only on $Q_0$, $\mathcal{K}_0$, $|\Sigma_0|$, $t_*$, $I_0$, and the initial data $\|\mathbf{R}(t_0)\|_{H^2}$.

Now we are ready to show that the quantity $\mathcal{R}_*$ defined by (10.1) is uniformly bounded for all $t_0 \le t < t_*$. Although the argument is standard, we include it here for completeness.

In view of the well-known equations

$$\nabla_i k_{jm} - \nabla_j k_{im} = \epsilon_{ij}{}^l H_{lm}, \tag{10.6}$$

$$R_{ij} - k_{ia}k^{aj} + \operatorname{Tr} k k_{ij} = E_{ij}. \tag{10.7}$$

We derive from Lemma 2.1 and Lemma 2.2 that

$$\|\mathrm{Ric}\|_{L^2} + \|k\|_{H^1} + \|E\|_{L^2} + \|H\|_{L^2} \le C, \tag{10.8}$$

where here and below all norms are taken over a fixed slice $\Sigma_t$, which is suppressed.

In order to obtain the derivative estimates, by straightforward calculation we have symbolically

$$\nabla_m E_{ij} = \mathbf{D}_m \mathbf{R}_{0i0j} - k \cdot H, \tag{10.9}$$

$$\nabla_m H_{ij} = \mathbf{D}_m{}^\star \mathbf{R}_{0i0j} - k \cdot E, \tag{10.10}$$

$$\nabla^2_{mn} E_{ij} = \mathbf{D}^2_{mn} \mathbf{R}_{0i0j} - k_{mn} \mathbf{D}_0 \mathbf{R}_{0i0j} - \nabla(k \cdot H), \tag{10.11}$$

$$\nabla^2_{mn} H_{ij} = \mathbf{D}^2_{mn}{}^\star \mathbf{R}_{0i0j} - k_{mn} \mathbf{D}_0{}^\star \mathbf{R}_{0i0j} - \nabla(k \cdot E). \tag{10.12}$$

From (10.9) and (10.10) it follows that

$$\|\nabla E\|_{L^2} + \|\nabla H\|_{L^2} \le \|\mathbf{DR}\|_{L^2} + \|k\|_{L^6} \|H\|_{L^3} + \|k\|_{L^6} \|E\|_{L^3}.$$

Applying Lemma 2.5 to $\|E\|_{L^3}$ and $\|H\|_{L^3}$, and using (10.5), (10.8), and Young's inequality, we obtain

$$\|\nabla E\|_{L^2} + \|\nabla H\|_{L^2} \le C. \tag{10.13}$$

Next we estimate $\|\nabla^2 k\|_{L^2}$. From $\operatorname{div} k = 0$ and (10.6) it follows $\triangle k = \mathrm{Ric} \cdot k + \nabla H$. Differentiating it, commuting $\nabla$ with $\Delta$, and using (10.7) yields

$$\triangle \nabla k = k \cdot k \cdot \nabla k + E \cdot \nabla k + \nabla E \cdot k + \nabla^2 H. \tag{10.14}$$

Multiplying (10.14) by $\nabla k$, integrating over $\Sigma_t$, and using the Hölder inequality gives

$$\begin{aligned} \|\nabla^2 k\|^2_{L^2} \lesssim{}& \|k\|^2_{L^6} \|\nabla k\|^2_{L^3} + \|E\|_{L^6} \|\nabla k\|^2_{L^{12/5}} \\ &+ \|\nabla E\|_{L^2} \|\nabla k\|_{L^3} \|k\|_{L^6} + \|\nabla H\|_{L^2} \|\nabla^2 k\|_{L^2}. \end{aligned}$$

By virtue of Lemma 2.5, (10.8), and (10.13), we have $\|\nabla^2 k\|^2_{L^2} \lesssim 1 + \|\nabla^2 k\|_{L^2}$, which implies $\|\nabla^2 k\|_{L^2} \le C$. By the Sobolev embedding we obtain

$$\|k\|_{L^\infty} + \|k\|_{H^2} \le C. \tag{10.15}$$

Using (10.15) and (10.5), it follows easily from (10.7), (10.11), and (10.12) that

$$\|\nabla \operatorname{Ric}\|_{L^2} + \|\nabla^2 \operatorname{Ric}\|_{L^2} + \|\nabla^2 E\|_{L^2} + \|\nabla^2 H\|_{L^2} \le C.$$

Finally, by differentiating (10.14), commuting $\nabla$ with $\triangle$, and using (10.7) we have

$$\begin{aligned}\triangle \nabla^2 k &= k \cdot k \cdot \nabla^2 k + k \cdot \nabla k \cdot \nabla k + E \cdot \nabla^2 k + \nabla E \cdot \nabla k \\ &\quad + \nabla^2 E \cdot k + \nabla^3 H.\end{aligned}$$

Multiplying this equation by $\nabla^2 k$ and integrating over $\Sigma_t$ yields

$$\begin{aligned}\|\nabla^3 k\|_{L^2}^2 &\lesssim \|k\|_{L^\infty}^2 \|\nabla^2 k\|_{L^2}^2 + \|k\|_{L^\infty} \|\nabla k\|_{L^4}^2 \|\nabla^2 k\|_{L^2} \\ &\quad + \|\nabla E\|_{L^4} \|\nabla k\|_{L^4} \|\nabla^2 k\|_{L^2} + \|E\|_{L^\infty} \|\nabla^2 k\|_{L^2}^2 \\ &\quad + \|k\|_{L^\infty} \|\nabla^2 E\|_{L^2} \|\nabla^2 k\|_{L^2} + \|\nabla^2 H\|_{L^2} \|\nabla^3 k\|_{L^2} \\ &\le C + C\|\nabla^3 k\|_{L^2}.\end{aligned}$$

Therefore $\|\nabla^3 k\|_{L^2} \le C$. The proof is thus complete.

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