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AN INTRODUCTION TO LOCAL BLACK HOLE HORIZONS IN THE 3+1 APPROACH TO GENERAL RELATIVITY

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We present an introduction to dynamical trapping horizons as quasi-local models for black hole horizons, from the perspective of an Initial Value Problem approach to the construction of generic black hole spacetimes. We focus on the geometric and structural properties of these horizons aiming, as a main application, at the numerical evolution and analysis of black hole spacetimes in astrophysical scenarios. In this setting, we discuss their dual role as an *a priori* ingredient in certain formulations of Einstein equations and as an *a posteriori* tool for the diagnosis of dynamical black hole spacetimes. Complementary to the first-principles discussion of quasi-local horizon physics, we place an emphasis on the *rigidity* properties of these hypersurfaces and their role as privileged geometric probes into near-horizon strong-field spacetime dynamics.

Keywords: Black holes; quasi-local horizons; Initial Value Problem.

1. Black Holes: Global vs (Quasi-)Local Approaches

1.1. *Establishment's picture of the gravitational collapse*

Our discussion is framed in the problem of gravitational collapse in General Relativity. The current understanding is summarized in what one could call the *establishment's picture* of gravitational collapse,[1] a heuristic chain of results and conjectures:

(1) *Singularity theorems*: If gravity is able to make all light rays locally *converge* (namely, if trapped surfaces exist), then a spacetime singularity forms.[2–5]
(2) (*Weak*) *Cosmic censorship* (Conjecture): In order to preserve predictability, the formed singularity is not visible for a distant observer.[6]
(3) *Black hole spacetimes stability* (Conjecture): General Relativity gravitational dynamics drives eventually the black hole spacetime to a stationary state.
(4) *Black hole uniqueness theorem*: The final state is a Kerr black hole spacetime.[7]

Light bending is a manifestation of spacetime curvature and black holes constitute a dramatic extreme case of this. The standard picture of gravitational collapse above suggests two (complementary) approaches to the characterization of black holes:

(a) *Global approach*: (weak) cosmic censorship suggests black holes as *no-escape* regions not extending to infinity. Its boundary defines the event horizon $\mathscr{E}$.
(b) *Quasi-local approach*: Singularity theorems suggest the characterization of a black hole as a spacetime *trapped region* where all light rays locally converge.

The *establishment's picture* of gravitational collapse depicts an intrinsically dynamical scenario. Hence, a systematic methodology to the study of dynamical spacetimes is needed. We adopt an *Initial* (*Boundary*) *Value Problem* approach, that offers a systematic avenue to the *qualitative* and *quantitative* aspects of generic spacetimes.

1.2. *The black hole region and the event horizon*

The traditional[5] approach to black holes involves global spacetime concepts, in particular a good control of the notion of infinity. Given a (strongly asymptotically predictable) spacetime $\mathcal{M}$, the *black hole region* $\mathscr{B}$ is defined as $\mathscr{B} = \mathcal{M} - J^-(\mathscr{I}^+)$, where $J^-(\mathscr{I}^+)$ is the causal past of future null infinity $\mathscr{I}^+$. That is, $\mathscr{B}$ is the spacetime region that cannot communicate with $\mathscr{I}^+$.

We are particularly interested in characterizing a notion of boundary *surface* of black holes. In this global context this is provided by the event horizon $\mathscr{E}$, defined as the boundary of $\mathscr{B}$, that is $\mathscr{E} = \partial J^-(\mathscr{I}^+) \cap \mathcal{M}$. Interesting geometric and physical properties of the event horizon are: (i) $\mathscr{E}$ is a null hypersurface in $\mathcal{M}$; (ii) it satisfies an *Area Theorem*,[8,9] so that the area of spatial sections $\mathcal{S}$ of $\mathscr{E}$ does not decrease in the evolution and, beyond that, (iii) a set of *black hole mechanics* laws are fulfilled.[10]

However, the global aspects of the event horizon also bring difficulties: (a) it is a *teleological* concept, i.e. the knowledge of the full (future) spacetime is needed in order to locate $\mathscr{E}$, and (b) the black hole region and the event horizon can enter into flat spacetime regions. In sum, the notion of event horizon is a too global one: it does not fit properly into the adopted Initial Value Problem approach.

1.3. *The trapped region and the trapping boundary*

The global approach requires controlling structures that are not accessible during the evolution. In this context, the seminal notion of *trapped surface*[2] plays a crucial role, capturing the idea that all light rays emitted from the surface locally converge. Through the singularity theorems and weak cosmic censorship, it offers a benchmark for the existence of a black hole region: in strongly predictable spacetimes with proper energy conditions, trapped surfaces lie inside the black hole region.[5] Moreover, their location does not involve a whole future spacetime development.

1.3.1. *Trapped and outer trapped surfaces. Apparent horizons*

Given a closed spatial surface $\mathcal{S}$ in the spacetime, we can consider the light emitted from it along outer and inner directions given, respectively, by future null vectors ℓ^a and k^a. Then, light locally converges (in the future) at $\mathcal{S}$ if the area of the emitted light-front spheres decreases in both directions (see though Ref. 11). Denoting the area element of $\mathcal{S}$ as $dA = \sqrt{q}d^2x$, the infinitesimal variations of the area along ℓ^a and k^a define outgoing and ingoing expansions $\theta^{(\ell)}$ and $\theta^{(k)}$ (see Sec. 2.1 for details)

$$\delta_\ell \sqrt{q} = \theta^{(\ell)} \sqrt{q}, \quad \delta_k \sqrt{q} = \theta^{(k)} \sqrt{q}. \tag{1}$$

A trapped surface is characterized by $\theta^{(\ell)}\theta^{(k)} > 0$. In the black hole context, in which the singularity occurs in the future, we refer to $\mathcal{S}$ as a future *trapped surface* (TS) if $\theta^{(\ell)} < 0, \theta^{(k)} < 0$ and as future *marginally trapped surface* (MTS) if one of the expansions, say $\theta^{(\ell)}$, vanishes: $\theta^{(\ell)} = 0, \theta^{(k)} \leq 0$. If a notion of *naturally expanding* direction for the light rays exists (e.g. in isolated systems, the *outer* null direction ℓ^μ pointing to infinity), a related notion of *outer trapped surface* is given[5] by $\theta^{(\ell)} < 0$. *Marginally outer trapped surfaces* (MOTS) are characterized by $\theta^{(\ell)} = 0$.

Before proceeding to a characterization of black holes in terms of trapped surfaces, let us consider trapped surfaces from the perspective of a spatial slice of spacetime Σ. The *trapped region in* Σ, $\mathcal{T}_\Sigma \subset \Sigma$, is the set of points $p \in \Sigma$ belonging to some (outer) trapped surface $\mathcal{S} \subset \Sigma$. The Apparent Horizon (AH) is then the outermost boundary of the trapped region $\mathcal{T}_\Sigma$. A crucial result is the following characterization[5,12,13] of AHs: if the trapped region $\mathcal{T}_\Sigma$ in a slice Σ has the structure of a manifold with boundary, the AH is a MOTS, i.e. $\theta^{(\ell)} = 0$.

Given a 3+1 foliation of spacetime $\{\Sigma_t\}$, let us consider the worldtube obtained by piling up the two-dimensional AHs $\mathcal{S}_t \subset \Sigma_t$. Such an AH-worldtube does not need to be a smooth hypersurface (it is not even necessarily continuous, as discussed in Sec. 5.1.1). This is our first encounter with the notion of a spacetime worldtube foliated by MOTS. Though these worldtubes are slicing-dependent, their characterization in terms of MOTSs makes them very useful from an operational perspective.

1.3.2. *The trapped region: Definition and caveats*

From a spacetime perspective, no reference to a slice Σ must enter into the characterization of the trapped region. The spacetime *trapped region* $\mathcal{T}$ is defined as the set of points $p \in \mathcal{M}$ belonging to some trapped surface $\mathcal{S} \subset \mathcal{M}$. Its boundary is referred[14] to as the *trapping boundary*. These concepts offer, in principle, an intrinsically quasi-local avenue to address the notion of black hole region and black hole horizon, with no reference to asymptotic quantities.

In spite of their appealing features, there are also important caveats associated with the *trapped region* and the *trapping boundary*. In particular, we lack an operational characterization of the *trapping boundary* (see also the contribution by

J. M. M. Senovilla[15]). A systematic attempt to address this issue is provided by the notion of *trapping horizon*,[14] namely smooth worldtubes of MOTS (see Sec. 2.2), as a model for the trapping boundary. Trapping horizons, that are nonunique, have led to important insights into the structure of the trapped region, though an operational characterization of the *trapping boundary* is still missing.

The difficulties are illustrated in the discussion of the relation between the trapping boundary and $\mathscr{E}$. In strongly predictable spacetimes with appropriate energy conditions (see, though Ref. 16), the trapped region $\mathcal{T}$ is contained in the black hole region $\mathscr{B}$. In attempts to refine this statement, support was found[17,18] suggesting that the trapping boundary actually coincides with the event horizon, though later work[19] showed that the trapped region not always extends up to $\mathscr{E}$. The question is still open for (*outer*) *trapped regions* constructed on *outer* trapped surfaces, rather than on TSs. Important insight into these issues has been gained in recent works[20,21] demonstrating truly global features of the trapped region $\mathcal{T}$. In particular:

(i) The trapping boundary cannot be foliated by MOTS.
(ii) Closed trapped surfaces can enter into the flat region. This is an important issue in this approach to black holes, since it was a main criticism in Sec. 1.2.
(iii) Closed trapped surfaces are *clairvoyant*, that is, they are *aware* of the geometry in noncausally connected spacetime regions. This nonlocal property challenges their applicability in an operational characterization of black holes.

1.4. *A pragmatic approach to quasi-local black hole horizons*

Trapping horizons offer a sound avenue towards the quasi-local understanding of black hole physics. They provide crucial insight in gravitational scenarios where a quasi-local notion of black hole horizon is essential, such as black hole thermodynamics beyond equilibrium, the characterization of physical parameters of strongly dynamical astrophysical black holes (notably in numerical simulations), semi-classical collapse, quantum gravity or mathematical relativity (cf. A. Nielsen's contribution[22]). But, on the other hand, issues like their nonuniqueness or the *clairvoyant* properties of trapped surfaces pose fundamental questions that cannot be ignored.

We do not aim here at addressing first-principles questions about the role of trapping horizons as a characterization of black hole horizons. We rather assume a *pragmatic approach* to the study of gravitational dynamics, which underlines the role of trapping horizons as hypersurfaces of remarkable geometric properties in black hole spacetimes. More specifically, our main interests are:

(i) The construction and diagnosis of black hole spacetimes in an Initial (Boundary) Value Problem approach.
(ii) Identification of a geometric probe into near-horizon spacetime dynamics.

Point (ii) is particularly important in the study of gravity in the strong-field regime, where the lack of *rigid structures* (e.g. symmetries, a *background* spacetime ...) is a generic and essential problem. Given our interests and the adopted pragmatic methodology, we look for a geometric object such that: (a) represents a footprint of black holes, providing a probe into their geometry; (b) is adapted, by construction, to an Initial-Boundary Value Problem approach; and (c) although not-necessarily unique, provides a geometric structure with some sort of *rigidity* property. As we shall see in the following, dynamical trapping horizons fulfill these requirements.

1.5. *General scheme*

In Sec. 2 we introduce the basics of the geometry of closed surfaces in a Lorentzian manifold and motivate quasi-local horizons in stationary and dynamical regimes. Section 3 reviews their geometric properties and their special features as *physical* boundaries. Sections 4 and 5 are devoted to applications in a 3+1 description of the spacetime. Section 4 shows the use of quasi-local horizons as inner boundary conditions for elliptic equations in General Relativity, whereas Sec. 5 discusses some applications to the analysis of spacetimes, in particular their role in a *correlation approach* to spacetime dynamics. In Sec. 6 a general overview is presented.

2. Quasi-Local Horizons: Concepts and Definitions

2.1. *Geometry of spacelike closed 2-surfaces $\mathcal{S}$*

2.1.1. *Normal plane: Outgoing and ingoing null vectors*

Let us consider a spacetime $(\mathcal{M}, g_{ab})$ with Levi-Civita connection ∇_a. Given a spacelike closed (compact without boundary) 2-surface $\mathcal{S}$ in $\mathcal{M}$ and a point $p \in \mathcal{S}$, the tangent space splits as $T_p\mathcal{M} = T_p\mathcal{S} \oplus T_p^{\perp}\mathcal{S}$. We span the normal plane $T_p^{\perp}\mathcal{S}$ either by (future-oriented) null vectors ℓ^a and k^a (defined by the intersection between $T_p^{\perp}\mathcal{S}$ and the null cone at p) or by any pair of normal timelike vector n^a and spacelike vector s^a. Let us denote conventionally ℓ^a to be the *outgoing* null normal and k^a the *ingoing* one. We choose normalizations:

$$\ell^a\ell_a = 0, \quad k^ak_a = 0, \quad \ell^ak_a = -1, \quad n^an_a = -1, \quad s^as_a = 1, \quad n^as_a = 0. \tag{2}$$

Directions ℓ^a and k^a are uniquely determined, but a *normalization-boost* freedom

$$\ell'^a = f\ell^a, \quad k'^a = f^{-1}k^a, \tag{3}$$

$$n'^a = \cosh(\sigma)n^a + \sinh(\sigma)s^a, \quad s'^a = \sinh(\sigma)n^a + \cosh(\sigma)s^a,$$

remains for some arbitrary rescaling positive function f on $\mathcal{S}$ (where $\sigma = \ln(f)$ and $\ell^a = \lambda(n^a + s^a)/\sqrt{2}$ and $k^a = \lambda^{-1}(n^a - s^a)/\sqrt{2}$, for some function λ on $\mathcal{S}$).

2.1.2. *Intrinsic geometry of $\mathcal{S}$*

The induced metric on $\mathcal{S}$ is given by

$$q_{ab} = g_{ab} + k_a\ell_b + \ell_ak_b = g_{ab} + n_an_b - s_as_b, \tag{4}$$

so that ${q^a}_b$ is the projector onto $\mathcal{S}$

$$ {q^a}_b {q^b}_c = {q^a}_c, \quad {q^a}_b v^b = v^a \ (\forall\, v^a \in T\mathcal{S}), \quad {q^a}_b w^b = 0 \ (\forall\, w^a \in T^\perp\mathcal{S}). \tag{5} $$

We denote the Levi-Civita connection associated with q_{ab} as ${}^2D_a$. The volume form on $\mathcal{S}$ will be denoted by ${}^2\epsilon = \sqrt{q}dx^1 \wedge dx^2$, i.e. ${}^2\epsilon_{ab} = n^c s^{d4}\epsilon_{cdab}$, though we will also employ the area measure notation $dA = \sqrt{q}d^2x$.

2.1.3. *Extrinsic geometry of $\mathcal{S}$ in $(\mathcal{M}, g)$*

We define the *second fundamental tensor* of $(\mathcal{S}, q_{ab})$ in $(\mathcal{M}, g_{ab})$ (also, *shape tensor* or *extrinsic curvature tensor*) as

$$ \mathcal{K}^c_{ab} = {q^d}_a {q^e}_b \nabla_d {q^c}_e, \tag{6} $$

where c is an index in the normal plane $T^\perp\mathcal{S}$, whereas a and b are indices in $T\mathcal{S}$. Given a vector v^a normal to $\mathcal{S}$, we can define the *deformation tensor* $\Theta^{(v)}_{ab}$ as

$$ \Theta^{(v)}_{ab} = {q^c}_a {q^d}_b \nabla_c v_d. \tag{7} $$

Then, using expression (4), the *second fundamental tensor* can be expressed as

$$ \mathcal{K}^c_{ab} = k^c\Theta^{(\ell)}_{ab} + \ell^c\Theta^{(k)}_{ab} = n^c\Theta^{(n)}_{ab} - s^c\Theta^{(s)}_{ab}. \tag{8} $$

We can express $\Theta^{(v)}_{ab}$ in terms of the variation of the intrinsic metric along v^a. Given a (tensorial) object ${A_{a_1\cdots a_n}}^{b_1\cdots b_m}$ tangent to $\mathcal{S}$ we denote by δ_v the operator ${(\delta_v A)_{a_1\cdots a_n}}^{b_1\cdots b_m} = {q_{a_1}}^{c_1}\cdots {q_{a_n}}^{c_n} {q_{d_1}}^{b_1}\cdots {q_{d_m}}^{b_m}\mathcal{L}_v {A_{c_1\cdots c_n}}^{d_1\cdots d_m}$, where $\mathcal{L}_v$ denote the Lie derivative along (some extension of) v^a. Then, it follows

$$ \delta_v q_{ab} = \frac{1}{2}\Theta^{(v)}_{ab}. \tag{9} $$

(a) *Shear and expansion associated with v^a*. Defining the expansion $\theta^{(v)}$ and shear tensor $\sigma^{(v)}_{ab}$ associated with the normal vector v^a as

$$ \theta^{(v)} \equiv q^{ab}\nabla_a v_b = \delta_v \ln\sqrt{q}, \quad \sigma^{(v)}_{ab} \equiv \Theta^{(v)}_{ab} - \frac{1}{2}\theta^{(v)} q_{ab}, \tag{10} $$

we express the deformation tensor $\Theta^{(v)}_{ab}$ in terms of his trace and traceless parts

$$ \Theta^{(v)}_{ab} = \sigma^{(v)}_{ab} + \frac{1}{2}\theta^{(v)} q_{ab}. \tag{11} $$

(b) *Mean curvature vector H^a*. Taking the trace of $\Theta^{(v)}_{ab}$ on $\mathcal{S}$ we define the *mean curvature vector*[a]

$$ H^c \equiv q^{ab}\mathcal{K}^c_{ab} = \theta^{(\ell)}k^c + \theta^{(k)}\ell^c. \tag{12} $$

[a]Note the opposite sign convention with respect to the contribution by J. M. M. Senovilla.[15]

The extrinsic curvature information of $(\mathcal{S}, q_{ab})$ in $(\mathcal{M}, g_{ab})$ is completed by the *normal fundamental forms* associated with normal vectors v^a. In particular[23]

$$\Omega_a^{(n)} = s^c q^d{}_a \nabla_d n_c, \quad \Omega_a^{(s)} = n^c q^d{}_a \nabla_d s_c,$$
$$\Omega_a^{(\ell)} = \frac{1}{k^b \ell_b} k^c q^d{}_a \nabla_d \ell_c, \quad \Omega_a^{(k)} = \frac{1}{k^b \ell_b} \ell^c q^d{}_a \nabla_d k_c. \tag{13}$$

All these normal fundamental forms are related up to a sign and a total derivative on $\mathcal{S}$. Using the normalizations (2) we get[b]: $\Omega_a^{(n)} = -\Omega_a^{(s)}, \Omega_a^{(\ell)} = -\Omega_a^{(k)}, \Omega_a^{(\ell)} = \Omega_a^{(n)} - {}^2D_a\lambda$. We choose to employ the 1-form $\Omega_a^{(\ell)}$ in the following.

2.1.4. *Transformation properties under null normal rescaling*

Under the rescaling (2) $\ell^a \to f\ell^a$, $k^a \to f^{-1}k^a$ the introduced fields transform as

$$\begin{aligned}
&q_{ab} \to q_{ab}, && {}^2D_a \to {}^2D_a, && \\
&\mathcal{K}^c_{ab} \to \mathcal{K}^c_{ab}, && H^a \to H^a, && \\
&\Theta^{(\ell)}_{ab} \to f\Theta^{(\ell)}_{ab}, && \theta^{(\ell)} \to f\theta^{(\ell)}, && \sigma^{(\ell)}_{ab} \to f\sigma^{(\ell)}_{ab}, \\
&\Theta^{(k)}_{ab} \to f^{-1}\Theta^{(k)}_{ab}, && \theta^{(k)} \to f^{-1}\theta^{(k)}, && \sigma^{(k)}_{ab} \to f^{-1}\sigma^{(k)}_{ab}. \\
&\Omega^{(\ell)}_a \to \Omega^{(\ell)}_a + {}^2D_a(\ln f), && &&
\end{aligned} \tag{14}$$

Finally, given an axial Killing vector ϕ^a on $\mathcal{S}$, we can write the angular momentum[c]

$$J = \frac{1}{8\pi} \int_{\mathcal{S}} \Omega_a^{(\ell)} \phi^a {}^2\epsilon. \tag{15}$$

The transformation rule of $\Omega_a^{(\ell)}$ in (14) together with the divergence-free property of ϕ^a (following from its Killing character) guarantee that the quantity J does not depend on the choice of null normals ℓ^a, k^a (i.e. J does not change under a boost).

2.2. *Trapping horizons*

2.2.1. *Worldtubes of marginally trapped surfaces*

A *trapping horizon*[14] is (the closure of) a hypersurface $\mathcal{H}$ foliated by closed marginal (outer) trapped surfaces: $\mathcal{H} = \bigcup_{t\in\mathbb{R}} \mathcal{S}_t$, with $\theta^{(\ell)}|_{\mathcal{S}_t} = 0$. Trapping horizons are classified according to the signs of $\theta^{(k)}$ and $\delta_k\theta^{(\ell)}$. In particular, the sign of $\theta^{(k)}$ controls if the singularity occurs either in the *future* or in the *past* of $\mathcal{S}$, whereas the sign of $\delta_k\theta^{(\ell)}$ controls the (local) *outer*- or *inner-most* character of $\mathcal{H}$. Then, a trapping horizon is said to be: (i) *future* (respectively, *past*) if $\theta^{(k)} < 0$ (respectively, $\theta^{(k)} > 0$), and (ii) *outer* (respectively, *inner*) if there exists[d] ℓ^a and k^a such that $\delta_k\theta^{(\ell)} < 0$ (respectively, $\delta_k\theta^{(\ell)} > 0$).

[b] When using $\ell^a k_a = -e^\sigma$ one gets: $\Omega_a^{(\ell)} = -\Omega_a^{(k)} - {}^2D_a\sigma$. This will be relevant later, in Eq. (39).
[c] The quantity J coincides with the Komar angular momentum in case that ϕ^a can be extended to an axial Killing in the neighborhood of $\mathcal{S}$.
[d] The sign of $\delta_k\theta^{(\ell)}$ is not invariant on the whole $\mathcal{S}$ under a rescaling (2). However, if there exists ℓ^a and k^a such that $\delta_k\theta^{(\ell)} < 0$ on $\mathcal{S}$, then there does not exist any choice of ℓ^a and k^a such that $\delta_k\theta^{(\ell)} > 0$ on $\mathcal{S}$; see Ref. 24 and also the marginally trapped surface stability condition in Ref. 25.

2.2.2. *Future outer trapping horizons*

In a black hole setting the singularity occurs in the future of sections $\mathcal{S}_t$ of $\mathcal{H}$, so that the related trapping horizon is of *future* type, $\theta^{(k)} < 0$. In addition, when considering displacements along k^a (*ingoing* direction) we should move into the trapped region, i.e. $\delta_k \theta^{(\ell)} < 0$, so that the trapping horizon should be *outer*.

The resulting characterization of quasi-local black hole horizons as *Future Outer Trapping Horizons* (FOTHs) is further supported by the following analysis of the area evolution. Hawking's area theorem for event horizons (cf. Sec. 1.2) captures a fundamental feature of classical black holes. It is natural to wonder about a quasi-local version of it. Let us consider an evolution vector h^a along the trapping horizon $\mathcal{H}$, characterized as: (i) h^a is tangent to $\mathcal{H}$ and orthogonal to $\mathcal{S}_t$, and (ii) h^a transports $\mathcal{S}_t$ onto $\mathcal{S}_{t+\delta t}$: $\delta_h t = 1$. We can write h^a and a *dual* vector τ^a orthogonal to $\mathcal{H}$ as

$$h^a = \ell^a - Ck^a, \quad \tau^a = \ell^a + Ck^a. \tag{16}$$

Then $h^a h_a = -\tau^a \tau_a = 2C$, i.e. h^a is spacelike for $C > 0$, null for $C = 0$ and timelike for $C < 0$. The evolution of the area $A = \int_{\mathcal{S}} dA = \int_{\mathcal{S}} {}^2\epsilon$ along h^a is given by

$$\delta_h A = \int_{\mathcal{S}} \theta^{(h)2}\epsilon = \int_{\mathcal{S}} (\theta^{(\ell)} - C\theta^{(k)})^2\epsilon = -\int_{\mathcal{S}} C\theta^{(k)2}\epsilon. \tag{17}$$

Considering for simplicity the spherical symmetric case ($C = \text{const}$; see discussion of Eq. (37) in 3.2.4, for the general case), the *trapping horizon* condition, $\delta_h \theta^{(\ell)} = 0$, writes $\delta_\ell \theta^{(\ell)} - C\delta_k \theta^{(\ell)} = 0$, so that $C = \frac{\delta_\ell \theta^{(\ell)}}{\delta_k \theta^{(\ell)}}$. Applying the *Raychaudhuri* equation for $\delta_\ell \theta^{(\ell)}$ [see later Eq. (21)], together with the $\theta^{(\ell)} = 0$ condition, we find

$$C = -\frac{\sigma^{(\ell)}_{ab}\sigma^{(\ell)ab} + 8\pi T_{ab}\ell^a\ell^b}{\delta_k \theta^{(\ell)}}. \tag{18}$$

Under the null energy and outer horizon conditions, it follows $C \geq 0$, so that the future condition guarantees the nondecrease of the area in (17). Therefore, FOTHs are *null* or *spacelike* hypersurfaces ($C \geq 0$), satisfying an area law result, and therefore providing appropriate models for quasi-local black hole horizons.

2.3. *Isolated and dynamical horizons*

The distinct geometric structure of null and spatial hypersurfaces suggests different strategies for the study of the stationary and dynamical regimes of quasi-local black holes, modeled as future outer trapping horizons. This has led to the parallel development of the *isolated horizon* and the *dynamical horizon* frameworks.[26–29]

In equilibrium, *Isolated Horizons* (IH) provide a hierarchy of geometric structures constructed on a null hypersurface $\mathcal{H}$ that is foliated by closed (outer) marginally trapped surfaces. They characterize different levels of stationarity for a black hole horizon in an otherwise dynamical environment:

(i) *Non-Expanding Horizons* (NEH). They represent the minimal notion of equilibrium by imposing the stationarity of the intrinsic geometry q_{ab}.

(ii) *Weakly Isolated Horizons* (WIH). They are NEHs endowed with an additional structure needed for a Hamiltonian analysis of the horizon and its related (thermo-)dynamics. They impose no additional constraints on the geometry of the NEH.

(iii) *Isolated Horizons* (IH). These are WIHs whose extrinsic geometry is also invariant along the evolution. They provide the strongest stationarity notion on $\mathcal{H}$.

The nonstationary regime can be characterized by *Dynamical Horizons* (DH), namely spacelike hypersurfaces $\mathcal{H}$ foliated by closed future marginally trapped surfaces, i.e. $\theta^{(\ell)} = 0$ and $\theta^{(k)} < 0$. Introduced in a 3+1 formulation, they provide a complementary perspective to the *dual-null foliation formulation*[14] of trapping horizons, making them naturally adapted for an Initial Value Problem perspective.

2.4. *IHs and DHs as stationary and dynamical sections of FOTHs*

A natural question when considering the transition from equilibrium to the dynamical regime is whether a section $\mathcal{S}_t$ of a FOTH can be partially stationary and partially dynamical. Or, in other words, whether the element of area dA can be non-expanding ($C = 0$) in a part of $\mathcal{S}_t$ whereas it already expands ($C > 0$) in another part. Namely, can h^a be both null and spacelike on a section $\mathcal{S}_t$ of a FOTH?

The answer is in the negative. Transitions between non-expanding and dynamical parts of a FOTH must happen *all at once*. More precisely, assuming the null energy condition, a FOTH can be completely partitioned into non-expanding and dynamical sections. For a section $\mathcal{S}_t$ to be completely dynamical ($C > 0$) it suffices that it has $\delta_\ell \theta^{(\ell)} < 0$ somewhere on it. Otherwise h^a is null ($C = 0$) all over $\mathcal{S}_t$.[24,30]

In more physical terms, it suffices that some *energy* crosses the horizon *somewhere*, and the *whole* horizon instantaneously grows as a whole. This nonlocal behavior is a consequence of the *elliptic* nature of quasi-local horizons. As shown in Sec. 3.2.3, the function C determining the metric type of h^a satisfies an elliptic equation [cf. Eq. (37)]. Under the outer condition $\delta_k \theta^{(\ell)} < 0$ one can apply a *maximum principle* to show that C is non-negative [generalization of Eq. (18)]. Moreover, it suffices that $\delta_\ell \theta^{(\ell)} \neq 0$ somewhere, for having $C > 0$ everywhere.

3. Quasi-Local Horizons: Properties from a 3+1 Perspective

3.1. *Equilibrium regime*

3.1.1. *Null hypersurfaces: Characterization and basic elements*

A hypersurface $\mathcal{H}$ is null if and only if the induced metric is degenerated. Equivalently, if and only if there is a tangent null vector ℓ^a orthogonal to all vectors tangent to $\mathcal{H}$: $\ell^a v_a = 0,\ \forall\, v^a \in T\mathcal{H}$.

Let us introduce some elements on the geometry of $\mathcal{H}$. Choosing a null vector k^a transverse to $\mathcal{H}$, we can write[e] the degenerated metric as $q_{ab} = g_{ab} + k_a \ell_b + \ell_a k_b$.

[e]We abuse notation and employ the same notation employed in sections $\mathcal{S}_t$ of $\mathcal{H}$, cf. Eq. (4).

A projector onto $\mathcal{H}$ can also be constructed as: $\Pi_a{}^b = \delta_a{}^b + \ell_a k^b = q_a{}^b - k_a \ell^b$. As a part of the extrinsic curvature of $\mathcal{H}$, a *rotation* 1*-form* can be introduced[31] on $\mathcal{H}$ as $\omega_a^{(\ell)} = \frac{1}{\ell^a k_a} k^c \nabla_a \ell_c$. This 1-form *lives* on $\mathcal{H}$, i.e. $k^a \omega_a^{(\ell)} = 0$. In particular, we can write $\Pi_a{}^c \nabla_c \ell^b = \omega_a^{(\ell)} \ell^b + \Theta^{(\ell)}{}_a{}^b$, where $\Theta_{ab}^{(\ell)}$ is given by expression (7) [cf. Eq. (5.23) in Ref. 28]. Contracting with ℓ^a we find: $\ell^c \nabla_c \ell^a = \kappa^{(\ell)} \ell^a$, a pre-geodesic equation where the non-affinity coefficient $\kappa^{(\ell)}$ is defined as $\kappa^{(\ell)} = \ell^a \omega_a^{(\ell)}$. If a foliation $\{\mathcal{S}_t\}$ of $\mathcal{H}$ is given, we can write [cf. Eq. (5.35) in Ref. 28]: $\omega_a^{(\ell)} = \Omega_a^{(\ell)} - \kappa^{(\ell)} k_a$.

Vectors ℓ^a and k^a can be completed to a tetrad $\{\ell^a, k^a, (e_1)^a, (e_2)^a\}$, where $(e_i)^a$ are tangent to sections $\mathcal{S}_t$. Normalizations given in (2) are then completed to

$$\ell \cdot (e_i)_a = 0, \quad k^a (e_i)_a = 0, \quad (e_i)^a (e_i)_b = \delta_{ab}. \tag{19}$$

Defining the complex null vector $m^a = \frac{1}{\sqrt{2}}[(e_1)^a + i(e_2)^a]$, the Weyl scalars are defined as the components of the Weyl tensor $C^a{}_{bcd}$ in the null tetrad $\{\ell^a, k^a, m^a, \overline{m}^a\}$

$$\begin{aligned} &\Psi_0 = C^a{}_{bcd}\, \ell_a m^b \ell^c m^d, \quad \Psi_3 = C^a{}_{bcd}\, \ell_a k^b \overline{m}^c k^d, \\ &\Psi_1 = C^a{}_{bcd}\, \ell_a m^b \ell^c k^d, \quad \Psi_4 = C^a{}_{bcd}\, \overline{m}_a k^b \overline{m}^c k^d. \\ &\Psi_2 = C^a{}_{bcd}\, \ell_a m^b \overline{m}^c k^d, \end{aligned} \tag{20}$$

3.1.2. *Null hypersurfaces: Evolution*

It is illustrative to give a 3+1 perspective on $\mathcal{H}$. Given a foliation $\mathcal{H} = \bigcup_{t\in\mathbb{R}} \mathcal{S}_t$ let us evaluate explicitly the evolution along ℓ^a of some quantities defined on sections $\mathcal{S}_t$.

(i) Expansion equation (null Raychaudhuri equation):

$$\delta_\ell \theta^{(\ell)} - \kappa^{(\ell)} \theta^{(\ell)} + \frac{1}{2} {\theta^{(\ell)}}^2 + \sigma_{ab}^{(\ell)} {\sigma^{(\ell)}}^{ab} + 8\pi T_{ab} \ell^a \ell^b = 0. \tag{21}$$

(ii) Tidal equation:

$$\delta_\ell \sigma_{ab}^{(\ell)} = \kappa^{(\ell)} \sigma_{ab}^{(\ell)} + \sigma_{cd}^{(\ell)} {\sigma^{(\ell)}}^{cd} q_{ab} - q^c{}_a q^d{}_b C_{ecfd} \ell^e \ell^f. \tag{22}$$

(iii) Evolution for Ω_a:

$$\delta_\ell \Omega_c^{(\ell)} + \theta^{(\ell)} \Omega_a^{(\ell)} = 8\pi T_{cd}\, \ell^c q^d{}_a + {}^2D_a \left(\kappa^{(\ell)} + \frac{\theta^{(\ell)}}{2} \right) - {}^2D_c {\sigma^{(\ell)}}^c{}_a. \tag{23}$$

3.1.3. *Non-expanding horizons*

A NEH[32] is a null-hypersurface $\mathcal{H} \approx S^2 \times \mathbb{R}$, on which the expansion associated with ℓ^a vanishes ($\theta^{(\ell)} = 0$), the Einstein equations hold and $-T^a_c \ell^c$ is future directed (*null dominant energy condition*). Note that any foliation $\mathcal{H} = \bigcup_{t\in\mathbb{R}} \mathcal{S}_t$ produces a foliation of $\mathcal{H}$ by MOTS $\mathcal{S}_t$.

(i) *NEH characterization.* Making $\theta^{(\ell)} = 0$ in the Raychaudhuri Eq. (21) we get

$$\sigma_{ab} \sigma^{ab} + 8\pi T_{ab} \ell^a \ell^b = 0. \tag{24}$$

Since the two terms are positive-definite, they vanish independently. This provides an *instantaneous* characterization of a NEH:

$$\theta^{(\ell)} = 0, \quad \sigma^{(\ell)}_{ab} = 0, \quad T_{ab}\ell^a\ell^b = 0. \tag{25}$$

From Eq. (11) with $v^a = \ell^a$, it follows $\Theta^{(\ell)}_{ab} = 0$. The NEH characterization is equivalent, cf. Eq. (9), to the *evolution independence* of the induced metric q_{ab}

$$\delta_\ell q_{ab} = \frac{1}{2}\Theta^{(\ell)}_{ab} = 0. \tag{26}$$

From Eq. (8), we conclude that a NEH fixes half of the degrees of freedom in the second fundamental form $\mathcal{K}^c_{ab}$ of $\mathcal{S}_t$ in $\mathcal{M}$. This will be relevant in Sec. 4.2.1.

(ii) *Connection* $\hat{\nabla}_a$ *on a NEH.* A null hypersurface has no unique (Levi-Civita) connection compatible with the metric. However, on a NEH $\mathcal{H}$ one can introduce a preferred connection as that one induced from the spacetime connection ∇_a: $u^c\hat{\nabla}_c w^a \equiv u^c\nabla_c w^a$, $\forall\, u^a, w^a \in T\mathcal{H}$. Indeed using NEH characterization (26), $u^c\nabla_c w^a$ is tangent to $\mathcal{H}$: $\ell_d(u^c\nabla_c w^d) = u^c\nabla_c(\ell_d w^d) - u^c w^d \Theta^{(\ell)}_{cd} = 0$.

(iii) *Geometry of a NEH.* We refer (cf. Ref. 33) to the pair $(q_{ab}, \hat{\nabla}_a)$ as the *geometry of a NEH.* Writing the components of the $\hat{\nabla}_a$ connection in terms of quantities on $\mathcal{S}_t$

$$\begin{aligned} q^c{}_a q^b{}_d \hat{\nabla}_c v^d &= {}^2\!D_a(q^b{}_c v^c), \\ q^c{}_a k_d \hat{\nabla}_c v^d &= {}^2\!D_a(v^c k_c) - q^c{}_a v^d \Theta^{(k)}_{cd}, \\ \ell^c \hat{\nabla}_c v^a &= \delta_\ell v^a + v^c \omega^{(\ell)}_c \ell^a, \end{aligned} \tag{27}$$

the free data on a NEH are given, from an evolution perspective, by $(q_{ab}|_{\mathcal{S}_t}, \Omega^{(\ell)}_a|_{\mathcal{S}_t}, \kappa^{(\ell)}|_{\mathcal{H}}, \Theta^{(k)}_{ab}|_{\mathcal{S}_t})$, where q_{ab} is time independent.

(iv) *Weyl tensor on a NEH.* Under the rescaling (3), the 1-form $\omega^{(\ell)}_a$ transforms as $\omega^{(\ell)}_a \to \omega^{(\ell)}_a + \hat{\nabla}_a \ln f$. Its exterior derivative $d\omega^{(\ell)}$ provides a gauge invariant object: understanding $\omega^{(\ell)}_a$ as a gauge connection, $d\omega^{(\ell)}$ is its gauge-invariant curvature. Using the NEH condition, $\Theta^{(\ell)}_{ab} = 0$, one can express (cf. Sec. 7.6.2. in Ref. 28)

$$d\omega^{(\ell)} = 2\,\mathrm{Im}\Psi_2\,{}^2\epsilon. \tag{28}$$

Hence, $\mathrm{Im}\Psi_2$ is gauge invariant on a NEH. Actually the full Ψ_2 is invariant, as it follows from its *boost* transformation rules and the values of Ψ_0 and Ψ_1 on a NEH,[28]

$$\Psi_0|_{\mathcal{H}} = \Psi_1|_{\mathcal{H}} = 0. \tag{29}$$

3.1.4. *Weakly isolated horizons*

A *Weakly Isolated Horizon* (WIH) $(\mathcal{H}, [\ell^a])$ is a NEH together with a class of null normals $[\ell^a]$ such that: $\delta_\ell \omega^{(\ell)}_a = 0$. This condition permits to set a well-posed variational problem for spacetimes containing stationary quasi-local horizons. This enables the development of a Hamiltonian analysis on the horizon $\mathcal{H}$ leading to the construction of conserved quantities under WIH-symmetries.[31] In particular, the expression for the angular momentum in Eq. (15) is recovered

$$J_{\mathcal{H}} = \frac{1}{8\pi}\int_{\mathcal{S}_t} \omega_c^{(\ell)}\phi^c\, {}^2\epsilon = \frac{1}{8\pi}\int_{\mathcal{S}_t} \Omega_c^{(\ell)}\phi^c\, {}^2\epsilon = -\frac{1}{4\pi}\int_{\mathcal{S}_t} f \mathrm{Im}\Psi_2\, {}^2\epsilon, \tag{30}$$

with $\phi^a = {}^2D_c f\, {}^2\epsilon^{ac}$ (ϕ^a is an axial Killing vector, in particular divergence-free).

The WIH structure is relevant for the discussion of IH thermodynamics (cf. A. Nielsen's contribution[22]). We do not address this issue here and just comment on the equivalence of the WIH condition with a thermodynamical *zeroth law*. Reminding $\omega_a^{(\ell)} = \Omega_a^{(\ell)} - \kappa^{(\ell)} k_a$, the (vacuum) evolution equation (23) for $\Omega_a^{(\ell)}$ leads to $\mathcal{L}_\ell \Omega_a^{(\ell)} = {}^2D_a\kappa^{(\ell)}$. More generally, $\delta_\ell \omega_a^{(\ell)} = \hat{\nabla}\kappa^{(\ell)}$ (cf. for example Eq. (8.5) in Ref. 28). That is, on WIHs the non-affinity coefficient (*surface gravity*) is constant: $\kappa^{(\ell)} = \kappa_o$.

WIHs and NEH geometry. WIHs do not constraint the underlying NEH geometry. In other words, every NEH admits a WIH structure. In fact, given $\kappa^{(\ell)} \neq$ const., the rescaling $\ell' = \alpha\ell$, with $\kappa_o = \text{const.} = \nabla_\ell \alpha + \alpha\kappa^{(\ell)}$, leads to a constant $\kappa^{(\ell')} = \kappa_o$. Finally, free data for a WIH are again $(q_{ab}|_{\mathcal{S}_t}, \Omega_a^{(\ell)}|_{\mathcal{S}_t}, \kappa^{(\ell)}|_{\mathcal{H}}, \Theta_{ab}^{(k)}|_{\mathcal{S}_t})$, but now $q_{ab}|_{\mathcal{S}_t}$, $\Omega_a^{(\ell)}|_{\mathcal{S}_t}$ and $\kappa^{(\ell)}|_{\mathcal{H}} = \kappa_o$ are time-independent.

3.1.5. (*Strongly*) *isolated horizons*

An isolated horizon (IH) is a WIH on which the whole extrinsic geometry is time-invariant: $[\delta_\ell, \hat{\nabla}_a] = 0$. This condition can be characterized[28,33] as $\delta_\ell \Theta^{(k)} = 0$, that leads to the geometric constraint

$$\kappa^{(\ell)}\Theta_{ab}^{(k)} = \frac{1}{2}({}^2D_a\Omega_b^{(\ell)} + {}^2D_b\Omega_a^{(\ell)}) + \Omega_a^{(\ell)}\Omega_b^{(\ell)} - \frac{1}{2}{}^2R_{ab} + 4\pi\left(q^c{}_a q^d{}_b T_{cd} - \frac{T}{2}q_{ab}\right). \tag{31}$$

With Eq. (26), this fixes completely the second fundamental form $\mathcal{K}^c_{ab}$. Free data of an IH, $(q_{ab}|_{\mathcal{S}_t}, \Omega_a^{(\ell)}|_{\mathcal{S}_t}, \kappa^{(\ell)}|_{\mathcal{H}} = \kappa_o)$, are time independent. Their geometric (gauge-invariant) content can be encoded in the pair[f]: $({}^2R, \mathrm{Im}\Psi_2)$. On the one hand, ${}^2R$ accounts for the gauge-invariant part of q_{ab}. Regarding $\Omega_a^{(\ell)}$, from $d\omega^{(\ell)} = 2\mathrm{Im}\Psi_2\, {}^2\epsilon$ and $\kappa^{(\ell)} = \text{const.}$, it follows $d\Omega^{(\ell)} = 2\mathrm{Im}\Psi_2\, {}^2\epsilon$. On a sphere $\mathcal{S}_t$ we can write $\Omega_a^{(\ell)} = \Omega_a^{\text{div-free}} + \Omega_a^{\text{exact}}$, so that $\Omega_a^{\text{exact}} = {}^2D_a g$ is gauge-dependent [cf. (14)]. From $d\Omega_a^{\text{div-free}} = 2\mathrm{Im}\Psi_2$, the gauge-invariant part of $\Omega_a^{(\ell)}$ is encoded in $\mathrm{Im}\Psi_2$.

IH multipoles of axially symmetric horizons. On an axially symmetric IH, the gauge-invariant part of the geometry, $({}^2R, \mathrm{Im}\Psi_2)$, can be decomposed onto spherical harmonics. On an axially symmetric section $\mathcal{S}_t$ of $\mathcal{H}$, a coordinate system can be

[f] Note the relation with the complex scalar $\mathcal{K}$ in Refs. 34 and 35.

canonically constructed,[36,37] such that [with $A_{\mathcal{H}} = 4\pi(R_{\mathcal{H}})^2$]

$$q_{ab}dx^a \otimes dx^b = (R_{\mathcal{H}})^2(F^{-1}\sin^2\theta d\theta \otimes d\theta + F d\phi \otimes d\phi). \tag{32}$$

In particular, $dA = (R_{\mathcal{H}})^2\sin\theta d\theta d\phi$ (*round sphere* area element). We can then use standard spherical harmonics $Y_{\ell m}(\theta)$, with $m = 0$ in this axisymmetric case

$$\int_{\mathcal{S}_t} Y_{\ell 0}(\theta)Y_{\ell' 0}(\theta)d^2A = (R_{\mathcal{H}})^2\delta_{\ell\ell'}, \tag{33}$$

to define the *IH geometric multipoles*[36] I_n and L_n

$$I_n = \frac{1}{4}\int_{\mathcal{S}_t} {}^2R\, Y_{n0}(\theta)\, d^2A,$$

$$L_n = -\int_{\mathcal{S}_t} \mathrm{Im}\Psi_2\, Y_{n0}(\theta)\, d^2A. \tag{34}$$

Then, *mass* M_n and *angular momentum* J_n multipoles are defined[36–39] by adequate dimensional rescalings of I_n and L_n.

3.1.6. *Gauge freedom on a NEH: Non-uniqueness of the foliation*

Before proceeding to the dynamical case, we underline the existence of a fundamental gauge freedom in the equilibrium (null) case: *any foliation* $\{\mathcal{S}_t\}$ *of a NEH* $\mathcal{H}$ *provides a foliation of* $\mathcal{H}$ *by marginally trapped surfaces*. This is equivalent to the rescaling freedom of the null normal $\ell^a \to f\ell^a$. Therefore, the amount of gauge freedom in the equilibrium case is encoded in one arbitrary function f on $\mathcal{S}_t$.

Note that in this equilibrium horizon context, the relevant spacetime geometric object (the hypersurface $\mathcal{H}$) is *unique*, whereas the gauge-freedom enters in its evolution description due to the *non-uniqueness* of its possible foliation by MOTS.

3.2. *Dynamical case*

3.2.1. *Existence and foliation uniqueness results*

Let us introduce two fundamental results following from the application of geometric analysis techniques to the study of dynamical trapping horizons.

Property 1 (Dynamical horizon foliation uniqueness).[40] *Given a dynamical FOTH* $\mathcal{H}$, *the foliation by marginally trapped surfaces is unique.*

This first result identifies an important *rigidity property* of DHs: the uniqueness of its evolution description. This is in contrast with the equilibrium null case, with its freedom in the choice of the foliation. In particular, on a dynamical FOTH the evolution vector is completely determined: h^a is unique up to *time* reparametrization.

Property 2 (Existence of DHs).[30,41] *Given a marginally trapped surface* $\mathcal{S}_0$ *satisfying an appropriate stability condition on a Cauchy hypersurface* Σ, *to each* 3+1 *spacetime foliation* $(\Sigma_t)_{t\in\mathbb{R}}$ *it corresponds a unique dynamical FOTHs* $\mathcal{H}$ *containing* $\mathcal{S}_0$ *and sliced by marginally trapped surfaces* $\{\mathcal{S}_t\}$ *such that* $\mathcal{S}_t \subset \Sigma_t$.

This second result addresses the Initial Value Problem of DHs, in particular the existence of an evolution for a given MOTS into a dynamical FOTH. The result requires a stability condition (namely, $\mathcal{S}_0$ is required to be *stably outermost*[25,30,41,42]), so that the sign of the variation of $\theta^{(\ell)}$ in the inward (outward) direction is under control. This is essentially the *outer* condition[14] in the FOTH characterization.

3.2.2. *"Gauge" freedom: Non-uniqueness of dynamical horizons*

The evolution of an AH into a DH is non-unique, as a consequence of combining Properties 1 and 2 above. Let us consider an initial AH $\mathcal{S}_0 \subset \Sigma_0$ and two different 3+1 slicings $\{\Sigma_{t_1}\}$ and $\{\Sigma_{t_2}\}$, compatible with Σ_0. From Property 2 there exist DHs $\mathcal{H}_1 = \bigcup_{t_1} \mathcal{S}_{t_1}$ and $\mathcal{H}_2 = \bigcup_{t_2} \mathcal{S}_{t_2}$, with $\mathcal{S}_{t_1} = \mathcal{H}_1 \cap \Sigma_{t_1}$ and $\mathcal{S}_{t_2} = \mathcal{H}_2 \cap \Sigma_{t_2}$ marginally trapped surfaces. Let us consider now the sections of $\mathcal{H}_1$ by $\{\Sigma_{t_2}\}$, i.e. $\mathcal{S}'_{t_2} = \mathcal{H}_1 \cap \Sigma_{t_2}$, so that $\mathcal{H}_1 = \bigcup_{t_2} \mathcal{S}'_{t_2}$. In the generic case, slicings $\{\mathcal{S}'_{t_2}\}$ and $\{\mathcal{S}_{t_1}\}$ of $\mathcal{H}_1$ are different (one can consider a deformation of the slicing $\{\Sigma_{t_2}\}$, if needed). Therefore, from the foliation uniqueness of Property 1, sections $\mathcal{S}'_{t_2}$ cannot be marginally trapped surfaces. It follows then that $\mathcal{H}_1$ and $\mathcal{H}_2$ are different as hypersurfaces in $\mathcal{M}$: if $\mathcal{H}_1 = \mathcal{H}_2$, sections $\mathcal{S}_{t_2}$ (MOTSs) and $\mathcal{S}'_{t_2}$ (non-MOTSs) would coincide by construction, leading to a contradiction. In addition to this non-uniqueness, DHs *interweave* in spacetime due to the existence of causal constraints[40]: a DH $\mathcal{H}_1$ cannot lay completely in the causal past of another DH $\mathcal{H}_2$ (cf. Fig. 1).

Comparing with the discussion in Sec. 3.1.6 on the uniqueness and gauge-freedom issues in the equilibrium case, we conclude from the previous geometric

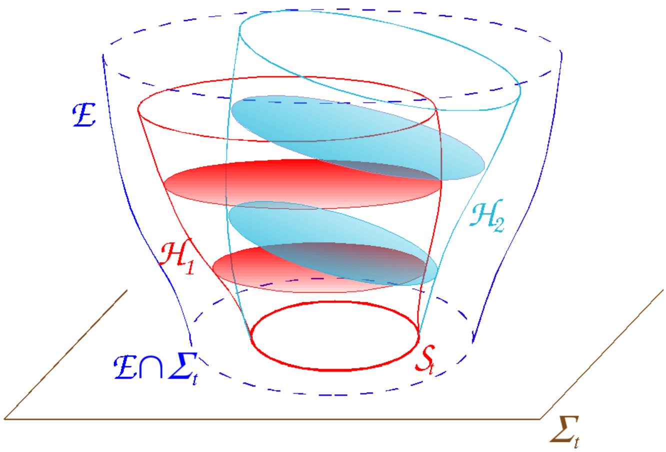

Fig. 1. (Color online) Illustration of the DH non-uniqueness. Dynamical horizons $\mathcal{H}_1$ and $\mathcal{H}_2$ represent evolutions from a given initial MOTS corresponding to different spacetime 3+1 slicings.

considerations that the dynamical and equilibrium cases contain the same amount of gauge freedom, namely a function on $\mathcal{S}$, although *dressed* in a different form. More specifically, whereas in the NEH case there is a fixed horizon, with a rescaling freedom ($\ell^a \to f\ell^a$, f function on $\mathcal{S}_t$), in the DH case the foliation is fixed, but a (gauge) freedom appears in the choice of the evolving horizon (lapse function N on $\mathcal{S}_t$). In other words, in the dynamical case the choice is among *distinct* spacetime geometric objects, $\mathcal{H}_1$ and $\mathcal{H}_2$, whereas in the equilibrium case the choice concerns the *description* (foliation) of a single spacetime geometric object $\mathcal{H}$.

3.2.3. *FOTH characterization*

As discussed in Subsec. 2.2, a FOTH with evolution vector $h^a = \ell^a - Ck^a$ is characterized by: (i) a *trapping horizon condition*: $\theta^{(\ell)} = 0, \delta_h\theta^{(\ell)} = 0$, (ii) a future condition $\theta^{(k)} < 0$, and (iii) an outer condition: $\delta_k\theta^{(\ell)} < 0$. These conditions can be made more explicit in terms of the variations[24,43]

$$\begin{aligned}
\delta_{\alpha\ell}\theta^{(\ell)} &= -\alpha(\sigma^{(\ell)}_{ab}\sigma^{(\ell)ab} - 8\pi T_{ab}\ell^a\ell^b),\\
\delta_{\beta k}\theta^{(\ell)} &= \beta\left[-{}^2D^c\Omega^{(\ell)}_c + \Omega^{(\ell)}_c\Omega^{(\ell)c} - \frac{1}{2}{}^2R + 8\pi T_{ab}k^a\ell^b\right] + {}^2\Delta\beta - 2\Omega^{(\ell)}_c{}^2D^c\beta, \quad (35)
\end{aligned}$$

with α and β functions on $\mathcal{S}_t$. Making $\beta = 1$, the outer condition writes

$$\delta_k\theta^{(\ell)} = -{}^2D^c\Omega^{(\ell)}_c + \Omega^{(\ell)}_c\Omega^{(\ell)c} - \frac{1}{2}{}^2R + 8\pi T_{ab}k^a\ell^b < 0, \quad (36)$$

for some ℓ^a and k^a, whereas the trapping horizon condition (with $\alpha = 1$, $\beta = C$) is

$$\delta_h\theta^{(\ell)} = \delta_\ell\theta^{(\ell)} - \delta_{Ck}\theta^{(\ell)} = \delta_\ell\theta^{(\ell)} - C\delta_k\theta^{(\ell)} - {}^2\Delta C + 2\Omega^{(\ell)}_c{}^2D^cC = 0, \quad (37)$$

that is

$$-{}^2\Delta C + 2\Omega^{(\ell)}_c{}^2D^cC - C\left[-{}^2D^c\Omega^{(\ell)}_c + \Omega^{(\ell)}_c\Omega^{(\ell)c} - \frac{1}{2}{}^2R\right] = \sigma^{(\ell)}_{ab}\sigma^{(\ell)ab} + 8\pi T_{ab}\tau^a\ell^b \quad (38)$$

This elliptic condition on C, in particular through the application of a maximum principle relying on the outer condition $\delta_k\theta^{(\ell)} < 0$, is at the heart of the nonlocal behavior of the worldtube $\bigcup_{t\in\mathbb{R}}\mathcal{S}_t$ discussed in Sec. 2.4.

Remark on the variation/deformation/stability operator $\delta_v\theta^{(\ell)}$. Before proceeding further, Eq. (35) requires some explanation. In Sec. 2.1.3, we have introduced δ_v in terms of the Lie derivative on a tensorial object. However, the expansion $\theta^{(\ell)}$ is not a scalar quantity in the sense of a point-like (tensorial) field defined on the manifold $\mathcal{M}$. The expansion is a quasi-local object whose very definition at a point $p \in \mathcal{M}$ requires the choice of a (portion of a) surface $\mathcal{S}$ passing through p. In this sense, $\delta_{\gamma v}$ (with γ a function on $\mathcal{S}$) cannot be in general evaluated as a Lie derivative. Consider a displacement of the surface $\mathcal{S}_t$ by a vector γv^a. The surface $\mathcal{S}_{t+\delta t}$ and therefore $\theta^{(\ell)}|_{t+\delta t}$ depend on the angular dependence of γ, so that $\delta_{\gamma v}\theta^{(\ell)} \neq \gamma\delta_v\theta^{(\ell)}$. The operator δ_v still satisfies a linear property for constant linear combinations, $\delta_{av+bw}\theta^{(\ell)} = a\delta_v\theta^{(\ell)} + b\delta_w\theta^{(\ell)}$ $(a, b \in \mathbb{R})$, and the Leibnitz rule, $\delta_v(\gamma\theta^{(\ell)}) = (\delta_v\gamma)\theta^{(\ell)} + \gamma\delta_v\theta^{(\ell)}$. Details about this operator can be found in Refs. 30, 24, 43.[g] Here we rather exploit a *practical trick* for the evaluation of $\delta_{\gamma v}\theta^{(\ell)}$, based on the remark that given the vector v^a normal to $\mathcal{S}$, and *not* multiplied by a function on $\mathcal{S}$, it still holds formally $\delta_v\theta^{(\ell)} = \mathcal{L}_v\theta^{(\ell)}$. Then, we can evaluate $\delta_{\gamma v}\theta^{(\ell)}$ as $\delta_{\gamma v}\theta^{(\ell)} = \delta_{\tilde{v}}\theta^{(\ell)} = \mathcal{L}_{\tilde{v}}\theta^{(\ell)}$, with $\tilde{v}^a = \gamma v^a$. In particular, the application of this strategy to the second line of (35) goes as follows. We write $\tilde{k}^a = \beta k^a$ and calculate $\delta_{\tilde{k}}\theta^{(\ell)}$ through a Lie derivative evaluation. This results in

$$\delta_{\tilde{k}}\theta^{(\ell)} = (-\tilde{k}^c\ell_c)\left[{}^2D^c\Omega_c^{(\tilde{k})} + \Omega_c^{(\tilde{k})}\Omega^{(\tilde{k})^c} - \frac{1}{2}{}^2R\right] + 8\pi T_{ab}\tilde{k}^a\ell^b. \tag{39}$$

Using $(-\tilde{k}^c\ell_c) = \beta$, $\Omega_a^{(\tilde{k})} = \Omega_a^{(k)} + {}^2D_a\ln\beta$ and $\Omega_a^{(k)} = -\Omega_a^{(\ell)}$ the expression for $\delta_{\tilde{k}}\theta^{(\ell)}$ in (35) follows (cf. footnote b).

3.2.4. *Generic properties of dynamical FOTHs*

We review some generic properties of dynamical trapping horizons.[14,24,26,32,44]

(i) *Topology law*: under the dominant energy condition, sections $\mathcal{S}_t$ are topological spheres. This can be shown by integrating $\delta_k\theta^{(\ell)} < 0$ on $\mathcal{S}_t$. Under the assumed energy condition, the Euler characteristic χ

$$\chi = \frac{1}{4\pi}\int_{\mathcal{S}} {}^2R\,{}^2\epsilon = \frac{1}{2\pi}\int_{\mathcal{S}}(-\delta_k\theta^{(\ell)} + \Omega_c^{(\ell)}\Omega^{(\ell)^c} + 8\pi T_{ab}k^a\ell^b)^2\epsilon,$$

is positive and, being $\mathcal{S}_t$ a closed 2-surface, its spherical topology follows.

(ii) *Signature law*: under the null energy condition, $\mathcal{H}$ is completely partitioned into null worldtube sections (where $\delta_\ell\theta^{(\ell)} = 0$) and space-like worldtube sections (where $\delta_\ell\theta^{(\ell)} \neq 0$ at least on a point). Applying a maximum principle to the trapping horizon constraint condition, Eq. (37), it follows that either $C =$ const ≥ 0, or C is a function $C > 0$ everywhere on $\mathcal{S}$ (cf. discussion in Sec. 2.4).

(iii) *Area law*: under the null energy condition, if $\delta_\ell\theta^{(\ell)} \neq 0$ somewhere on $\mathcal{S}_t$, the area grows locally everywhere on $\mathcal{S}_t$. Otherwise the area in constant along the evolution. This follows from applying the future condition, $\theta^{(k)} < 0$, and the signature law to $\delta_h{}^2\sqrt{q} = -C\theta^{(k)}\sqrt{q}$ [cf. Eq. (17)].

(iv) *Preferred choice of null tetrad on a DH.* According to the *foliation uniqueness* and *existence* results discussed in Sec. 3.2.1, there is a unique evolution vector

[g]See also the treatment in terms of Lie derivatives in the *double null foliations* treatment in Refs. 14 and 23.

h^a tangent to $\mathcal{H}$ and orthogonal to $\mathcal{S}_t$, such that h^a transports $\mathcal{S}_t \in \Sigma_t$ onto $\mathcal{S}_{t+\delta t} \in \Sigma_{t+\delta t}$: that is, $\delta_h t = 1$, for a given function t defining a 3+1 spacetime foliation $\{\Sigma_t\}$. Denoting the unit timelike normal to Σ_t by n^a, the lapse function by N, i.e. $n_a = -N\nabla_a t$, and the normal to $\mathcal{S}_t$ tangent to Σ_t by s^a, we can write on the horizon $\mathcal{H}_N$

$$h^a = Nn^a + bs^a, \tag{40}$$

for some b fixed from N and C in (16), as $2C = (b+N)(b-N)$. The expression of the evolution vector as $h^a = \ell^a - Ck^a$ [cf. Eq. (16)] links the scaling of ℓ^a and k^a to that of h^a. In particular, ℓ^a is singled out as the only null normal to $\mathcal{S}_t$ such that $h^a \to \ell^a$ as the trapping horizon is driven to stationarity ($C \to 0 \Leftrightarrow \delta_\ell \theta^{(\ell)} \to 0$). Writing generically the null normals at $\mathcal{H}_N$ as $\ell^a = f \cdot (n^a + s^a)$ and $k^a = (n^a - s^a)/(2f)$, Eqs. (40) and (16) lead to a preferred scaling of null normals on the DH $\mathcal{H}_N$

$$\ell^a_N = \frac{N+b}{2}(n^a + s^a), \quad k^a_N = \frac{1}{N+b}(n^a - s^a). \tag{41}$$

3.2.5. *Geometric balance equations*

One of the main motivations for the development of quasi-local horizon formalisms is the extension of the laws of black hole thermodynamics to dynamical regimes. This involves in particular finding balance equations to control the rate of change of *physical quantities* on the horizon, in terms of appropriate fluxes through the hypersurface. This is an extensive subject whose review is beyond our scope. In the spirit of the present discussion, we restrain ourselves to comment on the balance equations for two geometric quantities on $\mathcal{S}_t$: the area $A = \int_\mathcal{S} dA = \int_\mathcal{S} {}^2\epsilon$ and the angular momentum $J[\phi]$ in Eq. (15), for an axial Killing (or, more generally, divergence-free) vector ϕ^a . That is, we aim at writing

$$\frac{dA}{dt} = \int_{\mathcal{S}_t} F^A \, dA, \quad \frac{dJ[\phi]}{dt} = \int_{\mathcal{S}_t} F^J \, dA, \tag{42}$$

for appropriate area F^A and angular momentum F^J fluxes, with d/dt associated to the foliation Lie-transported by h^a. Eventually, one would aim at writing a *first law of thermodynamics* by appropriately combining the previous balance equations

$$\kappa_t \frac{dA}{dt} + \Omega_t \frac{dJ[\phi]}{dt} = \int_{\mathcal{S}_t} F^E \, dA, \tag{43}$$

for some functions κ_t and Ω_t on $\mathcal{S}_t$, so that F^E is interpreted as an energy flux.[24,26,44–50] As a first step towards (42) we write evolution equations for the expansion $\theta^{(h)}$ and the 1-form $\Omega^{(\ell)}_a$ along the evolution vector h^a. These equations are given by the projection of some of the components of the Einstein equations onto $\mathcal{H}$. Introducing a 4-*momentum current density* $p_a = -T_{ab}\tau^b$, with τ^a the vector orthogonal to $\mathcal{H}$ defined in (16), such equations provide three of the components of p_a. The fourth is given by the trapping horizon condition (38). In brief:

(i) Evolution element of area[51,52] ($p_a h^a = -T_{ab}\tau^b h^a$):

$$(\delta_h + \theta^{(h)})\theta^{(h)} = -\kappa^{(h)}\theta^{(h)} + \sigma^{(h)}_{ab}\sigma^{(\tau)\,ab} + \frac{(\theta^{(h)})^2}{2} - 2^2D^aQ_a + 8\pi T_{ab}\tau^a h^b - \frac{\theta^{(k)}}{8\pi}\delta_h C, \quad (44)$$

with $Q_a = \frac{1}{4\pi}[C\Omega^{(\ell)}_a - 1/2^2D_aC]$ and $\kappa^{(h)} = -h^b k^c \nabla_b \ell_c$.

(ii) Evolution normal (rotation) form[23,52] $\Omega^{(\ell)}_a$ ($p_b q^b{}_a = -T_{bc}\tau^c q^b{}_a$):

$$(\delta_h + \theta^{(h)})\Omega^{(\ell)}_a = {}^2D_a\kappa^{(h)} - {}^2D^c\sigma^{(\tau)}_{ac} - {}^2D_a\theta^{(h)} + 8\pi q^b{}_a T_{bc}\tau^c - \theta^{(k)\,2}D_aC. \quad (45)$$

(iii) Normal component ($p_a\tau^a = -T_{ab}\tau^b\tau^a$): linear combination, using $\tau^a = 2\ell^a - h^a$, of $T_{ab}\tau^a h^b$ (area element evolution) and $T_{ab}\tau^a\ell^b$ [trapping horizon constraint (38)].

In order to derive the evolution equation for A, we write $A = \int_{\mathcal{S}} dA = \int_{\mathcal{S}} {}^2\epsilon$ so that, using the transport of $\mathcal{S}_t$ into $\mathcal{S}_{t+\delta t}$ by h^a, we have $\frac{dA}{dt} = \int_{\mathcal{S}} \delta_h(dA) = \int_{\mathcal{S}} \theta^{(h)} dA$ and $\frac{d^2A}{dt^2} = \int_{\mathcal{S}} (\delta_h\theta^{(h)} + (\theta^{(h)})^2) dA$. From Eq. (44) it then follows

$$\frac{d^2A}{dt^2} + \bar{\kappa}'\frac{dA}{dt} = \int_{\mathcal{S}_t}\left[8\pi T_{ab}\tau^a h^b + \sigma^{(h)}_{ab}\sigma^{(\tau)\,ab} + \frac{(\theta^{(h)})^2}{2} + (\bar{\kappa}' - \kappa')\theta^{(h)}\right]{}^2\epsilon, \quad (46)$$

where $\kappa' \equiv \kappa - \delta_h \ln C$ and $\bar{\kappa}' = \bar{\kappa}(t) \equiv A^{-1}\int_{\mathcal{S}_t} \kappa'^2\epsilon$. Note that this is a second-order equation for the area.[51] Near equilibrium, the second time derivative as well as higher-order terms can be neglected leading to the Hawking and Hartle expression[53]

$$\bar{\kappa}'\frac{dA}{dt} = \int_{\mathcal{S}_t}[8\pi T_{ab}\ell^a\ell^b + \sigma^{(\ell)}_{ab}\sigma^{(\ell)\,ab}]dA.$$

Regarding the evolution equation for $J[\phi]$, we make use of Eq. (45) together with a divergence-free condition on ϕ^a (that relaxes the Killing condition) and the condition that ϕ^a is Lie-dragged by the evolution vector h^a. Then[23,54,55]

$$\frac{d}{dt}J(\phi) = -\int_{\mathcal{S}_t} T_{ab}\tau^a\phi^b\,{}^2\epsilon - \frac{1}{16\pi}\int_{\mathcal{S}_t}\sigma^{(\tau)}_{ab}\delta_\phi q^{ab}\,{}^2\epsilon, \quad (47)$$

with the second term on the right-hand side accounting for a non-Killing ϕ^a. Interestingly in dynamical (spacelike) horizons $\mathcal{H}$, the conditions ${}^2D_a\phi^a = 0$ and $\delta_h\phi^a$ completely fix[54] the form of the vector ϕ^a: $\phi^a = {}^2\epsilon^{ab\,2}D_b\theta^{(h)}$.

3.2.6. *Open geometric issues and physical remarks*

To close this generic section on geometric aspects of dynamical horizons, we list some relevant open geometric problems:

(i) *Canonical choice of dynamical trapping horizon*. DHs are highly non-unique in a given black hole spacetime. A natural question concerns the possibility of making a canonical choice. There has been some attempts in this direction based on *entropic* arguments.[51,56–58] A very interesting avenue lies on the recently introduced notion of the *core of the trapped region*[21] (see also J. M. M. Senovilla's contribution[15]).

(ii) *Asymptotics of dynamical horizons to the event horizon*. One would expect DHs to asymptote generically to the event horizon at late times. This is indeed a topic of active research.[16,59–61]

(iii) *Black hole singularity* covering *by dynamical horizons*. In addition to the asymptotics of DHs to the event horizon, it is also of interest to assess their behaviour at the *birth* of the black hole singularity, in particular their capability to separate (*dress*) singularities from the rest of the spacetime (see Sec. 5.4.4).

DHs as physical surfaces. Dynamical horizons are objects with very interesting geometric properties for the study of black hole spacetimes. In addition, from a physical perspective it is remarkable that they admit a nontrivial thermodynamical description (cf. A. Nielsen's contribution[22]). However, it is also important to underline that, if thought as boundaries of compact physical objects (in the sense we think, say, of the surface of a neutron star), then they have nonstandard physical properties:

(a) They are *non-unique*. From an Initial Value Problem perspective, the question about the evolution of a given AH is not well-posed, since it depends on the 3+1 slicing choice (such non-uniqueness in evolution is typical in gauge dynamics).

(b) Dynamical trapping horizons are *superluminal*, something difficult to reconciliate with the physical surface of an object.

(c) DHs show a *nonlocal behavior*. For instance, they grow globally (reacting *as a whole*) when *energy* crosses them at a given local region (even a point). This is a consequence of their intrinsic *elliptic*, rather than *hyperbolic*, behavior.

4. Black Hole Spacetimes in an Initial-Boundary Value Problem Approach

In the context of an Initial-Boundary Value Problem approach to the construction of spacetimes, dynamical trapping horizons play a role at two levels: (i) first, as an *a priori* ingredient to be incorporated into a given PDE formulation of Einstein equations, and (ii) as an *a posteriori* tool to extract information of the constructed spacetimes. In this section we address their application as an *a priori* ingredient.

4.1. *The initial value problem in general relativity: 3+1 formalism*

Our general basic problem is the control[62] of the qualitative and quantitative aspects of *generic* solutions to Einstein equations in dynamical scenarios involving a black hole spacetime. The Initial-Boundary Value Problem approach provides a powerful avenue to it. Such a strategy is well suited, on the one hand, to the use of global analysis and Partial Differential Equations (PDE) tools for controlling the qualitative aspects of the problem and, on the other hand, to the employment of numerical techniques to assess the quantitative ones. In particular, we focus here on the Cauchy (and hyperboloidal) Initial Value Problem.

4.1.1. *Einstein equations: Constraint and evolution system*

General Relativity is a geometric theory in which not all the fields constitute physical degrees of freedom (gauge theory), so that constraints among the fields are present. In the passage from the geometric formulation of the theory to an analytic problem in the form of a specific PDE system, several PDE subsystems enter into scene.[63] First, the *constraint system* is determined by the (Gauss–Codazzi) conditions that data on a three-dimensional Riemannian manifold must satisfy to be considered as initial data on a spacetime slice. The Hamiltonian and momentum constraints are determined by the $G_{ab}n^b$ components of the Einstein equation, where n^a is a unit timelike vector normal to the initial slice. Second, the *evolution system* is built from the rest of Einstein equation, including possible auxiliary fields. The *gauge system* determines the dynamical choice of coordinates in the spacetime. Finally, a *subsidiary system* controls the internal consistency of the previous systems.

4.1.2. *3+1 formalism*

We introduce some notation regarding the 3+1 formalism.[64] As in Sec. 3.2.4, given a 3+1 slicing of spacetime by spacelike hypersurfaces $\{\Sigma_t\}$, the unit timelike normal to Σ_t is denoted by n^a and the lapse function as N, $n_a = -N\nabla_a t$, with t the scalar function defining the 3+1 slicing. The 3+1 evolution vector is denoted by $t^a = Nn^a + \beta^a$, where β^a is the shift vector. The induced metric on Σ_t is denoted by γ_{ab}, i.e. $\gamma_{ab} = g_{ab} + n_a n_b$. We choose the following sign convention for the extrinsic curvature of Σ_t in $\mathcal{M}$: $K_{ab} = -\gamma^c{}_a\nabla_c n_b = -\frac{1}{2}\mathcal{L}_n\gamma_{ab}$. In particular, we can write $K_{ij} = \frac{1}{2N}(\gamma_{ik}D_j\beta^k + \gamma_{jk}D_i\beta^k - \dot{\gamma}_{ij})$, where the dot denotes the derivative $\mathcal{L}_t$. Indices $i, j, k, \ldots$ are used for objects leaving on Σ_t. For concreteness, we focus on a particular 3+1 decomposition of Einstein equations, namely involving the following conformal decomposition (*conformal Ansatz*[65]) for data (γ_{ij}, K^{ij}) on Σ_t:

$$\gamma_{ij} = \Psi^4\tilde{\gamma}_{ij}, \quad K_{ij} = \Psi^\zeta\tilde{A}_{ij} + \frac{1}{3}K\gamma_{ij}, \tag{48}$$

for several ζ choices. Denoting by $\tilde{D}_i$ the Levi-Civita connection associated with $\tilde{\gamma}_{ij}$ and inserting (48) into Einstein equations leads to a coupled elliptic–hyperbolic PDE system on the variables Ψ, β^i, N and $\tilde{\gamma}_{ab}$. The elliptic part has the form

$$\begin{aligned} \tilde{D}_k\tilde{D}^k\Psi - \frac{{}^3\tilde{R}}{8}\Psi &= S_\Psi[\Psi, N, \beta^i, K, \tilde{\gamma}, \ldots], \\ \tilde{D}_k\tilde{D}^k\beta^i + \frac{1}{3}\tilde{D}^i\tilde{D}_k\beta^k + {}^3\tilde{R}^i_k\beta^k &= S_\beta[\Psi, N, \beta^i, K, \tilde{\gamma}, \ldots], \\ \tilde{D}_k\tilde{D}^k N + 2\tilde{D}_k \ln\Psi\tilde{D}^k N &= S_N[N, \Psi, \beta^i, K, \tilde{\gamma}, \dot{K}, \ldots], \end{aligned} \tag{49}$$

where the equation on Ψ follows from the Hamiltonian constraint, the equation on β^i follows from the momentum constraint and the third equation on N follows from a (gauge) condition imposed on $\dot{K}$. If only solved on an initial slice with $\tilde{\gamma}_{ij}$, $\dot{\tilde{\gamma}}^{ij}$, K and $\dot{K}$ as free data, this system constitutes the *Extended Conformal Thin Sandwich* approach to initial data.[66,67] If we solve it during the whole evolution, together with

$$\frac{\partial^2\tilde{\gamma}^{ij}}{\partial t^2} - \frac{N^2}{\Psi^4}\Delta\tilde{\gamma}^{ij} - 2\mathcal{L}_\beta\frac{\tilde{\gamma}^{ij}}{\partial t} + \mathcal{L}_\beta\mathcal{L}_\beta\tilde{\gamma}^{ij} = S^{ij}_{\tilde{\gamma}}[N, \Psi, \beta^i, K, \tilde{\gamma}, \ldots], \tag{50}$$

for $\tilde{\gamma}_{ij}$, it defines a particular constrained evolution formalism.[68–70]

4.2. *Initial data: Isolated horizon inner boundary conditions*

There are two standard approaches to ensure that initial data on a slice Σ_0 correspond to a black hole spacetime. The *punctures* approach exploits the nontrivial topology[71,72] of Σ_0, whereas the *excision* approach removes a sphere from the initial slice and enforces it to be inside the black hole region. In a sense, they both reflect the *global* versus *quasi-local* discussion in Sec. 1. Here we discuss the use of inner boundary conditions derived from the IH formalism, when constructing initial data of black holes *instantaneously in equilibrium* in an *excision approach.*

4.2.1. *Non-expanding horizon conditions*

The NEH condition $\Theta^{(\ell)}_{ab} = 0$ in Eq. (26) [or (25)] provides three inner boundary conditions for the elliptic system (49). In particular, they enforce the excised surface $\mathcal{S}_0$ to be a section of a quasi-local horizon instantaneously in equilibrium.

For a given choice of free initial data in system (49), the geometric NEH inner boundary conditions, $\Theta^{(\ell)}_{ab} = 0$, must be complemented with two additional inner boundary (gauge) conditions. Denoting by s^i the normal vector to $\mathcal{S}_t$ tangent to Σ_t, we write $\beta^i = \beta^\perp s^i + \beta^i_\parallel$, with $\beta^\perp = \beta^i s_i$ and $\beta^i_\parallel s_i = 0$. Adapting the coordinate system to the horizon (i.e. $t^a = \ell^a + \beta^a_\parallel \Leftrightarrow \beta^\perp = N$) supplies a fourth gauge condition that, together with the $\theta^{(\ell)} = 0$ and $\sigma^{(\ell)}_{ab} = 0$ NEH conditions, reads[28,73–75]

$$\tilde{s}^i \tilde{D}_i \Psi + \tilde{D}_i \tilde{s}^i \Psi + \Psi^{-1} K_{ij} \tilde{s}^i \tilde{s}^j - \Psi^3 K = 0,$$
$$^2\tilde{D}_a \tilde{\beta}^{\parallel}_b + {}^2\tilde{D}_b \tilde{\beta}^{\parallel}_a - ({}^2\tilde{D}_c \beta^c_{\parallel}) \tilde{q}_{ab} = 0, \quad \beta^{\perp} = N, \tag{51}$$

where $\tilde{q}_{ab} = \Psi^4 q_{ab}$ and $\tilde{\beta}^{\parallel}_a = \tilde{q}_{ab} {\beta^{\parallel}}^b$. A fifth boundary condition, namely for N, can be obtained by choosing a slicing inner boundary condition. The (gauge) *weakly isolated horizon* structure can be used in this sense.[28,76]

4.2.2. (*Full*) *isolated horizon conditions*

The next geometric quasi-equilibrium horizon structure is a (full) IH (cf. Secs. 3.1.4 and 3.1.5). This involves three additional conditions that cannot be accommodated in system (49) for fixed free initial data. However, we can revert the argument and employ IH conditions to determine improved quasi-equilibrium free initial data $\tilde{\gamma}_{ab}$ and $\dot{\tilde{\gamma}}_{ab}$ by solving the full set of Einstein equations (49) and (50) under a *quasi-equilibrium Ansatz*. Namely, we can set $\partial_t \tilde{\gamma}^{ab}$ and $\frac{\partial^2 \tilde{\gamma}^{ab}}{\partial t^2}$ in (50) to prescribed functions f^{ab}_1 and f^{ab}_2 and consider the elliptic system formed by (49) together with

$$-\frac{N^2}{\Psi^4} \tilde{\Delta} \tilde{\gamma}^{ab} + \mathcal{L}_\beta \mathcal{L}_\beta \tilde{\gamma}^{ab} = S^{ab}_{\tilde{\gamma}} - f^{ab}_2 + 2\mathcal{L}_\beta f^{ab}_1. \tag{52}$$

This extended elliptic system is solved for ten fields: (Ψ, β^a, N) and the five $\tilde{\gamma}^{ab}$. Geometrically, we need to impose four gauge inner conditions, leaving exactly six inner conditions to be fixed. Remarkably, this fits exactly the six IH conditions[77]

$$\Theta^{(\ell)}_{ab} = 0, \quad \Theta^{(k)}_{ab} = \Theta^{(k)}_{ab}(\kappa_o, \tilde{q}_{ab}, \Omega^{(\ell)}_a) \Leftrightarrow F^{\Theta^{(k)}}_{ab}(\kappa_o, \Psi, \beta^a, N, \tilde{\gamma}_{ab}) = 0, \tag{53}$$

where $F^{\Theta^{(k)}}_{ab}$ is determined by the expression for $\Theta^{(k)}_{ab}$ in Eq. (31), fixed up to the value of the constant κ_o. It is interesting to remark that this IH prescription[77] completely fixes (up a κ_o one-parameter family) the extrinsic curvature tensor $\mathcal{K}^c_{ab} = k^c \Theta^{(\ell)}_{ab} + \ell^c \Theta^{(k)}_{ab}$ [cf. Eq. (8)] of $\mathcal{S}_0$ as embedded in the spacetime $\mathcal{M}$.

4.3. *Constrained evolutions: Trapping horizon inner boundary conditions*

The elliptic–hyperbolic system (49)–(50) provides a constrained evolution scheme for the dynamical construction of the spacetime. Adopting an excision approach to black holes, we need five inner boundary conditions for the elliptic part of the system. In principle, dynamical trapping horizon conditions on the inner boundary worldtube $\mathcal{H} = \cup_t \mathcal{S}_t$ provide a geometric prescription guaranteeing that $\mathcal{H}$ remains in the black hole region. However, imposing FOTH conditions on $\mathcal{H}$ can be *too stringent* in generic evolutions. The reason is that the constructed worldtube of MOTS $\mathcal{H}$, regarded as a hypersurface in spacetime, can change signature. This is in conflict with the outer condition in Sec. 2.2 (something related to *jumps* occurring generically[78–80] in AH evolutions; see Sec. 5.1.1) so that the resulting PDE system can become ill-posed. In this context, trapping horizon conditions together with the requirement of recovering NEH inner conditions at the equilibrium limit, provide an appropriate relaxed set of inner boundary conditions.[81] More specifically, trapping horizon conditions provides two geometric conditions $\theta^{(\ell)} = 0$ and $\delta_h\theta^{(\ell)} = 0$, whereas three additional gauge conditions guarantee the recovery of NEH at equilibrium.

As a first step, as in Sec. 4.2.1, we choose a coordinate system adapted to the horizon. This means that spacetime evolution t^a is tangent to $\mathcal{H}$. Decomposing the shift as $\beta^a = \beta^\perp s^a + \beta^a_\parallel$, then t^a is written as $t^a = Nn^a + \beta^a = (Nn^a + bs^a) + \beta^a_\parallel + (\beta^\perp - b)s^a = h^a + \beta^a_\parallel + (\beta^\perp - b)s^a$. Therefore t^a is tangent to $\mathcal{H}$ if and only if $\beta^\perp = b$.

(i) *Geometric trapping horizon conditions.* Condition $\theta^{(\ell)} = 0$ leads, in terms of the 3+1 quantities in Sec. 4.1.2, to the expression in the first line of Eq. (51). Condition $\delta_h\theta^{(\ell)} = 0$ in Eq. (38), using the adapted coordinate system $\beta^\perp = b$, leads to

$$[-{}^2D_a{}^2D^a - 2L^a{}^2D_a + A](\beta^\perp - N) = B(\beta^\perp + N), \tag{54}$$

where $L_a = K_{ij}s^iq^j{}_a$, $A = \frac{1}{2}{}^2R - {}^2D_aL^a - L_aL^a - 4\pi T_{ab}(n^a + s^a)(n^b - s^b)$, and $B = \frac{1}{2}\sigma^{(\hat{\ell})}_{ab}\sigma^{(\hat{\ell})ab} + 4\pi T_{ab}(n^a + s^b)(n^b + s^b)$, with $\hat{\ell}^a = n^a + s^a$.

(ii) *Gauge boundary conditions* I. Aiming at recovering NEH boundary conditions for $\beta^a_\parallel$, we first express $\delta_h q_{ab} = \theta^{(h)}q_{ab} + 2\sigma^{(h)}_{ab}$ in adapted coordinates ($h^a = t^a - \beta^a_\parallel$)

$$2\sigma^{(h)}_{ab} = \left(\frac{\partial q_{ab}}{\partial t} - \frac{\partial}{\partial t}\ln\sqrt{q}\;q_{ab}\right) - ({}^2D_a\beta^\parallel_b + {}^2D_b\beta^\parallel_a - {}^2D_c\beta^c_\parallel q_{ab}). \tag{55}$$

Then, the coordinate choice $\partial_t q_{ab} - \partial_t \ln\sqrt{q}\; q_{ab} = 0$ leads to the condition on $\beta^\parallel_a$

$${}^2D_a\beta^\parallel_b + {}^2D_b\beta^\parallel_a - {}^2D_c\beta^c_\parallel q_{ab} = -2\sigma^{(h)}_{ab}, \tag{56}$$

that is completed by using the evolution equation for $\sigma^{(h)}_{ab}$ on $\mathcal{H}$

$$\begin{aligned}\delta_h\sigma^{(h)}_{ab} &= -q^d{}_a q^f{}_b C^c{}_{def}\ell_c\ell^e - C^2 q^d{}_a q^f{}_b C^c{}_{def}k_ck^e \\ &\quad - 8\pi C\left[q^c{}_a q^d{}_b T_{cd} - \frac{1}{2}(q^{cd}T_{cd})q_{ab}\right] + \cdots.\end{aligned} \tag{57}$$

(iii) *Gauge boundary conditions* II. The slicing condition for N is essentially free. However, from Properties 1 and 2 in Sec. 3.2.1, such a choice is equivalent to choosing a dynamical horizon $\mathcal{H}$. Since each $\mathcal{H}$ is a genuine geometric object, this suggests the possibility of recasting into *geometric* terms the gauge choice of inner boundary condition for N, by selecting a trapping horizon $\mathcal{H}$ satisfying some specific *geometric criterion* for $\mathcal{H}$. As an example of this, maximizing the area growth rate $\dot{A}$ of $\mathcal{H}$ leads[51,81] to the condition $\beta^\perp - N = -\mathrm{const.} \cdot \theta^{(\hat{k})}$, with $\hat{k}^a = n^a - s^a$.

5. *A posteriori* Analysis of Black Hole Spacetimes

We address here the application of dynamical trapping horizons to the *a posteriori* analysis of spacetimes, their main application in the Initial Value Problem approach.

5.1. *"Tracking" the* black hole region*: AH finders*

As discussed in Sec. 1.2, event horizons cannot be located during the spacetime evolution. However, in applications such as numerical relativity, assessing if a region of spacetime lays inside the black hole region can be crucial during the evolution. Under the assumption of cosmic censorship, the location of AHs in spatial sections Σ_t and the worldtubes constructed by piling them up (see Sec. 1.3.1) are extremely useful to determine the evolutive properties of the black hole. In this sense, *AH finders* prove to be extraordinary practical tools. These are algorithms for searching surfaces $\mathcal{S}_t \subset \Sigma_t$ that satisfy the MOTS condition $\theta^{(\ell)} = 0$. There are many approaches to this problem,[82] but all of them aim at solving the condition $D_i s^i - K + K_{ij} s^i s^j = 0$. For instance, assuming spherical topology, we can characterize the surface in an adapted (spherical) coordinate system as $F(r, \theta, \varphi) = r - h(\theta, \varphi)$ with $F = \mathrm{const}$, so that the normal vector to $\mathcal{S}_t$ is given by $s_i = \frac{1}{\sqrt{D^i F \cdot D_i F}} D_i F$ with $D_i F = (1, -\partial_\theta h, -\partial_\varphi h)$ in the spherical coordinate system. The MOTS condition becomes then a nonlinear elliptic equation on h that can be solved very efficiently.

5.1.1. *Understanding AH jumps*

Noncontinuous *jumps* of AHs occur generically in 3+1 black hole evolutions. The dynamical trapping horizon framework sheds light[79,80,83] on these *AH jumps*, suggesting a spacetime picture where the jumps are understood as multiple spatial cuts of a single underlying spacetime MOTS worldtube. Jumps are associated with the change of metric type of the horizon hypersurface (see Fig. 2). This is particularly dramatic in binary black hole simulations, where at a given time t the two individual nonconnected horizons *jump* to a common one. A specific prediction of the dynamical horizon picture is that new (common) horizons form in *pairs*[37,83]: the outermost (apparent) horizon growing in area and a *dual* inner one whose area decrease in the time t. Apart from providing a better understanding of the underlying geometry of the trapped region, this spacetime picture can be of use in the study of flows interpolating between a given MOTS and the eventual event horizon, something of potential interest for studies of the Penrose inequality (see Sec. 5.2.2).

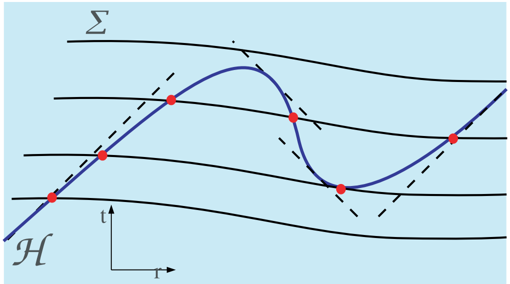

Fig. 2. (Color online) Illustration of AH jumps as multiple cuts of a single spacetime MOTS-worltube $\mathcal{H}$. In particular, timelike sections of $\mathcal{H}$ produce jumps (null hypersurfaces are represented with 45°).

5.2. *Horizon analysis parameters*

Assigning parameters to (individual) black holes can offer crucial insight into the dynamical evolution. These can be physical parameters like the mass or the angular momentum, or diagnosis parameters informing of relevant dynamical properties. Given the generic absence of background rigid structures, first-principles parameters are often out of reach and one must follow nonrigorous or *pragmatic* approaches.

5.2.1. *Mass and angular momentum. IH and DH multipoles*

In our discussion we have avoided entering into first-principles physical issues, stressing rather the geometric properties of dynamical trapping horizons and their applications. However, mass and angular momentum estimates for individual black holes, either fundamental or effective, are extremely important in the modeling of astrophysical systems involving matter or binary systems. The problem has two aspects. First, one must identify a surface to be associated with the black hole boundary. Discussion in Sec. 1 shows that this is a delicate question. In any case, AHs provide surfaces $\mathcal{S}_t \in \Sigma_t$ tracking the black hole region, that can be employed as preferred choices for pragmatic estimations. The second problem refers to the ambiguities in the quasi-local characterization of the gravitational field mass and angular momentum in General Relativity.[84,85] Regarding the angular momentum, the Komar expression (15) characterizes appropriately the axisymmetric case. Effective prescriptions[86–88] exist for generic horizons. Regarding the mass, the irreducible mass M_{irred} $A = 16\pi M_{\text{irred}}^2$ provides a purely geometric estimation in terms of the area. Its physical interpretation as the portion of the black hole mass that cannot be extracted by a Penrose process, together with its equivalence with the Hawking energy, $M_{\text{Hawking}} = \sqrt{A/(16\pi)}(1+1/(8\pi)\oint \theta^{(\ell)}\theta^{(k)}dA)$ for MOTSs, makes it useful in numerical applications and in thermodynamical treatments.[45,46] Given A and J one can also consider[32] the Christodoulou expression for the Kerr mass

$$M_{\text{Chris}} = \left(\frac{A}{16\pi} + \frac{4\pi J^2}{A}\right)^{\frac{1}{2}}. \quad (58)$$

There are many prescriptions for the *quasi-local mass*.[84,85] It is therefore crucial to choose and keep consistently a prescription when comparing different solutions. In this latter sense, the *mass* and *angular momentum* horizon geometric multipoles I_n and L_n in (34) offer a useful and refined diagnosis tool in numerical studies.[37,89]

5.2.2. *Useful diagnosis parameters*

Insight into the geometric properties of MOTS worldtubes leads to useful diagnosis parameters for monitoring dynamical evolutions. Geometric black hole inequalities provide a particular avenue. In particular, the conjectured Penrose's inequality $A \leq 16\pi M^2_{\text{ADM}}$ for asymptotically flat spacetimes provides a bound to the AH area (strictly speaking, the bound is on the area of a minimal surface enclosing the AH). A violation of $\epsilon_{\text{Penrose}} \equiv A/(16\pi M^2_{\text{ADM}}) \leq 1$ *indicates* a more exterior MOTS. In the axially symmetric case this can be refined in terms of a so-called[90,91] *Dain number*

$$\epsilon_{\text{Dain}} \equiv \frac{A}{8\pi(M^2_{\text{ADM}} + \sqrt{M^4_{\text{ADM}} - J^2})} \leq 1. \tag{59}$$

Moreover, the rigidity part of the conjecture provides an extremely simple characterization of Kerr as satisfying $\epsilon_{\text{Dain}} = 1$. In the same spirit, the geometric inequality[92] $J \leq M^2_{\text{ADM}}$ provides a characterization of (sub)extremality of black holes. However, these inequalities involve total quantities such as the ADM mass. It is remarkable that the dynamical horizon structure (actually the *outer* trapping horizon condition) provides exactly the needed conditions to prove the quasi-local inequality[42,93,94]

$$A \geq 8\pi|J|, \tag{60}$$

in generic spacetimes with matter satisfying the dominant energy condition. The validity of the area-angular momentum inequality (60) is equivalent to the non-negativity of the surface gravity κ of isolated and dynamical horizons,[32] supporting the internal consistency of their first law of black hole thermodynamics. Inequality (60) provides a quasi-local characterization of black hole (sub)extremality, that is directly related to changes in the horizon metric type[80] and jumps discussed in Sec. 5.1.1. This is also the context of the *Booth & Fairhurst extremality parameter*[80,95]

$$e \equiv 1 + \frac{1}{4\pi}\int_{\mathcal{S}} dA\ \delta_k \theta^{(\ell)} \leq 1. \tag{61}$$

5.3. *Heuristic and effective approaches in a posteriori spacetime analysis*

Hitherto we have discussed analysis tools to be applied in numerically constructed spacetimes, but related to sound geometric structures. However, when developing a qualitative understanding of the underlying dynamics, involving, e.g. a comparison with Newtonian or Special Relativity scenarios, the available geometric notions are often not enough. This is manifest in astrophysical contexts requiring estimations for linear, orbital angular momentum or binding energies. In some cases, a choice must be done between saying nothing at all or rather adopting a *heuristic* approach.

An example of the latter is the following *heuristic* proposal[96] for a quasi-local black hole linear momentum. Given a vector ξ^a transverse to a MOTS $\mathcal{S}$, applying on $\mathcal{S}$ the linear momentum ADM prescription at spatial infinity leads to

$$P(\xi) = \frac{1}{8\pi} \int_{\mathcal{S}_t} (K_{ab} - K\gamma_{ab})\xi^a s^b \, {}^2\epsilon. \tag{62}$$

In spite of its *ad hoc* nature, this quantity has been successfully applied in the analysis[96] of linear and orbital angular momentum in binary black hole orbits and in the recoil dynamics of the black hole resulting of asymmetric binary mergers.

5.4. *An effective correlation approach to the analysis of spacetime dynamics*

The qualitative and quantitative understanding of strong-field spacetime dynamics represents a challenge in gravitational physics both at a fundamental level and in applications. In astrophysical settings a natural strategy consists in extending to general relativistic scenarios the Newtonian *celestial mechanics* approach. This has indeed led to fundamental achievements in the understanding of the physics of compact objects. However, the focus on the properties of individual objects, in particular in multi-component systems, also meets fundamental obstacles in a gravitational theory (i) without *a priori* rigid structures providing canonical structures, and (ii) with global aspects playing a crucial role. The latter encompasses global causal issues and also the in-built elliptic character of certain objects, both aspects relevant in the characterization of black holes. In this context, an approach to spacetime analysis that explicitly emphasizes the global/quasi-local properties of the relevant fields, at the price of renouncing to a detailed tracking of the geometry and *trajectories* of small compact regions, can offer complementary insights to the *celestial mechanics* approach. Such a *coarse-grained effective* description is much in the spirit of the *correlation* approach in the analysis of complex condensed-matter systems or in quantum/statistical-field theory, where the functional structure of the (local) dynamical fields is encoded in the associated *n-point correlation functionals*.[h] Such an approach underlines the *relational aspects* of the theory, as a complementary methodology to the isolation of the dynamical properties a compact parts of the system. In sum, we can paraphrase the strategy as aiming at *a functional and coarse-grained description of the spacetime geometry, by importing functional tools for the analysis of condensed matter and quantum/statistical field theory systems.*

[h] N-point correlation functions encode the functional structure of the local fields. A coarse-grained description appear as a truncation to a finite number of n-point functions.

5.4.1. *Cross-correlations of geometric quantities at test screens*

The strategy outlined above is admittedly vague. We sketch now a particular implementation[97] of some of its aspects in a *cross-correlation* approach to the analysis of spacetime dynamics. Aiming at studying the gravitational dynamics in a given spacetime region $\mathcal{R}$, we consider an *outer* $\mathcal{B}_o$ and an *inner* $\mathcal{B}_i$ hypersurfaces lying in the causal future of $\mathcal{R}$. These hypersurfaces are taken as outer and inner boundaries of the bulk spacetime region of interest. The geometry of $\mathcal{B}_o$ and $\mathcal{B}_i$ is causally affected by the dynamics in $\mathcal{R}$, so that $\mathcal{B}_o$ and $\mathcal{B}_i$ can be understood as *balloon probes* into the spacetime geometry. In other words, $\mathcal{B}_o$ and $\mathcal{B}_i$ provide *test screens* (they do not *back-react* on the bulk dynamics) on which we can construct *geometric quantities* h_o and h_i to be cross-correlated. Choosing causally disconnected screens $\mathcal{B}_o$ and $\mathcal{B}_i$, a nontrivial correlation between h_o and h_i encodes geometric information about the common past region $\mathcal{R}$. We can think of this as the reconstruction of the interaction region from the debris in a *scattering* experiment (*inverse scattering* picture). Let us now restrict ourselves to the study of near-horizon spacetime dynamics.[97] In an (asymptotically flat) black hole spacetime setting, null infinity $\mathscr{I}^+$ and the (event) black hole horizon $\mathscr{E}$ provide *canonical* choices for $\mathcal{B}_o$ and $\mathcal{B}_i$, respectively (cf. Fig. 3). Retarded and advanced null coordinates u and v provide good parameters for quantities h_o and h_i calculated as integrals on sections $\mathcal{S}_u \subset \mathscr{I}^+$ and $\mathcal{S}_v \subset \mathcal{E}$. A meaningful notion for the cross-correlation between $h_o(u)$ and $h_i(v)$, considered as time series, requires the introduction of a (gauge-dependent) mapping between u and v at $\mathscr{I}^+$ and $\mathcal{E}$. We refer to this point as the *time-stretching issue*.

5.4.2. *Cross-correlations in an Initial Value Problem approach: Dynamical horizons as canonical inner probe screens*

The adopted Initial Value Problem approach has a direct impact in the *cross-correlation* picture above. In particular, the event horizon is not available during the evolution.[i] Instead, the (outermost) DH $\mathcal{H}$ fixed by the chosen 3+1 foliation stands as a natural spacetime inner boundary $\mathcal{B}_i$. Although *any* hypersurface *covering* the black hole singularity could be envisaged for the present cross-correlation purposes, the DH $\mathcal{H}$ provides a natural geometric prescription. Regarding the *time-stretching issue*, the time function t defining the 3+1 spacetime slicing automatically implements a (gauge) mapping between *retarded* and *advanced* times u and v. Cross-correlations between geometric quantities at $\mathcal{H}$ and $\mathscr{I}^+$ can then be calculated as standard time-series $h_i(t)$ and $h_o(t)$ (cf. Fig. 4). Due to the gauge nature of t, the geometric information in quantities $h_i(t)$ and $h_2(t)$ is not encoded in their *local* (arbitrary) time dependence, but rather in the *global* structure of successive maxima and minima. The calculation of cross-correlations must take this

[i] Regarding $\mathscr{I}^+$, a pragmatic choice in a Cauchy approach consists in substituting it by a timelike worldtube of large radii spheres. However, $\mathscr{I}^+$ can be kept if using a hyperboloidal foliation.

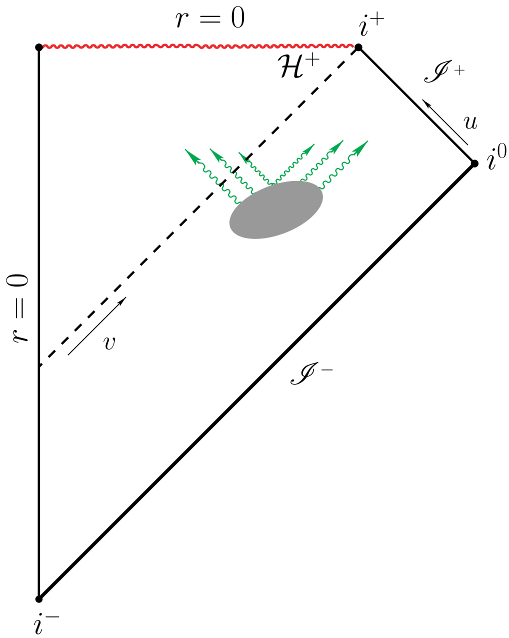

Fig. 3. (Color online) Carter–Penrose diagram representing a generic (spherically symmetric) collapse and illustrating the *cross-correlation* approach to near-horizon gravitational dynamics.

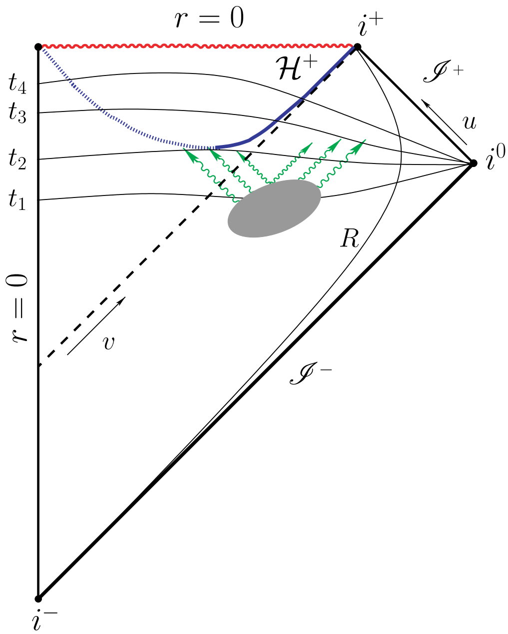

Fig. 4. (Color online) Carter–Penrose diagram for the *cross-correlation* picture in a Cauchy IVP approach.

into account.[97] This means, in particular, that quantities to be correlated must be scalars.

5.4.3. *Application to black hole recoil dynamics: Towards DH news functions*

In the context of the study of black hole recoil dynamics after an asymmetric merger, let us take $h_o(u)$ as the Bondi flux of linear momentum along a (preferred) direction

$$\frac{dP^{\rm B}[\xi]}{du}(u) = \lim_{(u,r\to\infty)} \frac{r^2}{8\pi}\oint_{\mathcal{S}_{u,r}} (\xi^i s_i)\, |\mathcal{N}(u)|^2 d\Omega, \quad \mathcal{N}(u) = \int_{-\infty}^{u} \Psi_4(u')du'. \tag{63}$$

Here $\mathcal{N}$ is the news function at $\mathscr{I}^+$, and ξ^a is a given spacelike transverse direction to $\mathcal{S}_{u,r}$, so that $(dP^{\rm B}/du)[\xi]$ is a scalar. A natural choice[j] for $h_i(v)$ would be given by the expression (63) with Ψ_4 at $\mathscr{I}^+$ substituted by some Ψ_0 at $\mathcal{H}$. A preferred null tetrad on $\mathcal{S}_v$ is then needed, something that for DHs is provided by ℓ^a_N and k^a_N in (41). Using them in (20), the preferred Weyl scalar Ψ^N_0 is employed to construct

$$\tilde{K}^N[\xi](v) = -\frac{1}{8\pi}\oint_{\mathcal{S}_v} (\xi^i s_i)|\tilde{\mathcal{N}}^{(0)}_N(v)|^2 dA, \quad \text{with } \tilde{\mathcal{N}}^{(0)}_N(v) = \int_{v_0}^{v} \Psi^N_0(v')dv'. \tag{64}$$

In spite of the formal similarity between (63) and (64) there is a fundamental difference: whereas $(dP^{\rm B}/du)[\xi]$ is an instantaneous flux through $\mathscr{I}^+$, this is not true for $\tilde{K}^N[\xi](v)$. The function $\mathcal{N}(u)$ can be written in terms of geometric quantities on sections $\mathcal{S}_u$. This *local-in-time* behavior is a crucial feature of any valid *news function* and it is not shared by $\tilde{\mathcal{N}}^{(0)}_N(v)$. However, it suffices to modify $\tilde{\mathcal{N}}^{(0)}_N(v)$ with terms completing the integrand $\Psi^N_0(v')$ to a total differential in time. Noting $q^c_a q^d_b C_{lcfd}\ell^l\ell^f = \Psi_0\overline{m}_a\overline{m}_b + \overline{\Psi}_0 m_a m_b$, inspection of Eq. (22) [actually its dynamical version with h^a instead of ℓ^a] suggests the identification of a correct *news-like* function at $\mathcal{H}$ as proportional to the shear $\sigma^{(h)}_{ab}$ (see also Refs. 99 and 100 for the discussion of the news in quasi-local contexts). In tensorial notation, we write

$$\frac{dP^N}{dv}[\xi](v) = -\frac{1}{8\pi}\oint_{\mathcal{S}_v} (\xi^i s_i)(\mathcal{N}^{\rm N,g}_{ab}\mathcal{N}^{ab}_{\rm N,g})dA, \quad \text{with } \mathcal{N}^{\rm N,g}_{ab} = -\frac{1}{\sqrt{2}}\sigma^{(h)}_{ab}, \tag{65}$$

where the coefficient in $\mathcal{N}^{\rm N,g}_{ab}$ guarantees the correct factor in the leading-term. This $(dP^N/dv)[\xi]$ provides a natural quantity to be correlated with $(dP^{\rm B}/du)[\xi]$. The notation underlines the local character in time as the *flux* of a quantity $P^N[\xi]$, but no physical meaning is given to the latter. It is worthwhile, though, to remark the

[j] We also mention an effective curvature vector[97,98] constructed from the Ricci scalar ${}^2R$ on sections $\mathcal{S}_v$ of $\mathcal{H}$, that provides an intrinsic prescription for $h_i(v)$ leading to nontrivial[97] cross-correlations with $(dP^{\rm B}/du)[\xi]$.

formal similarity of the monopolar part of the square of the news $\mathcal{N}_{ab}^{\mathrm{N,g}}$, i.e.

$$\frac{dE^N}{dv}(v) = \frac{1}{16\pi}\oint_{\mathcal{S}_v} \sigma^{(h)}_{ab}\sigma^{ab}_{(h)} dA$$
$$= \frac{1}{16\pi}\oint_{\mathcal{S}} [\sigma^{(\ell)}_{ab}\sigma^{(\ell)\,ab} - 2C\sigma^{(\ell)}_{ab}\sigma^{(k)\,ab} + C^2\sigma^{(k)}_{ab}\sigma^{(k)\,ab}] dA \tag{66}$$

with the expression of the flux of gravitational energy[26,44] through a DH, in particular with its *transverse* part.[45,46] The identification of $\sigma^{(h)}_{ab}$ as a news-like function suggests a further step, by introducing a heuristic notion of *Bondi-like* 4-momentum flux through $\mathcal{H}$. Considering the unit normal $\hat{\tau}^a$ to $\mathcal{H}$ $(\hat{\tau}^a = \tau^a/\sqrt{|\tau^b\tau_b|} = (\ell^a + Ck^a)/\sqrt{2C} = (bn^a + Ns^a)/\sqrt{2C})$, and for a generic spacetime vector η^a

$$\frac{dP^N_\tau}{dv}[\eta] = -\frac{1}{16\pi}\oint_{\mathcal{S}_v} (\eta^a\hat{\tau}_a)\sigma^{(h)}_{ab}\sigma^{ab}_{(h)} dA, \tag{67}$$

has formally the expression of a *Bondi-like* 4-momentum.[k] The flux of energy associated with an Eulerian observer n^a would be

$$\frac{dE^N_\tau}{dv}(v) \equiv \frac{dP^N_\tau}{dv}[n^a] = \frac{1}{16\pi}\oint_{\mathcal{S}} \frac{b}{\sqrt{2C}}(\sigma^{(h)}_{ab}\sigma^{(h)\,ab}) dA, \tag{68}$$

where $\frac{b}{\sqrt{2C}} = \sqrt{1 + N^2/2C}$. The flux of linear momentum for $\xi^a \in T\Sigma_t$ would be

$$\frac{dP^N_\tau}{dv}[\xi] = -\frac{1}{16\pi}\oint_{\mathcal{S}_v} \frac{N}{\sqrt{2C}}(\xi^a s_a)(\sigma^{(h)}_{ab}\sigma^{(h)\,ab}) dA. \tag{69}$$

Near equilibrium $(C \to 0)$, we have $\sigma^{(h)}_{ab}\sigma^{ab}_{(h)} \sim C$ on DHs [cf. Eq. (17)] so that expressions (68) and (69) are regular $(O(\sqrt{C}))$. Integrating (69) in time would lead to a Bondi-like counterpart[l] of the heuristic *ADM-like* linear momentum in (62).

Before finishing this section, let us mention that the present discussion on horizon news-like functions can be related[97] to a *viscous fluid analogy* for quasi-local horizons.[23,51] In particular, geometric *decay* and *oscillation timescales* (respectively, τ and T) can be constructed on the horizon[97] from the expansion $\theta^{(h)}$ and shear $\sigma^{(h)}_{ab}$, respectively related to bulk and shear viscosity terms. In the context of black hole recoil dynamics, this provides an instantaneous geometric prescription for a *slowness parameter*[101] $P = T/\tau$ controlling the qualitative aspects of the dynamics.

[k] An alternative expression would follow by using in (67), instead of $\sigma^{(h)}_{ab}\sigma^{(h)\,ab}$, the integrand in the DH energy flux,[26,44–46] that would also include the *longitudinal* part $\Omega^{(\ell)}_a\Omega^{(\ell)\,a}$.

[l] A related prescription for a DH linear momentum flux would be given by angular integration of the appropriate components in the *effective gravitational-radiation energy-tensor* of Ref. 46.

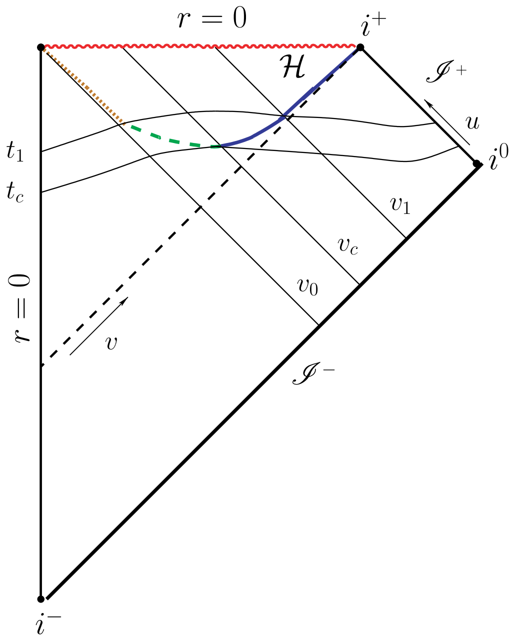

Fig. 5. (Color online) Illustration of the splitting of a DH into *internal* and *external* sections by a 3+1 slicing.

5.4.4. *The role of the inner horizon in the integration of fluxes along $\mathcal{H}$*

Flux integrations along $\mathcal{H}$ require appropriate parametrizations of $\mathcal{H}$, such as an *advanced time* v. Then, given the flux $F_Q(v)$ of a quantity $Q(v)$, we can write[m]

$$Q(v) = Q(v_0) + \text{sign}(C) \int_{v_0}^{v} F_Q(v')dv', \tag{70}$$

this requiring an initial value $Q(v_0)$. However, such coordinate v is not natural in an Initial Value Problem approach. As discussed in Sec. 5.1.1, the 3+1 slicing $\{\Sigma_t\}$ induces a splitting of the DH into *internal* and *external* sections. The integration in (70) can then be split into *external* and *internal* horizon parts (cf. Fig. 5)

$$Q(t) = Q(v_0) + \text{sign}(C)\int_{t_c}^{t} (F_Q)^{\text{int}}(t')dt' + \text{sign}(C)\int_{t_c}^{t} (F_Q)^{\text{ext}}(t')dt' + \text{Res}(t), \tag{71}$$

where the error $\text{Res}(t)$ is $\text{Res}(t) = \text{sign}(C)\int_t^{\infty}(F_Q)^{\text{int}}(t')dt'$.

If the growth of Q is understood as ultimately associated with some flow into the black hole singularity, the actual essential role of the horizon $\mathcal{H}$ would be that of capturing the associated fluxes. This assumes that the worldtube $\mathcal{H}$ *begins* at the *formation* of the singularity. More complex singularity structures (as those coming

[m]The coefficient sign(C), +1 for spacelike $\mathcal{H}$ and −1 for timelike $\mathcal{H}$, takes into account the possible integration of fluxes happening when timelike sections of $\mathcal{H}$ occur; cf. Fig. 2.

from a binary merger) would require a more detailed analysis of this point. From this perspective, there is nothing intrinsically special about dynamical horizons: *any* hypersurface separating the black hole singularity from past null infinity $\mathscr{I}^-$ (e.g. the event horizon) would be appropriate for fluxes evaluation. However, from a quasi-local perspective, if DHs are shown to *cover* systematically the black hole singularity (or, more generally, the inner Cauchy horizon), they actually provide excellent geometric prescriptions for such test screens (this is the motivation for the point (iii) in Sec. 3.2.6).

5.4.5. *Auxiliary test-field evolutions in curved backgrounds*

In Sec. 5.4.3 we have considered cross-correlations between different contractions of the Weyl tensor at distinct hypersurfaces. It is legitimate to question if such cross-correlations are meaningful at all, given the *a priori* different geometric content of the involved functions. Let us consider the following approach to this issue: evolve, together with the gravitational degrees of freedom in Einstein equations, an auxiliary (set of) scalar field(s) Φ_i without back-reaction on the geometry (i.e. *test fields*) and whose evolution on the dynamically evolving background spacetime closely tracks[n] its relevant geometric features. Then, the correlation approach outlined in Subsec. 5.4 for a (coarse-grained) extraction of geometric content, can be applied directly on Φ_i. We can paraphrase this approach as *pouring sand on a transparent surface*. On the one hand, this removes the ambiguity in the choice of quantities h_i and h_o at inner and outer hypersurfaces. On the other hand, and more importantly, it also permits to extend to the bulk spacetime the (cross-)correlation strategy between spacetime boundaries.

6. General Perspective

We have presented an introduction to some aspects of quasi-local black holes in an Initial Value Problem approach to the spacetime construction. From a fundamental perspective, quasi-local black hole horizons provide crucial insights into the geometry of the black hole and trapped regions and a sound avenue to black hole physics in generic scenarios. However, quasi-local black holes also meet challenges when considered as physical surfaces of a compact object. We have adopted a *pragmatic* or effective approach in which quasi-local black hole horizons are understood as hypersurfaces with remarkable geometric properties that provide worldtubes of canonical surfaces in a given 3+1 slicing of the spacetime. We have shown how they can be used as an *a priori* ingredient in evolution schemes to Einstein equations, where they provide inner boundary conditions for black hole spacetimes. Then we have illustrated their use as *a posteriori* analysis tools tracking and characterizing quasi-locally the black hole properties and providing, through their *rigidity*

[n]See Ref. 102 for a discussion of a similar approach in a binary black hole context, and Ref. 35 for a methodology sharing part of the spirit but directly tracking spacetime curvature quantities.

properties, excellent *test-screen* probes into the near-horizon black hole spacetime geometry.

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References

1. R. Penrose, *Annals N. Y. Acad. Sci.* **224** (1973) 125.
2. R. Penrose, *Phys. Rev. Lett.* **14** (1965) 57.
3. S. Hawking, *Proc. R. Soc. London, Ser. A* **300** (1967) 182.
4. S. Hawking and R. Penrose, *Proc. R. Soc. London, Ser. A* **314** (1970) 529.
5. S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space-Time* (Cambridge University Press, 1973).
6. R. Penrose, *Riv. Nuovo Cim.* **1** (1969) 252.
7. M. Heusler, *Liv. Rev. Relat.* **1** (1998) 6.
8. S. W. Hawking, *Phys. Rev. Lett.* **26** (1971) 1344.
9. S. W. Hawking, *Commun. Math. Phys.* **25** (1972) 152.
10. J. M. Bardeen, B. Carter and S. W. Hawking, *Commun. Math. Phys.* **31** (1973) 161.
11. V. Faraoni and A. B. Nielsen, arXiv:1103.2089.
12. M. Kriele and S. A. Hayward, *J. Math. Phys.* **38** (1997) 1593.
13. P. T. Chruściel, in *The Conformal Structure of Spacetime: Geometry, Analysis, Numerics*, J. Frauendiener and H. Friedrich (eds.), Lecture Notes in Physics (Springer, 2002), p. 61.
14. S. Hayward, *Phys. Rev. D* **49** (1994) 6467.
15. J. M. M. Senovilla, *Int. J. Mod. Phys. D* **22** (2011) 2139.
16. A. B. Nielsen, *Class. Quant. Grav.* **27** (2010) 245016.
17. D. M. Eardley, *Phys. Rev. D* **57** (1998) 2299.
18. E. Schnetter and B. Krishnan, *Phys. Rev.* **D73** (2006) 021502.
19. I. Ben-Dov, *Phys. Rev. D* **75** (2007) 064007.
20. J. E. Aman, I. Bengtsson and J. M. M. Senovilla, *J. Phys. Conf. Ser.* **229** (2010) 012004.
21. I. Bengtsson and J. M. M. Senovilla, (2010) arXiv:1009.0225.
22. A. B. Nielsen, *Int. J. Mod. Phys. D* **22** (2011) 2205.
23. E. Gourgoulhon, *Phys. Rev. D* **72** (2005) 104007.
24. I. Booth and S. Fairhurst, *Phys. Rev. D* **75** (2007) 084019.
25. I. Racz, *Class. Quant. Grav.* **25** (2008) 162001.
26. A. Ashtekar and B. Krishnan, *Phys. Rev. D* **68** (2003) 104030.
27. I. Booth, *Can. J. Phys.* **83** (2005) 1073.
28. E. Gourgoulhon and J. L. Jaramillo, *Phys. Rep.* **423** (2006) 159.
29. B. Krishnan, *Class. Quant. Grav.* **25** (2008) 114005.
30. L. Andersson, M. Mars and W. Simon, *Phys. Rev. Lett.* **95** (2005) 111102.
31. A. Ashtekar, C. Beetle and J. Lewandowski, *Phys. Rev. D* **64** (2001) 044016.

32. A. Ashtekar and B. Krishnan, *Liv. Rev. Relat.* **7** (2004) 10.
33. A. Ashtekar, C. Beetle and J. Lewandowski, *Class. Quant. Grav.* **19** (2002) 1195.
34. R. Penrose and W. Rindler, *Spinors and Space-Time*, Vol. 1: *Two-Spinor Calculus and Relativistic Fields* (Cambridge University Press, 1984).
35. R. Owen, J. Brink, Y. Chen, J. D. Kaplan, G. Lovelace, K. D. Matthews, D. A. Nichols, M. A. Scheel, F. Zhang, A. Zimmerman and K. S. Thorne, *Phys. Rev. Lett.* **106** (2011) 151101.
36. A. Ashtekar, J. Engle, T. Pawlowski and C. Van Den Broeck, *Class. Quant. Grav.* **21** (2004) 2549.
37. E. Schnetter, B. Krishnan and F. Beyer, *Phys. Rev. D* **74** (2006) 024028.
38. M. Jasiulek, *Class. Quant. Grav.* **26** (2009) 245008.
39. R. Owen, *Phys. Rev. D* **80** (2009) 084012.
40. A. Ashtekar and G. J. Galloway, *Adv. Theor. Math. Phys.* **9** (2005) 1.
41. L. Andersson, M. Mars and W. Simon, *Adv. Theor. Math. Phys.* **12** (2008) 853.
42. J. L. Jaramillo, M. Reiris and S. Dain, arXiv: 1106.3743.
43. L. M. Cao, *JHEP* **03** (2011) 112.
44. A. Ashtekar and B. Krishnan, *Phys. Rev. Lett.* **89** (2002) 261101.
45. S. Hayward, *Phys. Rev. Lett.* **93** (2004) 251101.
46. S. A. Hayward, *Phys. Rev. D* **70** (2004) 104027.
47. I. Booth and S. Fairhurst, *Phys. Rev. Lett.* **92** (2004) 011102.
48. S. A. Hayward, arXiv:0810.0923.
49. Y. H. Wu and C. H. Wang, *Phys. Rev. D* **80** (2009) 063002.
50. Y. H. Wu and C. H. Wang, *Phys. Rev. D* **83** (2011) 084044.
51. E. Gourgoulhon and J. L. Jaramillo, *Phys. Rev. D* **74** (2006) 087502.
52. E. Gourgoulhon and J. L. Jaramillo, *New Astron. Rev.* **51** (2008) 791.
53. S. W. Hawking and J. B. Hartle, *Commun. Math. Phys.* **27** (1972) 283.
54. S. A. Hayward, *Phys. Rev. D* **74** (2006) 104013.
55. S. A. Hayward, arXiv:gr-qc/0607081.
56. I. Booth, M. P. Heller and M. Spalinski, *Phys. Rev. D* **83** (2011) 061901.
57. A. B. Nielsen, M. Jasiulek, B. Krishnan and E. Schnetter, *Phys. Rev. D* **83** (2011) 124022.
58. I. Booth, M. P. Heller, G. Plewa and M. Spalinski, *Phys. Rev. D* **83** (2011) 106005.
59. C. Williams, *Annales Henri Poincare* **9** (2008) 1029.
60. C. Williams, *Commun. Math. Phys.* **293** (2010) 589.
61. C. Williams, arXiv:1005.5401.
62. H. Friedrich, *Ann. Phys.* (*Berlin*) **15** (2005) 84.
63. J. L. Jaramillo, J. A. Valiente Kroon and E. Gourgoulhon, *Class. Quant. Grav.* **25** (2008) 093001.
64. E. Gourgoulhon, Lectures delivered at Institut Henri Poincaré in 2006, arXiv:gr-qc/0703035.
65. A. Lichnerowicz, *J. Math. Pures Appl.* **23** (1944) 37. Reprinted in A. Lichnerowicz, *Choix d'œuvres mathématiques* (Hermann, Paris, 1982).
66. J. W. York Jr., *Phys. Rev. Lett.* **82** (1999) 1350.
67. H. P. Pfeiffer and J. W. York Jr., *Phys. Rev. D* **67** (2003) 044022.
68. S. Bonazzola, E. Gourgoulhon, P. Grandclément and J. Novak, *Phys. Rev. D* **70** (2004) 104007.
69. I. Cordero-Carrion, J. M. Ibanez, E. Gourgoulhon, J. L. Jaramillo and J. Novak, *Phys. Rev. D* **77** (2008) 084007.
70. I. Cordero-Carrion *et al.*, *Phys. Rev. D* **79** (2009) 024017.
71. D. Gannon, *J. Math. Phys.* **16** (1975) 2364.

72. D. Gannon, *Gen. Rel. Grav.* **7** (1976) 219.
73. J. L. Jaramillo and E. Gourgoulhon and G. A. Mena Marugán, *Phys. Rev. D* **70** (2004) 124036.
74. G. B. Cook and H. P. Pfeiffer, *Phys. Rev. D* **70** (2004) 104016.
75. S. Dain, J. L. Jaramillo and B. Krishnan, *Phys. Rev. D* **71** (2005) 064003.
76. J. L. Jaramillo, M. Ansorg and F. Limousin, *Phys. Rev. D* **75** (2007) 024019.
77. J. L. Jaramillo, *Phys. Rev. D* **79** (2009) 087506.
78. A. B. Nielsen and M. Visser, *Class. Quant. Grav.* **23** (2006) 4637.
79. I. Booth, L. Brits, J. A. Gonzalez and C. Van Den Broeck, *Class. Quant. Grav.* **23** (2006) 413.
80. I. Booth and S. Fairhurst, *Phys. Rev.* **D77** (2008) 084005.
81. J. L. Jaramillo, E. Gourgoulhon, I. Cordero-Carrion and J. M. Ibanez, *Phys. Rev. D* **77** (2008) 047501.
82. J. Thornburg, *Liv. Rev. Relat.* **10** (2007) 3.
83. J. L. Jaramillo, M. Ansorg and N. Vasset, *AIP Conf. Proc.* **1122** (2009) 308.
84. L. B. Szabados, *Liv. Rev. Relat.* **12** (2009) 4.
85. J. L. Jaramillo and E. Gourgoulhon, *Mass and Angular Momentum in General Relativity*, Springer Series, Fundamental Theories of Physics (Springer, 2010).
86. O. Dreyer, B. Krishnan, D. Shoemaker and E. Schnetter, *Phys. Rev. D* **67** (2003) 024018.
87. G. B. Cook and B. F. Whiting, *Phys. Rev. D* **76** (2007) 041501.
88. M. Korzynski, *Class. Quant. Grav.* **24** (5944) 5935.
89. N. Vasset, J. Novak and J. L. Jaramillo, *Phys. Rev. D* **79** (2009) 124010.
90. S. Dain, C. O. Lousto and R. Takahashi, *Phys. Rev. D* **65** (2002) 104038.
91. J. L. Jaramillo, N. Vasset and M. Ansorg, *EAS Publications Series* **30** (2008) 257.
92. S. Dain, *Phys. Rev. Lett.* **96** (2006) 101101.
93. J. Hennig, C. Cederbaum and M. Ansorg, *Commun. Math. Phys.* **293** (2010) 449.
94. S. Dain and M. Reiris, arXiv:1102.5215.
95. I. Booth, arXiv:0709.0934.
96. B. Krishnan, C. O. Lousto and Y. Zlochower, *Phys. Rev. D* **76** (2007) 081501.
97. J. L. Jaramillo, R. P. Macedo, P. Moesta and L. Rezzolla, arXiv:1108.0060, arXiv:1108.0061.
98. L. Rezzolla, R. P. Macedo and J. L. Jaramillo, *Phys. Rev. Lett.* **104** (2010) 221101.
99. S. A. Hayward, *Class. Quantum Grav.* **11** (1994) 3037.
100. S. A. Hayward, *Phys. Rev. D* **68** (2003) 104015.
101. R. H. Price, G. Khanna and S. A. Hughes, *Phys. Rev. D* **83** (2011) 124002.
102. E. Bentivegna, D. M. Shoemaker, I. Hinder and F. Herrmann, *Phys. Rev. D* **77** (2008) 124016.