Area – Angular momentum inequality for axisymmetric black holes

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We prove the local inequality $A \geq 8\pi|J|$, where A and J are the area and angular momentum of any axially symmetric closed stable minimal surface in an axially symmetric maximal initial data. From this theorem it is proved that the inequality is satisfied for any surface on complete asymptotically flat maximal axisymmetric data. In particular it holds for marginal or event horizons of black holes. Hence, we prove the validity of this inequality for all dynamical (not necessarily near equilibrium) axially symmetric black holes.

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Introduction. — Black holes in equilibrium are characterized by two parameters that can be chosen to be the area A (the "size" of the black hole) and the angular momentum J. Moreover, these Kerr black holes satisfy the well known lower bound $A \geq 8\pi|J|$. Fixing $|J| \neq 0$, then the smallest black hole satisfies $A = 8\pi|J|$ and is called extreme. In the realm of dynamical black holes, quantities like the quasi-local angular momentum, do not have at the moment a clear parallel. However, axially symmetric black holes form a relevant class of dynamical black holes for which the quasi-local angular momentum is, via Komar's formula, well defined. Based on heuristic physical arguments, it has been conjectured in [1] that the same lower bound $A \geq 8\pi|J|$ indeed holds for dynamical axially symmetric black holes. Interesting physical consequences on the evolution were discussed in [1]. In [2] [3] the bound was proved under several restrictive assumptions and in the stationary case with matter and charge, it has been proved in [4]. Numerical evidence was given in [5]. The purpose of this letter is to present a general proof of this inequality for axially symmetric black holes. Precisely, we extend the validity of the inequality from the unique stationary Kerr black hole to all dynamical, in principle even very far from equilibrium, dynamical axially symmetric vacuum black holes in the maximal gauge. The proof provides also new connections between black holes and stable minimal surfaces.

In order to describe the results we need to introduce some definitions. An initial data set of the Einstein vacuum equations, with cosmological constant Λ, consists in a Riemannian three-manifold S (possible with boundary), together with its first and second fundamental forms, h_{ij} and K_{ij} respectively, satisfying the constraints equations

$$R + K^2 - K_{ij}K^{ij} = 2\Lambda, \quad \nabla^i K_{ij} - \nabla_j K = 0. \qquad (1)$$

In these equations, $K = h^{ij}K_{ij}$ and R denotes the scalar curvature of h_{ij}. Initial data are called maximal if $K = 0$. The data are axially symmetric if there exists a vector field with closed orbits η^i such that

$$\mathcal{L}_\eta h_{ij} = 0, \qquad \mathcal{L}_\eta K_{ij} = 0, \qquad (2)$$

where $\mathcal{L}$ denotes the Lie derivative.

For axially symmetric data the angular momentum J associated to an arbitrary oriented closed surface Σ in S is defined by the surface integral

$$J(\Sigma) = \int_\Sigma \pi_{ij}\eta^i n^j dS_\Sigma, \qquad (3)$$

where $\pi_{ij} = K_{ij} - Kh_{ij}$ and n^i, dS_Σ are, respectively, the unit normal vector and the area element of Σ. Note that J represents the angular momentum intrinsic to the surface Σ and it depends only on the homology class of Σ. It coincides with the total angular momentum of an asymptotically flat end when the surface Σ is homologous to an sphere at infinity.

For axially symmetric data there exist two relevant scalars determined by the Killing field: the square of its norm $\eta = \eta^i\eta^j h_{ij}$ and the twist potential ω, which can be computed in terms of second fundamental form as follows (see [6] for details). Define the vectors S^i and K^i by

$$S_i = K_{ij}\eta^j - \eta^{-1}\eta_i K_{jk}\eta^j\eta^k, \quad K_i = \epsilon_{ijk}S^j\eta^k, \qquad (4)$$

where ϵ_{ijk} is the volume element. Then, the momentum constraint implies that the vector K^i is locally a gradient

$$K_i = \frac{1}{2}\nabla_i\omega. \qquad (5)$$

The twist potential ω evaluated at a surface Σ determines its angular momentum (see [2]).

We denote by γ_{AB} and χ_{AB} the intrinsic metric and the second fundamental form of a surface Σ. The surface is called minimal if its mean curvature (i.e. $\chi = \gamma^{AB}\chi_{AB}$) vanishes. A minimal surface is called stable if it is a local minimum of the area. A surface is axially symmetric if the Killing field η^i is tangent to it. For axially symmetric surfaces, we have (outside the axis) a canonical adapted triad defined by (n^i, ξ^i, η^i), where n^i is the unit normal vector to Σ and ξ^i is an unit vector tangent to the surface and orthogonal to η^i.

The local geometry near the horizon of an extreme Kerr black hole plays an important role as limit case in our result. This geometry is characterized by the concept of an extreme Kerr throat sphere, with angular momentum J, defined as follows (see [1]). The sphere is embedded in an initial data with intrinsic metric given by

$$\gamma_0 = 4J^2 e^{-\sigma_0} d\theta^2 + e^{\sigma_0} \sin^2\theta d\phi^2, \qquad (6)$$

where

$$\sigma_0 = \ln(4|J|) - \ln(1+\cos^2\theta). \qquad (7)$$

Moreover, the sphere must be totally geodesic (i.e. $\chi_{AB} = 0$), the twist potential evaluated at the surface must be given by

$$\omega_0 = -\frac{8J\cos\theta}{1+\cos^2\theta}, \qquad (8)$$

and the components of the second fundamental

$$K_{ij}\xi^i = K_{ij}n^j n^i = K_{ij}\eta^j\eta^i = 0, \qquad (9)$$

must vanish at the surface.

The following is the main result of this article.

Theorem 1. *Consider an axisymmetric, vacuum and maximal initial data, with a non-negative cosmological constant. Assume that the initial data contain an orientable closed stable minimal axially symmetric surface* Σ*. Then*

$$A \geq 8\pi|J|, \qquad (10)$$

where A *is the area and* J *the angular momentum of* Σ*. Moreover, if the equality in* (10) *holds then* $\Lambda = 0$ *and the local geometry of the surface* Σ *is an extreme Kerr throat sphere.*

Theorem 1 has a remarkable consequence. Namely, for every orientable and closed surface Σ in a (complete) axisymmetric datum with several asymptotically flat ends the inequality (10) holds. It is in particular satisfied by the event or marginal horizon of an axially symmetric black hole. This proves the conjecture raised in [1]. We will briefly outline this phenomenon. Further details will appear elsewhere. For such class of manifolds, it follows from a general result [7], that for every closed surface Σ there exist a finite set of possibly repeated stable minimal surfaces $\{\Sigma_i\}$, such the sum of its areas is equal to the infimum of the areas among all the isotopic variations of Σ. Furthermore, because $\cup\Sigma_i$ is the measure theoretical limit of isotopies of Σ ([7]), it is deduced that $|J(\Sigma)| \leq \sum |J(\Sigma_i^o)|$ were Σ_i^o are those Σ_i's that are orientable. Finally it is shown that, in our setting, each each Σ_i^o must be an axially symmetric sphere. Theorem 1 applies for each Σ_i and the claim follows.

Proof. — We first observe that if $J \neq 0$ then the surface Σ is diffeomorphic to S^2. This follows from a classical result of [8] since the integral of the scalar curvature is strictly positive on Σ. Let $F_t : \mathbb{R}\times S^2 \to S$ be a flow of surfaces parametrized by $t \in \mathbb{R}$, such that $F|_{t=0}(S^2) = \Sigma$. We impose that the family satisfies the equation $\dot{F}^i|_{t=0} = \alpha n^i$, where dot denotes derivatives with respect to t, n^i is the unit normal to Σ and α is an arbitrary function on Σ that will be fixed later on. As before, γ_{AB} and χ_{AB} denote the intrinsic metrics and the second fundamental forms of the surfaces $F_t(S^2)$.

The derivative of the mean curvature along the flow F is given by

$$\dot{\chi} = -\Delta_\Sigma\alpha - (\chi_{AB}\chi^{AB} + R_{ij}n^i n^j)\alpha, \qquad (11)$$

where Δ_Σ is the Laplacian with respect to γ_{AB}.

We use the relation

$$R_\Sigma = R - 2R_{ij}n^i n^j + \chi^2 - \chi_{AB}\chi^{AB}, \qquad (12)$$

to write $R_{ij}n^i n^j$ in terms of R_Σ (the scalar curvature of γ_{AB}) in equation (11). We obtain

$$\dot{\chi} = -\Delta_\Sigma\alpha - \frac{1}{2}(R - R_\Sigma + \chi^2 + \chi_{AB}\chi^{AB})\alpha. \qquad (13)$$

For a minimal surface $\chi = 0$ and the stability condition on Σ implies that

$$\ddot{A}|_{t=0} = \int \alpha\dot{\chi} dS_\Sigma \geq 0, \qquad (14)$$

where dS_Σ is the area element with respect to γ_{AB}.

We multiply equation (13) by α, integrate it over Σ and use condition (14) to obtain

$$\int (|D\alpha|^2 + \frac{1}{2}R_\Sigma\alpha^2) dS_\Sigma \geq \frac{1}{2}\int \left(R + \chi_{AB}\chi^{AB}\right)\alpha^2 dS_\Sigma. \qquad (15)$$

Note that only derivatives intrinsic to Σ appear on the left hand side of this inequality.

By assumption, the surface Σ is axially symmetric, therefore it intersects the axis of symmetry at two points, which we define to be the poles of Σ. A general axially symmetric metric on S^2 can be written in the form

$$\gamma = e^\sigma\left[e^{2q}d\theta^2 + \sin^2\theta d\phi^2\right], \qquad (16)$$

where σ, q are regular functions of θ. The coordinates (θ,ϕ) cover the sphere $\theta \in [0,\pi]$, $\phi \in [0, 2\pi)$. The poles are given by $\theta = 0, \pi$. The axial Killing vector is given by ∂_ϕ and the square of its norm is given by

$$\eta = e^\sigma \sin^2\theta. \qquad (17)$$

In these coordinates the determinant of the metric and the scalar curvature are given respectively by

$$\sqrt{\det(\gamma)} = e^{\sigma+q}\sin\theta, \tag{18}$$

$$R_\Sigma = \frac{e^{-\sigma-2q}}{\sin\theta}\left(2q'\cos\theta + \sin\theta\sigma' q' + 2\sin\theta - (\sin\theta\sigma')'\right), \tag{19}$$

where prime denotes derivative with respect to θ.

We want to find a change of coordinates $\tilde\theta(\theta)$ such that in the new coordinates the metric has the same form, namely

$$\gamma = e^{\tilde\sigma}\left[e^{2\tilde q}(d\tilde\theta^2) + \sin^2\tilde\theta d\phi^2\right]. \tag{20}$$

and such that

$$\tilde\sigma + \tilde q = c, \tag{21}$$

where c is a constant. Comparing (16) with (20) we obtain the following relations

$$e^{\sigma}\sin^2\theta = e^{\tilde\sigma}\sin^2\tilde\theta, \quad e^{\sigma/2+q} = e^{\tilde\sigma/2+\tilde q}\tilde\theta'. \tag{22}$$

Using these equations and the condition (21) we obtain

$$\tilde\theta'\sin\tilde\theta = e^{-c+\sigma+q}\sin\theta. \tag{23}$$

This equation can be integrated to obtain

$$\cos\tilde\theta - 1 = -e^{-c}\int_0^\theta e^{\sigma+q}\sin\bar\theta d\bar\theta. \tag{24}$$

Where we have fixed the integration constant with the condition $\tilde\theta(0) = 0$. The constant c is fixed with the condition $\tilde\theta(\pi) = \pi$. Using $\cos(\tilde\theta(\pi)) = -1$, from (24) we obtain

$$e^c = \frac{1}{2}\int_0^\pi e^{\sigma+q}\sin\bar\theta d\bar\theta. \tag{25}$$

The constant c is related to the area of the surface Σ by

$$A = \int dS_\Sigma = 2\pi\int_0^\pi e^{\sigma+q}\sin\theta d\theta = 4\pi e^c. \tag{26}$$

Note also that $dS_\Sigma = e^c dS_0$, where $dS_0 = \sin\tilde\theta d\tilde\theta d\phi$ is the area element of the standard metric in S^2.

The regularity conditions on the metric at the axis imply that $\tilde q(\tilde\theta = 0,\pi) = 0$. Hence, by equation (21), in these coordinates we have

$$\tilde\sigma(\tilde\theta = 0) = \tilde\sigma(\tilde\theta = \pi). \tag{27}$$

From now on, we assume that this coordinate system is used and we denote the functions and the coordinates without the tilde.

The key step in the proof is to chose the lapse function α to be

$$\alpha = e^{c-\sigma/2}. \tag{28}$$

Using this choice of α we can explicitly calculate the left hand side of inequality (15)

$$\int(|D\alpha|^2 + \frac{1}{2}R_\Sigma\alpha^2)dS_\Sigma = e^c\left(4\pi(c+1) - \int(\sigma + \frac{1}{4}\sigma'^2)dS_0\right), \tag{29}$$

where we have used the expression (19) for R_Σ, the condition (21) and the boundary condition (27). For the right hand side of (15) we use the Hamiltonian constraint (1) and the hypothesis that the data are maximal to write the scalar curvature as

$$R = K_{ij}K^{ij} + \Lambda. \tag{30}$$

Using the adapted triad (n^i, ξ^i, η^i), we write $K_{ij}K^{ij}$ as the following sum of positive terms

$$K_{ij}K^{ij} = \left(K_{ij}n^in^j\right)^2 + \left(K_{ij}\xi^i\xi^j\right)^2 + \eta^{-2}\left(K_{ij}\eta^i\eta^j\right)^2 + 2\left(K_{ij}\xi^in^j\right)^2 + 2\eta^{-1}\left(K_{ij}\eta^in^j\right)^2 + 2\eta^{-1}\left(K_{ij}\eta^i\xi^j\right)^2. \tag{31}$$

In [6], eq. (42), it has been proved that

$$\left(K_{ij}\eta^in^j\right)^2 = \frac{1}{4}\frac{\omega'^2}{\eta}e^{-\sigma-2q}. \tag{32}$$

Collecting these inequalities and discarding all the positive terms we obtain $8(c+1) \geq \mathcal{M}$, where the important mass functional $\mathcal{M}$ (see [1]) is defined by

$$\mathcal{M} = \frac{1}{2\pi}\int\left(\sigma'^2 + 4\sigma + \frac{\omega'^2}{\eta^2}\right)dS_0. \tag{33}$$

Using the relation between c and the area we finally obtain our main inequality

$$A \geq 4\pi e^{\frac{\mathcal{M}-8}{8}}. \tag{34}$$

Inequality (10) follows from the bound

$$2|J| \leq e^{\frac{\mathcal{M}-8}{8}}. \tag{35}$$

proved in lemma 4.1 in [2] for all σ, ω such that ω satisfies the boundary condition $\omega(0) = -\omega(\pi) = 4J$ which ensures that Σ has angular momentum J. Note that Lemma 4.1 in [2] has a larger scope (not used or required here) as it applies to the extension of the functional (33) to non-axisymmetric functions (σ, ω).

It remains to prove the rigidity statement. We will prove that, if equality in (35) holds, then $\sigma = \sigma_0$ and $\omega = \omega_0$, where σ_0 and ω_0 are given by (7) and (8). Having proved this, rigidity follows imposing $8\pi(c+1) = \mathcal{M}$, and, using equation (31) in the now equality (15), track down all the null terms.

The strategy to prove rigidity is the following. If equality in (35) is achieved for the pair (σ, ω) then, being a minimum of $\mathcal{M}$, it must be a solution of the Euler-Lagrange equations of $\mathcal{M}$. Interestingly, a solution of the Euler-Lagrange equations of $\mathcal{M}$ is also a solution of the Euler-Lagrange equations of the functional

$$\tilde{\mathcal{M}}_\epsilon = \int_\epsilon^{\pi-\epsilon} \frac{\eta'^2 + \omega'^2}{\eta^2} \sin\theta d\theta, \tag{36}$$

under smooth variations of compact support in $(\epsilon, \pi - \epsilon)$ for every $\frac{\pi}{2} > \epsilon > 0$ (see [2] for further discussions). Where η is given in terms of σ by (17). Further, making the change of variables $\bar{s} = \ln\tan\theta/2$ we have

$$\tilde{\mathcal{M}}_\epsilon = \int_{\ln\tan\epsilon/2}^{\ln\tan(\pi-\epsilon)/2} \frac{\eta'^2 + \omega'^2}{\eta^2} d\bar{s}, \tag{37}$$

where prime in this equation denotes derivative with respect to $\bar{s}$. It is well known that a critical point of this functional (for every ϵ and under variations of compact support), namely solutions of its Euler-Lagrange equations, are geodesics in the hyperbolic plane $\mathbb{H}^2$. Here we are identifying the hyperbolic plane to the half plane $\mathbb{R}^{2+} = \{(\eta, \omega)/\eta > 0\}$ together with the (hyperbolic metric) $(d\eta^2 + d\omega^2)\eta^{-2}$. The geodesics are parametrized by $\bar{s}$ where $\bar{s}$ and the arc-length are related by $s = c_1\bar{s} + c_2$ where c_1, c_2 are constants. Thus, the pair $(\eta(\theta), \omega(\theta))$ representing our minimizing solution will be a geodesic $\gamma(\bar{s}(\theta))$ in the hyperbolic plane. Now, geodesics of $\mathbb{H}^2$, are either half circles or half lines perpendicular to the axis $\{\eta = 0\}$. The boundary condition $\omega(0) = -\omega(\pi) = 4J$ on ω fixes the geodesic to be a centered half circle with radius fixed by the angular momentum and equal to $4|J|$. The only freedom left is thus that of the parametrization. This freedom, as shown below, is fixed using (27). The solution found in this way will be unique and extreme Kerr.

Following the line of reasoning described, we will compute explicitly the solution using a complex expression for geodesics in the hyperbolic plane. In complex notation $\gamma = \omega + i\eta$, it is $\gamma = \frac{ae^s i+b}{ce^s i+d}$. One has (suppose $a/c > 0$ but the analysis is the same otherwise) $4|J| = \omega(\pi) = a/c$ and $-4|J| = \omega(0) = b/d$ (note $c \neq 0$ and $d \neq 0$). Thus

$$\eta = \mathrm{Im}(\gamma) = \frac{cd(\frac{a}{c} - \frac{b}{d})e^s}{c^2e^{2s} + d^2} = \frac{8(\frac{c}{d})|J|e^s}{(\frac{c}{d})^2e^{2s} + 1}. \tag{38}$$

To find the general solution in terms of θ we need to find c_1 using the Euler-Lagrange equations of (36)

$$\left(\frac{\omega'\sin\theta}{\eta^2}\right)' = 0, \quad \left(\sin\theta\frac{\eta'}{\eta}\right)' + \frac{\omega'^2}{\eta^2}\sin\theta = 0. \tag{39}$$

The first equation implies $\omega' = \lambda_0\frac{\eta^2}{\sin^2\theta}$, where λ_0 is a constant and therefore $(\frac{\omega'}{\eta})^2 = \lambda_0^2(\frac{\eta}{\sin\theta})^2$. Inserting ω' into the second equation, multiplying it by $(\eta'\sin\theta)\eta^{-1}$ and, finally, integrating it in θ, brings us to the identity $\sin^2\theta(\eta'/\eta)^2 + \lambda_0^2\eta^2 = \lambda_1$, where λ_1 is a constant. From this identity, the expression $\eta'/\eta = \sigma' + 2\frac{\cos\theta}{\sin\theta}$ and the fact that σ is bounded at the poles we deduce that $\lambda_1 = 4$. Thus $(\eta'^2 + \omega'^2)\eta^{-2} = 4\sin^{-2}\theta$. It follows that $c_1 = 2$ and $s = c_2 + \ln\tan^2\frac{\theta}{2}$. The general solution of η is found making $\frac{c}{d}e^{c_2} = \beta$ in (38), explicitly

$$\eta = \ln 4|J| + \ln\frac{2\beta\tan^2\frac{\theta}{2}}{\beta^2\tan^2\frac{\theta}{2} + 1}, \quad \beta > 0. \tag{40}$$

The condition $\sigma(0) = \sigma(\pi)$ implies $\beta = 1$. In this case a trigonometric manipulation shows that $\eta = \ln 4|J| - \ln 1 + \cos^2\theta$ which is the expression for η of extreme Kerr.

Final remarks. — As shown in [9], any axially symmetric sphere has an adapted coordinate system as required in [2] to deduce inequality (10). It may seem thus, that part of Theorem 1 could be derived from [2] without further elaborations. In fact, the condition $\dot{\chi} \geq 0$ in [2] can be replaced by $\int \alpha\dot{\chi}dS_0 \geq 0$, which is similar to (14) but has different area element. It is not clear a priori how these two integral inequalities can be reciprocally implied. Our approach avoids the use of special coordinates.

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