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A fourth-order indirect integration method for black hole perturbations: even modes

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Abstract

On the basis of a recently proposed strategy of finite element integration in time domain for partial differential equations with a singular source term, we present a fourth-order algorithm for non-rotating black hole perturbations in the Regge–Wheeler gauge. Herein, we address even perturbations induced by a particle plunging in. The forward time value at the upper node of the (r^*, t) grid cell is obtained by an algebraic sum of (i) the preceding node values of the same cell, (ii) analytic expressions, related to the jump conditions on the wavefunction and its derivatives and (iii) the values of the wavefunction at adjacent cells. In this approach, the numerical integration does not deal with the source and potential terms directly, for cells crossed by the particle world line. This scheme has also been applied to circular and eccentric orbits and it will be the object of a forthcoming publication.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

In the scenario of the capture of compact objects by a supermassive black hole of mass M , the seized object is compared to a small mass m (henceforth the particle or the source) perturbing the background spacetime curvature and generating gravitational radiation. A comprehensive

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introduction to the general relativistic issues related to EMRI (extreme mass ratio inspiral) sources is contained in a topical volume [1].

Schwarzschild–Droste (SD) [2–5] (see [6] for a justification of this terminology) black hole perturbations have been hugely developed in the Regge–Wheeler (RW) gauge, before in vacuum [7] and after in the presence of a particle by Zerilli [8–11]. The first finite difference scheme in time domain was proposed by Lousto and Price [12]. The initial conditions, reflecting the past motion of the particle and the initial amount of gravitational waves, were parametrized by Martel and Poisson [13].

If the gravitational radiation emitted and the mass of the captured object are to be taken into account for the determination of the motion of the latter, it is necessary to compute the derivatives of the perturbations that imply the third derivative of the wavefunction $\Psi(r^*, t)$, see e.g. [14]. For a given accuracy $\mathcal{O}(h)$ of the third derivative of Ψ , the error on Ψ itself should be $\mathcal{O}(h^4)$. Effectively, the reminder ought to be $\mathcal{O}(h^5)$ due to the presence in the mesh of the particle that lowers by one more degree the convergence order of the code for geometrical effects [15]. We have therefore developed a fourth-order scheme.

The complexity in assessing the continuity of the perturbations at the position of the particle and the compatibility of the self-force to the harmonic (Lorenz–de Donder⁵) gauge [16, 17] has led researchers to convey their efforts to this gauge, as commenced by Barack and Lousto [19]. Conversely, work in harmonic gauge is made cumbersome by the presence of a system of ten coupled equations which replace the single wave equation of the RW gauge.

We have proposed [20, 21] a finite element method of integration, in RW gauge, based on the jump conditions that the wavefunction and its derivatives have to satisfy for the SD black hole perturbations to be continuous at the position of the particle. We first deal with the radial trajectory and the associated even parity perturbations, while in a forthcoming paper we present the circular and eccentric orbital cases, thus referring to both odd and even parity perturbations.

The main feature of this method consists in avoiding the direct and explicit integration of the wave equation (the potential and the source term with the associated singularities) whenever the grid cells are crossed by the particle. Indeed, the information on the wave equation is implicitly given by the jump conditions on the wavefunction and its derivatives. Conversely, for cells not crossed by the particle world line, the integrating method might retain the previous approach by Lousto [22] and Haas [15]. Among the efforts using jump discontinuities, although in a different context, it is worthwhile to mention those of Haas [15] and Sopena and co-workers [23–25] getting the self-force in a scalar case. For the geodesic gravitational case, like Sopena and co-workers, Jung *et al* [26] and Chakraborty *et al* [27] rely on spectral methods; Zumbusch [28] and Field *et al* [29] use a discontinuous Galerkin method; Hopper and Evans [30] work partially in the frequency domain. Among recent results not based on jump discontinuities but concerning fourth-order time domain codes, the one proposed by Thornburg [31] deals with an adaptive mesh refinement, while Nagar and co-workers replace the delta distribution with a narrow Gaussian [32, 33].

For the computation of the back-action, this method ensures a well-behaved wavefunction at the particle position, since the approach is governed by the analytical values of the jump conditions at the particle position.

In [21] we have provided waveforms at infinity and the wavefunction at the position of the particle at first order. Herein, we focus instead on the improvement of the algorithm at fourth order and refer to [21] for all complementary information. The features of this method can be summarized as follows.

⁵ FitzGerald is considered to have also identified the harmonic gauge [18].

- Avoidance of direct and explicit integration of the wave equation (the potential and the source term with the associated singularities) for the grid cells crossed by the particle.
- Improvement of the reliability, since analytic expressions partly replace numerical ones (the replacement is total at first order [20, 21]).
- Applicability of the method to generic orbits, assuming that the even and odd wave equations are satisfied by Ψ , respectively R , being C^{-16} .

Geometric units ($G = c = 1$) are used, unless stated otherwise. The metric signature is $(-, +, +, +)$.

2. The wave equation

The wavefunction (its dimension is such that the energy is proportional to $\int_0^\infty \Psi^2 dt$), in the Moncrief form [34] and RW gauge [7], is defined by

$$\Psi_l(t, r) = \frac{r}{\lambda + 1} \left[K^l + \frac{r - 2M}{\lambda r + 3M} \left(H_2^l - r \frac{\partial K^l}{\partial r} \right) \right], \quad (1)$$

where $K(t, r)$ and $H_2(t, r)$ are the perturbations, and the Zerilli [9] normalization is used for Ψ_l . The wave equation is given by the operator \mathcal{Z} acting on the wavefunction

$$\mathcal{Z}\Psi^l(t, r) = \partial_{r^*}^2 \Psi^l(t, r) - \partial_t^2 \Psi^l(t, r) - V^l(r) \Psi^l(t, r) = S^l(t, r), \quad (2)$$

where $r^* = r + 2M \ln(r/2M - 1)$ is the tortoise coordinate and the potential $V^l(r)$ is

$$V^l(r) = \left(1 - \frac{2M}{r} \right) \frac{2\lambda^2(\lambda+1)r^3 + 6\lambda^2Mr^2 + 18\lambda M^2r + 18M^3}{r^3(\lambda r + 3M)^2}, \quad (3)$$

being $\lambda = 1/2(l-1)(l+2)$. The source $S^l(t, r)$ includes the derivative of the Dirac distribution (the latter appears in the process of forming the wave equation out of the ten linearized Einstein equations)

$$S^l = \frac{2(r-2M)\kappa}{r^2(\lambda+1)(\lambda r+3M)} \times \left\{ \frac{r(r-2M)}{2U^0} \delta'[r-r_u(t)] - \left[\frac{r(\lambda+1)-3M}{2U^0} - \frac{3MU^0(r-2M)^2}{r(\lambda r+3M)} \right] \delta[r-r_u(t)] \right\}, \quad (4)$$

$U^0 = E/(1-2M/r_u)$ being the time component of the 4-velocity, $E = \sqrt{1-2M/r_{u0}}$ the conserved energy per unit mass and $\kappa = 4m\sqrt{(2l+1)\pi}$. The geodesic in the unperturbed SD metric $z_u(\tau) = \{t_u(\tau), r_u(\tau), \theta_u(\tau), \phi_u(\tau)\}$ assumes different forms according to the initial conditions. For radial infall of a particle starting from rest at finite distance r_{u0} , $r_u(t)$ is the inverse function in coordinate time t of the trajectory in the background field, corresponding to the geodesic in proper time τ (u stands for unperturbed):

$$\begin{aligned} \frac{t(r_u)}{2M} &= \sqrt{1 - \frac{2M}{r_{u0}}} \sqrt{1 - \frac{r_u}{r_{u0}}} \left(\frac{r_{u0}}{2M} \right) \left(\frac{r_u}{2M} \right)^{1/2} + 2 \operatorname{arctanh} \left(\frac{\sqrt{\frac{2M}{r_u} - \frac{2M}{r_{u0}}}}{\sqrt{1 - \frac{2M}{r_{u0}}}} \right) \\ &+ \sqrt{1 - \frac{2M}{r_{u0}}} \left(1 + \frac{4M}{r_{u0}} \right) \left(\frac{r_{u0}}{2M} \right)^{3/2} \arctan \left(\sqrt{\frac{r_{u0}}{r} - 1} \right). \end{aligned} \quad (5)$$

The above expressions correspond to those in [14], where some of the errors of the previously published literature on radial fall are indicated.

⁶ A C^{-1} continuity class element, like a Heaviside step distribution, may be seen as an element which after integration transforms into an element belonging to the C^0 class of functions.

3. Jump conditions

From the visual inspection of the Zerilli wave equation (2), it is evinced that the wavefunction Ψ is of C^{-1} continuity class (the second derivative of the wavefunction is proportional to the first derivative of the Dirac distribution, in itself a C^{-3} class element). Thus, the wavefunction and its derivatives can be written as (the l index is dropped henceforth for simplicity of notation)

$$\Psi = \Psi^+ \Theta_1 + \Psi^- \Theta_2, \quad (6)$$

$$\Psi_{,r} = \Psi_{,r}^+ \Theta_1 + \Psi_{,r}^- \Theta_2 + (\Psi^+ - \Psi^-) \delta, \quad (7)$$

$$\Psi_{,t} = \Psi_{,t}^+ \Theta_1 + \Psi_{,t}^- \Theta_2 - \dot{r}_u (\Psi^+ - \Psi^-) \delta, \quad (8)$$

$$\Psi_{,rr} = \Psi_{,rr}^+ \Theta_1 + \Psi_{,rr}^- \Theta_2 + 2(\Psi_{,r}^+ - \Psi_{,r}^-) \delta + (\Psi^+ - \Psi^-) \delta', \quad (9)$$

$$\Psi_{,tt} = \Psi_{,tt}^+ \Theta_1 + \Psi_{,tt}^- \Theta_2 - 2\dot{r}_u (\Psi_{,t}^+ - \Psi_{,t}^-) \delta - \ddot{r}_u (\Psi^+ - \Psi^-) \delta + \dot{r}_u^2 (\Psi^+ - \Psi^-) \delta', \quad (10)$$

$$\Psi_{,tr} = \Psi_{,tr}^+ \Theta_1 + \Psi_{,tr}^- \Theta_2 + (\Psi_{,t}^+ - \Psi_{,t}^-) \delta - \dot{r}_u (\Psi_{,r}^+ - \Psi_{,r}^-) \delta - \dot{r}_u (\Psi^+ - \Psi^-) \delta', \quad (11)$$

where in shortened notation $\Theta_1 = \Theta[r - r_u(t)]$, and $\Theta_2 = \Theta[r_u(t) - r]$ are two Heaviside step distributions, while $\delta = \delta[r - r_u(t)]$ and $\delta' = \delta'[r - r_u(t)]$ are the Dirac delta—and its derivative—distributions, respectively. The dot and the prime indicate time and space derivatives, respectively.

3.1. Jump conditions from the wave equation

For the computation of back-action effects, we need first-order derivatives of the perturbations and thus third-order wavefunction derivatives. To this end, we operate directly on the wave equation, equation (2). The source term is cast in the following form:

$$S(t, r) = G(t, r) \delta + F(t, r) \delta' = \tilde{G}_{r_u(t)} \delta + F_{r_u(t)} \delta', \quad (12)$$

where $\tilde{G}_{r_u(t)} = G_{r_u(t)} - F'_{r_u(t)}$ and one of the properties of the Dirac delta distribution, namely $\phi(r) \delta'[r - r_u(t)] = \phi_{r_u(t)} \delta'[r - r_u(t)] - \phi'_{r_u(t)} \delta[r - r_u(t)]$, has been used at the position of the particle. The subscript implies that we consider the value of a given function at a point or on the trajectory. The determination of the jump conditions imposes the transformation of equation (2) into the corresponding equation in the (r, t) domain (the tortoise coordinate can't be inverted). Turning to the r variable, we obtain ($f = 1 - 2M/r$)

$$\begin{aligned} \partial_{r^*}^2 \Psi &= f f' \partial_r \Psi + f^2 \partial_r^2 \Psi \\ &= [f f' \Psi_{,r}^+ + f^2 \Psi_{,rr}^+] \Theta_1 + [f f' \Psi_{,r}^- + f^2 \Psi_{,rr}^-] \Theta_2 + f f' (\Psi^+ - \Psi^-) \delta \\ &\quad + 2f^2 (\Psi_{,r}^+ - \Psi_{,r}^-) \delta + f^2 (\Psi^+ - \Psi^-) \delta', \end{aligned} \quad (13)$$

$$\partial_t^2 \Psi = \Psi_{,tt}^+ \Theta_1 + \Psi_{,tt}^- \Theta_2 - 2\dot{r}_u \partial_t (\Psi^+ - \Psi^-) \delta - \ddot{r}_u (\Psi^+ - \Psi^-) \delta + \dot{r}_u^2 (\Psi^+ - \Psi^-) \delta', \quad (14)$$

$$V \Psi = V \Psi^+ \Theta_1 + V \Psi^- \Theta_2. \quad (15)$$

The notation $[\Psi]$ stands for the difference $(\Psi^+ - \Psi^-)_{r_u}$ and a likewise notation is used for the derivatives at the point r_u . Equating the coefficients of δ' , and owing to the above-mentioned property of the delta derivative for which $(\Psi^+ - \Psi^-) \delta' = [\Psi] \delta' - [\Psi_{,r}] \delta$, we obtain the jump condition for Ψ :

$$[\Psi] = \frac{1}{f_{r_u}^2 - \dot{r}_u^2} F_{r_u}. \quad (16)$$

Equating the coefficients of δ , we obtain the jump condition on the space derivative:

$$[\Psi_{,r}] = \frac{1}{f_{r_u}^2 - \dot{r}_u^2} \left[\tilde{G}_{r_u} + (f_{r_u} f'_{r_u} - \dot{r}_u) [\Psi] - 2\dot{r}_u \frac{d}{dr_u} [\Psi] \right], \quad (17)$$

and therefore the jump condition on the first time derivative:

$$[\Psi_{,t}] = \dot{r}_u \frac{d}{dr_u} [\Psi] - \dot{r}_u [\Psi_{,r}]. \quad (18)$$

Since $\mathcal{Z}\Psi^\pm = 0$, the coefficients of Θ_1 and Θ_2 ought to be equal. We thus obtain

$$[\Psi_{,tt}] - f_{r_u} f'_{r_u} [\Psi_{,r}] - f_{r_u}^2 [\Psi_{,rr}] + V_{r_u} [\Psi] = 0, \quad (19)$$

which is an equation with two unknowns. We circumvent the difficulty by using (i) the commutativity of the derivatives, $[\Psi_{,tr}] = [\Psi_{,rt}]$, (ii) the transformation $d/dt = \dot{r}_u d/dr_u$, and write

$$\begin{aligned} [\Psi_{,tt}] &= \frac{d}{dt} [\Psi_{,t}] - \dot{r}_u [\Psi_{,tr}] = \frac{d}{dt} [\Psi_{,t}] - \dot{r}_u \left\{ \frac{d}{dt} [\Psi_{,r}] - \dot{r}_u [\Psi_{,rr}] \right\} \\ &= \dot{r}_u \frac{d}{dr_u} [\Psi_{,t}] - \dot{r}_u^2 \frac{d}{dr_u} [\Psi_{,r}] + \dot{r}_u^2 [\Psi_{,rr}]. \end{aligned} \quad (20)$$

The jump condition on the second space derivative can now be expressed by

$$[\Psi_{,rr}] = \frac{1}{f_{r_u}^2 - \dot{r}_u^2} \left\{ \dot{r}_u \frac{d}{dr_u} [\Psi_{,t}] - \dot{r}_u^2 \frac{d}{dr_u} [\Psi_{,r}] - f_{r_u} f'_{r_u} [\Psi_{,r}] + V_{r_u} [\Psi] \right\}. \quad (21)$$

The other second derivatives are obtained by

$$[\Psi_{,tr}] = [\Psi_{,rt}] = \frac{d}{dt} [\Psi_{,r}] - \dot{r}_u [\Psi_{,rr}], \quad (22)$$

$$[\Psi_{,tt}] = \frac{d}{dt} [\Psi_{,t}] - \dot{r}_u [\Psi_{,tr}]. \quad (23)$$

For the third-order derivatives, we derive the wave equation with respect to r and obtain

$$\begin{aligned} [\Psi_{,rrr}] &= \frac{1}{\dot{r}_u^2 - f_{r_u}^2} \left\{ \dot{r}_u^2 \frac{d}{dr_u} [\Psi_{,rr}] - \dot{r}_u \frac{d}{dr_u} [\Psi_{,rt}] \right. \\ &\quad \left. + (f_{r_u}^2 + f_{r_u} f''_{r_u} - V_{r_u}) [\Psi_{,r}] + 3f_{r_u} f'_{r_u} [\Psi_{,rr}] - V'_{r_u} [\Psi] \right\}, \end{aligned} \quad (24)$$

while deriving with respect to t , we obtain

$$[\Psi_{,ttt}] = \frac{\dot{r}_u^2}{\dot{r}_u^2 - f_{r_u}^2} \left\{ f_{r_u}^2 \frac{d}{dr_u} [\Psi_{,rt}] - \dot{r}_u^{-1} f_{r_u}^2 \frac{d}{dr_u} [\Psi_{,tt}] + f_{r_u} f'_{r_u} [\Psi_{,rt}] - V_{r_u} [\Psi_{,t}] \right\}, \quad (25)$$

$$[\Psi_{,ttr}] = [\Psi_{,trt}] = [\Psi_{,rtt}] = \frac{d}{dr_u} [\Psi_{,tr}] - \dot{r}_u^{-1} [\Psi_{,ttt}], \quad (26)$$

$$[\Psi_{,trr}] = [\Psi_{,rtr}] = [\Psi_{,rrt}] = \frac{d}{dr_u} [\Psi_{,tr}] - \dot{r}_u^{-1} [\Psi_{,ttr}], \quad (27)$$

$$[\Psi_{,rrr}] = \frac{d}{dr_u} [\Psi_{,rr}] - \dot{r}_u^{-1} [\Psi_{,trr}]. \quad (28)$$

Finally, we similarly proceed for the fourth derivatives

$$[\Psi_{,tttt}] = \frac{\dot{r}_u^2}{\dot{r}_u^2 - f_{r_u}^2} \times \left\{ f_{r_u}^2 \frac{d}{dr_u} [\Psi_{,ttr}] - \dot{r}_u^{-1} f_{r_u}^2 \frac{d}{dr_u} [\Psi_{,ttt}] + f_{r_u} f'_{r_u} [\Psi_{,ttr}] - V_{r_u} [\Psi_{,tt}] \right\}, \quad (29)$$

$$[\Psi_{,tttr}] = [\Psi_{,trtt}] = [\Psi_{,trtt}] = [\Psi_{,rttt}] \frac{d}{dr_u} [\Psi_{,ttt}] - \dot{r}_u^{-1} [\Psi_{,tttt}], \quad (30)$$

$$[\Psi_{,ttrr}] = [\Psi_{,trtr}] = [\Psi_{,trrt}] = [\Psi_{,rttr}] [\Psi_{,trtt}] = [\Psi_{,rrtt}] = \frac{d}{dr_u} [\Psi_{,ttr}] - \dot{r}_u^{-1} [\Psi_{,tttr}], \quad (31)$$

$$[\Psi_{,trrr}] = [\Psi_{,rttr}] = [\Psi_{,rrrt}] [\Psi_{,rrtt}] \frac{d}{dr_u} [\Psi_{,ttr}] - \dot{r}_u^{-1} [\Psi_{,tttr}], \quad (32)$$

$$[\Psi_{,rrrr}] = \frac{d}{dr_u} [\Psi_{,rrr}] - \dot{r}_u^{-1} [\Psi_{,rrrt}]. \quad (33)$$

3.1.1. Jump conditions in explicit form. We list hereafter the jump conditions in explicit form.

Jump conditions.

$$[\Psi] = \frac{\kappa E r_u}{(\lambda + 1)(3M + \lambda r_u)} \quad (34)$$

First-derivative jump conditions.

$$[\Psi_{,t}] = -\frac{\kappa E r_u \dot{r}_u}{(2M - r_u)(3M + \lambda r_u)} \quad (35)$$

$$[\Psi_{,r}] = \frac{\kappa E [6M^2 + 3M\lambda r_u + \lambda(\lambda + 1)r_u^2]}{(\lambda + 1)(2M - r_u)(3M + \lambda r_u)^2} \quad (36)$$

Second-derivative jump conditions.

$$[\Psi_{,rr}] = -\frac{\kappa E [3M^3(5\lambda - 3) + 6M^2\lambda(\lambda - 3)r_u + 3M\lambda^2(\lambda - 1)r_u^2 - 2\lambda^2(\lambda + 1)r_u^3]}{(\lambda + 1)(2M - r_u)^2(3M + \lambda r_u)^3} \quad (37)$$

$$[\Psi_{,tr}] = \frac{\kappa E (3M^2 + 3M\lambda r_u - \lambda r_u^2) \dot{r}_u}{(2M - r_u)^2(3M + \lambda r_u)^2} \quad (38)$$

$$[\Psi_{,tt}] = -\frac{\kappa E M}{r_u^2 (3M + r_u \lambda)} \quad (39)$$

Third-derivative jump conditions.

$$[\Psi_{,rrr}] = \frac{\kappa E}{r_u(\lambda + 1)(2M - r_u)^3(3M + r_u \lambda)^4} \left[81(\lambda + 1)M^5 + 9r_u(19\lambda^2 + 18E^2\lambda + 3\lambda + 18E^2)M^4 + 9r_u^2\lambda(7\lambda^2 + 24E^2\lambda - 14\lambda + 24E^2 + 3)M^3 + 3r_u^3\lambda^2(7\lambda^2 + 36E^2\lambda - 11\lambda + 36E^2 + 18)M^2 + 3r_u^4\lambda^3(8E^2\lambda - 7\lambda + 8E^2 - 1)M + 2r_u^5\lambda^3(\lambda + 1)(E^2\lambda + 3) \right] \quad (40)$$

$$[\Psi_{,trr}] = \frac{-\kappa E \dot{r}_u}{r_u(2M - r_u)^3(3M + r_u\lambda)^3} [27M^4 + 6r_u(5\lambda + 9E^2 - 3)M^3 + 3r_u^2\lambda(5\lambda + 18E^2 - 6)M^2 + 6r_u^3\lambda^2(3E^2 - 2)M + 2r_u^4\lambda^2(E^2\lambda + 1)] \quad (41)$$

$$[\Psi_{,ttr}] = \frac{\kappa E}{r_u^3(2M - r_u)(3M + r_u\lambda)^2} [39M^3 + 9r_u(3\lambda + 2E^2 - 2)M^2 + r_u^2\lambda(4\lambda + 12E^2 - 13)M + 2r_u^3\lambda^2(E^2 - 1)] \quad (42)$$

$$[\Psi_{,ttt}] = \frac{-\kappa E \dot{r}_u}{r_u^3(2M - r_u)(3M + r_u\lambda)} [9M^2 + 2r_u(2\lambda + 3E^2 - 2)M + 2r_u^2\lambda(E^2 - 1)] \quad (43)$$

Fourth-derivative jump conditions.

$$[\Psi_{,rrrr}] = \frac{-3\kappa E}{r_u^2(\lambda + 1)(2M - r_u)^4(3M + r_u\lambda)^5} [567(\lambda + 1)M^7 + 162r_u(\lambda + 1)(6\lambda + 16E^2 - 5)M^6 + 6r_u^2(139\lambda^3 + 738E^2\lambda^2 - 123\lambda^2 + 162E^4\lambda + 441E^2\lambda - 171\lambda + 162E^4 - 297E^2 + 27)M^5 + 12r_u^3\lambda(21\lambda^3 + 252E^2\lambda^2 - 85\lambda^2 + 135E^4\lambda - 24\lambda + 135E^4 - 252E^2 + 18)M^4 + 3r_u^4\lambda^2(21\lambda^3 + 344E^2\lambda^2 - 95\lambda^2 + 360E^4\lambda - 340E^2\lambda + 100\lambda + 360E^4 - 684E^2 + 24)M^3 + 2r_u^5\lambda^3 \cdot \times (88E^2\lambda^2 - 47\lambda^2 + 180E^4\lambda - 260E^2\lambda + 25\lambda + 180E^4 - 348E^2 - 24)M^2 + 2r_u^6\lambda^4(6E^2\lambda^2 + 30E^4\lambda - 53E^2\lambda + 23\lambda + 30E^4 - 59E^2 + 11)M + 4r_u^7\lambda^4(\lambda + 1)(E^4\lambda - 2E^2\lambda - 2)] \quad (44)$$

$$[\Psi_{,trrr}] = \frac{3\kappa E \dot{r}_u}{r_u^2(2M - r_u)^4(3M + r_u\lambda)^4} [135M^6 + 27r_u(7\lambda + 32E^2 - 6)M^5 + 3r_u^2 \cdot \times (35\lambda^2 + 396E^2\lambda - 75\lambda + 108E^4 - 144E^2 + 18)M^4 + r_u^3\lambda(35\lambda^2 + 612E^2\lambda - 120\lambda + 432E^4 - 594E^2 + 72)M^3 + r_u^4\lambda^2(140E^2\lambda - 45\lambda + 216E^4 - 306E^2 + 36)M^2 + 2r_u^5\lambda^3(6E^2\lambda + 24E^4 - 35E^2 + 9)M + 2r_u^6\lambda^3(2E^4\lambda - 3E^2\lambda - 1)] \quad (45)$$

$$[\Psi_{,ttrr}] = \frac{-\kappa E}{r_u^4(2M - r_u)^2(3M + r_u\lambda)^3} [1431M^5 + 6r_u(251\lambda + 234E^2 - 210)M^4 + 9r_u^2(59\lambda^2 + 160E^2\lambda - 148\lambda + 36E^4 - 66E^2 + 30)M^3 + 6r_u^3\lambda(10\lambda^2 + 82E^2\lambda - 79\lambda + 54E^4 - 102E^2 + 48)M^2 + 2r_u^4\lambda^2(28E^2\lambda - 27\lambda + 54E^4 - 105E^2 + 52)M + 12r_u^5\lambda^3(E^2 - 1)^2] \quad (46)$$

$$[\Psi_{,tttr}] = \frac{\kappa E \dot{r}_u}{r_u^4(2M - r_u)^2(3M + r_u\lambda)^2} [243M^4 + 3r_u(61\lambda + 132E^2 - 64)M^3 + 3r_u^2 \cdot \times (12\lambda^2 + 92E^2\lambda - 49\lambda + 36E^4 - 48E^2 + 12)M^2 + 2r_u^3\lambda(24E^2\lambda - 15\lambda + 36E^4 - 51E^2 + 14)M + 6r_u^4\lambda^2(E^2 - 1)(2E^2 - 1)] \quad (47)$$

$$[\Psi_{,tttt}] = \frac{-\kappa E}{r_u^6(3M + r_u\lambda)} \left[189M^3 + 2r_u(36\lambda + 84E^2 - 77)M^2 + 6r_u^2(E^2 - 1)(10\lambda + 6E^2 - 5)M + 12r_u^3\lambda(E^2 - 1)^2 \right]. \quad (48)$$

While heuristic arguments [35, 36] have been put forward to show that, for radial fall in the RW gauge, even metric perturbations belong to the C^0 continuity class at the position of the particle, in [20, 21] we have provided an analysis *vis à vis* the jump conditions that the wavefunction and its (first and second) derivatives have to satisfy for guaranteeing the continuity of the perturbations at the position of the particle. Therein, we have derived the same jump conditions (34–38) from the inverse relations (expressions giving the perturbations as a function of the wavefunction and its derivatives) by fulfilment of the continuity conditions (equal coefficients for the two Heaviside distributions, and null coefficients for the Dirac distribution and its derivative).

4. The algorithm

The integration method considers cells belonging to two groups: for cells never crossed by the world line, the integrating method may be drawn by previous approaches explored by Lousto [22] and Haas [15], whereas for cells crossed by a particle, we propose a new algorithm. The grid is in the r^*, t domain.

Initial conditions require knowledge of the situation prior to $t = 0$. At fourth order, the wavefunction may be Taylor-expanded around $t = 0$. For the boundary conditions, simplicity suggests a sufficiently huge grid to avoid unwanted reflections.

4.1. Empty cells

Empty cells are those cells which are not crossed by the particle. In this case, the cell upper point is obtained by performing an integration of the wave equation over the entire surface A of the cell, identified by the nodes $\alpha, \beta, \gamma, \delta$. We briefly recall the algorithm used by Haas [15]. Therein, the sole numerical computation to be carried out is represented by the product of the potential term and the wavefunction $V\Psi = g$. It is performed via a double Simpson integral, using points of the past light cone of the upper node α , figure 1. We set $g_q = g(r_q^*, t_q) = V(r_q)\Psi(r_q^*, t_q)$, $V_q = V(r_q)$ and $\Psi_q = \Psi(r_q^*, t_q)$, where q is one of the points shown in figure 1. The increment h is defined as $h = \frac{1}{2}\Delta r^* = \frac{1}{2}\Delta t$ where Δr^* is the spatial step and Δt is the time step.

We have

$$\int \int_{\text{Cell}} g dA = \left(\frac{h}{3}\right)^2 [g_\alpha + g_\beta + g_\gamma + g_\delta + 4(g_{\beta\gamma} + g_{\alpha\beta} + g_{\delta\gamma} + g_{\alpha\delta}) + 16g_\sigma] + \mathcal{O}(h^6), \quad (49)$$

where the sum of the intermediate terms between nodes is given by

$$g_{\beta\gamma} + g_{\alpha\beta} + g_{\delta\gamma} + g_{\alpha\delta} = 2V_\sigma\Psi_\sigma \left[1 - \frac{1}{2}\left(\frac{h}{2}\right)^2 V_\sigma \right] + V_{\beta\gamma}\Psi_\beta \left[1 - \frac{1}{2}\left(\frac{h}{2}\right)^2 V_{\beta\gamma} \right] + V_{\delta\gamma}\Psi_\delta \left[1 - \frac{1}{2}\left(\frac{h}{2}\right)^2 V_{\delta\gamma} \right] + \frac{1}{2}[V_{\beta\gamma} - 2V_\sigma + V_{\delta\gamma}](\Psi_\beta + \Psi_\delta) + \mathcal{O}(h^4). \quad (50)$$

The last intermediate term g_σ in equation (49) is evaluated using given nodes in the past light cone of α , figure 1:

$$g_\sigma = \frac{1}{16}[8g_\beta + 8g_\gamma + 8g_\delta - 4g_{\gamma_1} - 4g_{\gamma_2} + g_{\mu_1} - g_{\mu_2} - g_{\mu_3} + g_{\mu_4}] + \mathcal{O}(h^4). \quad (51)$$

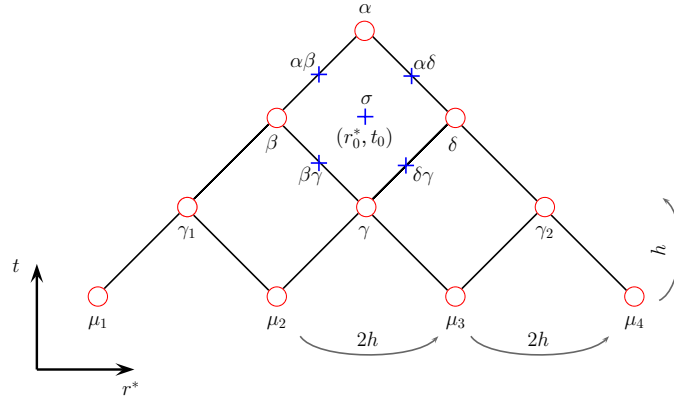


Figure 1. Set of points (circles and crosses) used for the integration of the $V\Psi = g$ term in the vacuum case. The crosses do not overlap with grid nodes; thus the field g at these points, equations (50, 51), is approximated by the field at the nodes on the past light cone of the grid node α .

For the differential operators, an exact integration simply leads to

$$\int \int_{\text{Cell}} (\partial_{r^*}^2 - \partial_t^2) \Psi(r^*, t) dA = -4[\Psi_\alpha - \Psi_\beta + \Psi_\gamma - \Psi_\delta]. \quad (52)$$

Finally, we obtain

$$\begin{aligned} \Psi_\alpha = & -\Psi_\gamma + \Psi_\beta \left[1 - \frac{1}{4} \left(\frac{h}{2} \right)^2 (V_\sigma + V_\beta) + \frac{1}{16} \left(\frac{h}{2} \right)^4 V_\sigma (V_\sigma + V_\beta) \right] \\ & + \Psi_\delta \left[1 - \frac{1}{4} \left(\frac{h}{2} \right)^2 (V_\sigma + V_\delta) + \frac{1}{16} \left(\frac{h}{2} \right)^4 V_\sigma (V_\sigma + V_\delta) \right] \\ & - \left(\frac{h}{2} \right)^2 \left[1 - \frac{1}{4} \left(\frac{h}{2} \right)^2 V_\sigma \right] [g_{\beta\gamma} + g_{\alpha\beta} + g_{\delta\gamma} + g_{\alpha\delta} + 4g_\sigma]. \end{aligned} \quad (53)$$

For cells adjacent to cells crossed by the particle, the requirement of good accuracy suggests a different dealing for the computation of g_σ , since the past light cone of an adjacent cell can cross the path of the particle. In such a case, g_σ is approximated by non-centred spatial finite difference expressions [15].

4.2. Cells crossed by the world line

For a given cell, our aim is the determination of the wavefunction value at the upper node, now rebaptized α_0 . As in the previous section, we consider 15 points both located in the past light cone of the point α_0 and lying around a chosen point on the discontinuity $r_u(t)$, with the intent of determining Ψ_{α_0} by their linear combination. The non-regularity of the wavefunction due to the discontinuity obviously entails a different value according to whether the discontinuity is approached from below (Ψ^- , left of the trajectory, figures 2–4) or above (Ψ^+ , right of the trajectory, figures 2–4) the particle in radial fall. The same stands for the wavefunction derivatives. The addition of the jump condition to the value of the e.g. Ψ^- (Ψ^+) wavefunction (or derivative of) allows us to equate this sum to the value Ψ^+ (Ψ^-) of the wavefunction (or derivative of). This straightforward property turns being helpful for the achievement of the just mentioned linear combination of 15 points. Incidentally, other linear combinations may

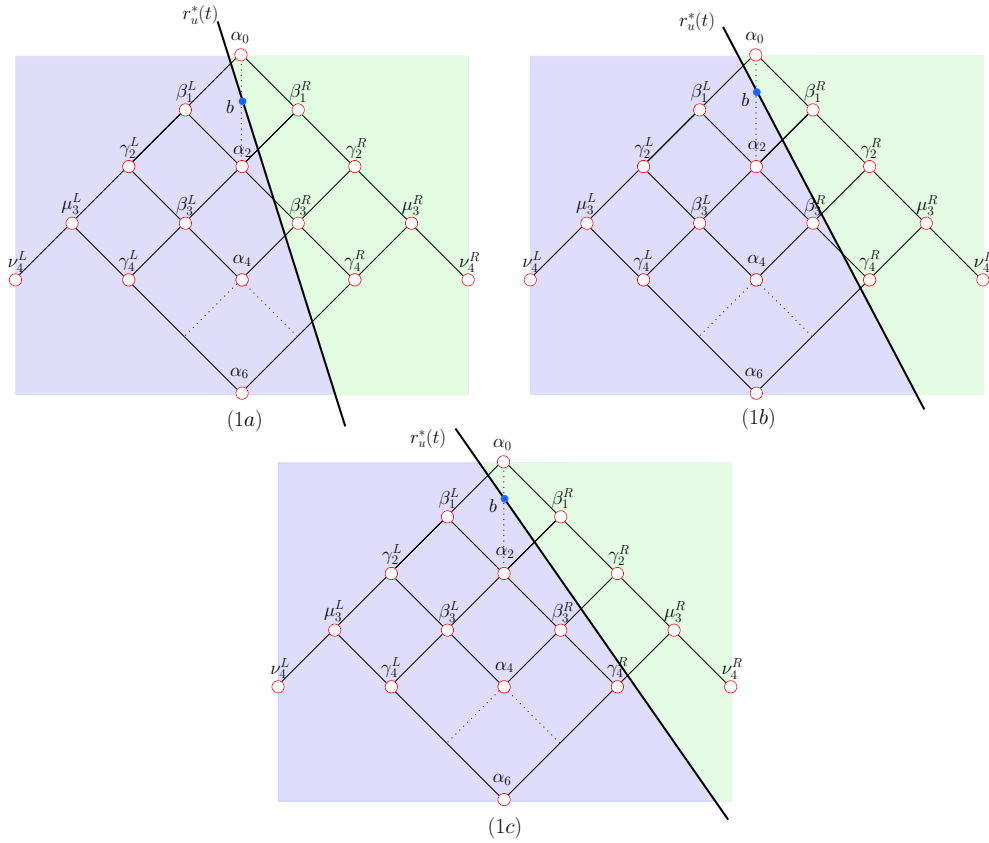


Figure 2. The three sub-cases for which the particle enters through the $[\alpha_2\beta_1^R]$ side and leaves through the $[\alpha_0\beta_1^L]$ side. The elimination of the Ψ_b^- derivatives demands, equation (60), the utilization of 15 points, represented by circles, in the light cone of α_0 . Numerical efficiency suggests that the points are taken at both left and right sides of the $r_u^*(t)$ trajectory. In the three cases, the particle crosses the line $[\alpha_0\alpha_2]$ at the point b . The background distinguishes two zones: one where $\Psi(r^* < r_u^*(t), t) = \Psi^-(r^*, t)$, and the other where $\Psi(r^* > r_u^*(t), t) = \Psi^+(r^*, t)$, the path $r_u^*(t)$ representing the separation between the two zones.

be envisaged, though combinations of points located solely on one side of the discontinuity are to be avoided.

With reference to figures 2–4, there are three different cases depending upon how the trajectory of the particle crosses the cell wherein α_0 lies. These three cases are further subdivided into three sub-cases, for a total of nine. In the following, we label by R the points on the right of the $[\alpha_0\alpha_6]$ line and by L the points on the left. Dealing with radial fall, and thereby with a 2D code, the up and down labels might be proper; nevertheless, we stick to right and left labels, given the orientation of the r^* axis in figures 2–4. For the first group of three, the trajectory crosses the $[\alpha_2\beta_1^R]$ and $[\alpha_0\beta_1^L]$ lines, figure 2; for the second group, the $[\alpha_2\beta_1^L]$ and $[\alpha_0\beta_1^L]$ lines, figure 3; finally for the third group, the $[\alpha_2\beta_1^R]$ and $[\alpha_0\beta_1^R]$ lines, figure 4.

We start considering the sub-case (1a) shown by figure 2, for which the trajectory crosses the line $[\alpha_0\alpha_2]$ at the point b . For compactness of the presentation of the final results, while

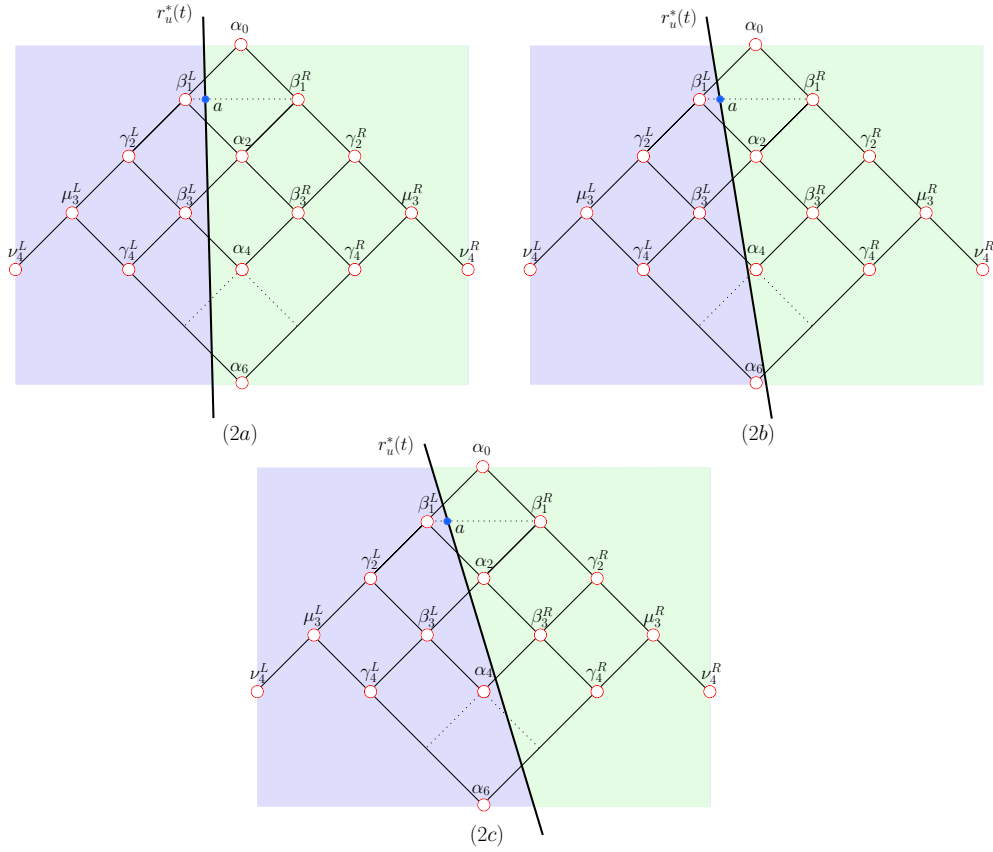


Figure 3. The three sub-cases for which the particle enters through the $[\alpha_2\beta_1^L]$ side and leaves through the $[\alpha_0\beta_1^L]$ side. The elimination of the Ψ_a^- derivatives demands the utilization of 15 points, represented by circles, in the light cone of α_0 . Numerical efficiency suggests that the points are taken at both left and right sides of the $r_u^*(t)$ trajectory. In the three cases, the particle crosses the line $[\beta_1^L\beta_1^R]$ at the point a . The background distinguishes two zones: one where $\Psi(r^* < r_u^*(t), t) = \Psi^-(r^*, t)$, and the other where $\Psi(r^* > r_u^*(t), t) = \Psi^+(r^*, t)$, the path $r_u^*(t)$ representing the separation between the two zones.

we still adopt the same notation for the jump conditions, namely $[\Psi]_q$ for the difference $(\Psi^+ - \Psi^-)_{r_u=r_u(t_q)}$, for the jump derivatives instead, we rely henceforth on the notation $[\partial_{r^*}^n \partial_t^m \Psi]_q = (\partial_{r^*}^n \partial_t^m \Psi^+ - \partial_{r^*}^n \partial_t^m \Psi^-)_{r_u=r_u(t_q)}$, where t_q is the coordinate time at the point $q = a, b$. We also define the lapse $\epsilon_b = t_{\alpha_0} - t_b$.

We recall that our aim is the determination of the value of $\Psi_{\alpha_0}^+$, knowing (i) ϵ_b , (ii) the jump (analytical) conditions on Ψ and its derivatives at the point b , and (iii) the values of Ψ on a set of 15 points $\{\alpha, \beta, \gamma, \mu, \nu\}$ at the left and right sides of the world line. A Taylor series is applied at each point around b up to fourth order, thereby obtaining

$$\Psi_{\alpha_0}^+ = \Psi^+(t_b + \epsilon_b, r_b^*) = \sum_{n=0}^4 \frac{\epsilon_b^n}{n!} \partial_t^n \Psi_b^+ + \mathcal{O}(\epsilon_b^5), \quad (54)$$

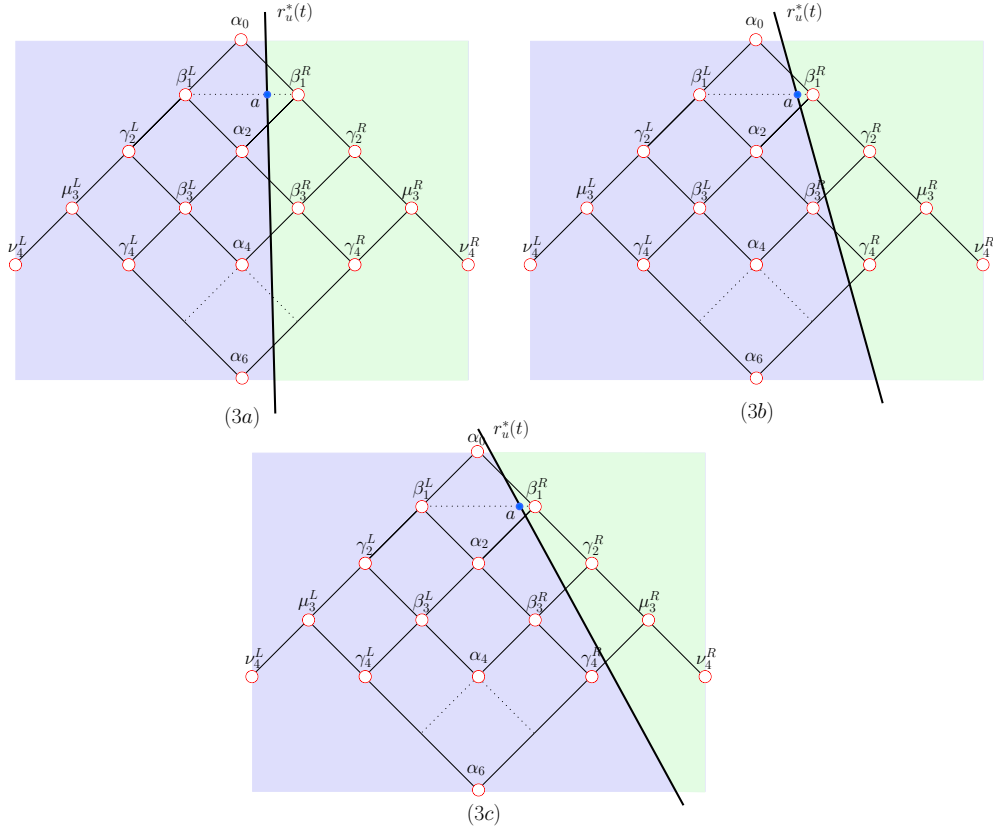


Figure 4. The three sub-cases for which the particle enters through the $[\alpha_2\beta_1^R]$ side and leaves through the $[\alpha_0\beta_1^L]$ side. The elimination of the Ψ_a^- derivatives demands the utilization of 15 points, represented by circles, in the light cone of α_0 . Numerical efficiency suggests that the points are taken at both left and right sides of the $r_u^*(t)$ trajectory. In the three cases, the particle crosses the line $[\beta_1^L\beta_1^R]$ at the point a . The background distinguishes two zones: one where $\Psi(r^* < r_u^*(t), t) = \Psi^-(r^*, t)$, the other where $\Psi(r^* > r_u^*(t), t) = \Psi^+(r^*, t)$, the path $r_u^*(t)$ representing the separation between the two zones.

$$\Psi_{\alpha_i}^- = \Psi^-(t_b - (ih - \epsilon_b), r_b^*) = \sum_{n=0}^4 (-1)^n \frac{(ih - \epsilon_b)^n}{n!} \partial_t^n \Psi_b^- + \mathcal{O}(h^5), \quad (55)$$

$$\Psi_{\beta_j^{\pm}} = \Psi^{\pm}(t_b - (jh - \epsilon_b), r_b^* \pm h) = \sum_{n+m \leq 4} (-1)^m (\pm 1)^n \frac{h^n}{n!} \frac{(jh - \epsilon_b)^m}{m!} \partial_{r^*}^n \partial_t^m \Psi_b^{\pm} + \mathcal{O}(h^5), \quad (56)$$

$$\Psi_{\gamma_k^{\pm}} = \Psi^{\pm}(t_b - (kh - \epsilon_b), r_b^* \pm 2h) = \sum_{n+m \leq 4} (-1)^m (\pm 1)^n \frac{(2h)^n}{n!} \frac{(kh - \epsilon_b)^m}{m!} \partial_{r^*}^n \partial_t^m \Psi_b^{\pm} + \mathcal{O}(h^5), \quad (57)$$

$$\Psi_{\mu_3^{\pm}} = \Psi^{\pm}(t_b - (3h - \epsilon_b), r_b^* \pm 3h) = \sum_{n+m \leq 4} (-1)^m (\pm 1)^n \frac{(3h)^n}{n!} \frac{(3h - \epsilon_b)^m}{m!} \partial_{r^*}^n \partial_t^m \Psi_b^{\pm} + \mathcal{O}(h^5), \quad (58)$$

$$\Psi_{v_4^{R,L}}^{\pm} = \Psi^{\pm}(t_b - (4h - \epsilon_b), r_b^* \pm 4h) = \sum_{n+m \leq 4} (-1)^m (\pm 1)^n \frac{(4h)^n}{n!} \frac{(4h - \epsilon_b)^m}{m!} \partial_{r^*}^n \partial_t^m \Psi_b^{\pm} + \mathcal{O}(h^5), \quad (59)$$

for the indexes running as $i = 2, 4, 6$, $j = 1, 3$ and $k = 2, 4$ and concerning the α , β and γ nodes, respectively. Our notation implies that the subscript R, L stands for R when the superscript \pm corresponds to $+$, whereas R, L stands for L when \pm corresponds to $-$. With reference to equation (54), we obtain

$$\begin{aligned} \Psi_{\alpha_0}^+ &= \sum_{n=0}^4 c_n \partial_t^n \Psi_b^+ + \mathcal{O}(h^5) = \sum_{n=0}^4 c_n (\partial_t^n \Psi_b^- + [\partial_t^n \Psi]_b) + \mathcal{O}(h^5) \\ &= c_0 \Psi_b^- + c_1 \partial_t \Psi_b^- + c_2 \partial_t^2 \Psi_b^- + c_3 \partial_t^3 \Psi_b^- + c_4 \partial_t^4 \Psi_b^- + \sum_{n=0}^4 c_n [\partial_t^n \Psi]_b + \mathcal{O}(h^5). \end{aligned} \quad (60)$$

For an accuracy at fourth order, all quantities $\mathcal{O}(h^5)$ are disregarded. The sum $\hat{S} = c_0 \Psi_b^- + c_1 \partial_t \Psi_b^- + c_2 \partial_t^2 \Psi_b^- + c_3 \partial_t^3 \Psi_b^- + c_4 \partial_t^4 \Psi_b^-$, equation (60), is composed by numerical derivatives of lower order than $\mathcal{O}(h^5)$, and therefore they cannot be neglected. However, the computation of high-order derivatives is often accompanied by numerical noise. Therefore, we replace this sum by a combination of wavefunction values in the α_0 light cone. This is attained in two steps. The former involves taking 15 wavefunction values on the two sides of the trajectory, that is $\{\Psi_{\alpha_i}^-, \Psi_{\beta_j}^-, \Psi_{\beta_j}^+, \Psi_{\gamma_k}^-, \Psi_{\gamma_k}^+, \Psi_{\mu_3}^-, \Psi_{\mu_3}^+, \Psi_{v_4}^-, \Psi_{v_4}^+\}$, figure 2. The latter employs the jump conditions to relate the 15 mentioned points with $\{\Psi_{\alpha_i}^-, \Psi_{\beta_j}^-, \Psi_{\beta_j}^R, \Psi_{\gamma_k}^-, \Psi_{\gamma_k}^R, \Psi_{\mu_3}^-, \Psi_{\mu_3}^R, \Psi_{v_4}^-, \Psi_{v_4}^R\}$. For the former step, we define the sum S

$$\begin{aligned} S &= \sum_i (\mathcal{A}_i \Psi_{\alpha_i}^-) + \sum_j (\mathcal{B}_j^L \Psi_{\beta_j}^- + \mathcal{B}_j^R \Psi_{\beta_j}^+) + \sum_k (\mathcal{G}_k^L \Psi_{\gamma_k}^- + \mathcal{G}_k^R \Psi_{\gamma_k}^+) \\ &\quad + \mathcal{M}_3^L \Psi_{\mu_3}^- + \mathcal{M}_3^R \Psi_{\mu_3}^+ + \mathcal{N}_4^L \Psi_{v_4}^- + \mathcal{N}_4^R \Psi_{v_4}^+, \end{aligned} \quad (61)$$

where $\{\mathcal{A}_i, \mathcal{B}_j^L, \mathcal{B}_j^R, \mathcal{G}_k^L, \mathcal{G}_k^R, \mathcal{M}_3^L, \mathcal{M}_3^R, \mathcal{N}_4^L, \mathcal{N}_4^R\}$ are constants.

We observe that the \hat{S} sum entails only wavefunction values at the left of the point b on the trajectory. The jump conditions are once more exploited to relate the two domains $r^* < r_u^*(t)$ and $r^* > r_u^*(t)$. This specifically concerns six points $\{\beta_j^R, \gamma_k^R, \mu_3^R, v_4^R\}$. For instance, at the point β_j^R , we can write

$$\begin{aligned} \Psi_{\beta_j^R}^+ &= \sum_{n+m \leq 4} (-1)^m \frac{h^n}{n!} \frac{(jh - \epsilon_b)^m}{m!} (\partial_{r^*}^n \partial_t^m \Psi_b^- + [\partial_{r^*}^n \partial_t^m \Psi]_b) + \mathcal{O}(h^5) \\ &= \Psi_{\beta_j^R}^- + \sum_{n+m \leq 4} (-1)^m \frac{h^n}{n!} \frac{(jh - \epsilon_b)^m}{m!} [\partial_{r^*}^n \partial_t^m \Psi]_b, \end{aligned} \quad (62)$$

where

$$\Psi_{\beta_j^R}^- = \sum_{n+m \leq 4} (-1)^m \frac{h^n}{n!} \frac{(jh - \epsilon_b)^m}{m!} (\partial_{r^*}^n \partial_t^m \Psi_b^-) + \mathcal{O}(h^5). \quad (63)$$

By the application of the same transformation to the quantities $\Psi_{\gamma_k}^+, \Psi_{\mu_3}^+, \Psi_{v_3}^+$, equation (61) becomes

$$\begin{aligned} S - \Phi_{r_u^*}^{\text{jump}} &= \sum_i (\mathcal{A}_i \Psi_{\alpha_i}^-) + \sum_j (\mathcal{B}_j^L \Psi_{\beta_j}^- + \mathcal{B}_j^R \Psi_{\beta_j}^-) + \sum_k (\mathcal{G}_k^L \Psi_{\gamma_k}^- + \mathcal{G}_k^R \Psi_{\gamma_k}^-) \\ &\quad + \mathcal{M}_3^L \Psi_{\mu_3}^- + \mathcal{M}_3^R \Psi_{\mu_3}^- + \mathcal{N}_4^L \Psi_{v_4}^- + \mathcal{N}_4^R \Psi_{v_4}^-, \end{aligned} \quad (64)$$

where $\Phi_{r_u^*}^{\text{jump}}$ is an analytic function, composed by the jump conditions at the point b , weighted by coefficients issued by equation (62) or similar equations.

Having only Ψ^- terms on the right-hand side of equation (64), we can finally search the coefficients $\{\mathcal{A}_i, \mathcal{B}_j^L, \mathcal{B}_j^R, \mathcal{G}_k^L, \mathcal{G}_k^R, \mathcal{M}_3^L, \mathcal{M}_3^R, \mathcal{N}_4^L, \mathcal{N}_4^R\}$ that satisfy the equation $\hat{S} = S - \Phi_{r_u^*}^{\text{jump}}$, that is

$$\begin{aligned} & c_0 \Psi_b^- + c_1 \partial_t \Psi_b^- + c_2 \partial_t^2 \Psi_b^- + c_3 \partial_t^3 \Psi_b^- + c_4 \partial_t^4 \Psi_b^- \\ & \times \sum_i (\mathcal{A}_i \Psi_{\alpha_i}^-) + \sum_j (\mathcal{B}_j^L \Psi_{\beta_j^L}^- + \mathcal{B}_j^R \Psi_{\beta_j^R}^-) + \sum_k (\mathcal{G}_k^L \Psi_{\gamma_k^L}^- + \mathcal{G}_k^R \Psi_{\gamma_k^R}^-) \\ & + \mathcal{M}_3^L \Psi_{\mu_3^L}^- + \mathcal{M}_3^R \Psi_{\mu_3^R}^- + \mathcal{N}_4^L \Psi_{\nu_4^L}^- + \mathcal{N}_4^R \Psi_{\nu_4^R}^-. \end{aligned} \quad (65)$$

Using the notation of equations (62) and (63), and by injection of equations (55)–(59), a Taylor expansion of fourth order at the point b is applied to the right-hand side of equation (65). The system can be cast in a matrix form

$$\mathbb{T} \cdot \mathbb{P} = \mathbb{C}, \quad (66)$$

where \mathbb{P} is the unknown 15-vector formed by the coefficients $\{\mathcal{A}_i, \mathcal{B}_j^L, \mathcal{B}_j^R, \mathcal{G}_k^L, \mathcal{G}_k^R, \mathcal{M}_3^L, \mathcal{M}_3^R, \mathcal{N}_4^L, \mathcal{N}_4^R\}$

$$\mathbb{P} = (\mathcal{A}_2, \mathcal{A}_4, \mathcal{A}_6, \mathcal{B}_1^L, \mathcal{B}_3^L, \mathcal{B}_1^R, \mathcal{B}_3^R, \mathcal{G}_2^L, \mathcal{G}_4^L, \mathcal{G}_2^R, \mathcal{G}_4^R, \mathcal{M}_3^L, \mathcal{M}_3^R, \mathcal{N}_4^L, \mathcal{N}_4^R)^t, \quad (67)$$

and \mathbb{C} is given by the 15-vector

$$\mathbb{C} = (c_0, c_1, c_2, c_3, c_4, 0, \dots, 0)^t, \quad (68)$$

while \mathbb{T} is the (15×15) matrix constructed from the Taylor coefficients in equations (55)–(59) (see the appendix). By inversion of \mathbb{T} , we obtain \mathbb{P} and specifically

$$\begin{aligned} \mathcal{A}_2 &= \frac{-27}{5}, & \mathcal{A}_4 &= \frac{-9}{5}, & \mathcal{A}_6 &= \frac{1}{5}, \\ \mathcal{B}_1^L &= \mathcal{B}_1^R = \frac{12}{5}, & \mathcal{B}_3^L &= \mathcal{B}_3^R = \frac{18}{5}, \\ \mathcal{G}_2^L &= \mathcal{G}_2^R = \frac{-9}{5}, & \mathcal{G}_4^L &= \mathcal{G}_4^R = \frac{-3}{5}, \\ \mathcal{M}_3^L &= \mathcal{M}_3^R = \frac{2}{5}, & \mathcal{N}_4^L &= \mathcal{N}_4^R = 0. \end{aligned}$$

The following equivalences path the last stretch of the way

$$\Psi_{\alpha_0}^+ = S - \Phi_{r_u^*}^{\text{jump}} + \sum_{n=0}^4 c_n [\partial_t^n \Psi]_b = S + \Phi_{r_u^*(t_b)}^{(1)}, \quad (69)$$

and explicitly, we obtain

$$\begin{aligned} \Psi_{\alpha_0}^+ &= -\frac{27}{5} \Psi_{\alpha_2}^- - \frac{9}{5} \Psi_{\alpha_4}^- + \frac{1}{5} \Psi_{\alpha_6}^- + \frac{12}{5} (\Psi_{\beta_1^L}^- + \Psi_{\beta_1^R}^-) + \frac{18}{5} (\Psi_{\beta_3^L}^- + \Psi_{\beta_3^R}^-) \\ & - \frac{9}{5} (\Psi_{\gamma_2^L}^- + \Psi_{\gamma_2^R}^-) + \frac{3}{5} (\Psi_{\gamma_4^L}^- + \Psi_{\gamma_4^R}^-) - \frac{2}{5} (\Psi_{\mu_3^L}^- + \Psi_{\mu_3^R}^-) + \Phi_{r_u^*(t_b)}^{(1)}, \end{aligned} \quad (70)$$

where $\Psi_{\beta_3^{\pm}}^{\pm} = \Psi_{\beta_3^{\pm}}^+$ for sub-case (1a), and $\Psi_{\beta_3^{\pm}}^{\pm} = \Psi_{\beta_3^{\pm}}^-$ for sub-cases (1b,1c); $\Psi_{\gamma_4^R}^{\pm} = \Psi_{\gamma_4^R}^+$ for sub-cases (1a,1b), and $\Psi_{\gamma_4^R}^{\pm} = \Psi_{\gamma_4^R}^-$ for sub-case (1c); and $\Phi_{r_u^*(t_b)}^{(1)}$ is an analytic function that for the (1a) sub-case takes the value

$$\begin{aligned} \Phi_{r_u^*(t_b)}^{(1a)} &= -3[\Psi]_b - \frac{3(5\epsilon_b - 14h)}{5} [\partial_t \Psi]_b - \frac{3(\epsilon_b - 2h)(5\epsilon_b - 18h)}{10} [\partial_t^2 \Psi]_b \\ & - \frac{5\epsilon_b^3 - 42h\epsilon_b^2 + 108h^2\epsilon_b - 96h^3}{10} [\partial_t^3 \Psi]_b \\ & - \frac{5\epsilon_b^4 - 56h\epsilon_b^3 + 216h^2\epsilon_b^2 - 384h^3\epsilon_b + 240h^4}{40} [\partial_t^4 \Psi]_b - \frac{12h}{5} [\partial_{r^*} \Psi]_b \end{aligned}$$

$$\begin{aligned}
& + \frac{2h^3}{5} [\partial_{r^*}^3 \Psi]_b - \frac{12h(\epsilon_b - 2h)}{5} [\partial_{r^*} \partial_t \Psi]_b - \frac{6h(\epsilon_b^2 - 4h\epsilon_b + 5h^2)}{5} [\partial_{r^*}^3 \partial_t \Psi]_b \\
& + \frac{2h^3(\epsilon_b - h)}{5} [\partial_{r^*}^2 \partial_t^2 \Psi]_b - \frac{2h(\epsilon_b^3 - 6h\epsilon_b^2 + 15h^2\epsilon_b - 11h^3)}{5} [\partial_{r^*} \partial_t^3 \Psi]_b.
\end{aligned} \tag{71}$$

The quantity $\Phi_{r_u^*}^{(1)}$ varies according to the different sub-cases: for the sub-case (1b) of figure 2, the point β_3^R , whereas for the case (1c) the points β_3^R and γ_4^R are in the $r^* < r_u^*$ domain. Therefore,

$$\begin{aligned}
\Phi_{r_u^*(t_b)}^{(1b)} &= \frac{6}{10} [\Psi]_b + \frac{3(\epsilon_b - 4h)}{5} [\partial_t \Psi]_b + \frac{3(\epsilon_b^2 - 8h\epsilon_b + 18h^2)}{10} [\partial_t^2 \Psi]_b \\
& + \frac{\epsilon_b^3 - 12h\epsilon_b^2 + 54h^2\epsilon_b - 66h^3}{10} [\partial_t^3 \Psi]_b \\
& + \frac{\epsilon_b^4 - 16h\epsilon_b^3 + 108h^2\epsilon_b^2 - 264h^3\epsilon_b + 246h^4}{40} [\partial_t^4 \Psi]_b \\
& + \frac{6h}{5} [\partial_{r^*} \Psi]_b + \frac{9h^2}{5} [\partial_{r^*}^2 \Psi]_b + h^3 [\partial_{r^*}^3 \Psi]_b + \frac{3h^4}{20} [\partial_{r^*}^4 \Psi]_b \\
& + \frac{6h(\epsilon_b - 5h)}{5} [\partial_{r^*} \partial_t \Psi]_b + \frac{9h^2(\epsilon_b - 3h)}{5} [\partial_{r^*}^2 \partial_t \Psi]_b \\
& + \frac{3h(\epsilon_b^2 - 10h\epsilon_b + 17h^2)}{5} [\partial_{r^*}^3 \partial_t \Psi]_b + \frac{9h^2(\epsilon_b - 3h)^2}{10} [\partial_{r^*} \partial_t^2 \Psi]_b \\
& + \frac{h^3(5\epsilon_b - 11h)}{5} [\partial_{r^*}^2 \partial_t^2 \Psi]_b + \frac{h(\epsilon_b^3 - 15h\epsilon_b^2 + 51h^2\epsilon_b - 59h^3)}{5} [\partial_{r^*} \partial_t^3 \Psi]_b,
\end{aligned} \tag{72}$$

$$\begin{aligned}
\Phi_{r_u^*(t_b)}^{(1c)} &= \frac{3h^2}{5} [\partial_t^2 \Psi]_b + \frac{h^2(3\epsilon_b - h)}{5} [\partial_t^3 \Psi]_b + \frac{h^2(6\epsilon_b^2 - 4h\epsilon_b - 5h^2)}{20} [\partial_t^4 \Psi]_b \\
& + \frac{3h^2}{5} [\partial_{r^*}^2 \Psi]_b + \frac{h^3}{5} [\partial_{r^*}^3 \Psi]_b - \frac{h^4}{4} [\partial_{r^*}^4 \Psi]_b - \frac{6h^2}{5} [\partial_{r^*} \partial_t \Psi]_b \\
& + \frac{3h^2(\epsilon_b - h)}{5} [\partial_{r^*}^2 \partial_t \Psi]_b - \frac{3h^2(2\epsilon_b - h)}{5} [\partial_{r^*}^3 \partial_t \Psi]_b \\
& + \frac{3h^2(\epsilon_b^2 - 2h\epsilon_b - 5h^2)}{10} [\partial_{r^*} \partial_t^2 \Psi]_b + \frac{h^3(\epsilon_b + 5h)}{5} [\partial_{r^*}^2 \partial_t^2 \Psi]_b \\
& - \frac{h^2(3\epsilon_b^2 - 3h\epsilon_b - 5h^2)}{5} [\partial_{r^*} \partial_t^3 \Psi]_b.
\end{aligned} \tag{73}$$

We thus have obtained, without direct integration of the singular source and the potential term, the value of the upper node. The equations show three types of terms: the preceding node values of the same cell, the jump conditions which are fully analytical quantities and the wavefunction values at adjacent cells. Incidentally, at first order [21], the latter type of terms disappear and a simpler expression is obtained.

Similar relations are found for the other two remaining cases. For case 2, figure 3, we obtain (having defined the shift $\epsilon_a = t_{\beta_1^R} - r_a^*$)

$$\begin{aligned}
\Psi_{\alpha_0}^+ &= -\frac{27}{5} \Psi_{\alpha_2}^+ - \frac{9}{5} \Psi_{\alpha_4}^\pm + \frac{1}{5} \Psi_{\alpha_6}^\pm + \frac{12}{5} (\Psi_{\beta_1^L}^- + \Psi_{\beta_1^R}^+) + \frac{18}{5} (\Psi_{\beta_3^L}^- + \Psi_{\beta_3^R}^+) \\
& - \frac{9}{5} (\Psi_{\gamma_2^L}^- + \Psi_{\gamma_2^R}^+) + \frac{3}{5} (\Psi_{\gamma_4^L}^- + \Psi_{\gamma_4^R}^+) - \frac{2}{5} (\Psi_{\mu_3^L}^- + \Psi_{\mu_3^R}^+) + \Phi_{r_u^*(t_a)}^{(2)},
\end{aligned} \tag{74}$$

where $\Psi_{\alpha_4}^{\pm} = \Psi_{\alpha_4}^{+}$ for sub-cases (2a, 2b), and $\Psi_{\alpha_4}^{\pm} = \Psi_{\alpha_4}^{-}$ for sub-case (2c); $\Psi_{\alpha_6}^{\pm} = \Psi_{\alpha_6}^{+}$ for sub-case (2a), and $\Psi_{\alpha_6}^{\pm} = \Psi_{\alpha_6}^{-}$ for sub-cases (2b, 2c). For the sub-case (2a), $\Phi_{r_u^*(t_a)}^{(2)}$ takes the following value:

$$\begin{aligned} \Phi_{r_u^*(t_a)}^{(2a)} = & 4[\Psi]_a - \frac{22h}{5}[\partial_t \Psi]_a + \frac{22h^2}{5}[\partial_t^2 \Psi]_a - \frac{7h^3}{3}[\partial_t^3 \Psi]_a + \frac{17h^4}{30}[\partial_t^4 \Psi]_a \\ & + \frac{4(5\epsilon_a - 8h)}{5}[\partial_{r^*} \Psi]_a + \frac{2(\epsilon_a - h)(5\epsilon_a - 11h)}{5}[\partial_{r^*}^2 \Psi]_a \\ & + \frac{2(5\epsilon_a^3 - 24h\epsilon_a^2 + 33h^2\epsilon_a - 11h^3)}{15}[\partial_{r^*}^3 \Psi]_a \\ & + \frac{(\epsilon_a - h)(5\epsilon_a^3 - 27h\epsilon_a^2 + 39h^2\epsilon_a - 5h^3)}{30}[\partial_{r^*}^4 \Psi]_a \\ & - \frac{2h(11\epsilon_a - 17h)}{5}[\partial_{r^*} \partial_t \Psi]_a - \frac{h(\epsilon_a - h)(11\epsilon_a - 23h)}{5}[\partial_{r^*}^2 \partial_t \Psi]_a \\ & + \frac{2h^2(11\epsilon_a - 17h)}{5}[\partial_{r^*} \partial_t^2 \Psi]_a - \frac{h(\epsilon_a - h)^2(11\epsilon_a - 29h)}{15}[\partial_{r^*}^3 \partial_t \Psi]_a \\ & + \frac{h^3(\epsilon_b + 5h)}{5}[\partial_{r^*}^2 \partial_t^2 \Psi]_b - \frac{h^3(35\epsilon_a - 41h)}{15}[\partial_{r^*} \partial_t^3 \Psi]_a. \end{aligned} \quad (75)$$

For the same preceding reason, the sub-cases (2b, 2c) differ as the points α_4 and α_6 are or are not in the $r^* > r_u^*$ domain. Therefore, we have

$$\begin{aligned} \Phi_{r_u^*(t_a)}^{(2b)} = & \frac{42}{10}[\Psi]_a - \frac{27h}{5}[\partial_t \Psi]_a + \frac{69h^2}{10}[\partial_t^2 \Psi]_a - \frac{13h^3}{2}[\partial_t^3 \Psi]_a + \frac{231h^4}{40}[\partial_t^4 \Psi]_a \\ & + \frac{3(7\epsilon_a - 11h)}{5}[\partial_{r^*} \Psi]_a + \frac{3(\epsilon_a - h)(7\epsilon_a - 15h)}{10}[\partial_{r^*}^2 \Psi]_a \\ & + \frac{7\epsilon_a^3 - 33h\epsilon_a^2 + 45h^2\epsilon_a - 15h^3}{10}[\partial_{r^*}^3 \Psi]_a \\ & + \frac{(\epsilon_a - h)(7\epsilon_a^3 - 37h\epsilon_a^2 + 53h^2\epsilon_a - 7h^3)}{40}[\partial_{r^*}^4 \Psi]_a \\ & - \frac{3h(9\epsilon_a - 13h)}{5}[\partial_{r^*} \partial_t \Psi]_a - \frac{3h(\epsilon_a - h)(9\epsilon_a - 17h)}{10}[\partial_{r^*}^2 \partial_t \Psi]_a \\ & + \frac{3h^2(23\epsilon_a - 31h)}{10}[\partial_{r^*} \partial_t^2 \Psi]_a - \frac{3h(\epsilon_a - h)^2(3\epsilon_a - 7h)}{10}[\partial_{r^*}^3 \partial_t \Psi]_a \\ & + \frac{h^2(\epsilon_a - h)(157\epsilon_a - 109h)}{20}[\partial_{r^*}^2 \partial_t^2 \Psi]_a - \frac{h^3(65\epsilon_a - 69h)}{10}[\partial_{r^*} \partial_t^3 \Psi]_a, \end{aligned} \quad (76)$$

$$\begin{aligned} \Phi_{r_u^*(t_a)}^{(2c)} = & \frac{24}{10}[\Psi]_a - \frac{6h^2}{5}[\partial_t^2 \Psi]_a + \frac{8h^3}{5}[\partial_t^3 \Psi]_a - \frac{3h^4}{10}[\partial_t^4 \Psi]_a \\ & + \frac{12(\epsilon_a - 2h)}{5}[\partial_{r^*} \Psi]_a + \frac{6(\epsilon_a - 3h)(\epsilon_a - h)}{5}[\partial_{r^*}^2 \Psi]_a \\ & + \frac{2(\epsilon_a^3 - 6h\epsilon_a^2 + 9h^2\epsilon_a - 3h^3)}{5}[\partial_{r^*}^3 \Psi]_a \\ & + \frac{(\epsilon_a - h)(\epsilon_a^3 - 7h\epsilon_a^2 + 11h^2\epsilon_a - h^3)}{10}[\partial_{r^*}^4 \Psi]_a + \frac{12h^2}{5}[\partial_{r^*} \partial_t \Psi]_a \\ & + \frac{12h^2(\epsilon_a - h)}{5}[\partial_{r^*}^2 \partial_t \Psi]_a - \frac{6h^2(\epsilon_a + h)}{5}[\partial_{r^*} \partial_t^2 \Psi]_a \end{aligned}$$

$$\begin{aligned}
& + \frac{6h^2(\epsilon_a - h)^2}{5} [\partial_{r^*}^3 \partial_t \Psi]_a + \frac{h^2(\epsilon_a - h)(19\epsilon_a - 7h)}{5} [\partial_{r^*}^2 \partial_t^2 \Psi]_a \\
& + \frac{2h^3(4\epsilon_a - 3h)}{5} [\partial_{r^*} \partial_t^3 \Psi]_a.
\end{aligned} \tag{77}$$

Finally for case 3, figure 4, we have

$$\begin{aligned}
\Psi_{\alpha_0}^- = & -\frac{27}{5}\Psi_{\alpha_2}^- - \frac{9}{5}\Psi_{\alpha_4}^- + \frac{1}{5}\Psi_{\alpha_6}^- + \frac{12}{5}(\Psi_{\beta_1^L}^- + \Psi_{\beta_1^R}^+) + \frac{18}{5}(\Psi_{\beta_3^L}^- + \Psi_{\beta_3^R}^\pm) \\
& - \frac{9}{5}(\Psi_{\gamma_2^L}^- + \Psi_{\gamma_2^R}^+) + \frac{3}{5}(\Psi_{\gamma_4^L}^- + \Psi_{\gamma_4^R}^\pm) - \frac{2}{5}(\Psi_{\mu_5^L}^- + \Psi_{\mu_5^R}^+) + \Phi_{r_a^*(t_a)}^{(3)},
\end{aligned} \tag{78}$$

where $\Psi_{\beta_3^R}^\pm = \Psi_{\beta_3^R}^+$ for sub-case (3a), and $\Psi_{\beta_3^R}^\pm = \Psi_{\beta_3^R}^-$ for sub-cases (3b, 3c); $\Psi_{\gamma_4^R}^\pm = \Psi_{\gamma_4^R}^+$ for sub-cases (3a, 3b), and $\Psi_{\gamma_4^R}^\pm = \Psi_{\gamma_4^R}^-$ for sub-case (3c); and $\Phi_{r_a^*(t_a)}^{(3)}$ takes the values

$$\begin{aligned}
\Phi_{r_a^*(t_a)}^{(3a)} = & -4[\Psi]_a + \frac{22h}{5}[\partial_t \Psi]_a - \frac{22h^2}{5}[\partial_t^2 \Psi]_a + \frac{7h^3}{3}[\partial_t^3 \Psi]_a - \frac{17h^4}{30}[\partial_t^4 \Psi]_a \\
& - \frac{4(5\epsilon_a - 2h)}{5}[\partial_{r^*} \Psi]_a - \frac{2(\epsilon_a - h)(5\epsilon_a + h)}{5}[\partial_{r^*}^2 \Psi]_a \\
& - \frac{2(5\epsilon_a^3 - 6h\epsilon_a^2 - 3h^2\epsilon_a + h^3)}{15}[\partial_{r^*}^3 \Psi]_a \\
& - \frac{(\epsilon_a - h)(5\epsilon_a^3 - 3h\epsilon_a^2 - 9h^2\epsilon_a - 5h^3)}{30}[\partial_{r^*}^4 \Psi]_a \\
& + \frac{2h(11\epsilon_a - 5h)}{5}[\partial_{r^*} \partial_t \Psi]_a + \frac{h(\epsilon_a - h)(11\epsilon_a + h)}{5}[\partial_{r^*}^2 \partial_t \Psi]_a \\
& - \frac{2h^2(11\epsilon_a - 5h)}{5}[\partial_{r^*} \partial_t^2 \Psi]_a + \frac{h(\epsilon_a - h)^2(11\epsilon_a + 7h)}{15}[\partial_{r^*}^3 \partial_t \Psi]_a \\
& + \frac{h^2(\epsilon_a - h)(11\epsilon_a + h)}{5}[\partial_{r^*}^2 \partial_t^2 \Psi]_a + \frac{h^3(35\epsilon_a - 29h)}{15}[\partial_{r^*} \partial_t^3 \Psi]_a,
\end{aligned} \tag{79}$$

$$\begin{aligned}
\Phi_{r_a^*(t_a)}^{(3b)} = & -\frac{2}{5}[\Psi]_a - \frac{14h}{5}[\partial_t \Psi]_a + \frac{14h^2}{5}[\partial_t^2 \Psi]_a - \frac{37h^3}{15}[\partial_t^3 \Psi]_a + \frac{11h^4}{6}[\partial_t^4 \Psi]_a \\
& - \frac{2(\epsilon_a - 4h)}{5}[\partial_{r^*} \Psi]_a - \frac{\epsilon_a^2 - 8h\epsilon_a - 2h^2}{5}[\partial_{r^*}^2 \Psi]_a \\
& - \frac{\epsilon_a^3 - 12h\epsilon_a^2 - 6h^2\epsilon_a + 2h^3}{15}[\partial_{r^*}^3 \Psi]_a \\
& - \frac{\epsilon_a^4 - 16h\epsilon_a^3 - 12h^2\epsilon_a^2 + 8h^3\epsilon_a + 10h^4}{60}[\partial_{r^*}^4 \Psi]_a \\
& - \frac{2h(7\epsilon_a + 5h)}{5}[\partial_{r^*} \partial_t \Psi]_a - \frac{h(7\epsilon_a^2 + 10h\epsilon_a + h^2)}{5}[\partial_{r^*}^2 \partial_t \Psi]_a \\
& + \frac{2h^2(7\epsilon_a + 5h)}{5}[\partial_{r^*} \partial_t^2 \Psi]_a - \frac{h(7\epsilon_a^3 + 15h\epsilon_a^2 + 3h^2\epsilon_a - 7h^3)}{15}[\partial_{r^*}^3 \partial_t \Psi]_a \\
& - \frac{h^2(7\epsilon_a^2 + 10h\epsilon_a + h^2)}{5}[\partial_{r^*}^2 \partial_t^2 \Psi]_a - \frac{h^3(37\epsilon_a + 29h)}{15}[\partial_{r^*} \partial_t^3 \Psi]_a,
\end{aligned} \tag{80}$$

$$\begin{aligned}
\Phi_{r_a^*(t_a)}^{(3c)} = & -[\Psi]_a - h[\partial_t \Psi]_a + \frac{h^2}{10}[\partial_t^2 \Psi]_a + \frac{7h^3}{30}[\partial_t^3 \Psi]_a - \frac{23h^4}{120}[\partial_t^4 \Psi]_a \\
& + (h - \epsilon_a)[\partial_{r^*} \Psi]_a - \frac{5\epsilon_a^2 - 10h\epsilon_a - h^2}{10}[\partial_{r^*}^2 \Psi]_a
\end{aligned}$$

$$\begin{aligned}
& - \frac{5\epsilon_a^3 - 15h\epsilon_a^2 - 3h^2\epsilon_a + 7h^3}{30} [\partial_{r^*}^3 \Psi]_a \\
& - \frac{5\epsilon_a^4 - 20h\epsilon_a^3 - 6h^2\epsilon_a^2 + 28h^3\epsilon_a + 23h^4}{120} [\partial_{r^*}^4 \Psi]_a \\
& - \frac{h(5\epsilon_a + h)}{5} [\partial_{r^*} \partial_t \Psi]_a - \frac{h(\epsilon_a - h)(5\epsilon_a + 7h)}{10} [\partial_{r^*}^2 \partial_t \Psi]_a \\
& + \frac{h^2(\epsilon_a - 7h)}{10} [\partial_{r^*} \partial_t^2 \Psi]_a - \frac{h(5\epsilon_a^3 + 3h\epsilon_a^2 - 21h^2\epsilon_a - 23h^3)}{30} [\partial_{r^*}^3 \partial_t \Psi]_a \\
& - \frac{h^2(\epsilon_a^2 - 14h\epsilon_a - 23h^2)}{20} [\partial_{r^*}^2 \partial_t^2 \Psi]_a + \frac{h^3(7\epsilon_a + 23h)}{30} [\partial_{r^*} \partial_t^3 \Psi]_a. \tag{81}
\end{aligned}$$

The jump conditions in the tortoise r^* relate to those previously computed in the r variable (the relations for mixed derivatives (r^*, t) are easily inferred)

$$[\Psi_{,r^*}] = f_{r_u} [\Psi_{,r}], \tag{82}$$

$$[\Psi_{,r^*r^*}] = f_{r_u} f'_{r_u} [\Psi_{,r}] + f_{r_u}^2 [\Psi_{,rr}], \tag{83}$$

$$[\Psi_{,r^*r^*r^*}] = f_{r_u} (f'^2 + f f'')_{r_u} [\Psi_{,r}] + 3 f_{r_u}^2 f'_{r_u} [\Psi_{,rr}] + f_{r_u}^3 [\Psi_{,rrr}], \tag{84}$$

$$\begin{aligned}
[\Psi_{,r^*r^*r^*r^*}] &= f_{r_u} (f'^3 + 4 f f' f'' + f^2 f''')_{r_u} [\Psi_{,r}] + f_{r_u}^2 (7 f'^2 + 4 f f'')_{r_u} [\Psi_{,rr}] \\
&+ 6 f_{r_u}^3 f'_{r_u} [\Psi_{,rrr}] + f_{r_u}^4 [\Psi_{,rrrr}]. \tag{85}
\end{aligned}$$

5. Numerical implementation

Waveforms at infinity and at the particle position at first order are to be found in [21], as well as comparisons with other methods. Herein we are concerned on the numerical improvement. To this end, we have considered a distant observer, located at $r^* = 400(2M)$. The observer is reached by a pulse produced by a Gaussian, time-symmetric perturbation

$$\Psi(r^*, t)_{t=0} = \exp[-(r^* - r_0^*)^2], \tag{86}$$

$$\partial_t \Psi(r^*, t)_{t=0} = 0. \tag{87}$$

Figure 5, obtained for $r_{u0} = 5(2M)$, shows the waveform produced in the homogeneous case. The convergence rate is computed as $(\epsilon^{(n)}(\xi))$ is the unknown error function of order $\approx 1)$

$$n = \log \left| \frac{\Psi(4h) - \Psi(2h)}{\Psi(2h) - \Psi(h)} \right| \bigg/ \log(2) + \log |\epsilon^{(n)}(\xi)| / \log(2). \tag{88}$$

Figure 6, obtained for $r_{u0} = 5(2M)$, shows the fourth- and second-order convergence rates (we remind that the first-order code [21] includes empty cells dealt at second order).

6. Conclusions

We have presented a fourth-order novel integration method in time domain for the Zerilli wave equation. We have focused our attention to the even perturbations produced by a particle plunging in a non-rotating black hole. For cells crossed by the particle world line,

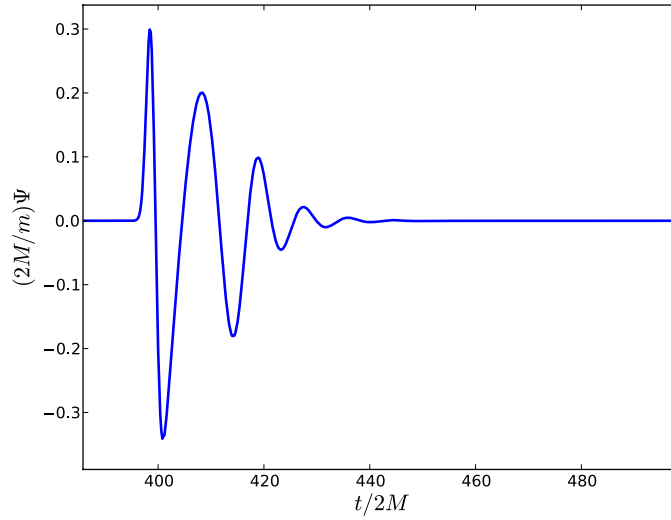


Figure 5. The waveform, $r_{u0} = 5(2M)$, of a Gaussian, time-symmetric initial pulse. The observer is located at $r^* = 400(2M)$.

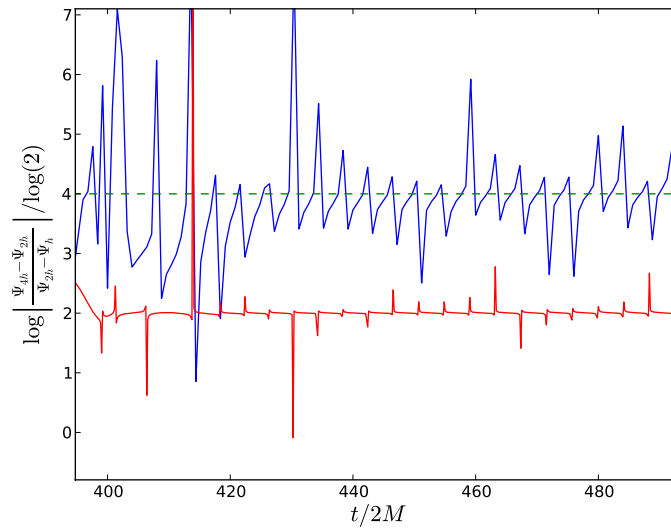


Figure 6. Convergence rates of the fourth- and second-order algorithms, $r_{u0} = 5(2M)$.

the forward time wavefunction value at the upper node of the (t, r^*) grid cell is obtained by the combination of the preceding node values of the same cell, analytic expressions related to the jump conditions, and the values of the wavefunction at adjacent cells. In this manner, the numerical integration does not deal directly nor with the source term and the associated singularities, nor with the potential term. In short, the direct integration of the wave equation is avoided. For empty cells, we refer instead to already published approaches [15].

The scheme has also been applied to circular and eccentric orbits and it will be the object of a forthcoming publication.

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Appendix

Through equation (60), we have determined the value of Ψ at the upper node of the cell as a function of the analytic jump conditions and of the time derivatives of the wavefunction up to fourth order. The derivatives are evaluated at the point b and weighted by five coefficients c_0, c_1, c_2, c_3 and c_4 . Afterwards, the derivatives are converted into a linear combination of the wavefunction values taken on points at the left and right sides of the trajectory. Indeed, equation (65) represents such a system of linear equations. By injection of equations (56)–(59) into equation (65), we obtain

$$\begin{aligned}
 & \mathcal{A}_2 T_{\alpha_2}^{(0,0)} \Psi_b^- + \mathcal{A}_2 T_{\alpha_2}^{(0,1)} \partial_t \Psi_b^- + \mathcal{A}_2 T_{\alpha_2}^{(0,2)} \partial_t^2 \Psi_b^- + \dots + \mathcal{A}_2 T_{\alpha_2}^{(1,3)} \partial_{r^*} \partial_t^3 \Psi_b^- & c_0 \Psi_b^- \\
 & & + \\
 & \mathcal{A}_4 T_{\alpha_4}^{(0,0)} \Psi_b^- + \mathcal{A}_4 T_{\alpha_4}^{(0,1)} \partial_t \Psi_b^- + \mathcal{A}_4 T_{\alpha_4}^{(0,2)} \partial_t^2 \Psi_b^- + \dots + \mathcal{A}_4 T_{\alpha_4}^{(1,3)} \partial_{r^*} \partial_t^3 \Psi_b^- & c_1 \partial_t \Psi_b^- \\
 & & + \\
 & & \vdots \\
 & & + \\
 & \mathcal{N}_4^R T_{v_4^R}^{(0,0)} \Psi_b^- + \mathcal{N}_4^R T_{v_4^R}^{(0,1)} \partial_t \Psi_b^- + \mathcal{N}_4^R T_{v_4^R}^{(0,2)} \partial_t^2 \Psi_b^- + \dots + \mathcal{N}_4^R T_{v_4^R}^{(1,3)} \partial_{r^*} \partial_t^3 \Psi_b^- & c_2 \partial_t^2 \Psi_b^- , \\
 & & + \\
 & & c_3 \partial_t^3 \Psi_b^- \\
 & & + \\
 & & c_4 \partial_t^4 \Psi_b^-
 \end{aligned} \tag{A.1}$$

where $T_p^{(n,m)}$ represents the Taylor series coefficients at p in the neighbourhood of b and the indexes correspond to n th space and m th time derivatives. The wavefunction at p is thus given by

$$\Psi_p^\pm = \sum_{n+m \leq 4} T_p^{(n,m)} \partial_{r^*}^n \partial_t^m \Psi_b^\pm + \mathcal{O}(h^5). \tag{A.2}$$

An example shows the procedure which is applicable to all cases. We pick the node α_2 , equation (55), where $T_{\alpha_2}^{(n,m)} = (-1)^n \frac{(2h-\epsilon_b)^n}{n!}$ and remind that $T_p^{(0,0)} = 1 \forall p$. By grouping the derivatives, we obtain

$$\begin{aligned}
 & (\mathcal{A}_2 T_{\alpha_2}^{(0,0)} + \mathcal{A}_4 T_{\alpha_4}^{(0,0)} + \dots + \mathcal{N}_4^R T_{v_4^R}^{(0,0)}) \Psi_b^- & c_0 \Psi_b^- \\
 & & + \\
 & (\mathcal{A}_2 T_{\alpha_2}^{(0,1)} + \mathcal{A}_4 T_{\alpha_4}^{(0,1)} + \dots + \mathcal{N}_4^R T_{v_4^R}^{(0,1)}) \partial_t \Psi_b^- & c_1 \partial_t \Psi_b^- \\
 & & + \\
 & (\mathcal{A}_2 T_{\alpha_2}^{(0,2)} + \mathcal{A}_4 T_{\alpha_4}^{(0,2)} + \dots + \mathcal{N}_4^R T_{v_4^R}^{(0,2)}) \partial_t^2 \Psi_b^- & = c_2 \partial_t^2 \Psi_b^- . \\
 & & + \\
 & & \vdots \\
 & & + \\
 & (\mathcal{A}_2 T_{\alpha_2}^{(1,3)} + \mathcal{A}_4 T_{\alpha_4}^{(1,3)} + \dots + \mathcal{N}_4^R T_{v_4^R}^{(1,3)}) \partial_{r^*} \partial_t^3 \Psi_b^- & c_3 \partial_t^3 \Psi_b^- \\
 & & + \\
 & & c_4 \partial_t^4 \Psi_b^-
 \end{aligned} \tag{A.3}$$

By identification, we obtain a linear system, that is cast in the form

$$\underbrace{\begin{pmatrix} 1 & \cdots & 1 & \cdots & 1 & \cdots & 1 & \cdots & 1 \\ T_{\alpha_1}^{(0,1)} & \cdots & T_{\beta_1^L}^{(0,1)} & \cdots & T_{\gamma_2^L}^{(0,1)} & \cdots & T_{\mu_5^L}^{(0,1)} & \cdots & T_{\nu_4^R}^{(0,1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ T_{\alpha_1}^{(0,4)} & \cdots & T_{\beta_1^L}^{(0,4)} & \cdots & T_{\gamma_2^L}^{(0,4)} & \cdots & T_{\mu_5^L}^{(0,4)} & \cdots & T_{\nu_4^R}^{(0,4)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ T_{\alpha_1}^{(1,0)} & \cdots & T_{\beta_1^L}^{(1,0)} & \cdots & T_{\gamma_2^L}^{(1,0)} & \cdots & T_{\mu_5^L}^{(1,0)} & \cdots & T_{\nu_4^R}^{(1,0)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ T_{\alpha_1}^{(1,3)} & \cdots & T_{\beta_1^L}^{(1,3)} & \cdots & T_{\gamma_2^L}^{(1,3)} & \cdots & T_{\mu_5^L}^{(1,3)} & \cdots & T_{\nu_4^R}^{(1,3)} \end{pmatrix}}_{\mathbb{T}} \underbrace{\begin{pmatrix} \mathcal{A}_2 \\ \vdots \\ \mathcal{B}_1^L \\ \vdots \\ \mathcal{G}_2^L \\ \vdots \\ \mathcal{M}_3^L \\ \vdots \\ \mathcal{N}_4^R \end{pmatrix}}_{\mathbb{P}} = \underbrace{\begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{pmatrix}}_{\mathbb{C}}, \tag{A.4}$$

where the upper indexes (n, m) cover all combinations such that $n + m \leq 4$. Finally, by inversion of the \mathbb{T} matrix, the unknown terms of the \mathbb{P} vector are identified.

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