

Maximum gravitational-wave energy emissible in magnetar flares

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Recent searches of gravitational-wave data raise the question of what maximum gravitational-wave energies could be emitted during gamma-ray flares of highly magnetized neutron stars (magnetars). The highest energies ($\sim 10^{49}$ erg) predicted so far come from a model [K. Ioka, *Mon. Not. R. Astron. Soc.* **327**, 639 (2001), <http://adsabs.harvard.edu/abs/2001MNRAS.327..639I>] in which the internal magnetic field of a magnetar experiences a global reconfiguration, changing the hydromagnetic equilibrium structure of the star and tapping the gravitational potential energy without changing the magnetic potential energy. The largest energies in this model assume very special conditions, including a large change in moment of inertia (which was observed in at most one flare), a very high internal magnetic field, and a very soft equation of state. Here we show that energies of 10^{48} – 10^{49} erg are possible under more generic conditions by tapping the magnetic energy, and we note that similar energies may also be available through cracking of exotic solid cores. Current observational limits on gravitational waves from magnetar fundamental modes are just reaching these energies and will beat them in the era of advanced interferometers.

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I. INTRODUCTION**A. Motivation**

Recent years have seen the publication of several searches for gravitational-wave (GW) bursts triggered by gamma-ray flares from soft gamma repeaters (SGRs) and anomalous x-ray pulsars [1–6], both of which are believed to be highly magnetized neutron stars (magnetars).

The most sensitive searches are from the Laser Interferometer Gravitational-wave Observatory (LIGO) and Virgo, targeting the 2004 giant flare from SGR 1806–20 as well as many smaller flares from up to six magnetars [2–5]. No GW signals were found, and thus the results are upper limits on the GW energy emitted as low as $\sim 10^{48}$ erg for fundamental or f -modes (frequencies above 10^3 Hz) or $\sim 10^{45}$ erg for frequencies of greatest LIGO and Virgo sensitivity ($\sim 10^2$ Hz) [5]. The best (lowest) energy limits on the 2004 giant flare (which emitted $\sim 10^{46}$ erg in photons) were $\sim 10^{51}$ erg for f -modes and 10^{48} erg at 10^2 Hz [3]. Similar best energy limits on the 2009 “ring” event (which is now believed to have been a giant flare emitting 10^{44} – 10^{45} erg in photons) were $\sim 10^{49}$ erg and 10^{46} erg [5]. In a few years, when LIGO and Virgo are upgraded to “advanced interferometer” status, their noise amplitudes will improve by an order of magnitude [7,8] and thus, energy sensitivities will improve by 2 orders of magnitude.

Present upper limits and predicted sensitivities raise the question of what maximum GW energies could possibly be

radiated during magnetar flares. In spite of its relevance for ongoing and rapidly improving searches for GWs from magnetars, there has been relatively little work on this question. The closely related question of what is the ratio of GW-emitted energy to electromagnetically (EM) emitted energy is not addressed at all in the literature, with searches therefore relying on possible correlations between observables [9]. We do note that a high GW/EM energy ratio, which is relevant to current GW observations, might be possible if most of the action takes place in the interior of the star, as suggested by recent work of Lander and Jones [10]. A high GW/EM energy ratio might also explain flares with high energy, but no initial spike or pulsations (typical of giant flares), as observed in SGR 1627–41 [11]. But in this article we concern ourselves only with the maximum available energies. In the rest of the Introduction we discuss the two major models: the crust-cracking model and the hydromagnetic deformation model.

B. Crust-cracking model

The now-standard interpretation of SGR flares within the magnetar model of a highly magnetized neutron star is that they involve the solid crust of the star cracking as it is strained by twisting magnetic field lines, with the field rearranging itself afterwards [12,13]. This is supported by the good fit of SGR flare gamma-ray energy and waiting time distributions to the universal power laws for brittle fracture, e.g. [14–17]. Some of the energy of the cracking event should excite quasinormal modes of the star. Indeed

there is evidence from quasi-periodic oscillations in x-ray tails of giant flares that shear modes or torsional modes of the solid crust are excited, possibly coupled to magneto-hydrodynamic modes in the core [18–20].

We note that, even under the hypothesis that the flare originates in the magnetosphere (see e.g. [21]), the magnetospheric reconfiguration exerts magnetic stress on the crust which can hydromagnetically couple to modes in the core [20,22,23].

In the above scenarios, the flare should excite to some extent the fundamental or f -modes of the star, which radiate GW with damping times of ~ 200 ms [24–26]. These time scales are shorter than other relevant ones, except for the Alfvén-wave crossing time of the star, to which they are comparable. Therefore, the f -modes are likely to radiate most of the energy they receive as GWs, even if other modes are excited to higher energies by the event that causes the gamma-ray flare. And, if much of the flare energy goes into exciting the f -modes, they might emit GW energy exceeding the emitted EM energy.

The details of which modes are most excited and what are likely GW to EM emission energy ratios are even more difficult to address than the total energy budget, and have not yet been investigated in the literature. Therefore we, like previous authors, restrict our attention to the total energy budget of the largest SGR flares, which serves as an upper limit to the GW energy emitted.

A natural estimate for the maximum GW energy radiated by the crust-cracking mechanism is the maximum elastic deformation energy of the crust, which should be at least comparable to largest gamma-ray energy emitted in a giant flare. The EM energy emitted in the 2004 giant flare of SGR 1806–20 [27], of order 10^{45} – 10^{46} erg, was greater than previous giant flare energies and hard to reconcile with the standard maximum crust elastic energy of order 10^{44} erg, e.g. [28]. The latter energy is proportional to the shear modulus of the solid part of the star, and thus the 2004 giant flare energy could be explained by solid quark matter. With a shear modulus exceeding that of a neutron-star crust by 3–4 orders of magnitude [29–31], energies of order 10^{47} – 10^{48} erg would become available.

The maximum crust elastic energy is also proportional to the square of the breaking strain of the material which, until recently, was usually assumed to be at most 10^{-2} , comparable to the best terrestrial alloys. Molecular dynamics simulations by Horowitz and Kadau [32], though strictly applicable only to the outer crust at densities below neutron drip, indicate that the breaking strain of dense solid matter can reach 10^{-1} as defects, domain walls, etc. are crushed away by the intense pressure. Using the above scaling, this brings the maximum elastic energy (and thus GW energy) up to 10^{46} erg for a normal neutron star, reconciling it with the EM energy emitted in the 2004 giant flare.

We note, apparently for the first time in the literature, that even higher energies are possible from the cracking

mechanism if the neutron star or at least its core is made of a solid form of quark matter, and the breaking strain of that matter is of order 10^{-1} . In fact, Horowitz and Kadau [32] restrict their simulations to the low-density outer layers of a normal nuclear matter crust, and do not speculate on the physics of exotic phases with or without strong magnetic fields. However, the crushing of defects under intense pressure which is responsible for a high breaking strain seems to be robust physics.

From the above mentioned scalings and from shear modulus calculations in the literature, we infer that, if the high breaking strain of Horowitz and Kadau [32] is generic, GW energies of order 10^{48} erg are possible for mixed baryon-meson or baryon-quark phases [30], and energies of order 10^{49} – 10^{50} erg are possible for solid quark phases [33,34]. A more detailed estimate of the former is forthcoming [35].

C. Hydromagnetic deformation model

The highest GW energies previously obtained in the literature and noted in the f mode searches [3–5] come from a model by Ioka [36] based on magnetic deformations of the star’s hydrostatic equilibrium. These can be 10^{48} – 10^{49} erg, comparable to the latest upper limits on GW emission from f -modes [5].

It may seem surprising that magnetar flares could be good candidates for GW detection given that supernovae, with a total EM energy emitted orders of magnitude above that emitted in giant flares, are difficult targets for GW searches even with improved instruments and algorithms [37]. Although the EM energy release in supernovae is large, the bulk motion of matter which generates GWs mainly involves material at densities lower than nuclear density and features relatively little quadrupolar motion. A rearrangement of the interior of a neutron star, on the other hand, involves matter at supernuclear densities, and the magnetic dipole couples directly to the mass quadrupole through the magnetic pressure.

While most neutron stars have external magnetic dipole fields less than $\sim 10^{12}$ – 10^{13} G, there is growing evidence for the existence of supermagnetized neutron stars with fields of $\sim 10^{14}$ – 10^{15} G [38,39]. Larger magnetic fields of $\sim 10^{16}$ G may be generated by the helical dynamo inside a newborn neutron star [12,40], and even the maximum field strength allowed by the virial theorem (10^{18} G) could be achieved if the central engines of gamma-ray bursts are magnetars [41–43]. Internal fields of order 10^{16} G are also suggested by lifetime energetics and cooling models and observations of persistent x-ray emission [44]. An internal field of 10^{16} G puts the ratio of magnetic potential energy ($\sim 10^{49}$ erg) to gravitational potential energy ($\sim 10^{53}$ erg) at 10^{-4} .

Ioka [36] noted that an increase in the spin period of SGR 1900 + 14 by a fraction 10^{-4} over an 80-day interval including its 1998 giant flare could have been produced by

a sudden 10^{-4} fractional increase in the moment of inertia at the time of the flare, which in turn could have been related to a reconfiguration of a toroidal internal magnetic field. The internal magnetic field is believed to be mainly toroidal due to dynamo action in the first few seconds of the star's life [12,38,40]. A mainly toroidal field makes the star prolate, leading to an increase in the moment of inertia when energy is released.

With some simplifying assumptions described below, Ioka [36] found a set of stellar equilibria with discrete energies and moments of inertia. For his most realistic equation of state (EOS), an $n = 1$ polytrope (see below), Ioka [36] found states separated by $\Delta I/I = 10^{-4}$ in moment of inertia and 10^{45} erg in energy, roughly the observed EM energy of the 1998 giant flare. In order to have energy differences between equilibria close to 10^{45} erg, being $\Delta I/I \sim \delta$, with δ being the magnetic/gravitational energy ratio, Ioka [36] chose flare models (i.e. jumps between equilibria) which kept the magnetic energy constant. This made the overall energy release second order in $\delta = 10^{-4}$: $10^{53} \text{ erg} \times (10^{-4})^2 = 10^{45} \text{ erg}$. Ioka [36] also gave energies for very soft EOS (high polytropic index) and high internal magnetic field (more than 10^{17} G) which were up to nearly 10^{49} erg, comparable to recent observational upper limits on GW emission.

Motivated by these high predicted energies, we reexamine the model of Ioka [36] with an eye toward exploring its broader applicability and robustness, and we push it to find under what conditions the highest GW energies are possible.

D. Outline

First to be addressed in generalizing the model by Ioka [36] are several simplifying assumptions such as Newtonian gravity, symmetry, lack of superconductivity, and polytropic equation of state. In Sec. II, we argue that the message to be drawn from the more detailed works appearing in the literature during the years since 2001 is that the physically simplified model of Ioka [36] well serves our present goal of estimating the order of magnitude of energy available.

In Sec. III, we describe Ioka's choice of magnetic field and the rest of the mathematical formalism (the first-order part of his calculation).

In Sec. IV, we show that the model by Ioka [36] has applicability beyond the 1998 giant flare. The biggest concern with such a model is, in fact, that it was built to explain the putative 10^{-4} change in spin period after the 1998 giant flare of SGR 1900 + 14. However, such changes are not observed associated with most flares; and indeed the data for the 1998 flare itself could be interpreted in other ways such as timing noise [45] or change of the external dipole field. We qualitatively discuss the broader possibilities for jumps between equilibria, and we give quantitative results for a particular family of jumps which

tends to produce larger energies with smaller moment of inertia changes.

In Sec. V, we summarize the results of our explorations and discuss their consequences for current and future GW searches by LIGO and Virgo.

II. PHYSICAL ASSUMPTIONS AND JUSTIFICATION

In this Section, we address the accuracy of a number of simplifying assumptions used in the analysis by Ioka [36], which we also adopt here. Most of them have been investigated further in recent years in the context of continuous GW emission from newborn magnetars.

A. Perturbative approach

Our first assumption is that the effect of the magnetic field on stellar equilibria is much greater than that of rotation, and much less than that of gravity. This is straightforward to check as in Ioka [36]: The magnetic field is a perturbative effect on the hydrostatic equilibrium of the star if the typical magnetic field strength satisfies $H \ll 10^{18}(R/10^6 \text{ cm})^{-4}(M/M_{\odot})^2 \text{ G}$, which it does even for the fields $H \sim 10^{16} \text{ G}$ predicted inside magnetars. The internal magnetic field induces a deformation which dominates the rotational one when $H \gg 10^{14}(P/1 \text{ s})^{-1} \text{ G}$, where P is the spin period. For SGRs, P is of the order of 5–10 s, see e.g. [46], and thus rotation can be neglected.

A recent calculation [47] including rotation and non-linear magnetic equilibrium confirms that these are negligible effects for the systems considered here. Neglecting these effects allows us to adopt a formalism similar to that developed by Chandrasekhar [48] and Chandrasekhar and Lebovitz [49] for slowly rotating polytropes, in which the perturbation parameter is the ratio of the rotational to gravitational energy. In cases where the magnetic field is the sole perturbation, the perturbation parameter becomes the ratio of magnetic-to-gravitational potential energy [50–56].

Like almost all other authors, we neglect the effect of stable stratification (nonbarotropic composition gradients) on the hydromagnetic equilibrium, although this may come into play on longer time scales, such as the cooling time scale [57].

B. Relativistic gravity corrections

The effects of relativistic gravity have also been investigated in recent years.

In Newtonian analyses such as Ioka [36], the magnetic stress of a toroidal field tends to make the star prolate, working like a rubber belt tightening up the equator of the star; and the analysis by Ioka and Sasaki [58] confirms the validity of this picture in relativistic stars.

More specifically, Ioka and Sasaki [59] and Ioka and Sasaki [58] extended the results of Ioka [36] to relativistic

gravity (for an $n = 1$ polytrope). They obtained stationary axisymmetric configurations of magnetized stars in the framework of general relativistic ideal magnetohydrodynamics, incorporating a toroidal magnetic field and meridional flow, in addition to a poloidal magnetic field. As in Ioka [36], Ioka and Sasaki [58] worked under the hypothesis of axisymmetry, boundary conditions so as to have the magnetic field vanishing at the stellar surface, and magnetic field weak compared to gravity, so that it can be treated as a small perturbation on an already-known nonmagnetized, nonrotating configuration. They found an eigenvalue problem with energies separated by nearly 10^{48} erg for internal fields of order 10^{16} G. (This is obtained from their Table 2, second group of rows—the first is unstable—multiplying column 3 by column 9 and keeping in mind that their \mathcal{R}_M is slightly greater than our δ .) These energies are nearly 2 orders of magnitude greater than the $n = 1$ jumps from Fig. 3 of Ioka [36], more comparable to the jumps for the nearly unstable $n = 2.5$ EOS. Relativity increases the central condensation of the star compared to Newtonian gravity and thus is expected to give numbers comparable to softer (higher- n) EOS. Therefore our Newtonian energy estimates for $n = 1$, in fact, should be somewhat conservative.

Other relativistic analyses, e.g. [60–62], change even more features of the analysis of Ioka [36], as we discuss in the next Sections.

C. Boundary condition

More important are the interlinked issues of magnetic field configurations, especially the toroidal-to-poloidal ratio and boundary conditions at the surface of the star, and the EOS.

The discrete energy spectrum at the heart of the model by Ioka [36], is due to the boundary condition imposed on the magnetic field at the surface. This may seem to be a very specialized condition, but we argue that it is more generally applicable.

Ioka [36] takes the toroidal part of the field to vanish at the stellar surface, which has the effect of forcing surface and magnetospheric currents to vanish. Ioka [36] also assumes a field configuration with a fixed toroidal-to-poloidal ratio, so that the poloidal field vanishes at the surface too. Both assumptions are common in the literature. The latter is an issue since the observed spin-downs of magnetars usually imply external dipole fields of 10^{14} – 10^{15} G just outside the surface. However this is small compared to the internal field, and there is now observational evidence for a magnetar with a large internal field and even smaller (less than 10^{13} G) external dipole field [63].

Invoking surface currents [60] can set the toroidal field discontinuously to zero just outside the star (compared to a finite value just inside) while letting the internal poloidal field be matched to an external dipole. However, there is

little to indicate what the surface currents on a neutron star should be, and thus they are neglected in most studies [47,58,61,62,64]. A barotropic EOS (dependent only on pressure) with density going to zero at the surface also forces the magnetic field to go to zero at the surface in ideal magnetohydrodynamics (MHD) [64]. However, magnetic diffusivity due to resistance can be invoked to get around that problem [65]: Neutron stars are not perfect conductors, and in moving from the superfluid interior to the crust and magnetosphere, the resistivity of the medium should increase and hence the boundary conditions should be adapted to reflect this behavior.

At any rate, if the internal field is matched to a much smaller external field the result should not differ greatly from matching to zero external field. Spin-down observations argue that the external dipole field does not change greatly even in giant flares [36,45]. Matching to any fixed external field will still result in discrete eigenvalues, so the mechanism should not be qualitatively changed and one would estimate is quantitatively changed by order of $H_{\text{ext}}/H_{\text{int}}$ or of order 10% for the scenario envisioned here.

The conclusion we draw from these works is that, while the no-external-field boundary condition is obviously a specialized simplification, the crucial property of discrete eigenvalues has greater generality.

D. Toroidal-to-poloidal field ratio

There has been much work on the toroidal-to-poloidal field ratio as well.

Recently Lander and Jones [47] studied the various stationary, axisymmetric equilibrium solutions for Newtonian fluid stars in perfect MHD, showing that the full equations of MHD reduce under these limits to two general cases: a mixed-field case (which includes purely poloidal fields as a special case) and purely toroidal fields.

In the mixed-field case, differently from the boundary condition of zero exterior field set by e.g. Ioka [36] and Haskell *et al.* [64], the toroidal field component is set to vanish outside the star (i.e., no currents exist on or outside the neutron star’s surface), while the poloidal field is matched through the stellar surface to an external dipole vanishing at infinity. Lander and Jones [47] find that the equilibrium configurations are poloidal-dominated.

The boundary condition being the main difference between Haskell *et al.* [64] and Lander and Jones [47], the latter authors conjecture that matching to an outside dipole field favors poloidal-dominated fields and oblate stars, while a vanishing magnetic field on the surface favors toroidal-dominated fields and prolate stars. As Lander and Jones [47] have emphasized, for a real neutron star the resistivity of the outer layers could resemble a boundary condition intermediate between the two cases.

Various studies dedicated to finding mixed-field equilibrium configurations with specific boundary conditions

[47,58,60–62,64,66–68] resulted in different poloidal-to-toroidal field ratios. The configurations obtained with the boundary condition set by Lander and Jones [47] all have no more than 7% of the magnetic energy stored in the toroidal field component. Ciolfi *et al.* [61,62] also found that in their configurations, although the amplitudes of both the poloidal and toroidal fields are of the same order of magnitude, and the toroidal field in the interior can be larger than the poloidal field at the surface, the contribution of the toroidal field to the total magnetic energy is $\lesssim 10\%$, because this field is non vanishing only in a finite region of the star. On the other hand, by setting the magnetic field to vanish outside the star, Ioka [36] (whose results agree with Haskell *et al.* [64] for the case of an $n = 1$ polytrope) obtains equilibrium configurations where up to $\sim 96\%$ of the magnetic energy is stored in the toroidal component (see line 10 in Table I).

An interesting point is how these results compare with those from studies aimed at evaluating the actual stability of magnetic equilibria in stars. These have shown that a stellar magnetic field in stable equilibrium must contain both poloidal (meridional) and toroidal (azimuthal) components, since both are unstable on their own [10,69–78]. Stars with purely poloidal magnetic fields suffer from a hydromagnetic instability, while the instabilities are suppressed if the toroidal magnetic fields in the star have comparable strength with the poloidal fields [67].

Numerical evolutions by Braithwaite [65] give indications that the toroidal field component should store 20%–90% of the total magnetic energy in order for the neutron star to be stable. Via MHD simulations, Braithwaite and Spruit [79] have found that purely poloidal magnetic fields in stars decay completely within a few Alfvén time scales, while “twisted-torus” poloidal-toroidal mixed configurations can survive much longer than the Alfvén time. These configurations are roughly axisymmetric; the poloidal field extends throughout the entire star and to the exterior, while the toroidal field is

confined in a torus-shaped region inside the star, where the field lines are closed [80].

In this paper, we follow Ioka [36] and consider equilibrium states where the contribution of the toroidal field energy is in between $\sim 65\%$ and $\sim 96\%$ of the total magnetic energy in the star (Table I).

We finally note that most of the above mentioned works have considered normal fluid stars, although neutron stars are believed to become superconducting superfluids over much of their volume shortly after birth. The latter case is much more complicated to treat, but see Akgun and Wasserman [82] for a recent careful calculation indicating that mostly toroidal fields may be stable in this case too.

E. Equation of state

A final issue is the dependence on the EOS.

Kiuchi and Kotake [83] have considered Newtonian magnetized stars with four kinds of realistic EOSs (Skyrme-Lyon (SLy) by [84]; Friedman-Pandharipande-Skyrme (FPS) by [85]; Shen by [86]; and LS by [87]). For the nonrotating sequences, they found that there exist nearly toroidal field configurations, irrespective of the EOSs. The magnetic energy stored in the stars increases with the degree of deformation being larger.

More recently, Kiuchi *et al.* [88] have investigated equilibrium sequences of relativistic stars containing purely toroidal magnetic fields, with the same four kinds of realistic EOSs. In the nonrotating case, it is found that for a SLy EOS, the toroidal magnetic field peaking in the vicinity of the equatorial plane acts through the Lorentz forces to pinch the matter around the magnetic axis, making the stellar shape prolate. Indeed, the toroidal magnetic field lines behave like a rubber belt that is wrapped around the waist of the star. This gross property is common to the other realistic EOSs [88].

For equal values of the central density, the profiles of stars with SLy and FPS EOSs are quite similar, while the density distribution of the star with the Shen EOS is less

TABLE I. For an $n = 1$ polytrope, and different state indices k (column 1) corresponding to the different eigenvalues λ_k (column 2), we give: the first-order (adimensional) magnetic energy (column 3), the (adimensional) poloidal field energy (column 4), the toroidal-to-poloidal magnetic energy ratio (column 5), the first-order corrections to the moment of inertia tensor (per unit magnetic energy, columns 6–8). The values in columns 2, 3, 5–7 are directly taken from Ioka [36].

k	λ_k	\mathcal{M}_1	$\mathcal{M}_{1,P}$	$\mathcal{M}_{1,T}/\mathcal{M}_{1,P}$	$I_{11;1}/\mathcal{M}_1$	$I_{33;1}/\mathcal{M}_1$	$I_{33;1}/\mathcal{M}_{1,P}$
1	2.361 933 0	1.307 07	0.456 554 54	1.862 90	4.094 18	2.272 35	6.505 510 9
2	3.407 865 0	0.307 662	0.078 0424 33	2.942 24	5.399 81	0.881 646	3.475 660 1
3	4.430 077 0	0.132 171	0.024 607 442	4.371 18	6.189 15	−0.314 410	−1.688 752 7
4	5.443 462 0	0.073 476 1	0.010 269 885	6.154 52	6.700 86	−1.186 14	−8.486 262 4
5	6.452 475 0	0.047 019 3	0.005 059 429 4	8.293 40	7.045 62	−1.805 98	−16.783 694
6	7.458 980 0	0.032 832 9	0.002 785 234 4	10.7882	7.285 85	−2.250 76	−26.532 409
7	8.463 904 0	0.024 316 7	0.001 661 067 5	13.6392	7.458 42	−2.576 05	−37.711 311
8	9.467 764 0	0.018 785 2	0.001 052 610 3	16.8463	7.585 83	−2.819 05	−50.309 612
9	10.470 870	0.014 978 4	0.000 699 608 12	20.4097	7.682 19	−3.004 33	−64.321 804
10	11.473 430	0.012 240 2	0.000 483 242 73	24.3293	7.756 63	−3.148 30	−79.744 235

prolate than SLy and FPS EOSs. The concentration of the magnetic field to the stellar center for the Shen EOS is weaker than that for the SLy or FPS EOS, the matter pressure stays relatively large up to the stellar surface, and the regions in which the magnetic pressure is dominant over the matter pressure appear rather in the outer regions. This implies that the magnetic fields for the Shen EOS are effectively less fastening to pinch the matter around the magnetic axis than those for the SLy or FPS EOS [88].

For the LS EOS, the regions in which the ratio of the magnetic pressure to the matter pressure is large also exist near the stellar surface. However the density distribution is found to become similar to that for SLy and FPS EOSs because the pressure ratio is sufficiently higher than that for the Shen EOS [88].

In conclusion, since relativistic corrections to gravity, boundary conditions and EOS do not seem to prevent the existence of prolate states of equilibrium sustained by strong toroidal fields, the simplified treatment by Ioka [36] is valid for the purpose of estimating the order of magnitude of the maximum GW energy that may be released in jumps between equilibria.

III. MATHEMATICAL FORMALISM

In this Section we review the mathematical formalism for the equilibria of magnetized polytropes and the particular choice of magnetic field configuration used by Ioka [36]. We basically follow his results, simplifying the presentation so as to concentrate only on the fundamental passages relevant for our work [but more details can be found in [89]], while giving all the necessary elements to understand the underlying physics. For an immediate comparison of our results with the ones by Ioka [36], we also keep his notation.

A. Equilibrium equations

Consider a nonrelativistic, one-component perfectly conducting fluid in hydrostatic equilibrium, with a magnetic field and vanishing net charge (as typical for astrophysical fluids or plasmas). The equations governing the equilibrium are

$$-\nabla p + \rho \nabla \Phi + \frac{1}{4\pi} (\nabla \times \vec{H}) \times \vec{H} = 0, \quad (1)$$

$$\nabla^2 \Phi = -4\pi G \rho, \quad (2)$$

$$\nabla \cdot \vec{H} = 0, \quad (3)$$

where ρ is the mass density, p is the pressure, and Φ is the gravitational potential. The first is the Euler equation for a nonrotating magnetized conducting fluid. The second is Poisson's equation and the last is one of Maxwell's equations. We further assume a polytropic EOS [90]

$$p = K \rho^{1+1/n}, \quad (4)$$

and use this equation and the corresponding length scale

$$\alpha = \left[\frac{(n+1)K\rho_c^{1/n-1}}{4\pi G} \right]^{1/2} \quad (5)$$

in terms of the central density ρ_c , to convert to dimensionless variables Θ , ξ , \vec{h} , ϕ , defined as follows:

$$\rho = \rho_c \Theta^n, \quad (6)$$

$$r = \alpha \xi, \quad (7)$$

$$\vec{H} = (4\pi G \delta)^{1/2} \rho_c \alpha \vec{h}, \quad (8)$$

$$\Phi = 4\pi G \alpha^2 \rho_c \phi. \quad (9)$$

Here δ is the ratio of magnetic-to-gravitational potential energy in physical units. In the dimensionless variables Eqs. (1)–(3) read [36]:

$$-\nabla \Theta + \nabla \phi + \frac{\delta}{4\pi \Theta^n} (\nabla \times \vec{h}) \times \vec{h} = 0, \quad (10)$$

$$\nabla^2 \phi = -\Theta^n, \quad (11)$$

$$\nabla \cdot \vec{h} = 0. \quad (12)$$

In the case of axisymmetry, \vec{h} can be conveniently expressed in terms of two scalar functions $P(\xi, \theta)$ and $T(\xi, \theta)$ as [91]

$$\begin{aligned} \frac{\vec{h}}{(4\pi)^{1/2}} &= -\frac{1}{\tilde{\omega}} \frac{\partial(\tilde{\omega}^2 P)}{\partial z} \hat{e}_{\tilde{\omega}} + \tilde{\omega} T \hat{e}_{\varphi} + \frac{1}{\tilde{\omega}} \frac{\partial(\tilde{\omega}^2 P)}{\partial \tilde{\omega}} \hat{e}_z \\ &= \nabla \times (\tilde{\omega} P \hat{e}_{\varphi}) + \tilde{\omega} T \hat{e}_{\varphi}, \end{aligned} \quad (13)$$

where $\hat{e}_{\tilde{\omega}}$, \hat{e}_{φ} , \hat{e}_z are unit vectors in the $\tilde{\omega}$, φ , z directions, and $(\tilde{\omega}, \varphi, z)$ are cylindrical coordinates, related to the spherical ones (ξ, θ, φ) by $\tilde{\omega} = \xi \sin \theta$ and $z = \xi \cos \theta$. Equation (10) implies that:

$$\nabla \times \left(\frac{(\nabla \times \vec{h}) \times \vec{h}}{4\pi \Theta^n} \right) = \nabla \times \vec{\mathcal{L}} = 0, \quad (14)$$

which is satisfied if $\vec{\mathcal{L}}$ is the gradient of a scalar function. It can be shown that for the case of an axisymmetric magnetic field one has

$$\vec{\mathcal{L}} = \nabla N_P(\omega^2 P) \quad (15)$$

if the following relations hold, see e.g. [91]:

$$\frac{\Delta_5 P}{\Theta^n} = -\frac{dN_P(\tau)}{d\tau} - \frac{T}{\Theta^n} \frac{dN_T(\tau)}{d\tau}, \quad (16)$$

$$\tilde{\omega}^2 T = N_T(\tau), \quad (17)$$

with the five-dimensional Laplacian

$$\Delta_5 = \frac{\partial^2}{\partial z^2} + \frac{3}{\tilde{\omega}} \frac{\partial}{\partial \tilde{\omega}} + \frac{\partial^2}{\partial \tilde{\omega}^2}. \quad (18)$$

Here $N_T(\tau)$ and $N_P(\tau)$ are arbitrary functions of their argument $\tau = \tilde{\omega}^2 P$. Assigning to such functions a specific form, the corresponding $P(\tilde{\omega}, \theta)$ and $T(\tilde{\omega}, \theta)$ are found (and thus the magnetic field configuration is fixed) by solving Eqs. (16) and (17) with appropriate boundary conditions. Once the magnetic field configuration is specified, the equilibrium density of the magnetized polytrope can be found solving (with appropriate boundary conditions) the modified Lane-Emden equation [36]

$$\nabla^2 \Theta = -\Theta^n + \delta \nabla^2 N_P(\tilde{\omega}^2 P), \quad (19)$$

which is obtained by combining Eq. (15) with Eqs. (10) and (11).

B. Perturbative approach

We assume that the solutions of Eqs. (16), (17), and (19) have the following form [92]:

$$P(\xi, \theta) = P_0(\xi, \theta) + \mathcal{O}(\delta), \quad (20)$$

$$T(\xi, \theta) = T_0(\xi, \theta) + \mathcal{O}(\delta), \quad (21)$$

$$\Theta(\xi, \theta) = \Theta_0(\xi) + \delta \Theta_1(\xi, \theta) + \mathcal{O}(\delta^2). \quad (22)$$

Substituting into Eqs. (16) and (17), one gets [36]

$$\frac{\Delta_5 P_0}{\Theta_0^n} = -\frac{dN_P(\tau_0)}{d\tau} - \frac{T_0}{\Theta_0^n} \frac{dN_T(\tau_0)}{d\tau}, \quad (23)$$

$$\tilde{\omega}^2 T_0 = N_T(\tau_0), \quad (24)$$

where $\tau_0 = \tilde{\omega}^2 P_0$. Next, suppose that a particular choice for the magnetic field configuration is made by specifying the functions $N_P(\tau)$ and $N_T(\tau)$ and assigning boundary conditions for the magnetic field. Then by solving Eqs. (23) and (24), $P_0(\xi, \theta)$ and $T_0(\xi, \theta)$ are found. Further, performing a Legendre expansion, we can write

$$N_P(\tau_0) = N_P(\tilde{\omega}^2 P_0(\xi, \theta)) = \sum_{m=0}^{\infty} \Psi_m(\xi) P_m(\cos\theta), \quad (25)$$

where $P_m(\cos\theta)$ denotes the Legendre polynomial of order m , and the coefficients $\Psi_m(\xi)$ are known once $P_0(\xi, \theta)$ is. To find the equilibrium configuration of the corresponding magnetized polytrope, one can then proceed as follows. We expand in Legendre polynomials the perturbed star density,

$$\Theta_1(\xi, \theta) = \sum_{m=0}^{\infty} \psi_m(\xi) P_m(\cos\theta), \quad (26)$$

where the coefficients ψ_m are to be found. It is possible to show that Eqs. (16), (17), and (19) imply [36]

$$\mathcal{D}_0 \Theta_0(\xi) = -\Theta_0^n(\xi), \quad (27)$$

$$\mathcal{D}_m(\psi_m(\xi) - \Psi_m(\xi)) = -n\Theta_0^{(n-1)}(\xi)\psi_m(\xi), \quad (28)$$

using the dimensionless radial Laplacian

$$\mathcal{D}_m = \left[\frac{1}{\xi^2} \frac{d}{d\xi} \left(\xi^2 \frac{d}{d\xi} \right) - \frac{m(m+1)}{\xi^2} \right]. \quad (29)$$

Equations (27) and (28) are to be solved by imposing the boundary conditions

$$\Theta_0(0) = 1, \quad \Theta_0'(0) = 0, \quad (30)$$

$$\psi_m(0) = 0, \quad \psi_m'(0) = 0, \quad (31)$$

which assure that, to first order in δ , the central density of the star is equal to ρ_c and the central pressure gradient vanishes. Moreover, it can be shown that the additional condition

$$(m+1)(\psi_m(\xi_0) - \Psi_m(\xi_0)) + \xi_0(\psi_m'(\xi_0) - \Psi_m'(\xi_0)) = 0 \quad (32)$$

for $m \geq 1$, where ξ_0 is the dimensionless radius of the unperturbed polytrope (i.e. $\Theta_0(\xi_0) = 0$), should be set in order to have Θ vanishing on the perturbed stellar surface [36]. Equation (27) with the boundary conditions (30) is simply the Lane-Emden equation for a polytrope of index n . Thus its solution $\Theta_0(\xi)$ is the density of the spherical, unmagnetized (i.e. unperturbed) star.

To summarize, the procedure to find the magnetically perturbed equilibrium of the star is as follows: Solve the unperturbed Lane-Emden Eq. (27) for Θ_0 . Choose the magnetic field's poloidal and toroidal components by specifying N_P and N_T inside the star and boundary conditions relating to the field just outside. Then obtain the perturbed density profile by solving Eq. (28) (see also [36,89] for more details), subject to the boundary conditions (30)–(32).

C. Perturbed quantities

Here we give several useful integrals related to global properties of the perturbed star.

In Newtonian gravity the addition of a magnetic field should not change the mass of the star. Therefore in general it changes the central density, for which we assume the form

$$\rho_c = \rho_0 + \delta\rho_1 + \mathcal{O}(\delta^2). \quad (33)$$

The first-order perturbed central density ρ_1 is found by writing the mass

$$M = C_M(M_0 + \delta M_1 + \mathcal{O}(\delta^2)) = \int_V \rho d^3r, \quad (34)$$

where we remove dimensions using the constant

$$C_M = 4\pi\rho_0\alpha_0^3. \quad (35)$$

Also, according to Eqs. (7) and (5), we reference an unperturbed characteristic length scale and radius of the star

$$\alpha_0 = R_0/\xi_0 = \left[\frac{K(n+1)\rho_0^{-1+1/n}}{4\pi G} \right]^{1/2}. \quad (36)$$

Using Eqs. (6), (7), (22), and (33) one has

$$\begin{aligned} M &= \int_V \rho_c \Theta^n \alpha^3 d^3 \xi \\ &= \alpha_0^3 \rho_0 \int_V d^3 \xi \left(1 + \delta \frac{\rho_1}{\rho_0} + \mathcal{O}(\delta^2) \right)^{3/2n-1/2} \\ &\quad \times (\Theta_0 + \delta \Theta_1 + \mathcal{O}(\delta^2))^n d\xi. \end{aligned} \quad (37)$$

Thus, comparing with Eq. (34), it can be shown that [36]

$$M_0 = \int_0^{\xi_0} \Theta_0^n(\xi) \xi^2 d\xi, \quad (38)$$

while imposing the mass conservation condition $M_1 = 0$, yields [36]

$$\frac{\rho_1}{\rho_0} = -\frac{2n^2}{M_0(3-n)} \int_0^{\xi_0} \xi^2 d\xi \psi_0(\xi) \Theta_0^{(n-1)}(\xi), \quad (39)$$

where ψ_0 is defined in Eq. (26).

In view of the axisymmetry of the problem, we can write components of the moment of inertia tensor (in units of $C_I = 4\pi\rho_0\alpha_0^5$)

$$I_{11} = I_{22} = \frac{1}{2}(I_{11} + I_{22}) = \frac{1}{2C_I} \int_V \rho(r^2 + z^2) dx dy dz, \quad (40)$$

where (x, y, z) are the usual Cartesian coordinates. Using Eqs. (6) and (7) we have the dimensionless-coordinate versions

$$I_{11} = I_{22} = \frac{1}{8\pi} \left(\frac{\rho_c}{\rho_0} \right)^{-(3/2)+(5/2n)} \int_V \Theta^n \xi^2 (1 + \cos^2 \theta) d^3 \xi, \quad (41)$$

$$I_{33} = \frac{1}{4\pi} \left(\frac{\rho_c}{\rho_0} \right)^{-(3/2)+(5/2n)} \int_V \Theta^n \xi^2 (1 - \cos^2 \theta) d^3 \xi. \quad (42)$$

Expanding up to first order in δ ,

$$I_{11} = I_0 + \delta I_{11,1} + \mathcal{O}(\delta^2), \quad (43)$$

$$I_{33} = I_0 + \delta I_{33,1} + \mathcal{O}(\delta^2), \quad (44)$$

it is possible to show that [36],

$$I_0 = \frac{2}{3} \int_0^{\xi_0} \Theta_0^n \xi^4 d\xi, \quad (45)$$

$$\begin{aligned} I_{11,1} &= \frac{2}{3} \left[\int_0^{\xi_0} n \Theta_0^{n-1} \left(\psi_0 + \frac{1}{10} \psi_2 \right) \xi^4 d\xi \right] \\ &\quad + \frac{5-3n}{2n} \frac{\rho_1}{\rho_0} I_0, \end{aligned} \quad (46)$$

$$\begin{aligned} I_{33,1} &= \frac{2}{3} \left[\int_0^{\xi_0} n \Theta_0^{n-1} \left(\psi_0 - \frac{1}{5} \psi_2 \right) \xi^4 d\xi \right] \\ &\quad + \frac{5-3n}{2n} \frac{\rho_1}{\rho} I_0, \end{aligned} \quad (47)$$

where ψ_0 and ψ_2 are defined according to Eq. (26) and found by solving Eq. (28).

The total energy of a polytropic star with a magnetic field can be written as, see e.g. [91,93],

$$\mathcal{E} = \mathcal{M} + \mathcal{U} + \mathcal{W}, \quad (48)$$

where \mathcal{M} is the magnetic energy, \mathcal{U} is the internal energy and \mathcal{W} is the gravitational potential energy, that read [93]:

$$\mathcal{M} = \frac{1}{8\pi C_E} \int_V |\vec{H}|^2 d^3 r, \quad (49)$$

$$\mathcal{U} = \frac{n}{C_E} \int_V p d^3 r, \quad (50)$$

$$\mathcal{W} = -\frac{1}{2C_E} \int_V \rho \Phi d^3 r, \quad (51)$$

where we remove dimensions with the characteristic energy

$$C_E = 4\pi K(n+1)\rho_0^{(1+1/n)}\alpha_0^3. \quad (52)$$

For polytropic configurations in hydromagnetic equilibrium, the virial theorem also holds [93]:

$$\mathcal{M} + \frac{3}{n} \mathcal{U} + \mathcal{W} = 0, \quad (53)$$

and thus the total energy of the configuration can be written as

$$\mathcal{E} = -\frac{3}{n} \mathcal{U} + \mathcal{U} = \frac{n-3}{n} \mathcal{U}. \quad (54)$$

Expanding to first order in δ

$$\mathcal{M} = \delta \mathcal{M}_1 + \mathcal{O}(\delta^2), \quad (55)$$

$$\mathcal{U} = \mathcal{U}_0 + \delta \mathcal{U}_1 + \mathcal{O}(\delta^2), \quad (56)$$

$$\mathcal{W} = \mathcal{W}_0 + \delta \mathcal{W}_1 + \mathcal{O}(\delta^2), \quad (57)$$

it is possible to show that [36]

$$\begin{aligned} \mathcal{M}_1 &= \frac{1}{4\pi} \int_V \left(-\frac{\tilde{\omega}^2}{2} P_0 \Delta_5 P_0 + \frac{\tilde{\omega}^2}{2} T_0^2 \right) d^3 \xi \\ &= \mathcal{M}_{1,P} + \mathcal{M}_{1,T}, \end{aligned} \quad (58)$$

with $\mathcal{M}_{1,P}$ and $\mathcal{M}_{1,T}$ being the energy in the poloidal and toroidal field components—respectively,

$$\mathcal{U}_0 = \frac{n}{5-n} \xi_0^3 \left(\frac{d\Theta_0}{d\xi}(\xi_0) \right)^2, \quad (59)$$

$$\mathcal{U}_1 = -\frac{n}{3-n} \mathcal{M}_1, \quad (60)$$

$$\mathcal{W}_0 = -\frac{3}{n} \mathcal{U}_0, \quad (61)$$

$$\mathcal{W}_1 = \frac{n}{3-n} \mathcal{M}_1. \quad (62)$$

Then the total energy of the equilibrium configuration to first order in δ reads

$$\begin{aligned} \mathcal{E} &= \frac{n-3}{n} \mathcal{U}_0 + \delta \mathcal{M}_1 \\ &= \frac{n-3}{n} \mathcal{U}_0 + \delta(\mathcal{M}_{1,P} + \mathcal{M}_{1,T}). \end{aligned} \quad (63)$$

The magnetic helicity $H = \int_V d^3r \vec{A} \cdot \vec{H}$ is also useful. (Here \vec{A} is the magnetic vector potential.) For the field configuration used here, $\vec{A} \cdot \vec{H} \propto \tilde{\omega}^2 P_0 T_0$ [91] and thus the helicity can be written (in physical units)

$$H = \frac{8\pi}{3} \alpha_0 C_E \delta \int_0^{\xi_0} d\xi \xi^4 P_0 T_0. \quad (64)$$

D. Choice of field configuration

Here we describe our special choice of magnetic field configuration and the consequent properties of equilibria.

Following [36], we choose all equilibria to have magnetic field configurations such that:

$$N_p(\tilde{\omega}^2 P_0) = -\tilde{\omega}^2 P_0, \quad N_T(\tilde{\omega}^2 P_0) = \lambda \tilde{\omega}^2 P_0, \quad (65)$$

where λ is a constant. With this choice, the solutions for $P_0(\xi)$ and $T_0(\xi)$ are functions of the radial coordinate only [56,94] and satisfy (see Eqs. (16) and (17)):

$$\Delta_5 P_0 + \lambda^2 P_0 = \Theta_0^n, \quad T_0 = \lambda P_0. \quad (66)$$

That is, the functions P and T specifying the poloidal and toroidal field components are proportional to each other, with their ratio being constant inside the star. This configuration is the polytropic version of the simplest choice of magnetic field (other than force-free) applicable to incompressible stars [69,91,94,95]. Although such a simple solution is unlikely to be perfectly realized in real magnetars, its study has long been considered useful to give rough

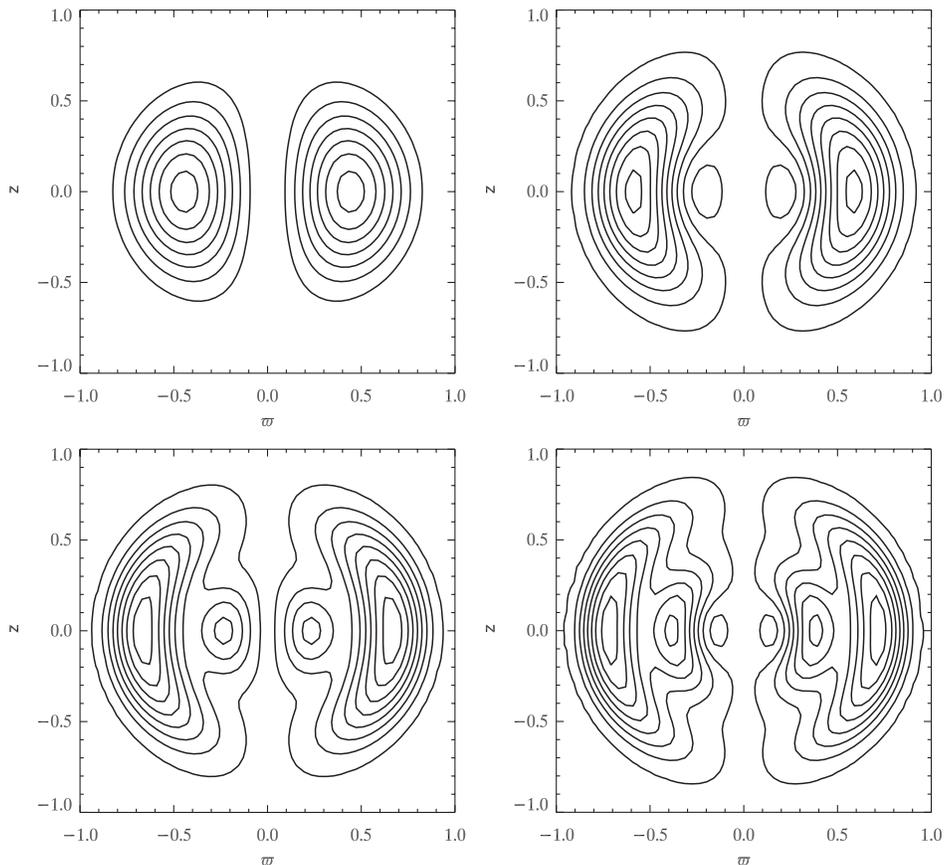


FIG. 1. Projection of the magnetic field lines on the meridional planes for the case of an $n = 1$ polytrope, and a magnetic field configuration characterized by an eigenvalue λ_1 (upper-left), λ_2 (upper-right), λ_3 (lower-left), λ_4 (lower right).

estimates of the influence of the density gradient on the magnetic field.

Since the external magnetic field is expected to be negligible with respect to the internal one, boundary conditions are set so as to have the magnetic field vanishing on the star's surface (see also Sec. II C):

$$P_0(\xi_0) = 0, \quad \frac{dP_0}{d\xi}(\xi_0) = 0. \quad (67)$$

Equation (66) with the boundary conditions in Eq. (67) gives [56,94]

$$P_0(\xi) = \frac{\lambda}{\xi} n_1(\lambda\xi) \int_0^\xi \Theta_0^n(\xi') j_1(\lambda\xi') \xi'^3 d\xi' + \frac{\lambda}{\xi} j_1(\lambda\xi) \int_\xi^{\xi_0} \Theta_0^n(\xi') n_1(\lambda\xi') \xi'^3 d\xi', \quad (68)$$

where j and n are spherical Bessel and Neumann functions, respectively, and with λ constrained to be a zero of the function:

$$F(\lambda) = \int_0^{\xi_0} \Theta_0^n(\xi') j_1(\lambda\xi') \xi'^3 d\xi'. \quad (69)$$

The first ten zeros of Eq. (69) are indicated in the second column of Table I. These correspond to different magnetic field configurations, as shown in Fig. 1, where we plot the magnetic field lines in the meridional planes (which run along the contours $\tilde{\omega}^2 P_0(\xi) = \text{const}$), for the first four λ roots of an $n = 1$ polytrope. It is evident that the higher λ_k is, the more complex are the magnetic field lines. As commented in Sec. II, also in light of the conditions required for the actual stability of the equilibrium state, in our analysis we consider configurations corresponding to the first ten λ roots, so as to deal with toroidal magnetic fields storing a ratio of the total magnetic energy which is in between $\sim 65\%$ and $\sim 96\%$.

Using Eq. (58), we can compute for each equilibrium state characterized by a given λ_k the corresponding values of the dimensionless total magnetic energy, the fraction of this energy going into the poloidal component, and the toroidal-to-poloidal energy ratio. These are listed in columns 3–5 of Table I. As evident from such a Table, the higher the value of λ_k , the higher the fraction of energy stored in the toroidal field component.

Looking at columns 6–7 in Table I, it is evident that the corrections in the moment of inertia normalized to the total magnetic energy of the state are such that the higher λ_k is, the more prolate is the star. This is equivalent to saying that states having the same total magnetic energy but a higher toroidal-to-poloidal field energy ratio, are more prolate. Physically, this is a consequence of the fact that the toroidal field tends to make the star prolate, working like a rubber belt tightening up the equator of the star.

IV. GENERALIZATION AND RESULTS

Ioka [36] has invoked jumps between the different equilibrium configurations of a magnetized neutron star to explain the properties of SGR flares. Here we explore the model [36] in terms of flare observables: jumps in energy and moment of inertia. First, we describe the choice of jumps considered by Ioka [36] (conserving the total magnetic energy and requiring $\Delta I/I = 10^{-4}$). Next, we present results for a new choice of jumps (conserving the energy of the poloidal field only). Finally, we discuss the dependence on the mean poloidal field strength for jumps that conserve the poloidal field energy, and describe the uncertainties associated with using a set of stellar models with $n = 1$ polytropic EOS.

A. Jump conditions

Equilibria of nonmagnetic polytropes can be characterized by one parameter, e.g. the gravitational potential energy; while equilibria of magnetic polytropes require two, e.g. the gravitational and magnetic potential energies. These 2 degrees of freedom also allow one to choose the two observables of SGR flares, total energy and moment of inertia, as parameters of the problem. Considering jumps between equilibria of a single star requires fixing the mass, leading to a sequence of equilibria characterized by a single parameter, e.g. the ratio of potential energies δ . Therefore jumps between equilibria, which are to model SGR flares, can trace various paths in the two-dimensional parameter space.

In Fig. 2, we plot the paths traced by the specific families of jumps considered by Ioka [36]. For this family,

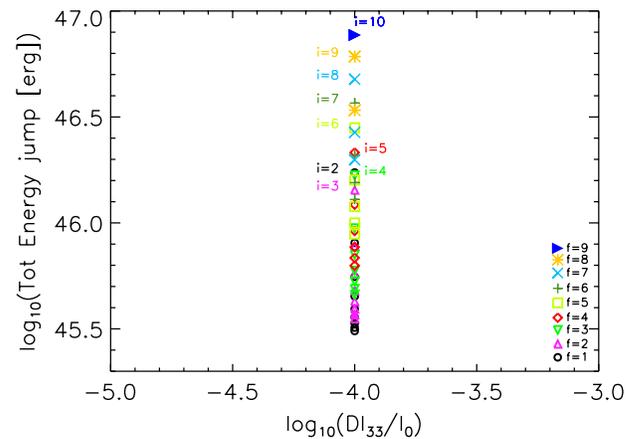


FIG. 2 (color online). Energy vs moment of inertia jumps for different final (f) and initial (i) state indices in jumps conserving the total magnetic energy (to first order), and with a change in moment of inertia of 10^{-4} (as possibly observed in the August giant flare of SGR 1900 + 14, see [36]). Jumps characterized by the same f are plotted with the same symbol (and color in the online version). For clarity, for each f , we mark on the plot the initial state index i of the jump with the highest energy, corresponding to $i = f + 1$.

$\Delta I/I = 10^{-4}$ and the total magnetic energy is kept constant in a jump. Because of this last requirement, since the contribution from the toroidal field decreases in a jump (see column 5 in Table I), the poloidal field increases. Because the toroidal fields make the star more prolate, and poloidal fields do the reverse, this allows a large change in the moment of inertia.

Note that in Ioka's model, for a given value of the final state index f , jumps from initial states with higher values of initial state index i release a smaller amount of energy. This is due to the fact that, for increasing i , the ratio of toroidal-to-poloidal field energy increases. Thus, higher values of i require a lower value of the total magnetic energy in the star if a fixed moment of inertia change is required in all jumps i -to- f with the same f . This in turn implies a smaller jump in total energy with higher values of i , the energy jump being proportional to the square of the magnetic-to-gravitational potential energy ratio of the initial state (see Eq. (94) of Ioka [36]).

In the present work, we modify the calculations by Ioka [36] by proposing a second family of higher-energy jumps based on keeping the potential energy of the poloidal magnetic field constant. The calculation of Ioka [36] is mainly modified in the fact that, since we allow the magnetic energy to change in jumps, we only need first-order perturbation theory, while Ioka [36] needed second-order. Besides the fact that larger energy jumps are obtained allowing the total magnetic energy to change, our choice is physically interesting for two reasons. First, in real magnetars, the internal poloidal field may remain matched to the outer poloidal field, which does not change by order of unity even in giant flares. Second, our choice is consistent with the standard theory that magnetic helicity is expelled from the star [39], since the helicity decreases through jumps (see below). This is more desirable than the behavior of Ioka's model, where the helicity increases in lower energy states.

Consider a transition (i, f) between two equilibrium states, the initial being characterized by an eigenvalue λ_i , the final by λ_f . This means that the magnetic-to-gravitational potential energy ratio will, unlike the case of Ioka [36], have different initial and final values δ_i and δ_f . If we make the hypothesis that the energy in the poloidal field is conserved in the transition, then the following relation holds:

$$\delta_i \mathcal{M}_{1,p}(\lambda_i) = \delta_f \mathcal{M}_{1,p}(\lambda_f). \quad (70)$$

As evident from Table I, $\mathcal{M}_{1,p}(\lambda_i) < \mathcal{M}_{1,p}(\lambda_f)$ for $i > f$. Thus, while in Eq. (64) the integral increases at lower-numbered states, the choice in Eq. (70) assures $\delta_f < \delta_i$ for $i > f$, making the overall helicity decrease in this family of jumps.

Using Eq. (63) and the above condition, the total energy change in the transition reads

$$\begin{aligned} \Delta \mathcal{E}_{(i,f)} &= \delta_f \mathcal{M}_{1,T}(\lambda_f) - \delta_i \mathcal{M}_{1,T}(\lambda_i) \\ &= \delta_i \mathcal{M}_{1,p}(\lambda_i) \left[\frac{\mathcal{M}_{1,T}(\lambda_f)}{\mathcal{M}_{1,p}(\lambda_f)} - \frac{\mathcal{M}_{1,T}(\lambda_i)}{\mathcal{M}_{1,p}(\lambda_i)} \right]. \end{aligned} \quad (71)$$

Looking at the 5th column in Table I, it is evident that to power an SGR flare (i.e. $\Delta \mathcal{E}_{(i,f)} < 0$), only jumps from higher to lower λ_k are permitted. As we will show in the next section, physically this corresponds to having the star becoming less prolate (i.e. more spherical) in the transition, thus passing from a more energetic to a less energetic equilibrium configuration.

To completely specify the energy (in physical units) of equilibria, three parameters are needed: two of them pertain to the EOS (e.g. the total mass $M = C_M M_0$ and the unperturbed radius $R_0 = \alpha_0 \xi_0$), while the third is the ratio δ between the physical unit in which we measure the gravitational potential energy (that is fixed by M and R_0) and the magnetic energy. For a star characterized by a given M and R_0 , a transition (i, f) leaves us with two parameters: the values of δ_i and δ_f . The requirement of having the poloidal field energy conserved in the jump fixes δ_f as a function of δ_i (see Eq. (70)) and leaves only δ_i free. Rather than specifying the last, we can equivalently specify the strength of the mean poloidal magnetic field inside the star,

$$\langle H_P \rangle = \sqrt{8\pi \frac{C_E \delta_i \mathcal{M}_{1,p}(\lambda_i)}{(4\pi R_0^3/3)}}, \quad (72)$$

and thus the energy jumps (in physical units) are given by

$$C_E \Delta \mathcal{E}_{(i,f)} = \frac{\langle H_P \rangle^2}{8\pi} \frac{4\pi R_0^3}{3} \left[\frac{\mathcal{M}_{1,T}(\lambda_f)}{\mathcal{M}_{1,p}(\lambda_f)} - \frac{\mathcal{M}_{1,T}(\lambda_i)}{\mathcal{M}_{1,p}(\lambda_i)} \right]. \quad (73)$$

In a transition (i, f) between two equilibrium states that conserves the poloidal field energy inside the star, the moment of inertia changes as

$$\begin{aligned} \frac{\Delta I_{33,(i,f)}}{I_0} &= \delta_f \frac{I_{33,1}(\lambda_f)}{I_0} - \delta_i \frac{I_{33,1}(\lambda_i)}{I_0} \\ &= \frac{\langle H_P \rangle^2}{8\pi} \frac{4\pi R_0^3}{3C_E I_0} \left[\frac{I_{33,1}(\lambda_f)}{\mathcal{M}_{1,p}(\lambda_f)} - \frac{I_{33,1}(\lambda_i)}{\mathcal{M}_{1,p}(\lambda_i)} \right]. \end{aligned} \quad (74)$$

In Fig. 3, we plot our family of fixed-poloidal magnetic energy jumps. As evident from this Figure, keeping the poloidal magnetic field constant, the resulting energy jumps range in between $\sim 2 \times 10^{47}$ erg and $\sim 4 \times 10^{48}$ erg, while moment of inertia jumps are always $\leq 10^{-4}$ (the upper limit observed in the 1998 giant flare of SGR 1900 + 14). Higher-energy jumps are possible, but require higher jumps in the moment of inertia. We note, however, that smaller changes in $\Delta I/I$ (i.e. in the observed spin period) could also be produced by a magnetic field axis misaligned with the rotation axis. Note also that,

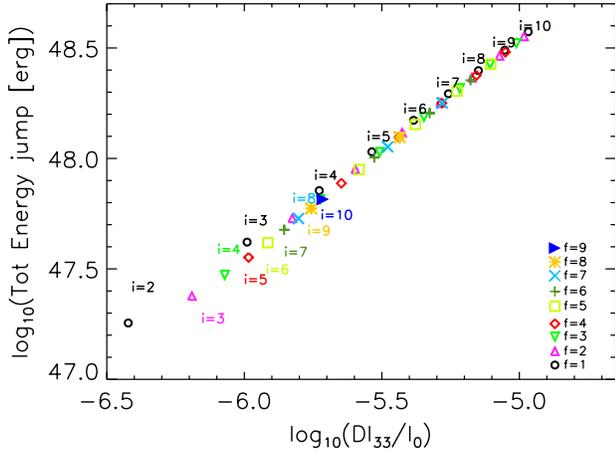


FIG. 3 (color online). Energy vs moment of inertia jumps for different final (f) and initial (i) state indices, for a family of jumps that conserves the poloidal field strength (10^{15} G). Symbols (and colors in the online version) are as in Fig. 2. We mark on the plot the initial state index of all the jumps with $f = 1$. For jumps with $f > 1$, we mark for clarity only the jump with lowest energy, corresponding to $i = f + 1$.

for the family of jumps we have considered here, the total internal magnetic field is of the order of $1\text{--}2 \times 10^{16}$ G, smaller than required by Ioka [36]. In his model, in fact, the total internal magnetic field is $\geq 10^{17}$ G for jumps with energies $\geq 10^{48}$ erg (see Figs. 3 and 4 in [36]).

The fundamental result here is that our choice of jumps is particularly large in energy and small in moment of inertia. In fact, allowing for a change in total magnetic energy produces energies larger than those of Ioka [36] (see Fig. 2) by $O(1/\delta)$. Moreover, since $\lambda > 1$ always, the toroidal field energy dominates in the equilibria. Because our family of jumps only conserves the poloidal field energy, they can change by a significant fraction the total magnetic energy. On the other hand, our moment of inertia changes are smaller than for Ioka's choice (see Fig. 2) since, as noted above, the decrease in toroidal field and increase in poloidal field in Ioka's model tend to add up their effect in increasing the moment of inertia (making the star less prolate).

B. Poloidal field energy dependence

To show the effect of the poloidal magnetic field strength on the family jumps introduced in the previous Section, in Fig. 4 we show, for an $n = 1$ polytrope with $R_0 = \alpha_0 \xi_0 = 10^6$ cm $M = C_M M_0 = 1.4 M_\odot$, the energy jumps $-C_E \Delta \mathcal{E}_{(i,f)}$ as a function of the index i of the initial state, for final states $f = 1\text{--}9$, and $\langle H_p \rangle = (10^{14}, 10^{14.5}, 10^{15}, 10^{15.3})$ G. These values of the mean poloidal field correspond to a total mean magnetic field inside the star lower than $\approx 10^{16}$ G, for transitions having $f < 10$ (see column 5 in Table I).

In Fig. 5 we show, for an $n = 1$ polytrope with $M = 1.4 M_\odot$ and $R_0 = 10^6$ cm, the moment of inertia jumps

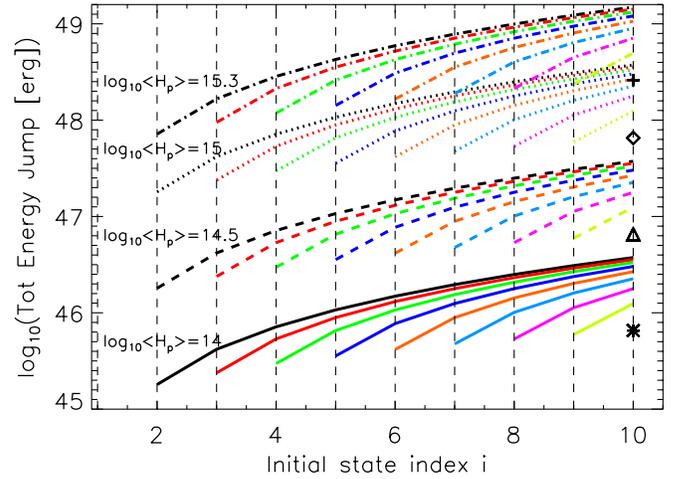


FIG. 4 (color online). Total energy jumps as a function of the initial state index i for final states having indices $f = 1$ to $f = 9$. The line describing $f = 1$ begins at $i = 2$, the line describing $f = 2$ begins at $i = 3$, and so on. The $f = 9$ case is described by a single symbol instead of a line. Colors in the online version are black, red, green, blue, orange, light blue, purple, yellow, and black for $f = 1$ to $f = 9$ respectively. The jumps are computed for different values of the mean poloidal field $\langle H_p \rangle$, which is conserved in the transition: from bottom to top, 10^{14} G (solid lines and asterisk), $10^{14.5}$ G (dashed lines and triangle), 10^{15} G (dotted lines and diamond), $10^{15.3}$ G (dot-dashed lines and cross). An $n = 1$ polytrope with $M = C_M M_0 = 1.4 M_\odot$ and $R_0 = 10^6$ cm is being considered.

$\Delta I_{33,(i,f)} / I_0$ as a function of the index i of the initial state, for final states $f = 1\text{--}9$, and $\langle H_p \rangle = (10^{14}, 10^{14.5}, 10^{15}, 10^{15.3})$ G. We note that in all cases $\Delta I_{33,(i,f)} / I_0 < 10^{-4}$, i.e. the transitions considered here are all associated with changes in moment of inertia smaller than the possible value inferred from the August

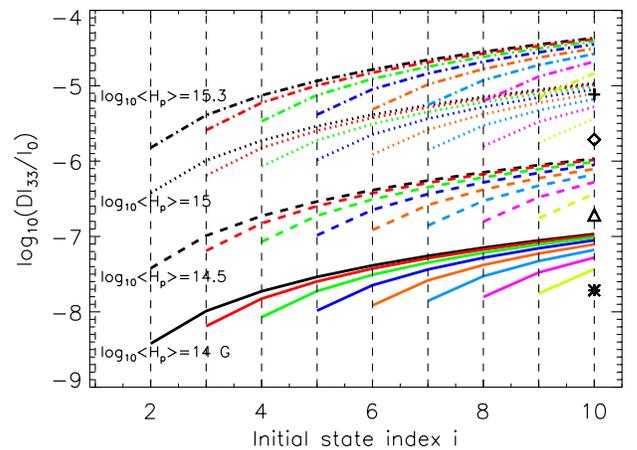


FIG. 5 (color online). Moment of inertia jumps, $\Delta I_{33} / I_0$, as a function of the initial state index i for different final states and different values of the mean poloidal field strength (see caption of Fig. 4 for symbols, and colors in the online version). An $n = 1$ polytrope with $M = 1.4 M_\odot$ and $R_0 = 10^6$ cm is being considered.

17, 1998 giant flare from SGR 1900 + 14 by Ioka [36]. Small jumps in moment of inertia can be hidden by the high timing noise and sparse observations of magnetar spin periods: For example, a jump of 5×10^{-6} could have happened in the 2004 giant flare of SGR 1806 – 20 [45].

C. Equation of state dependence

The EOS is the simplifying assumption which seems quantitatively most important in Ioka’s calculations. Fig. 3 of Ioka [36] shows that the highest jumps in energy are found for $n = 2.5$ polytropes, extremely soft EOS on the verge of being unstable to radial perturbations; and that energies for the more realistic $n = 1$ polytropes are orders of magnitude lower.

In contrast to Ioka [36], we schematically examine the EOS dependence of GW energy by restricting the polytropic index to $n = 1$ and varying the mass and radius of the star instead. For many problems this approach gives numbers which are comparable to those for more realistic EOS. This is because in Newtonian gravity for a given $n = 1$ polytrope, all stars have the same radius regardless of mass (e.g. [90]), a property which is approximately true of most realistic neutron-star EOS in relativistic gravity for most of the allowed mass range (e.g. [96]). Increasing the polytropic index as in Ioka [36] can lead to artificially large energy jumps since the star approaches instability to radial perturbations as $n \rightarrow 3$ (e.g. [90,97,98]). Also, as evident from Fig. 1, the magnetic field is usually concentrated in the outer core of the star (densities at or slightly above nuclear density), where all realistic EOS tend to be fit well by polytropes with $n = 0.5$ –1 [99]. As Ioka [36] showed, the energy jumps tend to rise with n , so $n = 1$ is good for conservatively estimating the maximum energy.

The choice $n = 1$ also makes the math very simple: we have [93]

$$\Theta_0(\xi) = \frac{\sin \xi}{\xi}, \quad \xi_0 = \pi. \quad (75)$$

Dimensionless unperturbed quantities are then easily calculated:

$$M_0 = \int_0^\pi (\sin \xi / \xi) \xi^2 d\xi = \pi, \quad (76)$$

from Eq. (38);

$$I_0 = \frac{2}{3} \int_0^\pi \xi^3 \sin \xi d\xi = \frac{2}{3}(\pi^3 - 6\pi), \quad (77)$$

from Eq. (45). For a choice of (dimensionful) mass $M = C_M M_0$ and radius $R_0 = \alpha_0 \xi_0$ of the unperturbed star, the dimensionful conversion factors are derived as

$$C_M = M / \pi, \quad (78)$$

from Eq. (34);

$$\alpha_0 = R_0 / \pi, \quad (79)$$

from Eq. (36);

$$K = 2\pi G \alpha_0^2, \quad (80)$$

from Eq. (36);

$$\rho_0 = C_M / (4\pi \alpha_0^3), \quad (81)$$

from Eq. (35). Inserting in Eq. (52) we derive the energy scale for our results in physical units,

$$C_E = \frac{G C_M^2}{\alpha_0} = \frac{G M^2}{\pi R_0} \quad (82)$$

which, for the canonical choice $M = 1.4 M_\odot$ and $R_0 = 10^6$ cm, yields $C_E \sim 1.6 \times 10^{53}$ erg.

To estimate the mass and radius dependence of the results we pick ranges of the parameters based on observations. The present observed mass range is roughly 1.2–2.0 M_\odot (see [96] for the highest mass) and predicted radii are roughly 9–15 km (for a summary see [99]).

The energy jumps scale differently between Ioka’s jump condition and the constant poloidal field condition. In this last case the magnetic energy, which for a given equilibrium state scales as $\langle H_p \rangle \times R_0^3$, is the source of the flare (i.e. jump, see Eq. (73)). Thus, for a fixed value of $\langle H_p \rangle$, the larger the star’s radius, the higher the energy jump. For $R_0 = 15$ km, the energy jumps shown in Fig. 4 would be a factor of ≈ 3.4 higher than for $R_0 = 10$ km.

On the other hand, the transitions considered by Ioka [36] involved no change in the total magnetic energy. In this case, the energy source is the gravitational potential energy, so the total energy jumps scaled as $C_E \propto M^2 / R_0$, with higher masses and smaller radii favoring more powerful flares. For $M = 2 M_\odot$ and $R_0 = 9$ km, this scaling increases the energy jumps by a factor of ≈ 2.3 with respect to a standard choice of $M = 1.4 M_\odot$ and $R_0 = 10$ km.

Concerning the (fractional) moment of inertia jumps, in Ioka [36] they were fixed to match the value derived for the August giant flare of SGR 1900 + 14. But using the fixed-poloidal jump condition they can change. We see from Eqs. (74) and (82) that, for a given $\langle H_p \rangle$, we have $\Delta I_{33,(i,f)} / I_0 \propto R_0^2 / M_0^3$. Thus, bigger masses and smaller radii help to keep the moment of inertia jumps small in a transition. In particular, for $M = 2 M_\odot$ and $R_0 = 9$ km, the jumps shown in Fig. 5 are reduced by a factor of ≈ 3.6 .

V. DISCUSSION

We have shown that changes in the hydromagnetic deformation of a magnetar can provide an energy reservoir of order 10^{48} – 10^{49} erg, comparable to LIGO and Virgo observational upper limits on f -mode GW emission, under more generic circumstances than considered in the original work by Ioka [36]. The key requirement is a change in the magnetic potential energy of the star, which causes the change in total energy of the star to be first order rather

than second order in the hydromagnetic perturbation parameter. Such an event can happen of order 10 times over the lifetime of the star, and such energies are then only applicable to (some of) the rare giant flares. However, in the family of jumps we proposed here to explain SGR flares, a large glitch in the magnetar's spin is not required, nor is an unrealistically soft EOS or extremely high internal field. Our family of jumps is also consistent with the idea that the helicity of the internal field is decreased rather than increased in giant flares.

We have briefly noted that such high energies are also available in the standard model of magnetar flares, crust cracking, if the solid part of the star is not limited to the crust but includes a core of solid quarks or mixed-phase material.

We have only considered equilibrium states and the total energy available. Our estimates are order of magnitude accuracy, and could be carried further by considering refinements such as relativity, field configurations, and realistic EOS. To establish high GW emission energies as a viable model also requires investigation of the dynamics to determine if the ratio of GW/EM energy emitted can be much higher than unity, for example, if most of the action takes place in the interior of the star.

We conclude by noting that the problem of GW emission from magnetar flares presents further opportunities: It is a problem that has received much less study than, for example, continuous GW emission from newborn magnetars.

Yet many of those results can be adapted to this problem. And, while newborn magnetars may become relevant to observations in the era of advanced interferometers, the flare problem is relevant right now. We hope that this will spur further work on the problem (and we refer the reader to Levin and van Hoven [100] and Kashiyama and Ioka [101], which were made public shortly after this paper).

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