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We present two methods for integrating forced geodesic equations in the Kerr spacetime. The methods can accommodate arbitrary forces. As a test case, we compute inspirals caused by a simple drag force, mimicking motion in the presence of gas. We verify that both methods give the same results for this simple force. We find that drag generally causes eccentricity to increase throughout the inspiral. This is a relativistic effect qualitatively opposite to what is seen in gravitational-radiation-driven inspirals, and similar to what others have observed in hydrodynamic simulations of gaseous binaries. We provide an analytic explanation by deriving the leading order relativistic correction to the Newtonian dynamics. If observed, an increasing eccentricity would thus provide clear evidence that the inspiral was occurring in a nonvacuum environment. Our two methods are especially useful for evolving orbits in the adiabatic regime. Both use the method of osculating orbits, in which each point on the orbit is characterized by the parameters of the geodesic with the same instantaneous position and velocity. Both methods describe the orbit in terms of the geodesic energy, axial angular momentum, Carter constant, azimuthal phase, and two angular variables that increase monotonically and are relativistic generalizations of the eccentric anomaly. The two methods differ in their treatment of the orbital phases and the representation of the force. In the first method, the geodesic phase and phase constant are evolved together as a single orbital phase parameter, and the force is expressed in terms of its components on the Kinnersley orthonormal tetrad. In the second method, the phase constants of the geodesic motion are evolved separately and the force is expressed in terms of its Boyer-Lindquist components. This second approach is a direct generalization of earlier work by Pound and Poisson [A. Pound and E. Poisson, *Phys. Rev. D* **77**, 044013 (2008).] for planar forces in a Schwarzschild background.

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I. INTRODUCTION

The two-body problem in relativity when one of the bodies is much more massive than the other is of great interest both theoretically and astrophysically. In this limit, the orbit of the smaller body is approximately geodesic on short time scales. Deviations from the geodesic trajectory arise from the back-reaction on the orbit of the spacetime perturbation created by the object, but can also arise from external factors such as gravitational interactions with other bodies, gaseous material in the spacetime and so forth. In all these situations, the orbit can be described as a geodesic acted on by a perturbing force, which is in general small. In this article, we describe techniques for integrating the Kerr geodesic equations in the presence of an arbitrary forcing term, which can be applied to any of these problems.

For the back-reaction on the orbit, the perturbing force, called the self-force, is of the order of the mass ratio μ/M and it can be computed by a perturbation expansion in this small parameter. Computing the linearized metric perturbation sourced by the compact object and hence the self-force is not an easy task and it has taken more than a decade

to solve this problem for a nonspinning compact object moving in a Schwarzschild background [1–4]. The conventional approach treats the compact object as a test mass which leads to a divergence of the field at the position of the particle and this must be dealt with using a regularization procedure. The extension to Kerr orbits is under way. The techniques described in this paper will be a useful tool in the future for constructing trajectories evolving under gravitational radiation reaction.

The problem of the motion of two bodies with very different masses is relevant for present and future gravitational wave detectors. Systems with mass ratios of 1:100 (intermediate-mass-ratio inspirals) could be detected by the advanced generation of ground-based detectors that are currently under construction [5]. The proposed space-based detector LISA [6] is expected to detect ~ 10 – 100 extreme-mass-ratio inspiral (EMRI) events per year [7]. These result from the capture of a compact stellar-mass object (a white dwarf, neutron star or black hole) by a massive black hole (MBH) from a surrounding cusp of stars in a galactic nucleus. The captured object generates a large number of gravitational wave cycles while it is orbiting in the strong field of the MBH, which makes these

very good sources to use as probes of strong-field gravity [8]. For both of these classes of source, techniques for evolving the orbit under the influence of both gravitational back-reaction and other perturbing forces are essential for constructing accurate waveform templates and for understanding how external perturbations can leave an imprint on the inspiral trajectory

We present two implementations that can be used to integrate geodesic motion in a Kerr background with an external force. We use the method of osculating elements extending previous work [9] for Schwarzschild orbits to the Kerr background. The problem of motion under a small perturbation is well studied in celestial mechanics and is regularly applied to model the motion of satellites and small planets. A geodesic in Newtonian mechanics, or relativity, is uniquely characterized either by the three components of the particle position vector, \mathbf{r} , and the three components of the particle velocity, $\dot{\mathbf{r}}$, at any time or by six orbital constants (three orbital constants of the motion and three initial phases). There is a one-to-one correspondence between the two characterizations. This means that *any* trajectory can be instantaneously identified with a geodesic that has the same values of \mathbf{r} and $\dot{\mathbf{r}}$. Of course, at two different instances of time, the geodesics will differ, but one can smoothly evolve the geodesic parameters to reproduce any nongeodesic trajectory. There are several approaches to do so and we describe these in the next subsection.

A. Osculating elements or variation of constants

As mentioned above we can describe a bound stable geodesic by six parameters, which we denote by I . In the nonrelativistic case these parameters are simply $I = (\mathbf{r}, \dot{\mathbf{r}})$, while for geodesic motion in Kerr we can take $I = \{E, L_z, Q, \psi_0, \phi_0, \chi_0\}$. Here E is the energy, L_z the azimuthal angular momentum, Q is the Carter constant, and the remaining phases are defined in Sec. III below.

At each instant we can therefore identify the true trajectory with a corresponding geodesic such that \mathbf{r} and $\dot{\mathbf{r}}$ are the same. This imposes a particular choice of parameters, I , at each instance of time, and the whole trajectory is thus described by a sequence in the geodesic phase space, e.g., $I(t) = \{E(t), L_z(t), \iota(t), \psi_0(t), \phi_0(t), \chi_0(t)\}$. These are referred to as the *osculating orbital elements* at the *osculating epoch* t [10]. Another name for this approach used in the Hamiltonian description is a *variation of constants*. We preserve the form of the equations of motion for a geodesic but slowly vary what used to be constants of motion in the unperturbed case. There are well known techniques for tackling such problems which are widely used in Newtonian celestial mechanics and can be extended to the relativistic regime. This was demonstrated by Pound and Poisson [9] for the trajectory of a particle in a Schwarzschild background under the action of (post-Newtonian) radiation reaction.

When we have a perturbed system of the form

$$\ddot{\mathbf{r}} = \mathbf{f}_{\text{geo}} + \delta\mathbf{f}, \quad (1.1)$$

we can describe the perturbed trajectory using the osculating elements referred to the orbits of the geodesic system $\dot{\mathbf{r}} = \mathbf{f}_{\text{geo}}$. From the chain rule, any one of the osculating elements evolves as

$$\dot{I} = \nabla_r I \cdot \dot{\mathbf{r}} + \nabla_v I \cdot \ddot{\mathbf{r}}, \quad (1.2)$$

in which the subscripts r and v denote derivatives with respect to the orbital position and velocity, respectively. In the absence of the perturbing force, each osculating element is constant, so $\dot{I} = \nabla_r I \cdot \dot{\mathbf{r}} + \nabla_v I \cdot \mathbf{f}_{\text{geo}} \equiv 0$. The perturbation equations thus take the rather simple form

$$\dot{I} = \nabla_v I \cdot \delta\mathbf{f}. \quad (1.3)$$

Given an explicit expression for the perturbing force we can integrate these equations.

The osculating element method can be formulated in several different ways. There is freedom in the parametrization of the geodesic solution that is used as a basis for deriving the osculating element equations, and in the basis used to prescribe the force. It is also possible to treat the orbital phase constants either as constants of the motion that are evolved explicitly or as part of a total phase variable which satisfies new equations that depend on the perturbation. We will describe two methods for treating the Kerr problem: (i) evolution of E, L_z, Q and the *full* orbital phases with the force prescribed with respect to the Kinnersley orthonormal tetrad; (ii) evolution of the orbital constants of motion E, L_z, Q and the initial phases, with the force prescribed by its Boyer-Lindquist components.

In the Hamiltonian approach we start with an unperturbed Hamiltonian, H_0 , and write the equations of motion in terms of the constant canonical coordinates and momenta, X^α, P_α (Hamilton-Jacobi approach), which are closely related, if not exactly the same, as the six constants of motion introduced above, I [11]. If we can describe the perturbation as a small addition δH to the unperturbed Hamiltonian, then we can describe the equations of motion in the same generalized coordinates and momenta, which are no longer constants. The derivatives of the perturbation δH give the equations for the evolution of X^α, P_α . Quite often those equations are solved iteratively starting with an assumption that the orbit is unperturbed in the right-hand side (in the δH). This is similar to the adiabatic solution to the osculating element equations which we will describe below. The Hamiltonian approach (if it can be formulated) would give equations equivalent to approach (ii) mentioned above.

We note that an obvious method of computing inspirals is to numerically integrate the second-order forced geodesic equations directly, taking the fundamental variables to be the Boyer-Lindquist coordinates and their derivatives with respect to proper time. The key advantage of the

methods discussed in this paper over second-order integrations is that they mesh much more naturally with the adiabatic approximation and more generally with two-time-scale approximation techniques [12]. For extreme-mass-ratio inspirals driven by radiation reaction, the orbital evolution time scale is much longer than the orbital time scale for most of the inspiral, until the orbit becomes close to the innermost stable orbit. The adiabatic approximation to the motion gives the motion as an expansion in the ratio of the time scales, and then there are various postadiabatic corrections to this. Although it is not possible yet to compute numerically the full first-order self-force for generic orbits in Kerr, it is possible to compute the averaged, dissipative piece of this force, which is sufficient to compute leading-order adiabatic inspirals [12]. The two-time-scale expansion also allows one to go beyond the adiabatic evolution and compute the small, rapidly oscillating perturbations to the evolution of the orbital variables, as well as the slow secular changes to higher order.

The two-time-scale method cannot be easily applied to the second-order, forced geodesic equations, but it can be applied to the equations derived in this paper, as we discuss in Secs. II and III below. In particular, the osculating elements method allows us to explicitly estimate the orbital average rate of change of the orbital elements. This gives us a physical insight into the effect of a perturbing force on the orbit which is otherwise obscured in the integration of the second-order equations of motion. Estimation of these secular changes also allows us to construct the adiabatic evolution of the orbit in the regime where it is applicable.

B. Numerical “kludge” waveform

Another application for the results described in this paper is for the construction of numerical kludge waveforms. The numerical kludge waveform for EMRIs is a fast and accurate way to compute the long waveforms [13] that will be needed for EMRI data analysis. These are built in a not entirely consistent way, but the basic philosophy is to model the underlying trajectory of the inspiralling object as accurately as possible in order to obtain the best possible phase match between the true and approximate waveforms. The approximation is based on geodesic motion in the MBH’s spacetime, combined with a flat spacetime waveform generation expression. In the most recent version of the numerical kludge [14], the instantaneous geodesic orbit was updated by evolving the three constants of the motion E , L_z , Q [15] only. The evolution of the constants was obtained by combining post-Newtonian results with fits to numerical fluxes obtained by solving the Teukolsky equation [14]. However this method of evolving the geodesics is not complete, as we described above, since we need to evolve the (initial) orbital phases together with the orbital constants E , L_z , Q . In particular, the natural (and incorrect) way to evolve the phase constants, which is to fix them at some initial point, leads to significantly different

evolutions in a time or frequency domain implementation of the kludge. The desire to resolve this apparent discrepancy between the two implementations was one of the initial motivations for the work described here. This article outlines the correct way to evolve geodesics under the self-force which could be used to further improve the numerical kludge waveforms in both time and frequency domain descriptions of them.

C. Main results and the structure of the paper

In this paper we will give a detailed description of the osculating elements approach applied to an arbitrary perturbing force acting on an object undergoing geodesic motion in the Kerr spacetime. As an introduction to the three dimensional relativistic problem of perturbed geodesic motion we will first consider a toy problem in Sec. II. We look at the one-dimensional nonlinear oscillator acted on by an external force. The external force is chosen to have two components: a dissipative part and a conservative part (which just redefines the energy of the system). As we will see later this problem is a very good model for the main problem of perturbed motion in the Kerr spacetime. We show how two implementations of the osculating elements approach work in this simplified model and compare the exact evolution with the adiabatic approximation. The second of these two implementations [in which we evolve the energy and the initial time defined as $x(t_0) = 0$] allows us to treat the problem analytically in terms of Jacobi elliptic functions. This one-dimensional example allows the reader to understand the main approach which we then extend to the problem of forced geodesic motion in the Kerr spacetime in Sec. III. We start that section with an introduction to our notation, before describing the osculating elements approach using the Kinnersley tetrad and “Hughes” variables (in terms of the orbital constants and the total phase variables). We then describe the forced geodesic equations in Boyer-Lindquist coordinates, evolving the orbital constants and the initial conditions, which is a direct extension to Kerr of the Schwarzschild results described in [9]. In both cases, we show how we can explicitly avoid the appearance of an apparent divergence in the osculating equations of motion at turning points.

In Sec. IV, we illustrate our techniques with a problem in which the perturbing force is a “gas-drag” force proportional to the velocity of the inspiralling compact object. This is a toy model for an object inspiralling in a gaseous environment around a MBH. We show that the different approaches give identical results, and once again compare the exact and adiabatic solutions to the problem. The influence of the drag force is to drive the inspiral of the object, but it also tends to increase the eccentricity of the orbit and decrease the orbital inclination. Although we primarily use this problem for illustrative purposes, the increase in eccentricity is an interesting result that could have observational consequences. The increase in

eccentricity is a purely relativistic effect, and is to be expected generically, as we discuss in more detail in Appendix D, in the context of a drag force acting on an object in a Schwarzschild background.

We summarize and discuss our findings in the concluding Sec. V. Some detailed mathematical calculations are included in additional appendixes.

II. A SIMPLE MODEL TO ILLUSTRATE METHODS USED: THE PERTURBED NONLINEAR OSCILLATOR

In this section we will study in detail the simple model of an anharmonic oscillator subject to an external perturbing force, in order to illustrate and explain in a simple context the methods that we use for Kerr inspirals in subsequent sections of the paper.

We take the equation of motion for the position $x(t)$ of the oscillator to be

$$\ddot{x} + x + \beta x^3 = \epsilon a_{\text{ext}}(x, \dot{x}). \quad (2.1)$$

Here the frequency of the oscillator is chosen to be unity for simplicity, $\beta > 0$ is a parameter governing the size of the nonlinear term, a_{ext} is an externally applied perturbing acceleration, which could be a function of both the position and the velocity, and ϵ is a small parameter. This simple system is similar in some respects to the system of a point particle in orbit about a Kerr black hole and subject to the gravitational self-force. The dimensionless small parameter ϵ in the system (2.1) plays the role of the mass ratio in the Kerr case, and the external acceleration a_{ext} is analogous to the self-force.

A. Analysis using simple phase and energy coordinates on phase space

Consider initially the situation where there is no external acceleration. It is useful for some purposes to use a set of phase space coordinates for the nonlinear oscillator which eliminate the turning points. We define coordinates a and ψ , functions of x and $v \equiv \dot{x}$, by the equations

$$\frac{1}{2}a^2 + \frac{1}{4}\beta a^4 = \frac{1}{2}\dot{x}^2 + \frac{1}{2}x^2 + \frac{1}{4}\beta x^4, \quad (2.2a)$$

$$x = a \cos \psi, \quad (2.2b)$$

$$\text{sgn}(\dot{x}) = -\text{sgn}(\sin \psi). \quad (2.2c)$$

The expression on the right-hand side of Eq. (2.2a) is just the conserved energy of the system, and a is the conserved amplitude of the oscillation. The variable ψ increases monotonically (but not linearly) with time. The equations of motion in these variables are

$$\dot{a} = 0, \quad (2.3a)$$

$$\dot{\psi} = \sqrt{1 + \beta a^2(1 + \cos^2 \psi)}/2. \quad (2.3b)$$

Now consider turning on the external force. Then the right-hand sides of the equations of motion (2.3) will acquire terms proportional to ϵ . If we differentiate the definition (2.2b) of ψ with respect to t , insert the result into the definition (2.2a) of a , and solve for $\dot{\psi}$ using also Eq. (2.2c), we obtain

$$\dot{\psi} = \frac{\dot{a}}{a} \cot \psi + \sqrt{1 + \beta a^2(1 + \cos^2 \psi)}/2, \quad (2.4)$$

which explicitly shows the extra forcing term. However this term contains an apparent divergence at $\psi = 0$. The divergence is only apparent, since \dot{a} will be constrained to vanish when $\psi = 0$, because the rate at which the force does work will vanish when the velocity of the particle is zero.

To see this explicitly, we substitute the definition (2.2b) of ψ into the definition (2.2a) of a , and solve for \dot{x} to get

$$\dot{x} = -a \sin \psi \sqrt{1 + \beta a^2(1 + \cos^2 \psi)}/2. \quad (2.5)$$

Next, we differentiate both sides of Eq. (2.2a) with respect to t , and simplify the right-hand side using the equation of motion (2.1). This gives

$$(a + \beta a^3)\dot{a} = \epsilon \dot{x} a_{\text{ext}}. \quad (2.6)$$

Now using the result (2.5) for \dot{x} and substituting into Eq. (2.4) gives the final results:

$$\dot{\psi} = \sqrt{1 + \beta a^2(1 + \cos^2 \psi)}/2 \left[1 - \epsilon \frac{\cos \psi a_{\text{ext}}}{a(1 + \beta a^2)} \right], \quad (2.7a)$$

$$\dot{a} = -\epsilon \sqrt{1 + \beta a^2(1 + \cos^2 \psi)}/2 \frac{\sin \psi a_{\text{ext}}}{1 + \beta a^2}, \quad (2.7b)$$

where $a_{\text{ext}}(x, v)$ is evaluated at $x = a \cos \psi$, and $v = v(a, \psi)$ given by the expression (2.5).

The final result (2.7) now casts the system of differential equations entirely in terms of the variables a and ψ , and as expected there are no divergences. Note however that Eq. (2.4) would show a divergence if one used an approximate, orbit-averaged version of \dot{a} instead of the exact expression for \dot{a} .

In the analogous problem in Kerr, it is very straightforward to compute the analog of the equation of motion (2.4) which contains the apparent divergence. For numerical work, this form of the equation would be problematic, since the right-hand side evaluates to $0/0$ at turning points. Our goal was to attempt to reformulate the equations in Kerr analytically, to achieve a form analogous to Eq. (2.7), where all the divergences have been removed. Although it was not clear *a priori* that this would be possible (because of the complexity of the Kerr-orbit dynamical system), we were successful in finding an explicitly finite form of the equations of motion in both sets of variables.

For the problem that we are really interested in, perturbed geodesics in the Kerr spacetime, it will be especially useful to consider the adiabatic limit $\epsilon \rightarrow 0$ of small

external perturbations. So we consider adiabatic perturbations in the context of our example problem. The equations of motion (2.7) for ψ and a can be written in the general form

$$\dot{\psi} = \omega(\psi, a) + \epsilon g^{(1)}(\psi, a) + O(\epsilon^2), \quad (2.8a)$$

$$\dot{a} = \epsilon G^{(1)}(\psi, a) + O(\epsilon^2). \quad (2.8b)$$

Here on the right-hand side, all the functions are periodic functions of ψ with period 2π . In Appendix B we derive the limiting form of the solutions in the limit $\epsilon \rightarrow 0$; see also Ref. [12]. The leading order or adiabatic solutions are given by the following set of steps:

- (1) We define the averaging operation, for any function $f(\psi)$ of ψ , by

$$\langle f \rangle_a \equiv \frac{\int_0^{2\pi} d\psi \frac{f(\psi)}{\omega(\psi, a)}}{\int_0^{2\pi} d\psi \frac{1}{\omega(\psi, a)}}. \quad (2.9)$$

The subscript a on the left-hand side is a reminder that the averaging operation depends on the value of a .

- (2) We define the averaged functions

$$\bar{\omega}(a) \equiv \langle \omega(\psi, a) \rangle_a, \quad (2.10)$$

and

$$\bar{G}^{(1)}(a) \equiv \langle G^{(1)}(\psi, a) \rangle_a. \quad (2.11)$$

- (3) We solve a pair of ordinary differential equations in the slow time parameter

$$\tilde{t} = \epsilon t, \quad (2.12)$$

for two auxiliary functions $\chi^{(0)}(\tilde{t})$ and $a^{(0)}(\tilde{t})$. This pair of ordinary differential equations is

$$\frac{d\chi^{(0)}}{d\tilde{t}} = \bar{\omega}(a^{(0)}(\tilde{t})), \quad (2.13a)$$

$$\frac{da^{(0)}}{d\tilde{t}} = \bar{G}^{(1)}(a^{(0)}(\tilde{t})). \quad (2.13b)$$

Note that, for this step, one does not need to specify a value of ϵ .

- (4) We can then write down the adiabatic solutions:

$$a(t, \epsilon) = a^{(0)}(\epsilon t), \quad (2.14a)$$

$$\psi(t, \epsilon) = \Xi \left[\frac{1}{\epsilon} \chi^{(0)}(\epsilon t), a^{(0)}(\epsilon t) \right], \quad (2.14b)$$

where the function $\Xi(\chi, a)$ is defined implicitly by the equation

$$\frac{\chi}{2\pi} = \frac{\int_0^{\Xi(\chi, a)} \frac{d\psi}{\omega(\psi, a)}}{\int_0^{2\pi} \frac{d\psi}{\omega(\psi, a)}} \quad (2.15)$$

and satisfies $\Xi(\chi + 2\pi, a) = \Xi(\chi, a) + 2\pi$. (The inverse of the mapping Ξ essentially maps the given phase space coordinates onto action-angle variables.)

Note that there is an asymmetry in how the forcing terms $g^{(1)}$ and $G^{(1)}$ in Eq. (2.8) enter into the adiabatic solution (2.14). The function $G^{(1)}$, which drives the energy evolution, does enter, but the function $g^{(1)}$, which drives the phase evolution, does not enter at all. It influences only the post-1-adiabatic solutions.

Note also that one cannot obtain the adiabatic solutions by any simple modification of the original differential equations.

B. Analysis exploiting analytic solution to un-forced motion

It is also possible to find an analytic solution to the unperturbed anharmonic oscillator in terms of Elliptic functions. Equation (2.2a) can be rearranged to give

$$\dot{x}^2 = \frac{\beta}{2} (x_+^2 - x^2)(x^2 - x_-^2), \quad (2.16)$$

where we have defined the turning points

$$x_{\pm}^2 = \frac{1}{\beta} (-1 \pm \sqrt{1 + 2E\beta}) \quad (2.17)$$

in terms of the energy E , which is set to be twice the conserved quantity on the right-hand side of Eq. (2.2a), and is related to the amplitude of motion a and the non-linearity parameter β by

$$E = a^2 + \frac{1}{2}\beta a^4. \quad (2.18)$$

For $\beta > 0$, all of the solutions are bound and oscillate periodically in the interval $-x_+^2 \leq x \leq x_+^2$. Without loss of generality we can set $x(t_0) = 0$, in which case Eq. (2.16) can be rearranged and integrated to give

$$\begin{aligned} & \int_0^x \frac{dy}{\sqrt{(y^2 - x_-^2)(x_+^2 - y^2)}} \\ &= \frac{1}{\sqrt{x_+^2 - x_-^2}} \mathbf{F} \left[\sin^{-1} \left(\frac{x}{x_+} \sqrt{\frac{x_+^2 - x_-^2}{x^2 - x_-^2}} \right); \frac{x_+}{\sqrt{x_+^2 - x_-^2}} \right] \\ &= \pm \sqrt{\frac{\beta}{2}} (t - t_0). \end{aligned} \quad (2.19)$$

Here $\mathbf{F}(\phi; k)$ denotes the Jacobi elliptic integral of the first kind [16]

$$\begin{aligned} \mathbf{F}(\phi; k) &= \int_0^{\sin \phi} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} \\ &= \int_0^{\phi} \frac{d\alpha}{\sqrt{1-k^2\sin^2\alpha}}. \end{aligned} \quad (2.20)$$

In the following, we will denote all elliptic integrals by bold capital letters. The inverse of this elliptic integral is given by the elliptic function $\text{sn}(u; k)$ such that

$$\mathbf{F}(\phi; k) = u \Leftrightarrow \phi(u; k) = \sin^{-1}[\text{sn}(u; k)]. \quad (2.21)$$

For the solutions (2.19), the parameter k and argument u are given by

$$k^2 = \frac{x_+^2}{x_+^2 - x_-^2} = \frac{\sqrt{1 + 2E\beta} - 1}{2\sqrt{1 + 2E\beta}}, \quad (2.22)$$

$$u = (1 + 2E\beta)^{1/4}(t - t_0). \quad (2.23)$$

The relation between $x(t)$ and the elliptic function is then

$$\frac{x}{\sqrt{x^2 - x_-^2}} = k \text{sn}(u; k). \quad (2.24)$$

Solving this gives an expression for x^2 that is somewhat unsatisfying since using it requires manually flipping the sign of x . We can instead get a simpler expression if we introduce the additional elliptic function

$$\text{dn}(u; k) = \sqrt{1 - k^2 \text{sn}^2(u; k)}. \quad (2.25)$$

The result is

$$x = k \sqrt{\frac{2(1 - k^2)}{\beta(1 - 2k^2)}} \text{sd}(u; k), \quad (2.26)$$

where $\text{sd}(u; k) = \text{sn}(u; k)/\text{dn}(u; k)$.

We will now derive the osculating element equations for the variables u and k . The physical variables E and t_0 can be obtained from these simply via

$$E = \frac{2k^2(1 - k^2)}{\beta(1 - 2k^2)^2}, \quad (2.27)$$

$$\frac{dE}{dt} = \frac{4k}{\beta(1 - 2k^2)^3} \frac{dk}{dt}, \quad (2.28)$$

$$t_0 = t - \sqrt{1 - 2k^2}u, \quad (2.29)$$

$$\frac{dt_0}{dt} = 1 - \sqrt{1 - 2k^2} \frac{du}{dt} + \frac{2ku}{\sqrt{1 - 2k^2}} \frac{dk}{dt}. \quad (2.30)$$

To derive the equations of motion in the osculating element form we need to differentiate $\text{sd}(u; k)$ with respect to u and k . This gives

$$\frac{\partial \text{sd}}{\partial u}(u; k) = \frac{\text{cn}(u; k)}{\text{dn}^2(u; k)}, \quad (2.31)$$

$$\begin{aligned} \frac{\partial \text{sd}}{\partial k}(u; k) &= \frac{u \text{cn}(u; k)}{k \text{dn}(u; k)} - \frac{\mathbf{E}[\phi(u; k); k] \text{cn}(u; k)}{k(1 - k^2) \text{dn}^2(u; k)} \\ &+ \frac{k \text{sn}(u; k)}{(1 - k^2) \text{dn}(u; k)}, \end{aligned} \quad (2.32)$$

where we have introduced the elliptic function $\text{cn}(u; k)$, which is defined by the analogue of Eq. (2.21) but with \sin^{-1} replaced by \cos^{-1} , and where $\mathbf{E}(\phi; k)$ is the elliptic integral of the second kind [16]:

$$\mathbf{E}(\phi; k) = \int_0^{\sin \phi} dx \sqrt{\frac{1 - k^2 x^2}{1 - x^2}} = \int_0^\phi d\alpha \sqrt{1 - k^2 \sin^2 \alpha}. \quad (2.33)$$

Since the parameter k depends only on the energy, the evolution equation can be derived directly from the equation for the energy evolution, which follows by differentiation of Eq. (2.2a) and use of Eq. (2.1):

$$\frac{dE}{dt} = 2\dot{x} \epsilon a_{\text{ext}}. \quad (2.34)$$

The evolution equation for u follows from differentiating the orbit equation with respect to time and setting this equal to the velocity of the unperturbed orbit, which is given by Eq. (2.26):

$$\frac{dx}{dt} = \sqrt{\frac{2(1 - k^2)k^2}{\beta(1 - 2k^2)^2}} \frac{\partial \text{sd}}{\partial u}. \quad (2.35)$$

Putting these elements together we find the equations for the osculating evolution of the orbit

$$\frac{dk}{dt} = \epsilon a_{\text{ext}}(x, \dot{x})(1 - 2k^2)^2 \sqrt{\frac{\beta}{2}(1 - k^2)} \frac{\partial \text{sd}}{\partial u}, \quad (2.36)$$

$$\begin{aligned} \frac{du}{dt} &= \frac{1}{\sqrt{1 - 2k^2}} - \epsilon a_{\text{ext}}(x, \dot{x})(1 - 2k^2) \sqrt{\frac{\beta}{2}(1 - k^2)} \\ &\times \left[\frac{1 - 2k^2 + 2k^4}{k(1 - k^2)(1 - 2k^2)} \text{sd}(u; k) + \frac{\partial \text{sd}}{\partial k} \right], \end{aligned} \quad (2.37)$$

where the perturbing force is to be evaluated for the geodesic position and velocity,

$$x = \sqrt{\frac{2k^2(1 - k^2)}{\beta(1 - 2k^2)}} \text{sd}(u; k), \quad (2.38)$$

$$\dot{x} = \sqrt{\frac{2k^2(1 - k^2)}{\beta(1 - 2k^2)^2}} \frac{\partial \text{sd}}{\partial u}. \quad (2.39)$$

We can now derive the adiabatic approximation to the solution of Eqs. (2.36) and (2.37) following the steps described at the end of Sec. II A. Eqs. (2.36) and (2.37) have the same general form as $d\psi/dt$ and da/dt in Sec. II A. That is, we can write them as

$$\dot{u} = \omega(u, k) + \epsilon g^{(1)}(u, k) + O(\epsilon^2), \quad (2.40a)$$

$$\dot{k} = \epsilon G^{(1)}(u, k) + O(\epsilon^2), \quad (2.40b)$$

where we have now redefined the functions ω , $g^{(1)}$, and $G^{(1)}$. By comparing against the formula for \dot{u} , we find

$$\omega(u, k) = \omega(k) = (1 - 2k^2)^{-1/2}. \quad (2.41)$$

As a result, the averaging operation is greatly simplified:

$$\langle f(u, k) \rangle_k = \frac{1}{\mathcal{U}(k)} \int_0^{\mathcal{U}(k)} du f(u, k), \quad (2.42)$$

where $\mathcal{U}(k)$ is the period in u for the general solution (2.38)

$$\mathcal{U}(k) = 4\mathbf{F}(\pi/2, k) = 4\mathbf{K}(k). \quad (2.43)$$

Here $\mathbf{K}(k)$ is the complete elliptic integral of the first kind. Note however that this period depends on k , whereas, in the previous parametrization, the period in ψ was simply 2π .

As before, the two functions we wish to average are ω and $G^{(1)}$. Since ω is independent of u ,

$$\bar{\omega} = \langle \omega \rangle_k = \omega. \quad (2.44)$$

To make further progress, we must specify the perturbing force. We take this to be

$$a_{\text{ext}} = -\gamma\dot{x} + \delta x^2. \quad (2.45)$$

By substituting the Eqs. (2.38) and (2.39) for $x(u, k)$ and $\dot{x}(u, k)$ into this expression, we find, omitting the arguments for the elliptic functions, all of which depend on both u and k ,

$$G^{(1)}(u, k) = -\gamma k(1-k^2)(1-2k^2) \left(\frac{\partial \text{sd}}{\partial u} \right)^2 + \delta k^2(1-k^2)^{3/2}(1-2k^2) \sqrt{\frac{2}{\beta}} \text{sd}^2 \frac{\partial \text{sd}}{\partial u}. \quad (2.46)$$

The second term in this expression is symmetric about zero, and has period \mathcal{U} , so it vanishes under the averaging operation. The first term is also periodic, but it does not vanish under averaging since it is always positive. Recalling the relation between $\partial \text{sd} / \partial u$ and the other elliptic functions (2.31),

$$\bar{G}^{(1)} = \langle G^{(1)} \rangle_k = -\gamma k(1-k^2)(1-2k^2) \left\langle \frac{\text{cn}^2}{\text{dn}^4} \right\rangle_k, \quad (2.47)$$

and exploiting the following identities,

$$\text{sn}^2 + \text{cn}^2 = 1, \quad (2.48)$$

$$\text{dn}^2 + k^2 \text{sn}^2 = 1, \quad (2.49)$$

we can rewrite $\bar{G}^{(1)}$ as

$$\bar{G}^{(1)} = \gamma(1-2k^2) \left[\frac{1-k^2}{k} [(1-k^2) \langle \text{dn}^{-4} \rangle_k - \langle \text{dn}^{-2} \rangle_k] \right]. \quad (2.50)$$

The averaging operations can be reduced to just one integral by using the identity [16]

$$\int dudn^m = \frac{1}{(m+1)(1-k^2)} \left[k^2 \text{dn}^{m+1} \text{sncn} + (m+2)(2-k^2) \times \int dudn^{m+2} - (m+3) \int dudn^{m+4} \right]. \quad (2.51)$$

The first term in square brackets vanishes on the ends of the interval $\{0, \mathcal{U}\}$, so

$$\int_0^{\mathcal{U}} dudn^m = \frac{1}{(m+1)(1-k^2)} \times \left[(m+2)(2-k^2) \int_0^{\mathcal{U}} dudn^{m+2} - (m+3) \times \int_0^{\mathcal{U}} dudn^{m+4} \right]. \quad (2.52)$$

which after using Eq. (2.51) gives

$$\bar{G}^{(1)} = \gamma(1-2k^2) \left[\left(\frac{2}{3} \frac{2-k^2}{k} - \frac{1}{k} \right) \langle \text{dn}^2 \rangle_k - \frac{1-k^2}{3k} \right]. \quad (2.53)$$

The average can be written as an elliptic integral using [16]

$$\int dudn^2 = \mathbf{E}[\phi(u; k); k], \quad (2.54)$$

where $\mathbf{E}(u; k)$ is the elliptic integral of the second kind given in Eq. (2.33), and the amplitude function $\phi(u; k)$ is given by Eq. (2.21). This leaves us with the final result

$$\bar{G}^{(1)} = \gamma(1-2k^2) \left[\left(\frac{2}{3} \frac{2-k^2}{k} - \frac{1}{k} \right) \frac{\mathbf{E}(k)}{\mathbf{K}(k)} - \frac{1-k^2}{3k} \right], \quad (2.55)$$

where we have used $\mathcal{U} = 4\mathbf{K}(k)$, and we use $\mathbf{E}(k) = \mathbf{E}(\pi/2, k)$ to denote the complete elliptic integral of the second kind.

C. Example force

We will illustrate the techniques described above for an oscillator subject to the forcing term given in Eq. (2.45), i.e.,

$$a_{\text{ext}} = -\gamma\dot{x} + \delta x^2, \quad (2.56)$$

with $\epsilon = 10^{-3}$, $\beta = 0.1$, $\gamma = 0.15$, $\delta = 0.2$ and initial conditions $x(0) = 1.0$, $\dot{x}(0) = 0$. The analytic solutions to the un-forced motion, as described in Secs. II A and II B, were found to be essentially identical over the full integration time (as we would expect since these are both exact solutions to the forced motion). In Fig. 1, we show the significant disagreements between (i) using the analytic solution to the un-forced motion as described in Sec. II B (labeled ‘‘exact’’); and (ii) using the adiabatic approximation to the evolution, given by Eqs. (2.40), (2.44), and (2.55) (labeled ‘‘adiabatic’’).

There was no significant disagreement when it came to predicting the scale parameter k^2 , but there was significant disagreement in what those formalisms predicted for the

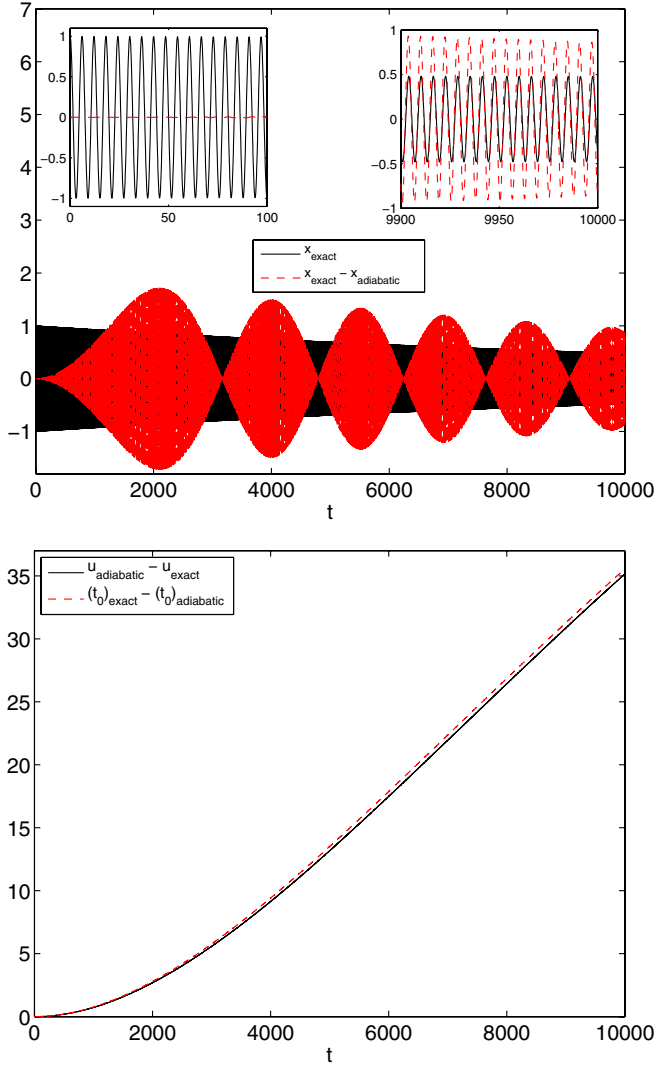


FIG. 1 (color online). Comparison between the exact and adiabatic approaches to evolving the orbit, both from Sec. II B. The top panel shows the exact solution $x(t)$ (solid lines), as well as the difference between the exact and adiabatic $x(t)$ (dashed lines). The insets show close-up views of the first/last 100s of the same curves. The bottom panel shows the disagreement between the exact and adiabatic predictions for the phase u (solid line) and time-offset t_0 (dashed line).

position $x(t)$, the phase u , and the time-offset t_0 . The top panel in Fig. 1 shows that the adiabatic and exact positions go completely out of phase around $t = 2000$, after which they then continue to pass in and out of phase with each other. This is to be expected, since the adiabatic solution is only an approximation. The bottom panel shows disagreements in both the phase u and the time-offset t_0 . These grow to several cycles by the end of the integration, while the error in k^2 (not shown) remains small throughout the integration. This is also to be expected. Because of the terms we omit, we would expect the error in k^2 to scale like ϵ^2 , while the error in phase u , and correspondingly the error in t_0 from Eq. (2.29), will scale like ϵ .

Given that the exact solutions obtained via the analytic and phase solutions are the same, the choice of which parametrization to use must be made on the basis of practicality. The integration of the analytic form of the equations is more computationally expensive, as the elliptic functions must be evaluated at each integration step, so the phase form of the equations is probably preferable if we are interested only in the exact solution to $x(t)$. However, the averaged functions required for the adiabatic approximation to the solution are most easily derived from the analytic form of the equations, so this approach is better when we are interested in deriving an approximate solution to the equations.

III. OSCULATING ELEMENTS FOR ORBITS IN THE KERR METRIC

A. Summary of notation

In Boyer-Lindquist coordinates (t, r, θ, ϕ) , the Kerr metric is

$$ds^2 = -\left(1 - \frac{2Mr}{\Sigma}\right)dt^2 - \frac{4a\sin^2\theta Mr}{\Sigma}dt d\phi + (\varpi^4 - \Delta a^2 \sin^2\theta) \frac{\sin^2\theta}{\Sigma} d\phi^2 + \Sigma d\theta^2 + \frac{\Sigma}{\Delta} dr^2. \quad (3.1)$$

Here,

$$\Sigma \equiv r^2 + a^2 \cos^2\theta, \quad (3.2a)$$

$$\Delta \equiv r^2 + a^2 - 2Mr, \quad (3.2b)$$

$$\varpi \equiv \sqrt{r^2 + a^2}, \quad (3.2c)$$

and M, a are the black hole mass and spin parameter. Throughout the rest of this paper we use units in which $M = 1$, for simplicity.

We will make use of the Kinnersley null tetrad $\vec{l}, \vec{n}, \vec{m}, \vec{m}^*$, which is given by

$$\vec{l} = \frac{\varpi^2}{\Delta} \partial_t + \partial_r + \frac{a}{\Delta} \partial_\phi, \quad (3.3)$$

$$\vec{n} = \frac{\varpi^2}{2\Sigma} \partial_t - \frac{\Delta}{2\Sigma} \partial_r + \frac{a}{2\Sigma} \partial_\phi, \quad (3.4)$$

and

$$\vec{m} = \frac{1}{\sqrt{2}(r + ia \cos\theta)} \left(ia \sin\theta \partial_t + \partial_\theta + \frac{i}{\sin\theta} \partial_\phi \right). \quad (3.5)$$

The corresponding one-forms are

$$l = -dt + a \sin^2\theta d\phi + \frac{\Sigma}{\Delta} dr, \quad (3.6)$$

$$\mathbf{n} = -\frac{\Delta}{2\Sigma} dt + \frac{a\Delta\sin^2\theta}{2\Sigma} d\phi - \frac{1}{2} dr, \quad (3.7)$$

and

$$\mathbf{m} = \frac{1}{\sqrt{2}(r + ia\cos\theta)} (-ia\sin\theta dt + \Sigma d\theta + i\varpi^2 \sin\theta d\phi). \quad (3.8)$$

The basis vectors obey the orthonormality relations $\vec{l} \cdot \vec{n} = -1$ and $\vec{m} \cdot \vec{m}^* = 1$, while all other inner products vanish. The metric can be written in terms of the basis one-forms as

$$g_{\alpha\beta} = -2l_{(\alpha}n_{\beta)} + 2m_{(\alpha}m_{\beta)}^*. \quad (3.9)$$

We define the conserved energy per unit rest mass μ :

$$E = -\vec{u} \cdot \frac{\partial}{\partial t}, \quad (3.10)$$

the conserved z -component of angular momentum divided by μM :

$$L_z = \vec{u} \cdot \frac{\partial}{\partial \phi}, \quad (3.11)$$

and Carter constant divided by $\mu^2 M^2$:

$$Q = u_\theta^2 - a^2 \cos^2\theta E^2 + \cot^2\theta L_z^2 + a^2 \cos^2\theta. \quad (3.12)$$

(From now on we will for simplicity call these dimensionless quantities ‘‘energy,’’ ‘‘angular momentum,’’ and ‘‘Carter constant.’’) The geodesic equations can then be written in the form [17]

$$\left(\frac{dr}{d\lambda}\right)^2 = [E(r^2 + a^2) - aL_z]^2 - \Delta[r^2 + (L_z - aE)^2 + Q] \equiv V_r(r), \quad (3.13)$$

$$\left(\frac{d\theta}{d\lambda}\right)^2 = Q - \cot^2\theta L_z^2 - a^2 \cos^2\theta(1 - E^2) \equiv V_\theta(\theta), \quad (3.14)$$

$$\frac{d\phi}{d\lambda} = \csc^2\theta L_z + aE\left(\frac{r^2 + a^2}{\Delta} - 1\right) - \frac{a^2 L_z}{\Delta} \equiv V_\phi(r, \theta), \quad (3.15)$$

$$\frac{dt}{d\lambda} = E\left[\frac{(r^2 + a^2)^2}{\Delta} - a^2 \sin^2\theta\right] + aL_z\left(1 - \frac{r^2 + a^2}{\Delta}\right) \equiv V_t(r, \theta). \quad (3.16)$$

Here λ is the Mino time parameter [18], related to proper time τ by

$$d\lambda = \frac{1}{\Sigma} d\tau, \quad (3.17)$$

and we use these equations to define the potentials $V_r(r)$, $V_\theta(\theta)$, $V_\phi(r, \theta)$ and $V_t(r, \theta)$. Sometimes it will be convenient to use, instead of the Carter constant Q , the quantity

$$K = Q + (L_z - aE)^2. \quad (3.18)$$

For the rest of this section we specialize to bound geodesics, which are periodic in r and θ .

B. Change of variables

Eqs. (3.13), (3.14), (3.15), and (3.16) form a complete set of equations that can be solved to obtain the geodesic motion. However, it is difficult in practice to use the variables r and θ due to sign flips that occur in, for example,

$$\frac{dr}{d\lambda} = \pm\sqrt{V_r(r)},$$

at turning points. Therefore we follow Drasco and Hughes in switching to an alternative set of variables [13,17].

1. Angular motion

We introduce the notation $z = \cos^2\theta$, and note that the effective potential can be written as

$$V_\theta(z) = \frac{1}{1-z} [Q(1-z) - zL_z^2 - \beta z(1-z)], \quad (3.19)$$

where $\beta = a^2(1 - E^2)$. We note that this β is different from the variable appearing in the forced oscillator in Sec. II. All subsequent references to β will assume this new definition. We define z_- and z_+ with $z_- < z_+$ to be the two roots, so that

$$V_\theta(z) = \frac{1}{1-z} \beta(z_- - z)(z_+ - z). \quad (3.20)$$

These roots z_- and z_+ are functions of E , L_z and Q , and are positive with $0 < z_- < 1$ and $z_+ > 1$. The motion takes place in the region $0 \leq z \leq z_-$.

We replace the angular variable θ , which oscillates, with another angular variable ψ_θ , which increases monotonically with time. The definition is given by

$$\cos\theta = \sqrt{z_-} \cos\psi_\theta. \quad (3.21)$$

Note that, for general forced motion, z_- will change with time, along with θ and ψ_θ .

2. Radial motion

We define r_1 , r_2 , r_3 and r_4 to be the roots of the radial potential $V_r(r)$:

$$V_r(r) = (1 - E^2)(r_1 - r)(r - r_2)(r - r_3)(r - r_4). \quad (3.22)$$

Here the roots are ordered as $0 < r_4 < r_3 < r_2 < r_1$, and the motion takes place in the region $r_2 < r < r_1$. The roots are functions of E , L_z and Q .

We replace the radial variable r , which oscillates, with an angular variable ψ_r , which increases monotonically with time. The definition is given by

$$r = \frac{p}{1 + e \cos \psi_r}, \quad (3.23)$$

where the semilatus rectum p and eccentricity e are defined by

$$r_1 = \frac{p}{1 - e}, \quad r_2 = \frac{p}{1 + e}. \quad (3.24)$$

C. Forced motion using tetrad components of acceleration and using convenient phase and energy coordinates on phase space

We now turn to the forced geodesic equation

$$\frac{d^2 x^\alpha}{d\tau^2} + \Gamma_{\beta\gamma}^\alpha \frac{dx^\beta}{d\tau} \frac{dx^\gamma}{d\tau} = a^\alpha, \quad (3.25)$$

where a^α is the external four acceleration. In this subsection we derive our first formulation for integrating this equation, which parametrizes the acceleration in terms of its components on the Kinnersley null tetrad, and which parametrizes the motion in terms of a convenient set of coordinates on phase space that includes E , L_z and Q . The formulation is analogous to that presented in Sec. II A for the nonlinear oscillator model. In particular, the phase variables used here are not conserved for geodesic motion. Our second formulation will be derived in the next subsection.

Eqs. (3.13), (3.14), (3.15), and (3.16) are still valid for the forced geodesic equation. However, they must now be supplemented with evolution equations for E , L_z and Q (or K). We decompose the four acceleration on the Kinnersley tetrad as

$$\vec{a} = -a_n \vec{l} - a_l \vec{n} + a_m^* \vec{m} + a_m \vec{m}^*. \quad (3.26)$$

These four components are not all independent, since the acceleration must be orthogonal to the four velocity. We define

$$R_a = \frac{1}{\sqrt{2}}(a_m + a_m^*), \quad I_a = \frac{i}{\sqrt{2}}(a_m - a_m^*), \quad (3.27)$$

and we take the three independent components to be a_n , R_a and I_a . In Sec. III E we will show how the tetrad components of the acceleration, (a_n, a_l, a_m^*, a_m) , relate to the acceleration components in Boyer-Lindquist coordinates, $(a_t, a_r, a_\theta, a_\phi)$.

We similarly decompose the four velocity in terms of the Kinnersley tetrad as

$$\vec{u} = -u_n \vec{l} - u_l \vec{n} + u_m^* \vec{m} + u_m \vec{m}^*, \quad (3.28)$$

and we define

$$R_u = \frac{1}{\sqrt{2}}(u_m + u_m^*), \quad I_u = \frac{i}{\sqrt{2}}(u_m - u_m^*). \quad (3.29)$$

The components are given by the expressions

$$u_l = u_r - \frac{F}{\Delta}, \quad (3.30a)$$

$$u_n = -\frac{F}{2\Sigma} - \frac{\Delta}{2\Sigma} u_r, \quad (3.30b)$$

$$R_u = \frac{r}{\Sigma} u_\theta + \frac{a\mathcal{H} \cos\theta}{\Sigma \sin\theta}, \quad (3.30c)$$

$$I_u = \frac{a \cos\theta}{\Sigma} u_\theta - \frac{r\mathcal{H}}{\Sigma \sin\theta}, \quad (3.30d)$$

where

$$\mathcal{H} = L_z - aE \sin^2\theta, \quad (3.31)$$

and

$$F = \varpi^2 E - aL_z. \quad (3.32)$$

The orthonormality condition $\vec{u} \cdot \vec{a} = 0$ allows us to solve for a_l :

$$a_l = -\frac{u_l}{u_n} a_n + \frac{1}{u_n} (R_a R_u + I_a I_u). \quad (3.33)$$

We also define the following three combinations of acceleration components:

$$\mathcal{A}_I = rR_a + aI_a \cos\theta, \quad (3.34a)$$

$$\mathcal{A}_{II} = rI_a - aR_a \cos\theta, \quad (3.34b)$$

$$\mathcal{A}_{III} = R_u R_a + I_u I_a. \quad (3.34c)$$

We can now write down the evolution equations for the energy, angular momentum and Carter constant. These are (see Appendix A)

$$\frac{dE}{d\lambda} = \frac{u_r a_n \Delta}{u_n} - \frac{\Delta \mathcal{A}_{III}}{2u_n} - a \sin\theta \mathcal{A}_{II}, \quad (3.35)$$

$$\frac{dL_z}{d\lambda} = \frac{a \sin^2\theta u_r a_n \Delta}{u_n} - \frac{a \sin^2\theta \Delta \mathcal{A}_{III}}{2u_n} - \varpi^2 \sin\theta \mathcal{A}_{II}, \quad (3.36)$$

and

$$\frac{dK}{d\lambda} = 2\Sigma^2 \mathcal{A}_{III}. \quad (3.37)$$

Equations of motion in terms of phase variables

We next replace the equations of motion (3.13) and (3.14) for r and θ with new equations of motion for ψ_θ and ψ_r , which are derived in Appendix A. The new equation for ψ_θ is

$$\begin{aligned} \frac{d\psi_\theta}{d\lambda} = & \sqrt{\beta(z_+ - z_-)} \left[1 + \frac{(1 - z_-)\Sigma\mathcal{A}_I \cos\psi_\theta}{\beta\sqrt{z_-}(z_+ - z_-)\sin\theta} \right] \\ & + \frac{\cos\psi_\theta \sin\psi_\theta \mathcal{H} a \Delta (\mathcal{A}_{III} - 2u_r a_n)}{2(z_+ - z_-)\beta u_n} \\ & + \frac{\cos\psi_\theta \sin\psi_\theta \mathcal{G} \mathcal{A}_{II}}{\beta(z_+ - z_-)}, \end{aligned} \quad (3.38)$$

where $z = z_- \cos^2 \psi_\theta$ and

$$\mathcal{G} = \frac{\varpi^2 L_z}{\sin\theta} - a^3(1 - z_-)\sin\theta E. \quad (3.39)$$

The new equation for ψ_r is

$$\begin{aligned} \frac{d\psi_r}{d\lambda} = & \mathcal{P} + \frac{C\mathcal{A}_{III} \sin\psi_r}{2(1 + e \cos\psi_r)u_n} + \frac{\mathcal{D}\Sigma\mathcal{A}_{III}\mathcal{P}}{2(1 + e \cos\psi_r)^2 u_n} \\ & - \frac{a\mathcal{E} \sin\theta \sin\psi_r \mathcal{A}_{II}}{1 + e \cos\psi_r} + \frac{\mathcal{P}a_n}{u_n(1 + e \cos\psi_r)^2} \\ & \times \left[(1 - e)^2(1 - \cos\psi_r) \frac{\Sigma_1 F_1}{\kappa_1} \right. \\ & \left. + (1 + e)^2(1 + \cos\psi_r) \frac{\Sigma_2 F_2}{\kappa_2} \right], \end{aligned} \quad (3.40)$$

where

$$\mathcal{P} = p\sqrt{\mathcal{J}}/(1 - e^2), \quad (3.41)$$

$$\begin{aligned} \mathcal{J} = & (1 - E^2)(1 - e^2) + 2\left(1 - E^2 - \frac{1 - e^2}{p}\right)(1 + e \cos\psi_r) \\ & + \left\{ (1 - E^2) \frac{3 + e^2}{1 - e^2} - \frac{4}{p} + [a^2(1 - E^2) + L_z^2 + Q] \frac{1 - e^2}{p^2} \right\} \\ & \times (1 + e \cos\psi_r)^2, \end{aligned} \quad (3.42)$$

$$C = \frac{\mathcal{Q}_1(1 - e)}{\kappa_1} - \frac{\mathcal{Q}_2(1 + e)}{\kappa_2}, \quad (3.43)$$

$$\mathcal{D} = (1 - e)^2(1 - \cos\psi_r) \frac{\Delta_1}{\kappa_1} + (1 + e)^2(1 + \cos\psi_r) \frac{\Delta_2}{\kappa_2}, \quad (3.44)$$

$$\begin{aligned} \mathcal{Q}_1 = & -2aL_z r r_1 - a^4 E(r + r_1) + a^3 L_z(r + r_1) \\ & - a^2 E(r^3 + r^2 r_1 + r_1^3 + r r_1(-2 + r_1)) \\ & - Err_1(r r_1(r + r_1) - 2(r^2 + r r_1 + r_1^2)) \\ & - a^2(2a^2 E - 2Err_1 + aL_z(-2 + r + r_1))\cos^2\theta, \end{aligned} \quad (3.45)$$

$$\begin{aligned} \mathcal{Q}_2 = & -2aL_z r r_2 - a^4 E(r + r_2) + a^3 L_z(r + r_2) \\ & - a^2 E(r^3 + r^2 r_2 + r_2^3 + r r_2(-2 + r_2)) \\ & - Err_2(r r_2(r + r_2) - 2(r^2 + r r_2 + r_2^2)) \\ & - a^2(2a^2 E - 2Err_2 + aL_z(-2 + r + r_2))\cos^2\theta, \end{aligned} \quad (3.46)$$

$$\mathcal{E} = \frac{F_1(1 - e)(r + r_1)}{\kappa_1} - \frac{F_2(1 + e)(r + r_2)}{\kappa_2}. \quad (3.47)$$

Here, $\kappa = V'_r(r)$, and subscripts 1 or 2 mean that a quantity is evaluated at $r = r_1$ or $r = r_2$ (except for \mathcal{Q}_1 and \mathcal{Q}_2).

D. Forced motion using Boyer-Lindquist coordinate components of acceleration and phase variables that are conserved for geodesic motion

In this subsection we derive our second formulation for integrating the forced geodesic equation, which is analogous to that presented in Sec. II B above for the nonlinear oscillator model. This formulation parametrizes the acceleration in terms of its Boyer-Lindquist coordinate components. It parametrizes the motion in terms of two phases ψ_0 and χ_0 defined below, which are conserved for geodesic motion, and three other parameters equivalent to E , L_z and Q , namely, the orbital eccentricity e , semilatus rectum p and angle of inclination ι (defined in Ref. [14]). This formulation is a generalization of the treatment of the Schwarzschild problem described by Pound and Poisson in [9].

In principle, we must evolve eight parameters, which are the four constants of motion and the four initial phase angles. However, one of these equations is eliminated by using the orthogonality condition $\dot{x}^\alpha a_\alpha = 0$, where a dot denotes differentiation with respect to proper time, τ , and a^α is the acceleration. This condition is discussed in [9] and comes from the definition of proper time.

In this section we will write the phase angles in the form $\psi_r = \psi - \psi_0$, $\psi_\theta = \chi - \chi_0$ and derive explicit equations for the time evolution of the initial-phase constants ψ_0 and χ_0 . The other parts of the phases, ψ and χ , are evolved using the standard geodesic expressions. While in practice we will need ψ_r and ψ_θ to evolve the orbit, we decompose the equations this way to facilitate comparison to [9] and to make it easier to identify the conservative contributions from the perturbing force, which are essentially $\langle \dot{\psi}_0 \rangle$, $\langle \dot{\chi}_0 \rangle$.

1. Contravariant formulation

The Gaussian perturbation equations (1.3) carry over to the relativistic case and gives Eqs. (27)–(32) in [9]. In the Kerr case, we have two additional equations as the θ motion is no longer trivial. In [9], the equations were integrated with respect to the anomaly. In the Kerr case, as we have two anomalies, it will be more convenient to

integrate the equations with respect to the coordinate time, t . The equations of motion that are independent of the force terms are

$$\frac{\partial r}{\partial p} p' + \frac{\partial r}{\partial e} e' + \frac{\partial r}{\partial \iota} \iota' + \frac{\partial r}{\partial \psi_0} \psi_0' + \frac{\partial r}{\partial \chi_0} \chi_0' = 0, \quad (3.48)$$

$$\frac{\partial \theta}{\partial p} p' + \frac{\partial \theta}{\partial e} e' + \frac{\partial \theta}{\partial \iota} \iota' + \frac{\partial \theta}{\partial \psi_0} \psi_0' + \frac{\partial \theta}{\partial \chi_0} \chi_0' = 0, \quad (3.49)$$

$$\frac{\partial \phi}{\partial p} p' + \frac{\partial \phi}{\partial e} e' + \frac{\partial \phi}{\partial \iota} \iota' + \frac{\partial \phi}{\partial \psi_0} \psi_0' + \frac{\partial \phi}{\partial \chi_0} \chi_0' + \Phi' = 0, \quad (3.50)$$

$$\frac{\partial t}{\partial p} p' + \frac{\partial t}{\partial e} e' + \frac{\partial t}{\partial \iota} \iota' + \frac{\partial t}{\partial \psi_0} \psi_0' + \frac{\partial t}{\partial \chi_0} \chi_0' + T' = 0. \quad (3.51)$$

Here Φ and T denote the phase offsets for the evolution of ϕ and t . We can ignore these equations if we evolve t and ϕ explicitly using the geodesic expressions evaluated along the instantaneous orbit, which amounts to evolving $t - T$ and $\phi - \Phi$ directly, as in the tetrad formulation. In the above, we use a dash to denote differentiation with respect to the parameter we use to define our orbit, which we take to be t . We will use a dot to denote differentiation with respect to the proper time τ . The remaining four equations of motion are

$$\frac{\partial \dot{t}}{\partial p} p' + \frac{\partial \dot{t}}{\partial e} e' + \frac{\partial \dot{t}}{\partial \iota} \iota' + \frac{\partial \dot{t}}{\partial \psi_0} \psi_0' + \frac{\partial \dot{t}}{\partial \chi_0} \chi_0' = a^t \tau', \quad (3.52)$$

$$\frac{\partial \dot{r}}{\partial p} p' + \frac{\partial \dot{r}}{\partial e} e' + \frac{\partial \dot{r}}{\partial \iota} \iota' + \frac{\partial \dot{r}}{\partial \psi_0} \psi_0' + \frac{\partial \dot{r}}{\partial \chi_0} \chi_0' = a^r \tau', \quad (3.53)$$

$$\frac{\partial \dot{\theta}}{\partial p} p' + \frac{\partial \dot{\theta}}{\partial e} e' + \frac{\partial \dot{\theta}}{\partial \iota} \iota' + \frac{\partial \dot{\theta}}{\partial \psi_0} \psi_0' + \frac{\partial \dot{\theta}}{\partial \chi_0} \chi_0' = a^\theta \tau', \quad (3.54)$$

$$\frac{\partial \dot{\phi}}{\partial p} p' + \frac{\partial \dot{\phi}}{\partial e} e' + \frac{\partial \dot{\phi}}{\partial \iota} \iota' + \frac{\partial \dot{\phi}}{\partial \psi_0} \psi_0' + \frac{\partial \dot{\phi}}{\partial \chi_0} \chi_0' = a^\phi \tau'. \quad (3.55)$$

The terms $\partial \dot{r}/\partial p$ etc. denote differentiation of the geodesic equations given earlier with respect to the various orbital parameters. Following [9], we can use the orthogonality condition to get rid of one of these equations, specifically Eq. (3.52), and we will directly integrate ϕ and t , which means we do not need to consider Eqs. (3.50) and (3.51).

We can rearrange Eqs. (3.48) and (3.49) to give

$$\psi_0' = -\frac{1}{\partial r/\partial \psi_0} \left(\frac{\partial r}{\partial p} p' + \frac{\partial r}{\partial e} e' + \frac{\partial r}{\partial \iota} \iota' \right), \quad (3.56)$$

$$\chi_0' = -\frac{1}{\partial \theta/\partial \chi_0} \left(\frac{\partial \theta}{\partial p} p' + \frac{\partial \theta}{\partial e} e' + \frac{\partial \theta}{\partial \iota} \iota' \right), \quad (3.57)$$

where we have made use of the fact that the equation for r , (3.24), is independent of χ_0 and the equation for θ , (3.21), is independent of ψ_0 . The partial derivative $\partial r/\partial \iota$ also vanishes, but we include this term explicitly for simplicity of notation in the following. We generalize [9] by writing

$$\mathcal{L}_a(x) = \frac{\partial \dot{x}}{\partial a} - \frac{\partial r/\partial a}{\partial r/\partial \psi_0} \frac{\partial \dot{x}}{\partial \psi_0} - \frac{\partial \theta/\partial a}{\partial \theta/\partial \chi_0} \frac{\partial \dot{x}}{\partial \chi_0}. \quad (3.58)$$

Substitution into Eqs. (3.53), (3.54), and (3.55) then gives

$$\begin{aligned} p' = \frac{\tau'}{D} & \left((\mathcal{L}_e(\theta) \mathcal{L}_\iota(\phi) - \mathcal{L}_e(\phi) \mathcal{L}_\iota(\theta)) a^r \right. \\ & + (\mathcal{L}_\iota(r) \mathcal{L}_e(\phi) - \mathcal{L}_\iota(\phi) \mathcal{L}_e(r)) a^\theta \\ & \left. + (\mathcal{L}_e(r) \mathcal{L}_\iota(\theta) - \mathcal{L}_e(\theta) \mathcal{L}_\iota(r)) a^\phi \right), \end{aligned} \quad (3.59)$$

$$\begin{aligned} e' = \frac{\tau'}{D} & \left((\mathcal{L}_\iota(\theta) \mathcal{L}_p(\phi) - \mathcal{L}_\iota(\phi) \mathcal{L}_p(\theta)) a^r \right. \\ & + (\mathcal{L}_p(r) \mathcal{L}_\iota(\phi) - \mathcal{L}_p(\phi) \mathcal{L}_\iota(r)) a^\theta \\ & \left. + (\mathcal{L}_\iota(r) \mathcal{L}_p(\theta) - \mathcal{L}_\iota(\theta) \mathcal{L}_p(r)) a^\phi \right), \end{aligned} \quad (3.60)$$

$$\begin{aligned} \iota' = \frac{\tau'}{D} & \left((\mathcal{L}_p(\theta) \mathcal{L}_e(\phi) - \mathcal{L}_p(\phi) \mathcal{L}_e(\theta)) a^r \right. \\ & + (\mathcal{L}_e(r) \mathcal{L}_p(\phi) - \mathcal{L}_e(\phi) \mathcal{L}_p(r)) a^\theta \\ & \left. + (\mathcal{L}_p(r) \mathcal{L}_e(\theta) - \mathcal{L}_p(\theta) \mathcal{L}_e(r)) a^\phi \right), \end{aligned} \quad (3.61)$$

$$\begin{aligned} D = \mathcal{L}_p(r) & (\mathcal{L}_e(\theta) \mathcal{L}_\iota(\phi) - \mathcal{L}_\iota(\theta) \mathcal{L}_e(\phi)) \\ & - \mathcal{L}_e(r) (\mathcal{L}_p(\theta) \mathcal{L}_\iota(\phi) - \mathcal{L}_\iota(\theta) \mathcal{L}_p(\phi)) \\ & + \mathcal{L}_\iota(r) (\mathcal{L}_p(\theta) \mathcal{L}_e(\phi) - \mathcal{L}_p(\phi) \mathcal{L}_e(\theta)). \end{aligned} \quad (3.62)$$

The correct evolution equations for the phase constants ψ_0 and χ_0 may be found by substituting the preceding equations into (3.56) and (3.57). In the next section we will describe an alternative form of these equations which greatly simplifies the evolution of the constants of the motion. We include the above equations for completeness and to allow a direct comparison to the Schwarzschild results described in [9].

2. Covariant formulation

The preceding section presented the equations in a contravariant formulation. We note that the equations for the evolution of the phase constants, (3.56) and (3.57), appear to be singular at turning points where $\partial r/\partial \psi_0 = 0$ or $\partial \theta/\partial \chi_0 = 0$. These are not real singularities, as the numerator also vanishes at the turning points, but it requires significant simplification to make this explicit. It is also possible to derive an alternative set of equations to (3.52), (3.53), (3.54), and (3.55) from a covariant formulation of the equations. Pound and Poisson [9] chose the

contravariant formulation in the Schwarzschild case, since they found it easier to eliminate the singularities at turning points in that formulation. However, there are advantages to using the covariant formulation, since two of the covariant velocity components are then equal to conserved quantities, $u_t = E$, $u_\phi = L_z$. The osculation conditions become

$$\frac{\partial x_G^\alpha}{\partial I^A} \dot{I}^A = 0, \quad \frac{\partial v_\alpha^G}{\partial I^A} \dot{I}^A = f_\alpha, \quad (3.63)$$

where

$$v_\alpha^G = g_{\alpha\beta} \frac{dz_\alpha^G}{d\lambda}, \quad (3.64)$$

in which I^A denotes the orbital elements, including the phase constants. The first equation is the same as Eqs. (3.48), (3.49), (3.50), and (3.51) which reduce to (3.56) and (3.57). The second equation is the equivalent of Eqs. (3.52), (3.53), (3.54), and (3.55), but in this case two of the equations simplify significantly, namely,

$$\dot{E} = f_r, \quad \dot{L}_z = f_\phi. \quad (3.65)$$

In the Schwarzschild case, there is no equation for the θ motion and the radial equation follows from (3.65) through the constraint $\dot{z}^\alpha f_\alpha = 0$. In the Kerr case, we do need to solve one of the radial or θ equations, or some combination of them. Alternatively, using the definition of the Carter constant in terms of the Killing tensor, we can derive the evolution equation for Q straightforwardly. The time evolution of the related constant K defined in Eq. (3.18), can be found from Eq. (A9) in Appendix A as $\dot{K} = K^{\alpha\beta} u_\alpha a_\beta$. The Killing tensor $K^{\alpha\beta}$ can be written in terms of l^α and n^α as

$$K^{\alpha\beta} = 2\Sigma l^{(\alpha} n^{\beta)} + r^2 g^{\alpha\beta}, \quad (3.66)$$

from which we obtain

$$\begin{aligned} \dot{K} = & \dot{E} \frac{2}{\Delta} (\varpi^4 E - a\varpi^2 L_z) + \dot{L}_z \frac{2}{\Delta} (a^2 L_z - a\varpi^2 E) \\ & - 2\Delta u_r a_r, \end{aligned} \quad (3.67)$$

where we have used $\dot{E} = -a_r$, and $\dot{L}_z = a_\phi$. An alternative expression for $K^{\alpha\beta}$ in terms of m^α and $m^{*\alpha}$ exists and is given in Appendix A as Eq. (A3). If we had used this definition we would have found an equivalent expression for \dot{K} that was a linear combination of \dot{E} , \dot{L}_z and a_θ . The two expressions are equivalent, since the orthogonality relation between the perturbation force and four velocity always allows the elimination of one component of the force.

These three equations provide an alternative way to evolve the constants of the motion, E , L_z and Q , but we must still evolve ψ_0 and χ_0 using (3.56) and (3.57) and therefore we still need to deal with the turning points.

It is possible to derive an alternative form of these expressions that is manifestly finite at turning points by starting with the radial geodesic equation in the form

$$\Sigma^2 \dot{r}^2 = V_r(r, L_z, E, Q). \quad (3.68)$$

We need to show that the term

$$\frac{\partial r}{\partial E} \dot{E} + \frac{\partial r}{\partial L_z} \dot{L}_z + \frac{\partial r}{\partial Q} \dot{Q}, \quad (3.69)$$

that appears in an alternative version of Eq. (3.56), is proportional to r' . Differentiation of Eq. (3.68) with respect to E yields

$$\begin{aligned} 2\Sigma^2 \dot{r} \frac{\partial \dot{r}}{\partial E} + 2\Sigma \left(2r \frac{\partial r}{\partial E} - 2a^2 \cos\theta \sin\theta \frac{\partial \theta}{\partial E} \right) \dot{r}^2 \\ = \frac{\partial V_r}{\partial E} + \frac{\partial V_r}{\partial r} \frac{\partial r}{\partial E}. \end{aligned} \quad (3.70)$$

Similar equations may be obtained by differentiating with respect to L_z and Q . Multiplying the E equation by \dot{E} etc. and adding the equations together, all terms on the left-hand side are proportional to \dot{r} , while on the right-hand side we get the expression (3.69) multiplied by $\partial V_r / \partial r$ plus the term

$$\frac{\partial V_r}{\partial E} \dot{E} + \frac{\partial V_r}{\partial L_z} \dot{L}_z + \frac{\partial V_r}{\partial Q} \dot{Q} = 2\dot{r} \Sigma^2 \left(\dot{r} - \frac{1}{2\Sigma^2} \frac{\partial V_r}{\partial r} + \frac{\dot{\Sigma}}{\Sigma} \dot{r} \right), \quad (3.71)$$

where the second equality follows from differentiation of Eq. (3.68) with respect to time. The term in parentheses on the right-hand side is what we would obtain if we were on a geodesic, and therefore it necessarily equals a^r in the evolving case. The final expression is

$$\begin{aligned} \dot{\psi}_0 = & 2 \frac{\dot{\psi}_{\text{geo}}}{\partial V_r / \partial r} \left[\Sigma^2 \left(\dot{E} \frac{\partial \dot{r}}{\partial E} + \dot{L}_z \frac{\partial \dot{r}}{\partial L_z} + \dot{Q} \frac{\partial \dot{r}}{\partial Q} \right) \right. \\ & + 2\Sigma r \dot{r} \left(\dot{E} \frac{\partial r}{\partial E} + \dot{L}_z \frac{\partial r}{\partial L_z} + \dot{Q} \frac{\partial r}{\partial Q} \right) \\ & \left. - 2\Sigma a^2 \cos\theta \sin\theta \dot{r} \left(\dot{E} \frac{\partial \theta}{\partial E} + \dot{L}_z \frac{\partial \theta}{\partial L_z} + \dot{Q} \frac{\partial \theta}{\partial Q} \right) - \Sigma^2 a^r \right], \end{aligned} \quad (3.72)$$

in which $\dot{\psi}_{\text{geo}}$ denotes the geodesic expression for $d\psi/d\tau$ which we use to evolve ψ . It is clear that this expression is indeed finite at radial turning points, provided that the radial self-force is finite. In the Schwarzschild case, it may be easily verified that Eq. (3.72) gives the same expression as Eq. (3.56) when they are explicitly simplified.

One important caveat is that, although expression (3.72) is finite at radial turning points, it appears to diverge where $\partial V_r / \partial r = 0$, and this condition will be satisfied once between each consecutive turning point. This is not a real divergence, either, which is clear from the fact that the original form of the equations did not show such a

divergence. Therefore, if we were to substitute the various terms into the above expression, we would find that the necessary cancellations would occur to eliminate these divergences. This simplification is a nontrivial calculation. However, an alternative approach that is easier to implement numerically is to use both Eqs. (3.56) and (3.72) without any attempt to simplify the expressions. By switching from one expression to the other near turning points we can avoid numerical round-off problems. This is the implementation that we use in practice and from which the results presented in Sec. IV were derived. We have verified in practice that both expressions do yield the same results at points where neither V_r nor $\partial V_r/\partial r$ vanish.

3. Action-angle formulation

The method described above for evolving the equations of motion in the covariant formulation can be readily adapted to other problems and to other formulations of the Kerr geodesic solutions. In particular, an action-angle formulation of the Kerr solution exists [19], in which the equations take the form

$$X = A_X(E, L_z, Q)F_X(\psi_X - \psi_{X0}; E, L_z, Q), \quad (3.73)$$

$$\frac{d\psi_X}{d\lambda} = \Omega_X(E, L_z, Q), \quad (3.74)$$

where X denotes (t, r, θ, ϕ) and λ is ‘‘Mino time.’’ The function F_X is periodic for r and θ , with a period of 2π ,¹ and for t and ϕ it is the sum of a secular piece and an oscillatory term. The osculating element conditions give

$$\begin{aligned} & \left(F_X \frac{\partial A_X}{\partial E} + A_X \frac{\partial F_X}{\partial E} \right) \frac{dE}{d\lambda} + \left(F_X \frac{\partial A_X}{\partial L_z} + A_X \frac{\partial F_X}{\partial L_z} \right) \frac{dL_z}{d\lambda} \\ & + \left(F_X \frac{\partial A_X}{\partial Q} + A_X \frac{\partial F_X}{\partial Q} \right) \frac{dQ}{d\lambda} = A_X F_X' \frac{d\psi_{X0}}{d\lambda}, \end{aligned} \quad (3.75)$$

where the dash denotes differentiation of F_X with respect to the phase argument $\psi_X - \psi_{X0}$. As before, this expression appears to be singular at turning points, where $F_X' = 0$. However, we can obtain an alternative expression by considering the potential

$$\left(\frac{dX}{d\lambda} \right)^2 = V_X(X; E, L_z, Q). \quad (3.76)$$

Adding the derivative of this expression with respect to E multiplied by $dE/d\lambda$ to the derivative with respect to L_z multiplied by $dL_z/d\lambda$ and the derivative with respect to Q multiplied by $dQ/d\lambda$ gives

¹The choice of periodicity is in a sense arbitrary, and different periodicities could be obtained by rescaling the angular variable ψ . We specify a period of 2π for convenience.

$$\begin{aligned} & \frac{\partial V_X}{\partial X} \left(\left(F_X \frac{\partial A_X}{\partial E} + A_X \frac{\partial F_X}{\partial E} \right) \frac{dE}{d\lambda} + \left(F_X \frac{\partial A_X}{\partial L_z} + A_X \frac{\partial F_X}{\partial L_z} \right) \frac{dL_z}{d\lambda} \right. \\ & \quad \left. + \left(F_X \frac{\partial A_X}{\partial Q} + A_X \frac{\partial F_X}{\partial Q} \right) \frac{dQ}{d\lambda} \right) \\ & + \frac{\partial V_X}{\partial E} \frac{dE}{d\lambda} + \frac{\partial V_X}{\partial L_z} \frac{dL_z}{d\lambda} + \frac{\partial V_X}{\partial Q} \frac{dQ}{d\lambda} \\ & = 2 \frac{dX}{d\lambda} \left(\frac{\partial}{\partial E} \left(\frac{dX}{d\lambda} \right) \frac{dE}{d\lambda} + \frac{\partial}{\partial L_z} \left(\frac{dX}{d\lambda} \right) \frac{dL_z}{d\lambda} + \frac{\partial}{\partial Q} \left(\frac{dX}{d\lambda} \right) \frac{dQ}{d\lambda} \right). \end{aligned} \quad (3.77)$$

The derivative of Eq. (3.76) with respect to Mino time is

$$2 \frac{dX}{d\lambda} \frac{d^2 X}{d\lambda^2} = \frac{\partial V_X}{\partial X} \frac{dX}{d\lambda} + \frac{\partial V_X}{\partial E} \frac{dE}{d\lambda} + \frac{\partial V_X}{\partial L_z} \frac{dL_z}{d\lambda} + \frac{\partial V_X}{\partial Q} \frac{dQ}{d\lambda}, \quad (3.78)$$

which thus allows us to replace Eq. (3.75) with

$$\begin{aligned} \frac{\partial V_X}{\partial X} A_X F_X' \frac{d\psi_{X0}}{d\lambda} & = \frac{dX}{d\lambda} \left(\left[\frac{\partial V_X}{\partial X} - 2 \frac{d^2 X}{d\lambda^2} \right] + 2 \left(\frac{\partial}{\partial E} \left(\frac{dX}{d\lambda} \right) \frac{dE}{d\lambda} \right. \right. \\ & \quad \left. \left. + \frac{\partial}{\partial L_z} \left(\frac{dX}{d\lambda} \right) \frac{dL_z}{d\lambda} + \frac{\partial}{\partial Q} \left(\frac{dX}{d\lambda} \right) \frac{dQ}{d\lambda} \right) \right). \end{aligned} \quad (3.79)$$

The term in square brackets vanishes for geodesics and is therefore proportional to the X component of the force when the orbit is perturbed. At turning points F_X' and $dX/d\lambda$ are both zero and cancel, so we obtain a new form of the equation that is manifestly finite at turning points, albeit singular where $\partial V_X/\partial X = 0$. As in the Boyer-Lindquist case, these two alternative formulations for the equations allow us to evolve the osculating element equations directly without worrying about singular behavior, just by switching between the two equivalent expressions in the vicinity of the turning points.

E. Connection between Boyer-Lindquist and tetrad formulations

The tetrad formulation of the osculation equations, described in Sec. III C, is written in terms of acceleration components, \mathcal{A}_I etc., that are adapted to the Kinnersley tetrad, while the Boyer-Lindquist coordinate formulation, described in Sec. III D, is written in terms of the Boyer-Lindquist components of the acceleration. To identify the accelerations between the two approaches, we first write down the tetrad components of the acceleration in terms of the Boyer-Lindquist components:

$$a_n = \frac{\varpi^2}{2\Sigma} a_t - \frac{\Delta}{2\Sigma} a_r + \frac{a}{2\Sigma} a_\phi, \quad (3.80)$$

$$a_l = \frac{\varpi^2}{\Delta} a_t + a_r + \frac{a}{\Delta} a_\phi, \quad (3.81)$$

$$a_m = \frac{1}{\sqrt{2}(r + ia \cos\theta)} \left(ia \sin\theta a_t + a_\theta + \frac{i}{\sin\theta} a_\phi \right), \quad (3.82)$$

$$a_m^* = \frac{1}{\sqrt{2}(r - ia \cos\theta)} \left(-ia \sin\theta a_t + a_\theta - \frac{i}{\sin\theta} a_\phi \right). \quad (3.83)$$

The acceleration functions $R_a = (a_m + a_m^*)/\sqrt{2}$ and $I_a = i(a_m - a_m^*)/\sqrt{2}$ introduced in Sec. III C have components

$$R_a = \frac{a^2 \sin\theta \cos\theta}{\Sigma} a_t + \frac{r}{\Sigma} a_\theta + \frac{a \cot\theta}{\Sigma} a_\phi, \quad (3.84)$$

$$I_a = -\frac{ar \sin\theta}{\Sigma} a_t - \frac{a \cos\theta}{\Sigma} a_\theta - \frac{r}{\sin\theta \Sigma} a_\phi, \quad (3.85)$$

from which we obtain the tetrad acceleration components in terms of the Boyer-Lindquist components of the acceleration

$$\mathcal{A}_I = \frac{r^2 - a^2 \cos^2\theta}{\Sigma} a_\theta, \quad (3.86)$$

$$\mathcal{A}_{II} = -a \sin\theta a_t - \frac{2ra \cos\theta}{\Sigma} a_\theta - \frac{1}{\sin\theta} a_\phi, \quad (3.87)$$

$$\begin{aligned} \mathcal{A}_{III} = & -\alpha a \sin\theta a_t + \frac{u_\theta(r^2 - a^2 \cos^2\theta)/\Sigma - 2\alpha r \cos\theta}{\Sigma} a_\theta \\ & - \frac{\alpha}{\sin\theta} a_\phi, \end{aligned} \quad (3.88)$$

in which

$$\alpha = \frac{aE \sin^2\theta - L_z}{\Sigma \sin\theta}. \quad (3.89)$$

In Sec. IV below we will consider a toy problem as an illustration of the two methods. The force will be specified in Boyer-Lindquist coordinates, and the preceding expressions can be used to obtain the corresponding tetrad components.

F. Features and drawbacks of the two formulations

In this final subsection we discuss some of the advantages and disadvantages of our two formulations.

First, as discussed in the Introduction, earlier work on methods of computing radiation reaction driven inspirals focused on the adiabatic limit [13–15,20,21]. In this limit, it is sufficient to use orbit-averaged forces, or, equivalently, orbit-averaged proper time derivatives of the first integrals, \dot{E} , \dot{L}_z and \dot{Q} . These quantities can be computed as functions of E , L_z and Q , both in post-Newtonian expansions and exactly using numerical black hole perturbation theory. In this paper our focus is on developing methods that allow going beyond the adiabatic limit. For this purpose, orbit-averaged quantities are insufficient; one must use a

prescription for the perturbing force that depends on the two nontrivial orbital phases. One could, in principle, continue to use the quantities \dot{E} , \dot{L}_z and \dot{Q} to parametrize the force, if these quantities are taken to be functions of E , L_z and Q and of two additional phases. This would be the most natural way to generalize the analyses of Refs. [13–15,20,21].

However, such a parametrization turns out to have a significant disadvantage compared to the parametrizations used in this paper, when one is attempting to compute approximate inspirals. Specifically, there are constraints that the fluxes must satisfy at radial and polar turning points, in order to ensure that the four acceleration be finite. Approximate versions of the fluxes may violate the constraints and lead to cusps in the motion at the turning points. (This will be true, in particular, for orbit-averaged fluxes.) The existence of these constraints can be seen from the expression for the square of the four acceleration in terms of \dot{E} , \dot{L}_z and \dot{K} , which is

$$\bar{a}^2 = \frac{1}{\Sigma \Delta u_r^2} \left(\frac{1}{2} \dot{K} - \frac{F\bar{F}}{\Delta} \right)^2 + \frac{1}{\Sigma u_\theta^2} \left(\frac{1}{2} \dot{K} - \mathcal{G}\bar{\mathcal{G}} \right)^2 - \frac{\bar{F}^2}{\Sigma \Delta} + \frac{\bar{\mathcal{G}}^2}{\Sigma}. \quad (3.90)$$

Here $F = \varpi^2 E - aL_z$, $\bar{F} = \varpi^2 \dot{E} - a\dot{L}_z$, $\mathcal{G} = a \sin\theta E - \csc\theta L_z$, and $\bar{\mathcal{G}} = a \sin\theta \dot{E} - \csc\theta \dot{L}_z$. It can be seen that, at radial turning points where $u_r = 0$, the fluxes must satisfy the constraint $\dot{K} = 2F\bar{F}/\Delta$, while at polar turning points the constraint is $\dot{K} = 2\mathcal{G}\bar{\mathcal{G}}$.

By contrast, in the tetrad formulation used here, the magnitude of the acceleration is automatically finite. The independent components of the four acceleration are taken to be three of the four components on the Kinnersley null tetrad, namely, a_n , a_m and a_m^* , with the fourth component being determined by the orthogonality of the four acceleration and the four velocity. In terms of these three components, the square of the four acceleration is

$$\bar{a}^2 = \frac{4u_l a_n}{1 + 2|u_m|^2} [u_l a_n - u_m^* a_m - u_m a_m^*] + 2|a_m|^2, \quad (3.91)$$

which is clearly always finite.³

Similarly, in our Boyer-Lindquist formulation, the acceleration is again always finite, except in some special

²A similar phenomenon occurred in the nonlinear oscillator model of Sec. II, where the time derivative of the energy was constrained to vanish at turning points.

³A related issue is that the time derivative of the orbital eccentricity e can diverge $\propto 1/e$ as $e \rightarrow 0$, for forces parametrized in terms of \dot{E} , \dot{L}_z and \dot{Q} , unless the fluxes obey certain constraints at $e = 0$. This issue is discussed in detail in Ref. [14]. Again, this divergence is automatically excluded if one parametrizes the force in terms of its tetrad components: The eccentricity can be written as a smooth function $e = e(x^\alpha, p^\beta)$ on phase space. Taking a proper time derivative gives $de/d\tau = ma^\alpha \partial e / \partial p^\alpha$, which is finite for finite accelerations.

cases in the ergosphere. The independent components of the acceleration are taken to be the spatial, contravariant components $a^i = (a^r, a^\theta, a^\phi)$, with a^t being determined by orthonormality. The square of the four acceleration is then

$$\vec{a}^2 = \left(g_{ij} - 2 \frac{g_{ti}u_j}{u_t} + g_{tt} \frac{u_i u_j}{u_t^2} \right) a^i a^j, \quad (3.92)$$

which is always finite except in the ergosphere where it is possible for u_t to vanish.

We now turn to a discussion of a second issue, which is a significant advantage of the Boyer-Lindquist formulation over the tetrad formulation. This advantage is its simple behavior under the discrete symmetries of the Kerr space-time. Specifically, note that any four acceleration $\vec{a} = \vec{a}(x^\alpha, u^\beta)$ can be uniquely decomposed as the sum of a dissipative piece and a conservative piece. For the dissipative piece, the components a^r and a^θ are odd under $u^r \rightarrow -u^r$, $u^\theta \rightarrow -u^\theta$, while the components a^t and a^ϕ are even. For the conservative piece, the components a^r and a^θ are even, while the components a^t and a^ϕ are odd [12]. It follows that, in the Boyer-Lindquist formulation, wherein one specifies the components a^r , a^θ and a^ϕ of the four acceleration, it is straightforward to independently specify the dissipative and conservative pieces.

By contrast, in the tetrad formulation presented here, the independent variables are taken to be a_n , a_m and a_m^* , and the decomposition into conservative and dissipative pieces in terms of these variables is somewhat involved. In particular, if one is attempting to find useful approximations to the conservative self-force, for example, by naively using conservative post-Newtonian approximations to the quantities a_n , a_m and a_m^* , the errors in the approximation will generically lead to a self-force with both conservative and dissipative pieces. This can be a problem since in the adiabatic limit the effect of the dissipative self-force on the motion is boosted relative to the conservative self-force.

There are alternative parametrizations of the self-force that combine the advantages of our two formulations, for example,

$$a^\alpha = a^{\hat{r}} e_{\hat{r}}^\alpha + a^{\hat{\theta}} e_{\hat{\theta}}^\alpha + a_\perp \epsilon_{\beta\gamma\delta}^\alpha u^\beta e_{\hat{r}}^\gamma e_{\hat{\theta}}^\delta + (a^{\hat{r}} u_{\hat{r}} + a^{\hat{\theta}} u_{\hat{\theta}}) u^\alpha, \quad (3.93)$$

where $\vec{e}_{\hat{r}}$ and $\vec{e}_{\hat{\theta}}$ are unit vectors in the directions of ∂_r and ∂_θ . Here the dissipative and conservative pieces of the quantities $a^{\hat{r}}$, $a^{\hat{\theta}}$ and a_\perp have simple transformation properties under discrete symmetries, and moreover the magnitude of the four acceleration is

$$\vec{a}^2 = (a^{\hat{r}})^2 [1 + u_{\hat{r}}^2] + (a^{\hat{\theta}})^2 [1 + u_{\hat{\theta}}^2] + a_\perp^2 [1 + u_{\hat{r}}^2 + u_{\hat{\theta}}^2], \quad (3.94)$$

which is always finite. Useful approximation schemes can be obtained by (i) formulating approximations in terms of the three variables $a^{\hat{r}}$, $a^{\hat{\theta}}$ and a_\perp ; (ii) using the exact, Kerr

relations to compute a_n , a_m and a_m^* in terms of $a^{\hat{r}}$, $a^{\hat{\theta}}$ and a_\perp ; and (iii) using the resulting expressions in the tetrad formulation equations of motion (3.15), (3.16), (3.35), (3.36), (3.37), (3.38), and (3.40). See Ref. [22] for an application of this approach.

IV. EXAMPLE OF PERTURBED KERR GEODESICS: “GAS-DRAG”

As an example problem, we will suppose that the small mass experiences a drag force proportional to velocity, which could represent the behavior of an EMRI occurring in the presence of gas. Here we derive the four acceleration for such a force.

In a given frame of reference, the relativistic analog of this simple drag force will have a term proportional to the spatial part of the velocity, plus a term proportional to the frame velocity constructed so that the force remains orthogonal to the total velocity. Let \vec{u}_{ZAMO} be the velocity of zero-angular-momentum observers (ZAMOs), and let \vec{u} be the velocity of the small mass. In the frame of a ZAMO, the spatial part of the velocity of the small mass is

$$\vec{u}_\perp = \vec{u} + \Gamma \vec{u}_{\text{ZAMO}}, \quad (4.1)$$

where $\Gamma = \vec{u} \cdot \vec{u}_{\text{ZAMO}}$. The drag force then has the form

$$\vec{a} = -\gamma \vec{u}_\perp + \kappa \vec{u}_{\text{ZAMO}} - \gamma \vec{u} + (\kappa - \gamma \Gamma) \vec{u}_{\text{ZAMO}}, \quad (4.2)$$

where γ is the linear drag coefficient. Enforcing the condition $\vec{a} \cdot \vec{u} = 0$ then determines the value of κ :

$$\kappa = \frac{\gamma(\Gamma^2 - 1)}{\Gamma}. \quad (4.3)$$

Inserting this into the formula for the acceleration due to drag (4.2) gives

$$\vec{a} = -\gamma \left(\vec{u} + \frac{\vec{u}_{\text{ZAMO}}}{\vec{u} \cdot \vec{u}_{\text{ZAMO}}} \right). \quad (4.4)$$

Writing this explicitly in terms of Boyer-Lindquist coordinates, we have

$$a^\alpha = -\gamma \left(u^\alpha + \frac{u_Z^\alpha}{u_Z^t u^t} \right), \quad (4.5)$$

where u^α denotes the four velocity of the inspiraling object and

$$u_Z^t = \sqrt{\frac{(r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta}{\Sigma \Delta}}, \quad (4.6)$$

$$u_Z^\phi = \frac{2ar}{\sqrt{\Sigma \Delta ((r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta)}}, \quad (4.7)$$

$$u_Z^r = u_Z^\theta = 0, \quad (4.8)$$

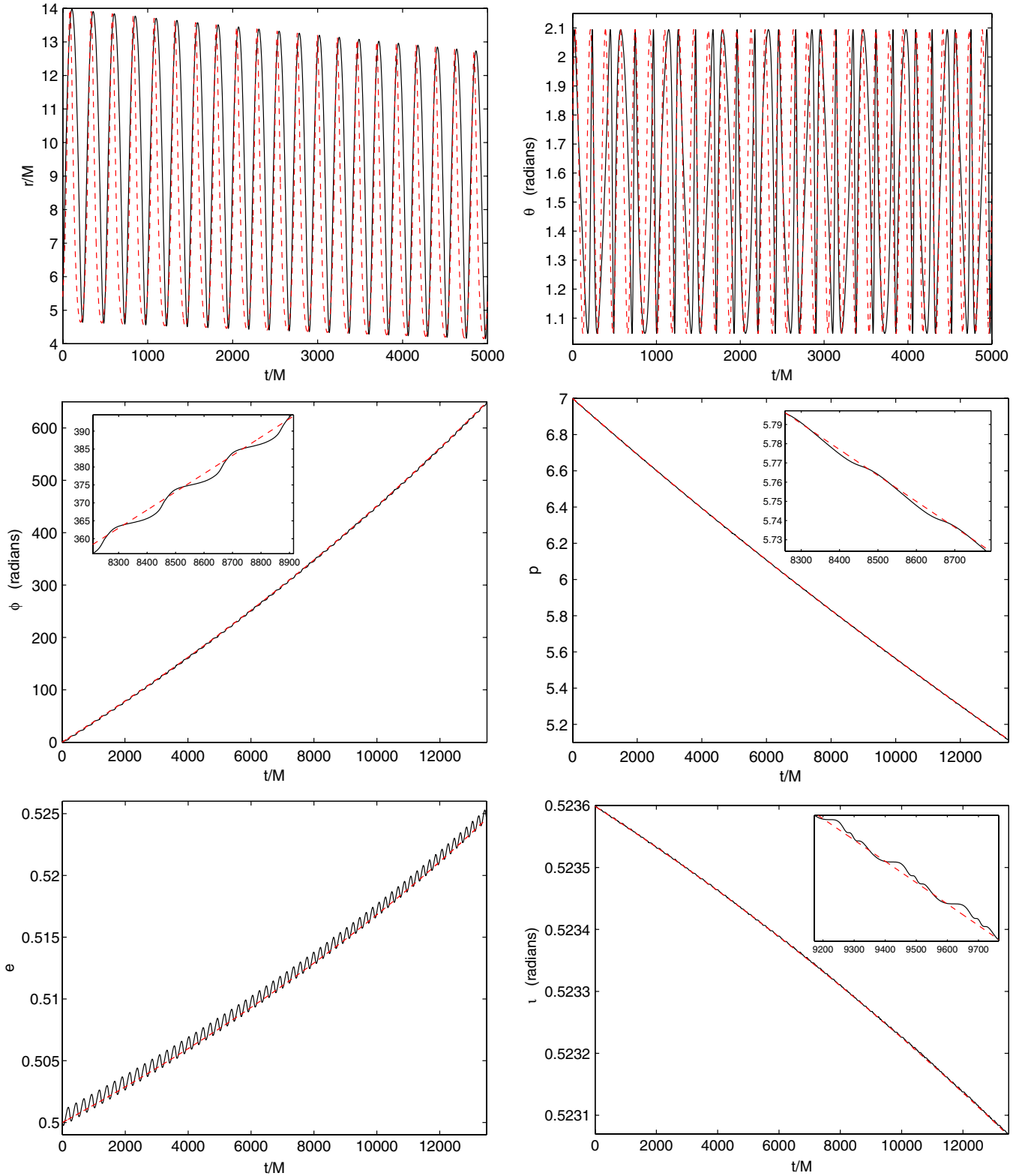


FIG. 2 (color online). Evolution of the orbit under the influence of the “gas-drag” force. The panels show, as functions of Boyer-Lindquist time t , the particle coordinates r (top left), θ (top right) and ϕ (middle left) and the orbital constants p (middle right), e (bottom left) and l (bottom right). In each panel the solid curve was computed using the exact evolution, while the dashed curve was computed using the adiabatic evolution. The plots showing ϕ , p , and l have insets which show close-up views of the same data so that the difference between the adiabatic and exact results can be seen.

$$u_{Zt} = -\left(1 - \frac{2r}{\Sigma}\right)u_Z^t - \frac{2a\sin^2\theta r}{\Sigma}u_Z^\phi, \quad (4.9)$$

in which $\Delta = r^2 - 2Mr + a^2$, $\Sigma = r^2 + a^2\cos^2\theta$ as before.

As a test case, we constructed an inspiral into a central black hole with spin $a = 0.9$ under the influence of this gas-drag force with $\gamma = 10^{-5}$. We took the initial orbital parameters to be $p/M = 7$, $e = 0.5$, $\iota = \pi/6$, $\phi = t = 0$, $\psi_r = 1$ and $\psi_\theta = 2$. The inspiral trajectory was constructed using both the Boyer-Lindquist and the tetrad formulations. The evolutions were found to be identical, as we would hope, and this gives us confidence that our results are correct. In the following discussion, we will not distinguish between the results obtained using the different formulations as they differed only at the level of numerical noise.

In Fig. 2 we show the evolution of the orbit under the influence of the gas-drag force and initial conditions given above. The six panels show the three Boyer-Lindquist coordinates, (r, θ, ϕ) , and the three constants that describe the orbital shape, (p, e, ι) , as functions of the Boyer-Lindquist time t . We see that the influence of the drag force is to drive the inspiral of the object, but also to increase the eccentricity of the orbit and decrease the orbital inclination, i.e., to make the orbit more prograde. The trajectories of the geodesic constants of motion, (p, e, ι) , show oscillations on the orbital time scale, superimposed on a monotonic secular evolution on the radiation reaction time scale. The secular part is the analog of the averaged evolution, $\bar{G}^{(1)}$, described for the perturbed non-linear oscillator in Sec. II. The panels in Fig. 2 also show the solution to the adiabatic equations of motion for this problem computed using Eqs. (2.8), (2.9), (2.10), (2.11), (2.12), (2.13), (2.14), and (2.15). We see that the adiabatic solution is a good approximation to the average evolution along the inspiral, as expected, and, as we saw for the toy problem in Sec. II, it provides a closer fit to the change in the constants of the motion, in this case p , e and ι , than for the phase. In this case, although the adiabatic solution remains in phase on average over the whole of the evolution seen in Fig. 2, within each cycle the adiabatic solution goes in and out of phase with the true evolution. This arises because the forcing term in this case is relatively large, and so the orbit changes significantly between periapse and apoapse as a result of the forcing term. The orbit evolved using the instantaneous force therefore looks quite different from the orbit evolved continuously by the orbital-averaged force. Note that the orbits do come back into phase after each complete cycle, as expected.

This example illustrates the application of the osculating elements formalism to the computation of inspiral evolutions in the Kerr spacetime, and it serves to demonstrate that the two alternative formulations do indeed yield the same results. However, even though the prescription for the drag force was rather simple, Fig. 2 also illustrates some

qualitative effects of the drag force that could be used infer the presence of such a drag from observations. For an orbit evolving under the influence of gravitational radiation reaction only, the eccentricity tends to decrease, except toward the end of the inspiral just prior to plunge [23,24], while the inclination tends to increase, i.e., the orbit becomes more retrograde [20,21,24]. We see here that the effect of the drag force is qualitatively different, as it drives increasing eccentricity and decreasing inclination. If observed, this would provide a robust observational signature for an orbit that was evolving under the influence of drag. A decrease in orbital inclination due to hydrodynamic drag was also seen in [25], in which a more sophisticated model for the drag force was employed. It is gratifying that this simple model produces this expected feature qualitatively. The same paper [25] found that the eccentricity would increase in parts of the parameter space and decrease in other parts. For Newtonian orbits, the osculating element equations predict that the eccentricity will remain constant under the action of a simple drag force of this type (see Appendix C). Increasing eccentricity has, however, been seen in Newtonian simulations of binaries embedded in a realistic disc [26–28]. In Appendix C we show that the increase in eccentricity is an expected post-Newtonian effect and give an explanation in the context of a Schwarzschild black hole (BH). It is clear from that discussion that this increase in eccentricity is a generic feature of relativistic drag, and so this is an observational prediction. If observed, an increasing eccentricity or decreasing inclination would be a clear signature that the observed inspiral was not occurring in a vacuum Kerr background.

V. DISCUSSION

We have described two methods for integrating the equations of motion for bound, accelerated orbits in the Kerr spacetime, which are based on identifying the orbit with a geodesic at each point. The first method parametrizes the position and velocity of the orbit in terms of the conserved quantities (energy, axial angular momentum and Carter constant) in addition to three angular variables which increase monotonically and correspond to relativistic generalizations of the anomalies of Keplerian motion. The second method is the traditional “osculating element” technique which parametrizes the position and velocity of the orbit in terms of the geodesic with the same position and velocity. Practically, the second method differs from the first only in the treatment of the three phase variables, which are split up into a geodesic piece and a “phase offset” piece that is constant for geodesics.

To illustrate the methods, we first analyzed, as a simpler model, a forced anharmonic oscillator. This was written in terms of a set of phase space coordinates. The forced equations of motion contained an apparent divergence at the turning points, but it was possible to reformulate the

equations to eliminate the problematic terms and thus obtain equations of motion in a form without divergences. We discussed the adiabatic prescription for computing the leading order motion, which corresponds to a gradual evolution of the oscillator's amplitude and fundamental frequency driven by the phase space averaged forcing function for the amplitude. We presented an alternative analysis of this toy problem analogous to the osculating orbit method in terms of the analytic solution to the unforced motion. By numerically integrating the equations, we verified that both parametrizations gave the same results and compared these to the adiabatic approximation to the solution.

Next, we showed that the equation of forced motion in the Kerr spacetime could be reformulated in a similar fashion. For the first method, it was advantageous to parametrize the force in terms of its components on the Kinnersley tetrad instead of using the instantaneous time derivatives of the conserved quantities. We derived a formulation of the equations of motion in terms of phase variables that was manifestly divergence-free at the turning points. We then generalized the second method, of osculating orbits, to generic orbits in the Kerr spacetime and showed how we could write down a divergent-free form of equations of this type without explicit simplification.

As an application of our results, we considered the case of a simple force that could represent a gas drag. Numerical integrations of the equations of motion for a choice of parameters verified that the two methods of parametrizing the motion gave the same results. We identified a key observational signature of the presence of a drag force, namely, a decrease in the orbital inclination and an increase in eccentricity, which is opposite to the increase in inclination and decrease in eccentricity characteristic of the gravitational radiation reaction forces during the early stage of an inspiral.

The first of our two methods has been applied to the study of transient resonances that occur in the radiation-reaction-driven inspirals of point particles into spinning black holes, using approximate post-Newtonian expressions for the self-force [22]. Other applications of this work will include the construction of accurate trajectories for orbits evolving under the action of the self-force, once self-force data for generic orbits are available. This will be essential for the construction of accurate gravitational waveforms for EMRIs, which will be needed for LISA data analysis. The formalism can also be used to estimate the magnitude of any secular changes in the orbital parameters that arise from the action of external perturbing forces. These could arise from gravitational perturbations from distant objects, such as stars or a second massive black hole, or from the presence of other material in the spacetime, such as the gas drag which we considered in a simple way here. It will be very important to have a quantitative understanding of the importance of all these

effects if intermediate-mass-ratio inspirals or EMRIs are to be used to carry out high-precision mapping of the space-time around Kerr black holes and for tests of general relativity. Finally, the results described here will be useful to augment existing kludge models for inspiral waveforms. In particular, these methods will allow us to extract the secular part of the evolution of both the orbital constants of the motion and the phase constants, from self-force calculations. It is straightforward to include secular changes to the orbital parameters in the kludge framework [13], and by doing this it should be possible to ensure that the kludge waveform stays in phase with the true waveform for long stretches of the inspiral. It will be important to have accurate but cheap-to-calculate waveform models available when the data from gravitational wave detectors are analyzed, as this data analysis will rely heavily on matched filtering using template waveforms.

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APPENDIX A: DERIVATION OF TETRAD EQUATIONS OF MOTION IN TERMS OF RADIAL AND POLAR ANGULAR VARIABLES

In this Appendix we derive the forms Eqs. (3.35), (3.36), (3.37), (3.38), and (3.40) of the equations of motion for forced motion in Kerr, in the tetrad formulation, and using the angular variables ψ_r and ψ_θ instead of r and θ .

1. Evolution equations for first integrals E , L_z , K

The evolution equations (3.35), (3.36), and (3.37) for the conserved quantities E , L_z and K are obtained as follows. We start from the standard expressions for the first integrals in terms of the Killing vectors and Killing tensors:

$$\xi^\alpha = -\delta_t^\alpha, \quad (\text{A1})$$

$$\eta^\alpha = \delta_\phi^\alpha, \quad (\text{A2})$$

$$K^{\alpha\beta} = 2\Sigma m^{(\alpha} m^{\star\beta)} - a^2 \cos^2\theta g^{\alpha\beta}, \quad (\text{A3})$$

and take proper time derivatives. Using the tetrad decomposition (3.26) of the acceleration together with the expressions (3.6), (3.7), and (3.8) for the basis covectors gives

$$\begin{aligned} \frac{dE}{d\tau} &= -\vec{a} \cdot \frac{\partial}{\partial t} \\ &= -\left(a_n + \frac{\Delta}{2\Sigma} a_l\right) - \frac{iar \sin\theta}{\sqrt{2}\Sigma} (a_m - a_m^*) \\ &\quad + \frac{a^2 \sin\theta \cos\theta}{\sqrt{2}\Sigma} (a_m + a_m^*). \end{aligned} \quad (\text{A4})$$

Here, we have used the fact that \mathbf{m} given in Eq. (3.8) can be written as

$$\begin{aligned} \mathbf{m} &= \frac{1}{\sqrt{2}\Sigma} \left[-(ir + a \cos\theta) \sin\theta dt + (r - ia \cos\theta) \Sigma d\theta \right. \\ &\quad \left. + (ir + a \cos\theta) d\phi \right]. \end{aligned} \quad (\text{A5})$$

Noting that $a_m = (R_a - iI_a)/\sqrt{2}$, and eliminating a_l with the aid of Eq. (3.33) transforms Eq. (A4) to the form

$$\begin{aligned} \frac{dE}{d\tau} &= -\frac{a_n}{u_n} \left(u_n - \frac{\Delta}{2\Sigma} u_l \right) - \frac{\Delta}{2\Sigma u_n} (R_a R_u + I_a I_u) \\ &\quad - \frac{a \sin\theta}{\Sigma} (r I_a - a \cos\theta R_a). \end{aligned} \quad (\text{A6})$$

Using the expressions (3.30) for the tetrad components u_n and u_l of the four velocity and converting from τ derivatives to λ derivatives using Eq. (3.17) then leads to the final form given in Eq. (3.35).

Similarly, we obtain Eq. (3.36) by starting from

$$\begin{aligned} \frac{dL_z}{d\tau} &= \vec{a} \cdot \frac{\partial}{\partial \phi} \\ &= -a \sin^2\theta \left(a_n + \frac{\Delta}{2\Sigma} a_l \right) - \frac{\varpi^2 r \sin\theta}{\Sigma} I_a \\ &\quad + \frac{a \varpi^2 \sin\theta \cos\theta}{\Sigma} R_a, \end{aligned} \quad (\text{A7})$$

and eliminating a_l using Eq. (3.33) to obtain

$$\begin{aligned} \frac{dL_z}{d\tau} &= -a \sin^2\theta \frac{a_n}{u_n} \left(u_n + \frac{\Delta}{2\Sigma} u_l \right) \\ &\quad - \frac{a \sin^2\theta}{2\Sigma u_n} (R_a R_u + I_a I_u) - \frac{\varpi^2 r \sin\theta}{\Sigma} I_a \\ &\quad + \frac{a \varpi^2 \sin\theta \cos\theta}{\Sigma} R_a. \end{aligned} \quad (\text{A8})$$

The form quoted in Eq. (3.36) is then obtained from this using Eqs. (3.30) as before.

The evolution of the Carter constant is obtained very simply from the expression for the Killing tensor to be

$$\frac{dK}{d\tau} = 2K^{\alpha\beta} u_\alpha a_\beta = 2\Sigma (R_u R_a + I_u I_a), \quad (\text{A9})$$

where we have used the orthogonality relation $g^{\alpha\beta} u_\alpha a_\beta = 0$ and written the combination $(u_m^* a_m + u_m a_m^*)$ in terms of R_u and R_a . Combining this with the definition (3.34c) of \mathcal{A}_{III} yields Eq. (3.37).

2. Polar motion

To obtain the equation of motion (3.38) for ψ_θ , we start by differentiating its definition (3.21) with respect to λ :

$$\begin{aligned} &\sin\theta \cos\theta \left(\frac{d\theta}{d\lambda} \right) \\ &= z_- \sin\psi_\theta \cos\psi_\theta \left[\left(\frac{d\psi_\theta}{d\lambda} \right) - \frac{1}{2z_-} \cot\psi_\theta \left(\frac{dz_-}{d\lambda} \right) \right]. \end{aligned} \quad (\text{A10})$$

The equation of motion (3.14) for θ can be rewritten in the form

$$\left(\frac{d\theta}{d\lambda} \right)^2 = \beta z_- \sin^2\psi_\theta \frac{(z_+ - z_- \cos^2\psi_\theta)}{(1 - z_- \cos^2\psi_\theta)}. \quad (\text{A11})$$

We now take the square root of this equation. From the definition (3.21) of ψ_θ and noting that ψ_θ monotonically increases, we see that $(d\theta/d\lambda) > 0$ for $0 < \psi_\theta < \pi$ and $(d\theta/d\lambda) < 0$ for $\pi < \psi_\theta < 2\pi$, so we must choose the positive square root on the right-hand side. Combining Eq. (A10) and the square root of Eq. (A11) now leads to an equation of motion for ψ_θ in the form

$$\frac{d\psi_\theta}{d\lambda} = \sqrt{\beta(z_+ - z_- \cos^2\psi_\theta)} + \frac{\cot\psi_\theta}{2z_-} \frac{dz_-}{d\lambda}. \quad (\text{A12})$$

Next, we can obtain an expression for $dz_-/d\lambda$ in terms of $dP_i/d\tau$, where $P_i = (E, L_z, K)$ are the constants of motion, as follows. Using the chain rule gives $dz_-/d\lambda = (\partial z_-/\partial P_i)(dP_i/d\lambda)$. Differentiating V_θ in the form given in Eq. (3.20) with respect to P_i at fixed z and evaluating the result at z_- relates dz_-/dP_i to $(\partial V_\theta/\partial P_i)_{z_-}$, which can be computed from Eq. (3.19). This yields

$$\begin{aligned} \frac{\beta(z_+ - z_-)}{(1 - z_-)} \frac{dz_-}{d\lambda} &= \frac{dQ}{d\lambda} - 2L_z \left(\frac{z_-}{1 - z_-} \right) \frac{dL_z}{d\lambda} \\ &\quad + 2a^2 E z_- \frac{dE}{d\lambda}. \end{aligned} \quad (\text{A13})$$

Now switching from the Carter constant Q to $K = Q + (L_z - aE)^2$ and using that $d\lambda = d\tau/\Sigma$, we obtain

$$\begin{aligned} \frac{\beta(z_+ - z_-)}{\Sigma} \frac{dz_-}{d\lambda} &= (1 - z_-) \frac{dK}{d\tau} - 2(L_z - a(1 - z_-)E) \\ &\quad \times \left[\frac{dL_z}{d\tau} - a(1 - z_-) \frac{dE}{d\tau} \right]. \end{aligned} \quad (\text{A14})$$

The expressions for the evolution of P_i in Eqs. (3.35), (3.36), and (3.37) can now be used to obtain the explicit dependence on ψ_θ of $dz_-/d\lambda$ given by Eq. (A14) by direct substitution. This gives

$$\begin{aligned}
\beta(z_+ - z_-) \frac{dz_-}{d\lambda} &= \frac{2\Delta\mathcal{H}_- u_r a_n}{u_n} a z_- \sin^2 \psi_\theta \\
&+ \left[2(1 - z_-)(r^2 + a^2 z_- \cos^2 \psi_\theta)^2 - a z_- \mathcal{H}_- \frac{\Delta \sin^2 \psi_\theta}{u_n} \right] \\
&\times (R_u R_a + I_u I_a) + 2\mathcal{H}_- \sqrt{1 - z_- \cos^2 \psi_\theta} (r^2 + a^2 z_-) \\
&\times (r I_a - a \cos \theta R_a), \quad (\text{A15})
\end{aligned}$$

where $\mathcal{H}_- = \mathcal{H}(z_-) = L_z - a(1 - z_-)E$. This can be written as

$$\begin{aligned}
\beta(z_+ - z_-) \frac{dz_-}{d\lambda} &= \frac{\Delta\mathcal{H}_-}{u_n} a z_- \sin^2 \psi_\theta [(R_u R_a + I_u I_a) - 2u_r a_n] \\
&+ 2[(1 - z_-)(r^2 + a^2 z_- \cos^2 \psi_\theta)^2 R_u \\
&- \mathcal{H}_- a \sqrt{z_-} (r^2 + a^2 z_-) \sqrt{1 - z_-^2 \cos^2 \psi_\theta} \cos \psi_\theta] R_a \\
&+ 2[(1 - z_-)(r^2 + a^2 z_- \cos^2 \psi_\theta)^2 I_u \\
&+ \mathcal{H}_- r (r^2 + a^2 z_-) \sqrt{1 - z_-^2 \cos^2 \psi_\theta}] I_a. \quad (\text{A16})
\end{aligned}$$

Substituting the expressions for the four velocity components Eqs. (3.30) and the definition (3.31) of \mathcal{H} into the coefficients of R_a and I_a inside the square brackets and expanding them out gives

$$\begin{aligned}
\beta(z_+ - z_-) \frac{dz_-}{d\lambda} &= \frac{\Delta\mathcal{H}_-}{u_n} a z_- \sin^2 \psi_\theta [\mathcal{A}_{III} - 2u_r a_n] \\
&+ 2(1 - z_-) \Sigma u_\theta (a \cos \theta I_a + r R_a) \\
&+ \frac{2r z_- \sin^2 \psi_\theta}{\sqrt{1 - z_- \cos^2 \psi_\theta}} \\
&\times [\mathfrak{w}^3 L_z - a^3 E (1 - z_-) (1 - z_- \cos^2 \psi_\theta)] \\
&\times (r I_a - a \cos \theta R_a). \quad (\text{A17})
\end{aligned}$$

Using that $u_\theta = (d\theta/d\lambda)$, with $(d\theta/d\lambda)$ given by the positive square root of Eq. (A11) and inserting Eq. (A17) into the equation of motion for ψ_θ of Eq. (A12) leads to the final result quoted in Eq. (3.38).

3. Radial motion

We now give a derivation of the radial equation of motion (3.40) which is similar to the above derivation of the equation (3.38) of polar motion. From the definitions

(3.13) and (3.22) of the radial potential we have

$$\begin{aligned}
\left(\frac{dr}{d\lambda}\right)^2 &= F^2 - \Delta(r^2 + K) \\
&= (1 - E^2)(r_1 - r)(r - r_2)(r - r_3)(r - r_4), \quad (\text{A18})
\end{aligned}$$

where F was defined in Eq. (3.32). We parametrize the roots of the right-hand side by Eq. (3.24) for the turning points r_1 and r_2 of the bound motion, and by

$$r_3 = \frac{p_3}{1 - e}, \quad r_4 = \frac{p_4}{1 + e} \quad (\text{A19})$$

for the other two roots. Substituting the definition (3.23) of ψ_r into Eq. (A18) and using Eqs. (3.24) and (A19) gives, after some algebra,

$$\begin{aligned}
\left(\frac{dr}{d\lambda}\right)^2 &= \frac{(1 - E^2)p^2 e^2 \sin^2 \psi_r}{(1 - e^2)^2 (1 + e \cos \psi_r)^4} \\
&\times [p(1 - e) - p_3(1 + e \cos \psi_r)] \\
&\times [p(1 + e) - p_4(1 + e \cos \psi_r)]. \quad (\text{A20})
\end{aligned}$$

By differentiating the definition (3.23) of ψ_r we obtain

$$\begin{aligned}
\frac{d\psi_r}{d\lambda} &= \frac{(1 + e \cos \psi_r)^2}{ep \sin \psi_r} \left(\frac{dr}{d\lambda}\right) + \frac{\cot \psi_r}{e} \left(\frac{de}{d\lambda}\right) \\
&- \frac{1 + e \cos \psi_r}{ep \sin \psi_r} \left(\frac{dp}{d\lambda}\right). \quad (\text{A21})
\end{aligned}$$

We note that ψ_r is chosen to monotonically increase, which means $d\psi_r/d\lambda > 0$. We specialize to the convention that $\psi_r = 0$ at $r = r_2$ and $\psi_r = \pi$ at $r = r_1$, so that r increases for $0 < \psi_r < \pi$ and decreases for $\pi < \psi_r < 2\pi$, and we choose the positive square root in Eq. (A20). Substituting Eq. (A20) for $dr/d\lambda$ in Eq. (A21) shows that the geodesic term becomes

$$\begin{aligned}
\frac{d\psi_r}{d\lambda} \Big|_{\text{geodesic}} &= \frac{\sqrt{1 - E^2}}{(1 - e^2)} [p(1 - e) - p_3(1 + e \cos \psi_r)]^{1/2} \\
&\times [p(1 + e) - p_4(1 + e \cos \psi_r)]^{1/2} \mathcal{P}. \quad (\text{A22})
\end{aligned}$$

Here one can check that Eq. (A22) is just a reparametrization of Eq. (3.42) by substituting the radial potential in the form given in Eq. (3.13) in terms of P_i into Eq. (A21), since $\mathcal{P} = (dr/d\psi_r)^{-1} \sqrt{V_r}$, expressed in terms of ψ_r .

The nongeodesic terms in Eq. (A21) are obtained as follows. From Eq. (3.24) for r_1 and r_2 it follows that $2p^{-1} = r_1^{-1} + r_2^{-1}$ and $2(1 - e)^{-1} = r_1/r_2 + 1$, and thus

$$2 \frac{dp}{d\lambda} = p^2 \left(\frac{dr_1/d\lambda}{r_1^2} + \frac{dr_2/d\lambda}{r_2^2} \right) = (1 - e)^2 \frac{dr_1}{d\lambda} + (1 + e)^2 \frac{dr_2}{d\lambda}, \quad (\text{A23a})$$

$$2 \frac{de}{d\lambda} = p^2 \left(\frac{dr_1/d\lambda}{r_1^2 r_2} - \frac{dr_2/d\lambda}{r_2^2 r_1} \right) = \frac{(1 - e^2)}{p} \left[(1 - e) \frac{dr_1}{d\lambda} - (1 + e) \frac{dr_2}{d\lambda} \right]. \quad (\text{A23b})$$

Substituting Eqs. (A23) into Eq. (A21) gives

$$\frac{d\psi_r}{d\lambda} = \mathcal{P} + \frac{1}{2ep \sin\psi_r} \left[(1-e)^2 (\cos\psi_r - 1) \frac{dr_1}{d\lambda} - (1+e)^2 (1 + \cos\psi_r) \frac{dr_2}{d\lambda} \right]. \quad (\text{A24})$$

Next, expressions for the derivatives of the turning points r_1 and r_2 can be computed in terms of $dP_i/d\lambda$ by using that $(dr_{1,2}/d\lambda) = (\partial r_{1,2}/\partial P_i) dP_i/d\lambda$. Differentiating the radial potential with respect to P_i at fixed r and evaluating the result at r_1 and r_2 gives

$$\left. \frac{\partial V_r}{\partial P_i} \right|_{r_1} = (1-E^2)(r_1-r_2)(r_1-r_3)(r_1-r_4) \frac{\partial r_1}{\partial P_i}, \quad (\text{A25})$$

$$\left. \frac{\partial V_r}{\partial P_i} \right|_{r_2} = -(1-E^2)(r_1-r_2)(r_2-r_3)(r_2-r_4) \frac{\partial r_2}{\partial P_i}. \quad (\text{A26})$$

We note that one can see from Eqs. (3.22), (A18), (A25), and (A26) that the coefficients of $\partial r_{1,2}/\partial P_i$ can be expressed in terms of the r -derivative of V_r at fixed P_i evaluated at the turning points as

$$\left. \frac{\partial V_r}{\partial P_i} \right|_{r_{1,2}} = - \left. \frac{\partial V_r}{\partial r} \right|_{r_{1,2}} \frac{\partial r_{1,2}}{\partial P_i}, \quad (\text{A27})$$

$$= -\kappa(r_{1,2}) \frac{\partial r_{1,2}}{\partial P_i}. \quad (\text{A28})$$

Here $\kappa(r) \equiv V'_r(r)$, which can be computed from Eq. (A18) to be

$$\kappa(r) = 4EFr - 2r\Delta - 2(r-M)(r^2 + K), \quad (\text{A29})$$

where the definition (3.32) of F has been used. Using the derivatives of Eq. (A18) with respect to P_i then results in the following expressions for $dr_{1,2}/d\lambda$:

$$\frac{dr_{1,2}}{d\lambda} = -\frac{2F_{1,2}}{\kappa_{1,2}} \left(\varpi_{1,2}^2 \frac{dE}{d\lambda} - a \frac{dL_z}{d\lambda} \right) + \frac{\Delta_{1,2}}{\kappa_{1,2}} \frac{dK}{d\lambda}. \quad (\text{A30})$$

With this, Eq. (A21) becomes

$$\begin{aligned} \frac{d\psi_r}{d\lambda} = & \mathcal{P} + \frac{1}{2ep \sin\psi_r} \left\{ (1-e)^2 (\cos\psi_r - 1) \right. \\ & \times \left[-\frac{2F_1}{\kappa_1} \left(\varpi_1^2 \frac{dE}{d\lambda} - a \frac{dL_z}{d\lambda} \right) + \frac{\Delta_1}{\kappa_1} \frac{dK}{d\lambda} \right] \\ & - (1+e)^2 (\cos\psi_r + 1) \\ & \left. \times \left[-\frac{2F_2}{\kappa_2} \left(\varpi_2^2 \frac{dE}{d\lambda} - a \frac{dL_z}{d\lambda} \right) + \frac{\Delta_2}{\kappa_2} \frac{dK}{d\lambda} \right] \right\}. \quad (\text{A31}) \end{aligned}$$

The next step is to substitute the expressions (3.35), (3.36), and (3.37) for the derivatives of the first integrals into Eq. (A31). After some algebra we obtain

$$\begin{aligned} 2ep \sin\psi_r \left(\frac{d\psi_r}{d\lambda} - \mathcal{P} \right) = & 2\Delta \frac{u_r}{u_n} a_n \left[(1-e)^2 (1 - \cos\psi_r) \frac{\Sigma_1 F_1}{\kappa_1} + (1+e)^2 (1 + \cos\psi_r) \frac{\Sigma_2 F_2}{\kappa_2} \right] \\ & + R_a (1-e)^2 (1 - \cos\psi_r) \left[R_u \left(\frac{\Sigma_1 F_1 \Delta}{\Sigma \kappa_1 u_n} + \frac{2\Sigma \Delta_1}{\kappa_1} \right) + \frac{2F_1 a^2 \sin\theta \cos\theta (r^2 - r_1^2)}{\kappa_1 \Sigma} \right] \\ & + I_a (1-e)^2 (1 - \cos\psi_r) \left[I_u \left(\frac{\Sigma_1 F_1 \Delta}{\Sigma \kappa_1 u_n} + \frac{2\Sigma \Delta_1}{\kappa_1} \right) - \frac{2F_1 a r \sin\theta (r^2 - r_1^2)}{\kappa_1 \Sigma} \right] \\ & + R_a (1+e)^2 (1 + \cos\psi_r) [(1 \leftrightarrow 2)] + I_a (1+e)^2 (1 + \cos\psi_r) [(1 \leftrightarrow 2)], \quad (\text{A32}) \end{aligned}$$

where $\Sigma_1 = \varpi_1^2 - a^2 \sin^2\theta$. Noting that $u_r = \Delta^{-1}(dr/d\lambda)$ and using the definition (A22) of \mathcal{P} gives an explicit expression for u_r :

$$u_r = \frac{pe \sin\psi_r \mathcal{P}}{\Delta(1 + e \cos\psi_r)^2}. \quad (\text{A33})$$

Also, from the definitions (3.23) and (3.24), we have that

$$(r - r_1) = -\frac{pe(1 + \cos\psi_r)}{(1-e)(1 + e \cos\psi_r)}, \quad (\text{A34})$$

$$(r - r_2) = \frac{pe(1 - \cos\psi_r)}{(1+e)(1 + e \cos\psi_r)}. \quad (\text{A35})$$

Substitution of Eq. (A33) and (3.30) together with further algebraic manipulations on Eq. (A32) lead to

$$\begin{aligned}
\frac{d\psi_r}{d\lambda} = & \mathcal{P} \left[1 + \frac{a_n}{u_n(1+e\cos\psi_r)^2} \left[(1-e)^2(1-\cos\psi_r) \frac{\Sigma_1 F_1}{\kappa_1} + (1+e)^2(1+\cos\psi_r) \frac{\Sigma_2 F_2}{\kappa_2} \right] \right] \\
& + \frac{(1-e)^2(\cos\psi_r-1)}{2ep\sin\psi_r} \left[\frac{1}{\kappa_1 u_n} (\Sigma_1 F_1 \Delta - \Sigma \Delta_1 F) - \frac{\Delta \Sigma \Delta_1 u_r}{\kappa_1 u_n} \right] (R_u R_a + I_u I_a) \\
& + \frac{(1-e)^2(\cos\psi_r-1)F_1 a \sin\theta}{\kappa_1 e p \sin\psi_r} (r+r_1)(r-r_1)(a\cos\theta R_a - r I_a) \\
& + \frac{(1+e)^2(1+\cos\psi_r)}{2ep\sin\psi_r} \left[\frac{1}{\kappa_2 u_n} (\Sigma_2 F_2 \Delta - \Sigma \Delta_2 F) - \frac{\Delta \Sigma \Delta_2 u_r}{\kappa_2 u_n} \right] (R_u R_a + I_u I_a) \\
& + \frac{F_2 a \sin\theta(1+e)^2(1+\cos\psi_r)}{\kappa_2 e p \sin\psi_r} (r+r_2)(r-r_2)(a\cos\theta R_a - r I_a).
\end{aligned} \tag{A36}$$

We can simplify the coefficients of R_a and I_a by expanding the term $(\Sigma_1 F_1 \Delta - \Sigma \Delta_1 F)$ using the explicit expressions in Eq. (3.2) to obtain an explicit factor of $(r-r_1)$:

$$\begin{aligned}
\frac{(\Sigma_1 F_1 \Delta - \Sigma \Delta_1 F)}{(r-r_1)} &= -(r+r_1)[a^3(aE+zL_z) + E(r^2+r_1^2)(a^2-2Mr) - 2a^2EMr - 4a^2EMr + 2aL_zMr + Er^2r_1^2] \\
&\equiv Q_1,
\end{aligned} \tag{A37}$$

where $z = \cos^2\theta$, as before. We similarly define Q_2 by replacing $1 \rightarrow 2$ in Eq. (A37). Substituting Eqs. (A37) as well as Eqs. (A33) and (A35) into Eq. (A36) and using the definitions (3.34) yields after simplifications

$$\begin{aligned}
\frac{d\psi_r}{d\lambda} = & \mathcal{P} \left[1 + \frac{a_n}{u_n(1+e\cos\psi_r)^2} \left[(1-e)^2(1-\cos\psi_r) \frac{\Sigma_1 F_1}{\kappa_1} + (1+e)^2(1+\cos\psi_r) \frac{\Sigma_2 F_2}{\kappa_2} \right] \right] \\
& + \frac{(1-e)^2(1-\cos\psi_r)}{2\sin\psi_r} \left\{ \frac{\Sigma \Delta_1 \mathcal{P} \mathcal{A}_{III} \sin\psi_r}{\kappa_1 u_n(1+e\cos\psi_r)^2} + \frac{Q_1 \mathcal{A}_{III}(1+\cos\psi_r)}{(1+e\cos\psi_r)(1-e)} - \frac{2F_1 a \sin\theta(r+r_1) \mathcal{A}_{II}(1+\cos\psi_r)}{\kappa_1(1-e)(1+e\cos\psi_r)} \right\} \\
& + \frac{(1+e)^2(1+\cos\psi_r)}{2\sin\psi_r} \left\{ \frac{\Sigma \Delta_2 \mathcal{P} \mathcal{A}_{III} \sin\psi_r}{\kappa_2 u_n(1+e\cos\psi_r)^2} - \frac{Q_2 \mathcal{A}_{III}(1-\cos\psi_r)}{(1+e\cos\psi_r)(1+e)} + \frac{2F_2 a \sin\theta(r+r_2) \mathcal{A}_{II}(1-\cos\psi_r)}{\kappa_2(1+e)(1+e\cos\psi_r)} \right\}.
\end{aligned}$$

This can be further simplified to be

$$\begin{aligned}
\frac{d\psi_r}{d\lambda} = & \mathcal{P} \left[1 + \frac{a_n}{u_n(1+e\cos\psi_r)^2} \left[(1-e)^2(1-\cos\psi_r) \frac{\Sigma_1 F_1}{\kappa_1} + (1+e)^2(1+\cos\psi_r) \frac{\Sigma_2 F_2}{\kappa_2} \right] \right] \\
& + \frac{\mathcal{A}_{III} \sin\psi_r}{2(1+e\cos\psi_r)u_n} \left[\frac{Q_1(1-e)}{\kappa_1} - \frac{Q_2(1+e)}{\kappa_2} \right] \\
& + \frac{\Sigma \mathcal{A}_{III} \mathcal{P}}{2(1+e\cos\psi_r)^2 u_n} \left[(1-e)^2(1-\cos\psi_r) \frac{\Delta_1}{\kappa_1} + (1+e)^2(1+\cos\psi_r) \frac{\Delta_2}{\kappa_2} \right] \\
& - \frac{a \sin\theta \sin\psi_r \mathcal{A}_{II}}{1+e\cos\psi_r} \left[\frac{F_1(1-e)(r+r_1)}{\kappa_1} - \frac{F_2(1+e)(r+r_2)}{\kappa_2} \right].
\end{aligned} \tag{A38}$$

APPENDIX B: ADIABATIC LIMIT

In this Appendix, we derive our method of obtaining the leading order, adiabatic solutions to the forced geodesic equations in Kerr. This method was used to obtain the numerical adiabatic solutions that are plotted and discussed in Sec. IV above. The starting point is the specific form (3.35), (3.36), (3.37), (3.38), (3.39), and (3.40) of the forced

geodesic equations derived in Sec. III C above, which have the general form

$$\begin{aligned}
\dot{\psi}_\alpha &= \omega_\alpha(\psi_\alpha, \mathbf{J}) + \epsilon g_\alpha^{(1)}(\boldsymbol{\psi}, \mathbf{J}) + O(\epsilon^2), \quad 1 \leq \alpha \leq N, \\
\dot{J}_\lambda &= \epsilon G_\lambda^{(1)}(\boldsymbol{\psi}, \mathbf{J}) + \epsilon^2 G_\lambda^{(2)}(\boldsymbol{\psi}, \mathbf{J}) + O(\epsilon^3), \quad 1 \leq \lambda \leq M.
\end{aligned} \tag{B1}$$

Here $\boldsymbol{\psi} = (\psi_1, \dots, \psi_N)$ are a set of angular variables, and $\mathbf{J} = (J_1, \dots, J_M)$ are a set of quantities that are conserved for the unperturbed system. Dots denote derivatives with respect to λ . The functions ω_α determine the frequencies of the unperturbed motion (geodesic motion for the Kerr application), and the functions $g_\alpha^{(1)}$, $G_\lambda^{(1)}$ and $G_\lambda^{(2)}$ represent the external perturbations on the system.⁴ These functions are all periodic in each phase variable with period 2π . In the special case when the frequencies ω_α are independent of the phase variables $\boldsymbol{\psi}$, the variables ψ_α and J_λ are (generalized versions of) action-angle variables. This special case is actually fully general; one can always perform a redefinition of the phase variables to achieve this. This case of action-angle variables was studied in detail in Ref. [12], where the form of the adiabatic and postadiabatic solutions were derived.

Here we will generalize the analysis of Ref. [12] to the more general system of Eqs. (B1), since our system of Eqs. (3.35), (3.36), (3.37), (3.38), (3.39), and (3.40) in Kerr is of this form. We start by describing the result for the adiabatic limit, and then we outline its derivation. The adiabatic solutions are given by the following set of steps:

- (1) We define the averaging operation, for any function $f(\boldsymbol{\psi})$ of $\boldsymbol{\psi}$, by

$$\langle f \rangle_{\mathbf{J}} \equiv \frac{\int_0^{2\pi} \frac{d\psi_1}{\omega_1(\boldsymbol{\psi}_1, \mathbf{J})} \cdots \int_0^{2\pi} \frac{d\psi_N}{\omega_N(\boldsymbol{\psi}_N, \mathbf{J})} f(\psi_1, \dots, \psi_N)}{\int_0^{2\pi} \frac{d\psi_1}{\omega_1(\boldsymbol{\psi}_1, \mathbf{J})} \cdots \int_0^{2\pi} \frac{d\psi_N}{\omega_N(\boldsymbol{\psi}_N, \mathbf{J})}}. \quad (\text{B2})$$

The subscript \mathbf{J} on the left-hand side is a reminder that the averaging operation depends on the value of \mathbf{J} .

- (2) We define the averaged frequencies and forcing functions

$$\bar{\omega}_\alpha(\mathbf{J}) \equiv \langle \omega_\alpha(\boldsymbol{\psi}_\alpha, \mathbf{J}) \rangle_{\mathbf{J}}, \quad (\text{B3})$$

and

$$\bar{G}_\lambda^{(1)}(\mathbf{J}) \equiv \langle G_\lambda^{(1)}(\boldsymbol{\psi}, \mathbf{J}) \rangle_{\mathbf{J}}. \quad (\text{B4})$$

- (3) We solve a set of ordinary differential equations in the slow time parameter

$$\dot{\tilde{\lambda}} = \epsilon \lambda, \quad (\text{B5})$$

for two sets of auxiliary functions $\bar{\chi}_\alpha(\tilde{\lambda})$ and $\mathcal{J}_\lambda(\tilde{\lambda})$. This set of ordinary differential equations is

$$\frac{d\bar{\chi}_\alpha}{d\tilde{\lambda}} = \bar{\omega}_\alpha(\mathcal{J}(\tilde{\lambda})), \quad (\text{B6a})$$

$$\frac{d\mathcal{J}_\lambda}{d\tilde{\lambda}} = \bar{G}_\lambda^{(1)}(\mathcal{J}(\tilde{\lambda})). \quad (\text{B6b})$$

Note that for this step, one does not need to specify a value of ϵ .

- (4) We can then write down the adiabatic solutions:

$$J_\lambda(\lambda, \epsilon) = \mathcal{J}_\lambda(\epsilon \lambda), \quad (\text{B7a})$$

$$\psi_\alpha(\lambda, \epsilon) = \Xi_\alpha \left[\frac{1}{\epsilon} \bar{\chi}_\alpha(\epsilon \lambda), \mathcal{J}(\epsilon \lambda) \right], \quad (\text{B7b})$$

where the function $\Xi_\alpha(\chi, \mathbf{J})$ is defined implicitly by the equation

$$\frac{\chi}{2\pi} = \frac{\int_0^{\Xi_\alpha(\chi, \mathbf{J})} \frac{d\psi}{\omega_\alpha(\boldsymbol{\psi}, \mathbf{J})}}{\int_0^{2\pi} \frac{d\psi}{\omega_\alpha(\boldsymbol{\psi}, \mathbf{J})}} \quad (\text{B8})$$

and satisfies

$$\Xi_\alpha(\chi + 2\pi, \mathbf{J}) = \Xi_\alpha(\chi, \mathbf{J}) + 2\pi. \quad (\text{B9})$$

We now turn to the derivation of this result. We start by rewriting the differential Eqs. (B1) in terms of the new variables (χ_α, J_λ) , defined implicitly by the relation

$$\psi_\alpha(\chi_\alpha, \mathbf{J}) \equiv \Xi_\alpha(\chi_\alpha, \mathbf{J}). \quad (\text{B10})$$

All of the functions appearing in the differential equations are expressed as functions of the new phases χ_α ; they must be periodic functions of each χ_α by virtue of the property (B9). Using the definitions (B8), (B2), and (B3), the result can be written in the form

$$\dot{\chi}_\alpha = \bar{\omega}_\alpha(\mathbf{J}) + \epsilon \frac{\bar{\omega}_\alpha(\mathbf{J})}{\omega_\alpha(\chi_\alpha, \mathbf{J})} g_\alpha^{(1)}(\boldsymbol{\chi}, \mathbf{J}) + O(\epsilon^2), \quad 1 \leq \alpha \leq N, \quad (\text{B11a})$$

$$\dot{J}_\lambda = \epsilon G_\lambda^{(1)}(\boldsymbol{\chi}, \mathbf{J}) + \epsilon^2 G_\lambda^{(2)}(\boldsymbol{\chi}, \mathbf{J}) + O(\epsilon^3), \quad 1 \leq \lambda \leq M. \quad (\text{B11b})$$

This system of equations is now in a form to which the results of Ref. [12] can be applied; the variables (χ_α, J_λ) are generalized action-angle variables. The averaging operation defined in [12], a straightforward averaging with respect to the phases χ_α , coincides with the definition (B2) used here, because of the definition (B10). The results of Ref. [12] now imply that the leading order solution for J_λ is of the form given by Eqs. (B7a) and (B6b). They also imply that the leading order solution for χ_α is of the form $\chi_\alpha(\lambda, \epsilon) = \bar{\chi}_\alpha(\epsilon \lambda)/\epsilon$, where $\bar{\chi}_\alpha$ satisfies the differential equation (B6a). Combining this with the definition (B10) now yields the result (B7b).

⁴Note that the notation $\omega_\alpha(\boldsymbol{\psi}_\alpha, \mathbf{J})$ means that each ω_α depends on only a single phase variable ψ_α , and does not depend on the phase variables ψ_β with $\beta \neq \alpha$. The adiabatic limit of the more general system of equations with $\omega_\alpha = \omega_\alpha(\boldsymbol{\psi}, \mathbf{J})$ would be considerably more complicated.

APPENDIX C: PERTURBATION OF KEPLERIAN ORBITS

Here we derive the osculating element equations for a Keplerian orbit experiencing a force in the plane of the orbit, $\mathbf{f} = -\mu\mathbf{r}/r^3$. In this case, we can take the orbital plane to be the x-y plane. The orbit is described by four parameters—the semimajor axis, a , the eccentricity, e , the argument of perihelion, ω , and the time of pericenter passage, T_0 . (The restriction to a plane gets rid of the other two orbital constants.) The orbit is elliptical and described by

$$r = \frac{a(1 - e^2)}{1 + e \cos(u - \omega)} = a(1 - e \cos E), \quad (\text{C1})$$

$$\dot{u} = \sqrt{\frac{\mu}{a^3(1 - e^2)^3}}(1 + e \cos(u - \omega))^2, \quad (\text{C2})$$

in which u is the argument. It is usual to call $v = u - \omega$ the *true anomaly* and E defined by the first equation above is the *eccentric anomaly*. The time of pericenter passage is given implicitly by

$$\int_0^{v_0} \frac{dv'}{(1 + e \cos v')^2} = \sqrt{\frac{\mu}{a^3(1 - e^2)^3}}(t_0 - T_0), \quad (\text{C3})$$

where $v_0 = v(t_0)$.

Under the action of a force in the orbital plane with radial component R' and tangential component S' , the Gaussian perturbation equations predict the following evolution equations for the four orbital elements [10]:

$$\dot{a} = \sqrt{\frac{a(1 - e^2)}{\mu}} \frac{2a}{1 - e^2} \left(e \sin v R' + \frac{p}{r} S' \right), \quad (\text{C4})$$

$$\dot{e} = \sqrt{\frac{a(1 - e^2)}{\mu}} [\sin v R' + (\cos v + \cos E) S'], \quad (\text{C5})$$

$$\dot{\omega} = \frac{1}{e} \sqrt{\frac{a(1 - e^2)}{\mu}} \left[-\cos v R' + \left(1 + \frac{r}{p} \right) \sin v S' \right], \quad (\text{C6})$$

$$\begin{aligned} \dot{T}_0 = & -\frac{a^2(1 - e^2)}{\mu e} \left[\left(\cos v - 2e \frac{r}{p} \right) R' - \left(1 + \frac{r}{p} \right) \sin v S' \right] \\ & - \frac{3}{2} \frac{\dot{a}}{a} (t - T_0). \end{aligned} \quad (\text{C7})$$

If we consider the true anomaly, v , then, since $v = u - \omega$, $\dot{v} = \dot{u} - \dot{\omega}$. By the definition of the osculating elements, the value of \dot{u} is always given by the geodesic value, and so we see that the evolution of the true anomaly differs from integrating the instantaneous time-evolving geodesic equation by the $\dot{\omega}$ term. This can also be seen by differentiating the orbit equation and using that both r and \dot{r} are consistent with the instantaneous geodesic to obtain

$$\dot{v} = \frac{\sqrt{\mu a(1 - e^2)}}{r^2} + \frac{\dot{e} \cos v}{e \sin v} - \left(\frac{\dot{a}}{a} - \frac{2\dot{e}e}{1 - e^2} \right) \left(\frac{1 + e \cos v}{e \sin v} \right). \quad (\text{C8})$$

The first term is the geodesic \dot{v} , while the other terms arise as a result of the perturbation. Although this equation looks singular at turning points, $\sin v = 0$, substitution of the expressions for \dot{a} , \dot{e} and the geodesic equations gives the necessary calculations, and the expression reduces to $-\dot{\omega}$, as it should.

APPENDIX D: DRAG FORCE IN SCHWARZSCHILD GEOMETRY

In order to understand the effect that leads to an increase of eccentricity we can consider a Schwarzschild BH system, in which the same effect is seen, but which is easier to analyze and to understand. The osculating element equation for the evolution of the eccentricity, Eq. (3.60), in the case of a nonrotating BH reduces to Eq. (37) in [9] and has the form

$$\frac{de}{dv} = \mathcal{R}(p, e, v) a^r + \mathcal{T}(p, e, v) a^\phi. \quad (\text{D1})$$

We use a drag force to perturb the orbit which takes a very simple form $a^r = -\gamma u^r$, $a^\phi = -\gamma u^\phi$. The velocities, in Schwarzschild coordinates, are

$$u^r = e \sin v \sqrt{\frac{p - 6 - 2e \cos v}{p(p - 3 - e^2)}}, \quad (\text{D2})$$

$$u^\phi = \frac{(1 + e \cos v)^2}{pM\sqrt{p - 3 - e^2}}. \quad (\text{D3})$$

The equation for de/dv is integrable for this perturbing force if changes to e and p are ignored over the orbit and the result can expressed in terms of elliptic integrals. However, this is quite messy and we are primarily interested in the leading order correction to the orbit. We make a weak field expansion ($M/p \ll 1$) of the terms entering this equation:

$$\mathcal{R} \approx \frac{p^2}{M} \left(\mathcal{R}_0(e, v) + \frac{M}{p} \mathcal{R}_1(e, v) + O(M^2/p^2) \right), \quad (\text{D4})$$

$$\mathcal{T} \approx p^3 M \left(\mathcal{T}_0(e, v) + \frac{M}{p} \mathcal{T}_1(e, v) + O(M^2/p^2) \right); \quad (\text{D5})$$

here we do not go beyond the first correction to the Keplerian term. Similarly, we find for the velocities

$$u^r = \sqrt{\frac{M}{p}} u_0^r(e, v) \left(1 + \frac{M}{p} u_1^r(e, v) + O(M^2/p^2) \right), \quad (\text{D6})$$

$$u^\phi = \sqrt{\frac{M^3}{p^3}} u_0^\phi(e, \nu) \left(1 + \frac{M}{p} u_1^\phi(e, \nu) + O(M^2/p^2) \right). \quad (\text{D7})$$

The explicit form of the terms in these expansions is

$$u_0^r = e \sin \nu, \quad u_1^r = \left(-\frac{3}{2} - e \cos \nu + \frac{1}{2} e^2 \right), \quad (\text{D8})$$

$$u_0^\phi = (1 + e \cos \nu)^2, \quad u_1^\phi = \left(\frac{3}{2} + \frac{1}{2} e^2 \right), \quad (\text{D9})$$

$$\mathcal{R}_0 = \frac{\sin \nu}{(1 + e \cos \nu)^2}, \quad \mathcal{R}_1 = 3\mathcal{R}_0(1 - e^2), \quad (\text{D10})$$

$$\mathcal{T}_0 = \frac{(e \cos \nu + 2) \cos \nu + e}{(1 + e \cos \nu)^4}, \quad (\text{D11})$$

$$\mathcal{T}_1 = \frac{2e + 6 \cos \nu + 2e \cos^2 \nu - e^2 \cos^3 \nu - 5e^2 \cos \nu - 3e^3 \cos^2 \nu - e^3}{(1 + e \cos \nu)^4}. \quad (\text{D12})$$

The leading order terms give us the Newtonian perturbation of the eccentricity (C5) with perturbing force components $R' = -\gamma \dot{r}$, $S' = -\gamma r \dot{\phi}$. Overall, the Newtonian term is

$$\left(\frac{de}{dv} \right)_{00} = -2\gamma p^{3/2} \frac{e + \cos \nu}{(1 + e \cos \nu)^2}. \quad (\text{D13})$$

This equation can be integrated over an orbit, keeping e , p on the right-hand side constant, to give

$$\delta e(\nu) = -2\gamma p^{3/2} \frac{\sin \nu}{1 + e \cos \nu}. \quad (\text{D14})$$

It is clear that in the Newtonian case there is no a secular change in the eccentricity. Note also that the individual components (radial and azimuthal) of the perturbation are *not* zero after integration over one orbit, but they are exactly equal and opposite in sign. We now consider the first relativistic corrections. First, we note that the perturbations \mathcal{R}_1 and u_1^ϕ are independent of ν , and so we can reabsorb these into a redefinition of $\gamma \rightarrow \gamma'$ where

$$\gamma' = \left[1 + \frac{1}{2p}(3 + e^2) \right] \left[1 + \frac{3}{p}(1 - e^2) \right] \gamma,$$

and so the rescaled leading order term still averages to zero, as it is proportional to the Newtonian expression. There remain two perturbations, one that comes from the radial velocity perturbation, $\mathcal{R}_0 u_1^r$, and one that comes from the relativistic correction to the orbit's response to the azimuthal perturbation, $u_0^\phi \mathcal{T}_1$. The velocity perturbation contributes

$$\begin{aligned} \left(\frac{de}{dv} \right)_{01} &= -\gamma' \frac{p^{1/2} e \sin^2 \nu}{(1 + e \cos \nu)^2} \\ &\times \left[\frac{1}{2}(-3 - 2e \cos \nu + e^2) - \frac{1}{2}(3 + e^2) \right] \\ &= \gamma' p^{1/2} \frac{e \sin^2 \nu (3 + e \cos \nu)}{(1 + e \cos \nu)^2}. \end{aligned} \quad (\text{D15})$$

Note that this term is always positive, and so it will lead to an increase in the eccentricity. This can be interpreted as an additional radial force which acts at each point of the orbit in the direction of motion slowing down the effective radial velocity in the force, which leads to the increase of eccentricity.

The second part of the perturbation, $u_0^\phi \mathcal{T}_1$, contributes

$$\begin{aligned} \left(\frac{de}{dv} \right)_{10} &= \gamma' e p^{1/2} [1 - e^2 + \cos^2 \nu (1 + e \cos \nu) \\ &- e(\cos \nu + e)] (1 + e \cos \nu)^{-2}. \end{aligned} \quad (\text{D16})$$

Note that the last term is proportional to the Newtonian term and therefore averages to zero, so we can ignore this term. The remaining part is always positive and also drives an increase in eccentricity. This time the extra term can be interpreted as an additional azimuthal force which further boosts the effective azimuthal velocity in the force and once again leads to the increase in the eccentricity.

We note that both of these perturbations, and also the Keplerian term, are proportional to eccentricity, and they will not drive a circular orbit to become eccentric. In fact,

the contribution from the relativistic correction to the velocity is equal to that coming from the correction to the orbital response. Taking the difference,

$$\left(\frac{de}{dv}\right)_{01} - \left(\frac{de}{dv}\right)_{10} = \gamma' \frac{ep^{1/2}}{(1+e\cos v)^2} [e(\cos v + e) + 2\sin^2 v - 2\cos^2 v(1+e\cos v)]. \quad (\text{D17})$$

The first term in the square bracket is proportional to the Newtonian term and therefore vanishes after averaging. The remaining term can be integrated analytically,

$$\int_0^v dv' \frac{\sin^2 v' - \cos^2 v'(1+e\cos v')}{(1+e\cos v')^2} = -\frac{\sin v \cos v}{1+e\cos v}, \quad (\text{D18})$$

which is also zero after integration over one orbit. We conclude that the leading order relativistic correction in

the perturbation equation predicts the increase in eccentricity that we observe numerically. This secular change comes equally from the first order correction to the radial velocity and the first order correction to the orbital response to an azimuthal perturbation. The relativistic corrections can be thought of as an extra force which slows down the effective radial motion and accelerates the effective azimuthal motion that enter the drag force. The radial drag force is correspondingly reduced, while the azimuthal drag force is increased and both drive a secular increase in eccentricity. The equality of the two parts of the force may reflect some hidden symmetry in the equations. The response of the orbit to a perturbation depends on the velocity at each point along the orbit, and we are using that same velocity to prescribe the perturbation in this case, which might explain why the net contribution from the two terms is equal. However, the osculating element equations are not explicit in how they depend on the instantaneous velocity, so this is only a speculation.

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