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Horizon-entropy increase laws for spherically symmetric horizons in Brans-Dicke theory.

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Abstract. We derive the horizon-entropy increase law for spherically symmetric quasi-local horizons in Brans-Dicke theory. The quasi-local horizons used do not marginally trap null rays, and hence are not apparent horizons or foliated by marginally trapped surfaces, but instead have instantaneously constant gravitational entropy in the outgoing null direction. The relation derived has a very direct comparison with the horizon-entropy increase law for event horizons.

Perhaps one of the first to suggest black hole thermodynamics be applied to quasi-local horizons such as apparent horizons rather than event horizons was Hajicek [1] who conjectured that Hawking radiation originates from the region close to the apparent horizon independently of whether an event horizon exists or not. This idea was further examined by Hiscock [2] who proposed identifying the entropy with one quarter the area of the apparent horizon and Collins [3] who obtained a $T\mathrm{d}S = \mathrm{d}Q$ like relation for apparent horizons. An important contribution was made by Hayward [4] who defined an outer condition for apparent horizons and was able to show that the area of a future outer trapping horizon is non-decreasing if the null energy condition is satisfied on the horizon. These ideas received further attention when it was shown that the microstates of black hole entropy can be counted in loop quantum gravity for a constant area apparent horizon [5] and in analogue models where an event horizon is not necessary for the production of Hawking radiation [6]. The first tentative observations of this effect in analogue models may already have been observed [7].

Here we consider the horizon-entropy increase law for quasi-local horizons. We will consider the case of spherically symmetric horizons in a spherically symmetric spacetime using a Brans-Dicke action. In this case horizon generators can be written as $r^a = Bl^a + Cn^a$ with l^a and n^a the ingoing and outgoing radial null vectors respectively and $B > 0$ and C spherically symmetric functions on the horizon. Generalisations to non-spherically symmetric situations in more general theories can be found in [8].

Brans-Dicke theory is the prototype alternative theory of gravity with scalar and tensor modes. The action is given by

$$S = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{1}{16\pi}\left(\phi R - \frac{\omega}{\phi}\nabla^a\phi\nabla_a\phi\right) + \mathcal{L}_{matter}\right] . \tag{1}$$

ω is the Brans-Dicke parameter and we assume the scalar field satisfies $\phi > 0$. Variation of this action with respect to the metric gives the gravitational field equations, which, when contracted

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twice with a null vector l^a give

$$R_{ab}l^a l^b = \frac{\omega}{\phi^2}\left(l^a \nabla_a \phi\right)^2 + \frac{8\pi}{\phi} T_{ab} l^a l^b + \frac{l^a l^b \nabla_a \nabla_b \phi}{\phi}\,, \tag{2}$$

where T_{ab} is the energy-momentum tensor of the matter fields. The last term arises from an integration by parts in the variation of the ϕR term. Because of this last term the sign of $R_{ab}l^a l^b$ is indeterminate. Even in situations where the null energy condition holds, $T_{ab}l^a l^b \geq 0$, the null curvature condition $R_{ab}l^a l^b \geq 0$ may not be satisfied. This is the condition that appears in the Raychaudhuri equation and on which the results of Hawking [9] and Hayward [4] depend. It was noticed for example in [10] that the apparent horizon can appear outside the event horizon in dynamical black hole collapse models in Brans-Dicke gravity.

Under a conformal transformation Brans-Dicke theory can be put into the form of Einstein gravity plus a scalar field non-minimally coupled to matter. Transforming

$$g_{ab} \to \tilde{g}_{ab} = \phi(x) g_{ab}\,, \tag{3}$$

transforms the action to

$$S = \int \mathrm{d}^4x \sqrt{-\tilde{g}} \left[\frac{1}{16\pi}\left(\tilde{R} - \frac{(3+\omega)}{2\phi^2}\tilde{\nabla}^a\phi\tilde{\nabla}_a\phi\right) + \frac{\mathcal{L}_{matter}}{\phi^2}\right]. \tag{4}$$

In this Einstein frame the contracted Ricci tensor is

$$\tilde{R}_{ab}l^a l^b = \frac{(3+\omega)}{2\phi^2}\left(l^a\tilde{\nabla}_a\phi\right)^2 + \frac{8\pi}{\phi^2}T_{ab}l^a l^b\,. \tag{5}$$

The theory automatically satisfies the null curvature condition in the Einstein frame if it satisfies the null energy condition. In the Einstein frame the apparent horizon appears inside the event horizon [10]. The reason why the apparent horizon can be outside the event horizon in one frame, but inside the event horizon in another is because the location of the trapping horizon is not invariant under a conformal transformation. The conformal transformation leaves the spacetime coordinates, causal structure and null rays unchanged. The location of the event horizon remains the same. But the area of a two-dimensional surface will change as

$$A \to \tilde{A} = \phi A\,. \tag{6}$$

In spherical symmetry the expansion of a congruence of rays normal to a spherically symmetric surface can be found by calculating the change of the area. Under a conformal transformation the expansion changes as

$$\theta_l \to \tilde{\theta}_l = \frac{l^a\nabla_a\tilde{A}}{\tilde{A}} = \theta_l + \frac{l^a\nabla_a\phi}{\phi}\,. \tag{7}$$

The vanishing of θ_l for a given surface is therefore not necessarily invariant under a conformal transformation. And thus the location of an apparent horizon satisfying $\theta_l = 0$ is not necessarily invariant.

A general result for the stationary entropy of a diffeomorphism invariant gravity theory was given in [11].

$$S_g = -2\pi \int_H \frac{\partial \mathscr{L}}{\partial R_{abcd}}\,\hat{\varepsilon}_{ab}\hat{\varepsilon}_{cd}\,\sqrt{q}\,\mathrm{d}^2x + \text{higher derivative terms}, \tag{8}$$

where $\hat{\varepsilon}_{ab}$ is the antisymmetric binormal form for the surface H, $\hat{\varepsilon}_{ab} = l_a n_b - n_a l_b$ and $\mathscr{L}$ is the full Lagrangian density. For ordinary Einstein gravity this formula gives the familiar result $S_g = A/4$ and for Brans-Dicke theory it gives the gravitational entropy as $S_g = \phi A/4$.

Consider now the following conditions

$$\begin{aligned} l^a\nabla_a S_g &= 0\,, \\ n^a\nabla_a S_g &< 0\,, \\ n^a\nabla_a\left(l^c\nabla_c S_g\right) &< 0\,. \end{aligned} \tag{9}$$

The conditions relate to how S_g is changing in the null directions normal to the horizon surface. In Brans-Dicke theory the first condition would be satisfied where $\theta_l + l^a\nabla_a\phi/\phi = 0$. Therefore these conditions are not satisfied by apparent horizons in Brans-Dicke theory. In ordinary Einstein gravity however, these would reduce to the requirements on the null expansions for a trapping horizon given in [4] since, in this case, $S_g = A/4$. The variation of the generalised entropy is now

$$r^a\nabla_a S_g = Cn^a\nabla_a S_g. \tag{10}$$

Since we require the tangent r^a to generate evolution along the generalised trapping horizon, on which $l^a\nabla_a S_g = 0$, we have $r^a\nabla_a(l^b\nabla_b S_g) = 0$ and thus

$$C = -\frac{Bl^a\nabla_a\left(l^b\nabla_b S_g\right)}{n^c\nabla_c\left(l^d\nabla_d S_g\right)}\,. \tag{11}$$

Putting this together with the Raychaudhuri equation and the equations of motion (2), the final result for the change of S_g along the horizon is then

$$r^a\nabla_a S_g = \frac{A\phi B n^a\nabla_a S_g}{n^c\nabla_c\left(l^d\nabla_d S_g\right)}\left(\frac{\theta_l^2}{2} + \frac{\omega+1}{\phi^2}\left(l^a\nabla_a\phi\right)^2 + \frac{8\pi}{\phi}T_{ab}l^a l^b\right)\,. \tag{12}$$

Since these horizons satisfy $\theta_l + l^a\nabla_a\phi/\phi = 0$ the term $\frac{1}{2}\,\theta_l^2 + \frac{\omega+1}{\phi^2}\left(l^a\nabla_a\phi\right)^2$ is guaranteed positive if $\omega > -3/2$. For a Brans-Dicke theory of this form and matter obeying the null energy condition $T_{ab}l^a l^b \geq 0$, the generalised entropy is guaranteed to be non-decreasing on horizons satisfying the conditions (9).

A similar result can be derived for causal horizons such as event horizons. Because causal horizons are generated by null rays we have $r^a = l^a$ and immediately

$$\begin{aligned} l^b\nabla_b\left(l^a\nabla_a S_g\right) &= A\phi\Bigg[\left(\theta_l + \frac{l^a\nabla_a\phi}{\phi}\right)^2 + \kappa\left(\theta_l + \frac{l^a\nabla_a\phi}{\phi}\right) \\ &\quad -\frac{\theta_l^2}{2} - \frac{\omega+1}{\phi^2}\left(l^a\nabla_a\phi\right)^2 - \frac{8\pi}{\phi}T_{ab}l^a l^b\Bigg]\,. \end{aligned} \tag{13}$$

This is now the second derivative of the horizon-entropy. But if the causal horizon settles down to a Killing horizon at late times, then it can be shown that the first derivative of the horizon-entropy cannot ever be negative [8]. Causal horizons will not coincide in general dynamical situations with the quasi-local horizons considered here [12].

Similar results can be extended to other theories such as scalar-tensor and $f(R)$ theories and non-spherically symmetric horizons [8]. These models cover a wide variety of cases from low-energy effective string theory actions, Kaluza-Klein compactifications of higher dimensions and quantum loop corrections. The quasi-local horizons used coincide with trapping horizons in Einstein gravity, but are generally different in all these other theories and models. Even if one restricts attention purely to Einstein gravity, the use of modified theories shows that the instantaneous constancy of gravitational entropy, rather than the focussing of light rays, is what is required to prove the horizon-entropy increase law. This has obvious implications for our understanding of entropy of the gravitational field and its relation to horizons.

The gravitational entropy used here was first derived for purely stationary systems and its extension to dynamical situations is known to contain some ambiguities [11]. The extension of other gravitational-thermodynamic properties to dynamical situations also contains some ambiguities [13, 14, 15]. But we have explicitly applied this horizon-entropy to dynamical situations here and found that a horizon-entropy increase law can be derived for different gravity models and for both quasi-local and causal horizons.

The definition also impacts our understanding of black holes. Ideally we would like to define a black hole in a way that indicates the presence of certain properties that are absent from objects such as neutron stars. Such properties might be the production of Hawking radiation transporting energy from the gravitational field to null infinity or regions with large deviations from classical metric geometry or some measure of black hole thermodynamics with horizon-entropy [16]. Black hole thermodynamics needs a surface in order to compute an area to equate with entropy. The non-uniqueness of trapping horizons [17] will also apply to the surfaces proposed here. But the horizon-entropy will only be conformally invariant for certain types of surfaces, typically requiring the scalar field to be constant on the horizon slice, such as the spherically symmetric surfaces considered here. This may pick out a preferred horizon for computing the horizon-entropy, or it may be that there is no unique definition of such a surface. The physical effects we commonly associate with black hole spacetimes may not be easily associated to a single surface [18], in which case the implications of the area-entropy relation and related holographic principles will need to be revisited.

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