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Null polygonal Wilson loops in full $\mathcal{N} = 4$ superspace

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Abstract

We compute the one-loop expectation value of light-like polygonal Wilson loops in $\mathcal{N} = 4$ super-Yang–Mills theory in full superspace. When projecting to chiral superspace, we recover the known results for the tree-level next-to-maximally-helicity-violating scattering amplitude. The one-loop maximally-helicity-violating amplitude is also included in our result but there are additional terms which do not immediately correspond to scattering amplitudes. We finally discuss different regularizations and their Yangian anomalies.

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Contents

1. Introduction

The $\mathcal{N} = 4$ super-Yang–Mills (SYM) theory exhibits integrable features in the planar limit [1]. This integrability has been used very successfully for finding the spectrum of anomalous dimensions of single-trace local operators.

One would like to go beyond this, and compute other physical quantities. The $\mathcal{N} = 4$ SYM theory being conformal, the correlation functions of local, gauge-invariant operators are natural quantities to consider. However, while some partial results have been obtained concerning the correlation functions, we are still very far from having an all-order understanding.

Part of the problem is that, even after using the superconformal symmetry, the correlation functions depend on a large number of invariants. One can consider special limits in which the kinematics simplify. For example, one can take the operators in the correlation functions to be pairwise light-like separated. In that limit, the correlation functions are essentially squares of the Wilson loop in the fundamental representation of the $SU(N)$ gauge group (or the Wilson loop in the adjoint representation, which is the same in the large N limit), defined on a polygonal light-like contour [2–5].

One can also consider scattering amplitudes in $\mathcal{N} = 4$ SYM. In the vacuum where all the scalars have zero expectation values, these amplitudes are infrared (IR) divergent and need to be regularized. There are two favored options for performing the regularization. One is a supersymmetry-preserving variant of dimensional regularization and the other is the so-called mass regularization, which consists in giving vacuum expectation values (VEV) to some of the scalars [6, 7].

The correlation functions in the light-like limit and the polygonal light-like Wilson loops have ultraviolet (UV) divergences so they also need to be regularized. How to perform this regularization is not entirely obvious, and some difficulties have been reported in the literature (see [8]), concerning the use of dimensional regularization.

Even though the scattering amplitudes, the light-like polygonal Wilson loops and the correlation functions in the light-like limit seem to be very different, it has been shown that, in fact, they contain essentially the same information[4]. There are several arguments that strongly support this. At strong coupling, this can be understood from supersymmetric T-duality (see [6, 12]), which maps Wilson loops to scattering amplitudes and at the same time exchanges the UV and IR regimes. At weak coupling, this is supported by explicit perturbative computations [9, 10, 13–19]. Note that, when relating Wilson loops to correlation functions, there is no need to exchange UV and IR. This is only needed when relating them to the scattering amplitudes.

The interchange of UV and IR makes it more challenging to match the answers for Wilson loops (or correlation functions) and scattering amplitudes. For example, in dimensional regularization, one has to match ϵ_{UV}, which is used for regularizing the UV divergences of the Wilson loop with ϵ_{IR}, which is used for regularizing the IR divergences of the scattering amplitudes.

In the planar limit, one can unambiguously define a notion of integrand [20, 21] for scattering amplitudes/Wilson loops. The integrand is a rational differential form which is well defined even in the absence of a regulator. Because of this, it has been more fruitful to compare the integrands of scattering amplitudes and Wilson loops and in [22, 23] it was shown that the integrands coincide.

So far, all of these quantities have been mostly studied in chiral superspace ([24] is an exception). The motivation was that, for describing the on-shell states used in scattering amplitudes, one uses an on-shell superspace which is very naturally described chirally. However, the chiral superspace has a big downside: it obscures some of the symmetries of the answers. This goes beyond just the obvious breaking of manifest parity symmetry since for chiral Wilson loops, the $\bar{\mathrm{Q}}$ operator is broken as well. However, as has recently been shown (see [25, 26]), one can repair the non-invariance under the $\bar{\mathrm{Q}}$ operator and use it to build higher loop answers from lower loop ones.

It has been shown in [27, 28] that the tree-level scattering amplitudes in $\mathcal{N} = 4$ SYM are invariant under a hidden dual superconformal symmetry. In [29], the superconformal symmetry and the dual superconformal symmetry were shown to generate an infinite-dimensional Yangian symmetry. In the case of $\mathcal{N} = 4$ SYM, this is the Yangian $\mathrm{Y}[\mathfrak{psu}(2, 2|4)]$. At loop level, this Yangian symmetry is broken by IR divergences for scattering amplitudes or by UV divergences for Wilson loops. The integrands are Yangian invariant up to total derivatives [30, 20].

In the chiral formulation, the momentum twistors [31, 32] $W_i = (w_i|\chi_i) = (\lambda_i, \mu_i|\chi_i)$, which are points in $\mathbb{CP}^{3|4}$, play an important role. They provide unconstrained variables for the kinematics and the superconformal group acts linearly on their homogeneous

[4] To be more precise, the scattering amplitudes and the correlation functions contain parts which are odd under parity transformations [9, 10]. As shown in [10], the scattering amplitudes also contain so-called μ terms which are curious integrals such that the integrand vanishes when the dimensional regularization parameter $\epsilon = (4 - D)/2$ goes to zero, but the *integral* diverges. However, it turns out that when taking the logarithm, all of these complicated contributions disappear and the result matches the Wilson loop result. The μ terms also cancel for the two-loop next-to-maximally-helicity-violating (NMHV) amplitudes, as shown in [11].

coordinates. The results are expressed in terms of two kinds of basic objects: four-brackets $\langle ijkl\rangle = \varepsilon_{abcd} w_i^a w_j^b w_k^c w_l^d$ and R-invariants [27, 32],

$$[ijklm] = \frac{\delta^{0|4}(\langle ijkl\rangle\chi_m + \langle jklm\rangle\chi_i + \langle klmi\rangle\chi_j + \langle lmij\rangle\chi_k + \langle mijk\rangle\chi_l)}{\langle ijkl\rangle\langle jklm\rangle\langle klmi\rangle\langle lmij\rangle\langle mijk\rangle}. \tag{1.1}$$

The R-invariants are superconformal invariant and in fact Yangian invariant, but the four-brackets are only conformally invariant. In the answer for scattering amplitudes, the R-invariants enter somewhat trivially, as global multiplicative factors, but the four-brackets enter in a much more non-trivial way, as arguments of transcendental functions. Therefore, the superconformal symmetry is much less obvious in this presentation.

Motivated by these shortcomings of the chiral formalism, in this paper we study the Wilson loops in full superspace, where both Q and $\bar{\mathrm{Q}}$ symmetry operators play the same role.

In the non-chiral approach, which we will present in more detail below, we have two sets of momentum twistors, W_i and their conjugate $\bar{W}_i$. Using them, one can easily form superconformal invariants $W_i \cdot \bar{W}_j$ (see appendix E for a discussion of superconformal invariants). In the chiral formulation, the answers are written in terms of twistor four-brackets. These four-brackets are conformal but not superconformal invariant. If we want to make superconformal symmetry manifest, we need to use quantities like $W_i \cdot \bar{W}_j$ instead. When performing the Grassmann expansion of the superconformal invariants $W_i \cdot \bar{W}_j$, we recover the usual four-brackets at the first order.

The momentum twistors and their conjugates are not unconstrained, but they satisfy some relations $W_i \cdot \bar{W}_i = W_i \cdot \bar{W}_{i+1} = W_i \cdot \bar{W}_{i-1} = 0$.

We define and compute a Wilson loop in full superspace to one-loop order. At this order, the answer contains a rational part which is the same as the tree-level NHMV scattering amplitude[5], and a transcendental piece which is similar to the one-loop maximally-helicity-violating (MHV) scattering amplitude. The transcendental part of the answer is of transcendentality 2 and it contains dilogarithms and products of logarithms of superconformal invariants $W_i \cdot \bar{W}_j$.[6] We believe that this form of the answer is more satisfactory than the chiral presentation, since the superconformal symmetry is manifest, except for some 'boundary' cases which appear when the propagator approaches a null edge. So the breaking of the symmetry is localized to the regions where the UV divergences arise.

We should note that the transcendentality-2 part of the answer, when expanded out in powers of Grassmann variables, yields the one-loop answer at zeroth order in the expansion. In [24], Caron–Huot also considered the next order in the $\bar{\theta}$ expansion.

The answer we obtain is not in the form that is usually presented in the literature (see [33] for the original computation), but it is related to it via dilogarithm identities. Another noteworthy feature is that the rational and transcendental parts are computed by two kinds of propagators, which are related by a Grassmann Fourier transform.

We have also studied the superconformal and Yangian anomalies of the answer. In order to avoid dealing with divergent quantities, we have used a framing regularization, conjectured a super-Poincaré invariant expression and defined a finite quantity from the Wilson loop which is similar to one defined in [34] for studying the near collinear limit of Wilson loops. Then we defined and computed the action of the Yangian on this quantity.

It is important to stress that our computation applies only to non-chiral Wilson loops $W(x_i, \theta_i, \bar{\theta}_i)$ but not to scattering amplitudes. One can obtain the scattering amplitudes by

[5] The tree-level NMHV amplitudes can be written in several different forms. The form we obtained is the same as the CSW-like form of Mason and Skinner [22].

[6] As we will show, its divergent parts in a certain regularization contain terms like $\langle ij\rangle$, or $[ij]$, which break superconformal symmetry.

setting $\bar{\theta}_i = 0$ but there is no obvious way to define non-chiral scattering amplitudes such that they are dual to the non-chiral Wilson loops.

The organization of the paper is as follows. In section 2, we review $\mathcal{N} = 4$ SYM theory in $\mathcal{N} = 4$ superspace. In section 3, we introduce some prepotentials for the gauge connection and compute their two-point functions in light-cone gauge. This puts us in the position to carry out simple computations in this quantum field theory. In section 4, we perform the one-loop computations in momentum space and in section 5, we perform the same computations in momentum twistor space. In section 6, we present the regularizations we use. In section 7, we compute the Yangian anomalies. We end in section 8 with some conclusions. Our conventions and some computational details can be found in the appendices.

2. $\mathcal{N} = 4$ SYM in superspace

We would like to compute the Wilson loop expectation value with as much manifest supersymmetry as possible. The obvious choice is to use the $\mathcal{N} = 4$ superspace. We therefore review a formulation of classical on-shell $\mathcal{N} = 4$ SYM theory in this full (non-chiral) superspace [35].

2.1. $\mathcal{N} = 4$ superspace

Superspace has the coordinates $z^A = \left(x^\mu, \theta^{a\alpha}, \bar{\theta}^{\dot{\alpha}}_a\right)$. Here $\alpha, \dot{\alpha}$ are Lorentz indices, and a are flavor-symmetry indices ranging from 1 to 4 and transforming in the **4** or $\bar{\mathbf{4}}$ representations of $\mathrm{SU}(4)$.

The supersymmetry transformations are

$$\mathrm{Q}_{a\alpha} = \frac{\partial}{\partial\theta^{a\alpha}} - \mathrm{i}\bar{\theta}^{\dot{\alpha}}_a\sigma^\mu_{\alpha\dot{\alpha}}\frac{\partial}{\partial x^\mu}, \qquad \bar{\mathrm{Q}}^a_{\dot{\alpha}} = -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}_a} + \mathrm{i}\theta^{a\alpha}\sigma^\mu_{\alpha\dot{\alpha}}\frac{\partial}{\partial x^\mu}. \tag{2.1}$$

Under a supersymmetry transformation $\zeta^{a\alpha}\mathrm{Q}_{a\alpha} + \bar{\mathrm{Q}}^a_{\dot{\alpha}}\bar{\zeta}^{\dot{\alpha}}_a$, the superspace coordinates transform like

$$\delta x^\mu = \mathrm{i}\left(\theta^{a\alpha}\sigma^\mu_{\alpha\dot{\alpha}}\bar{\zeta}^{\dot{\alpha}}_a - \zeta^{a\alpha}\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}_a\right), \tag{2.2}$$

$$\delta\theta^{a\alpha} = \zeta^{a\alpha}, \qquad \delta\bar{\theta}^{\dot{\alpha}}_a = \bar{\zeta}^{\dot{\alpha}}_a. \tag{2.3}$$

The supersymmetry covariant derivatives are

$$D_{a\alpha} = \frac{\partial}{\partial\theta^{a\alpha}} + \mathrm{i}\bar{\theta}^{\dot{\alpha}}_a\sigma^\mu_{\alpha\dot{\alpha}}\frac{\partial}{\partial x^\mu}, \quad \bar{D}^a_{\dot{\alpha}} = -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}_a} - \mathrm{i}\theta^{a\alpha}\sigma^\mu_{\alpha\dot{\alpha}}\frac{\partial}{\partial x^\mu}, \quad \partial_{\alpha\dot{\alpha}} = \sigma^\mu_{\alpha\dot{\alpha}}\frac{\partial}{\partial x^\mu}. \tag{2.4}$$

These supersymmetry covariant derivatives form the following algebra:

$$\{D_{a\alpha}, D_{b\beta}\} = 0, \quad \left\{\bar{D}^a_{\dot{\alpha}}, \bar{D}^b_{\dot{\beta}}\right\} = 0, \quad \left\{D_{a\alpha}, \bar{D}^b_{\dot{\alpha}}\right\} = -2\mathrm{i}\delta^b_a\partial_{\alpha\dot{\alpha}}. \tag{2.5}$$

These derivatives have the following behavior under Hermitian conjugation:

$$\partial^\dagger_{\alpha\dot{\alpha}} = -\partial_{\alpha\dot{\alpha}}, \quad D^\dagger_{a\alpha} = \bar{D}^a_{\dot{\alpha}}. \tag{2.6}$$

A naive interval $x_1 - x_2$ is invariant under translations, but not under superspace translations. A quantity which is invariant under superspace translations is

$$x^\mu_{j,k} \equiv x^\mu_k - x^\mu_j - \mathrm{i}\theta^{a\alpha}_k\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}_{ja} + i\theta^{a\alpha}_j\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}_{ka}. \tag{2.7}$$

We emphasize here that our notation $x_{j,k}$ does *not* stand for $x_k - x_j$.

It is usual to define chiral and antichiral combinations as $x^{\pm\mu} = x^\mu \pm \mathrm{i}\theta^{a\alpha}\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}_a$. The chiral/antichiral combinations satisfy $D_{a\alpha}x^{-\mu} = 0, \bar{D}^a_{\dot{\alpha}}x^{+\mu} = 0$. There are chiral and antichiral versions of the above superspace interval defined simply by $x^{\pm}_{j,k} = x^{\pm}_{k} - x^{\pm}_{j}$. We can also define a mixed-chiral interval $x^{+-\mu}_{j,k} \equiv x^{-\mu}_{k} - x^{+\mu}_{j} + 2\mathrm{i}\theta^{a\alpha}_{j}\sigma^{\mu}_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}_{ka}$, which has the property that $\bar D_j x^{+-}_{j,k} = 0$, $D_k x^{+-}_{j,k} = 0$. Here we have schematically denoted by $\bar D_j$ the antichiral derivative with respect to the superspace coordinates $(x_j, \theta_j, \bar\theta_j)$ and by D_k the chiral derivative with respect to the superspace coordinates $(x_k, \theta_k, \bar\theta_k)$. The chiral–antichiral interval can also be written as

$$x^{+-\mu}_{j,k} = x^{\mu}_{j,k} + \mathrm{i}\theta^{a\alpha}_{jk}\sigma^{\mu}_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}_{jka}, \tag{2.8}$$

where $\theta_{jk} = \theta_k - \theta_j$, $\bar\theta_{jk} = \bar\theta_k - \bar\theta_j$. This writing makes it clear that the chiral–antichiral interval is invariant under superspace translations.

2.2. *Superspace vielbein*

The supersymmetry covariant derivatives can be written more compactly as

$$D_M = E_M{}^A \frac{\partial}{\partial z^A}, \tag{2.9}$$

where $E_M{}^A$ is called the inverse supervielbein

$$E_M{}^A = \begin{array}{c|ccc} M\backslash A & \boxed{\nu} & \boxed{b\beta} & \boxed{\begin{smallmatrix} b \\ \dot\beta \end{smallmatrix}} \\ \hline \boxed{\mu} & \delta^{\nu}_{\mu} & 0 & 0 \\ \boxed{a\alpha} & i\bar\theta^{\dot\alpha}_{a}\sigma^{\nu}_{\alpha\dot\alpha} & \delta^{b}_{a}\delta^{\beta}_{\alpha} & 0 \\ \boxed{\begin{smallmatrix} a \\ \dot\alpha \end{smallmatrix}} & -i\theta^{a\alpha}\sigma^{\nu}_{\alpha\dot\alpha} & 0 & -\delta^{a}_{b}\delta^{\dot\beta}_{\dot\alpha} \end{array}. \tag{2.10}$$

The supervielbein is

$$E_A{}^N = \begin{array}{c|ccc} A\backslash N & \boxed{\rho} & \boxed{c\gamma} & \boxed{\begin{smallmatrix} c \\ \dot\gamma \end{smallmatrix}} \\ \hline \boxed{\nu} & \delta^{\rho}_{\nu} & 0 & 0 \\ \boxed{b\beta} & -i\bar\theta^{\dot\beta}_{b}\sigma^{\rho}_{\beta\dot\beta} & \delta^{c}_{b}\delta^{\gamma}_{\beta} & 0 \\ \boxed{\begin{smallmatrix} b \\ \dot\beta \end{smallmatrix}} & -i\theta^{b\beta}\sigma^{\rho}_{\beta\dot\beta} & 0 & -\delta^{b}_{c}\delta^{\dot\gamma}_{\dot\beta} \end{array}. \tag{2.11}$$

Now we can define the supervielbein as a differential form by $E^M = \mathrm{d}z^A E_A{}^M$. In components, this reads

$$E^{\rho} = \mathrm{d}x^{\rho} - \mathrm{i}\mathrm{d}\theta^{a\alpha}\sigma^{\rho}_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}_{a} - \mathrm{i}\mathrm{d}\bar\theta^{\dot\alpha}_{a}\sigma^{\rho}_{\alpha\dot\alpha}\theta^{a\alpha}, \quad E^{a\alpha} = \mathrm{d}\theta^{a\alpha}, \quad E^{\dot\alpha}_{a} = -\mathrm{d}\bar\theta^{\dot\alpha}_{a}. \tag{2.12}$$

Our conventions for differential calculus with Grassmann numbers are such that $\mathrm{d}z^M \wedge \mathrm{d}z^N = -(-)^{MN}\,\mathrm{d}z^N \wedge \mathrm{d}z^M$, where $(-)^{MN}$ is the product of gradings of z^M and z^N. Therefore, $\mathrm{d}\theta \wedge \mathrm{d}\bar\theta = \mathrm{d}\bar\theta \wedge \mathrm{d}\theta$ and $\mathrm{d}\theta \wedge \mathrm{d}x = -\mathrm{d}x \wedge \mathrm{d}\theta$.

Putting together the covariant derivatives and the supervielbein, there are two alternative forms for the exterior derivative,

$$d = E^{\rho}\partial_{\rho} + E^{a\alpha}D_{a\alpha} + E^{\dot\alpha}_{a}\bar D^{a}_{\dot\alpha} \tag{2.13}$$

$$= \mathrm{d}x^{\rho}\frac{\partial}{\partial x^{\rho}} + \mathrm{d}\theta^{a\alpha}\frac{\partial}{\partial\theta^{a\alpha}} + \mathrm{d}\bar\theta^{\dot\alpha}_{a}\frac{\partial}{\partial\bar\theta^{\dot\alpha}_{a}}. \tag{2.14}$$

Finally, note that the supervielbein has the following torsion components:

$$\mathrm{d}E^{\rho} = 2\mathrm{i}\sigma^{\rho}_{\alpha\dot\alpha}E^{a\alpha} \wedge E^{\dot\alpha}_{a}, \quad \mathrm{d}E^{a\alpha} = 0, \quad \mathrm{d}E^{\dot\alpha}_{a} = 0. \tag{2.15}$$

2.3. Superspace connection

We introduce a gauge connection one-form A on superspace. It is conveniently expanded in a basis of the supervielbein

$$A = E^{\rho}A_{\rho} + E^{a\alpha}A_{a\alpha} + E^{\dot\alpha}_{a}\bar A^{a}_{\dot\alpha}. \tag{2.16}$$

The components $A_{\alpha\dot\alpha}(x,\theta,\bar\theta)$, $A_{a\alpha}(x,\theta,\bar\theta)$ and $\bar A^{a}_{\dot\alpha}(x,\theta,\bar\theta)$ are used to define gauge- and supersymmetry-covariant derivatives, as follows:

$$\nabla_{a\alpha}\bullet = D_{a\alpha}\bullet + [A_{a\alpha},\bullet\},\quad \bar\nabla^{a}_{\dot\alpha}\bullet = \bar D^{a}_{\dot\alpha}\bullet + \big[\bar A^{a}_{\dot\alpha},\bullet\big\},\quad \nabla_{\alpha\dot\alpha}\bullet = \partial_{\alpha\dot\alpha}\bullet + [A_{\alpha\dot\alpha},\bullet]\,. \tag{2.17}$$

We take the gauge connection to be antiHermitian, $A = -A^{\dagger}$, and the components satisfy the following reality conditions:

$$(A_{\alpha\dot\beta})^{\dagger} = -A_{\beta\dot\alpha},\qquad (A_{a\alpha})^{\dagger} = \bar A^{a}_{\dot\alpha}. \tag{2.18}$$

The gauge potentials have infinitesimal gauge transformations given by

$$\delta A_{a\alpha} = \nabla_{a\alpha}\Lambda,\qquad \delta\bar A^{a}_{\dot\alpha} = \bar\nabla^{a}_{\dot\alpha}\Lambda,\qquad \delta A_{\alpha\dot\alpha} = \nabla_{\alpha\dot\alpha}\Lambda, \tag{2.19}$$

where $\Lambda(x,\theta,\bar\theta)$ is some antiHermitian superfield ($\Lambda^{\dagger} = -\Lambda$).

Starting with the gauge connections, one can define gauge-covariant field strengths as the components of $F = \mathrm{d}A + A\wedge A$ in the expansion in terms of the vielbeins $E^{\dot\alpha\alpha}$, $E^{a\alpha}$ and $E^{\dot\alpha}_{a}$. We find

$$\begin{aligned}F = \quad & E^{a\alpha}\wedge E^{\dot\alpha}_{b}\big(2\mathrm{i}\delta^{b}_{a}A_{\alpha\dot\alpha} + \bar D^{b}_{\dot\alpha}A_{a\alpha} + D_{a\alpha}\bar A^{b}_{\dot\alpha} + \{A_{a\alpha},\bar A^{b}_{\dot\alpha}\}\big)\\
&+\tfrac12 E^{a\alpha}\wedge E^{b\beta}\big(D_{b\beta}A_{a\alpha} + D_{a\alpha}A_{b\beta} + \{A_{a\alpha},A_{b\beta}\}\big)\\
&+\tfrac12 E^{\dot\alpha}_{a}\wedge E^{\dot\beta}_{b}\big(\bar D^{b}_{\dot\beta}\bar A^{a}_{\dot\alpha} + \bar D^{a}_{\dot\alpha}\bar A^{b}_{\dot\beta} + \big\{\bar A^{a}_{\dot\alpha},\bar A^{b}_{\dot\beta}\big\}\big)\\
&+E^{\dot\alpha\alpha}\wedge E^{b\beta}\big(D_{b\beta}A_{\alpha\dot\alpha} - \partial_{\alpha\dot\alpha}A_{b\beta} + [A_{b\beta},A_{\alpha\dot\alpha}]\big)\\
&+E^{\dot\alpha\alpha}\wedge E^{\dot\beta}_{b}\big(\bar D^{b}_{\dot\beta}A_{\alpha\dot\alpha} - \partial_{\alpha\dot\alpha}\bar A^{b}_{\dot\beta} + [\bar A^{b}_{\dot\beta},A_{\alpha\dot\alpha}]\big)\\
&+\tfrac12 E^{\dot\alpha\alpha}\wedge E^{\dot\beta\beta}\big(\partial_{\beta\dot\beta}A_{\alpha\dot\alpha} - \partial_{\alpha\dot\alpha}A_{\beta\dot\beta} + [A_{\beta\dot\beta},A_{\alpha\dot\alpha}]\big).\end{aligned} \tag{2.20}$$

When expanded in components, the gauge connections defined above contain too many fields to match the degrees of freedom (d.o.f.) in $\mathcal{N} = 4$ SYM. Said differently, these superfields form reducible representations of the supersymmetry algebra and we will have to impose constraints on them in order to obtain irreducible representations. The constraints are imposed by demanding that certain components of the field strength F vanish (see [35] as well as [36, chapter 12] for a textbook treatment of the $\mathcal{N} = 3$ extended supersymmetry),

$$D_{a\alpha}A_{b\beta} + D_{b\beta}A_{a\alpha} + \{A_{a\alpha},A_{b\beta}\} = \varepsilon_{\alpha\beta}\bar W_{ab}, \tag{2.21a}$$

$$\bar D^{a}_{\dot\alpha}\bar A^{b}_{\dot\beta} + \bar D^{b}_{\dot\beta}\bar A^{a}_{\dot\alpha} + \big\{\bar A^{a}_{\dot\alpha},\bar A^{b}_{\dot\beta}\big\} = \varepsilon_{\dot\alpha\dot\beta}W^{ab}, \tag{2.21b}$$

$$2\mathrm{i}\delta^{b}_{a}A_{\alpha\dot\alpha} + D_{a\alpha}\bar A^{b}_{\dot\alpha} + \bar D^{b}_{\dot\alpha}A_{a\alpha} + \big\{A_{a\alpha},\bar A^{b}_{\dot\alpha}\big\} = 0. \tag{2.21c}$$

These are at the same time definitions for the scalar superfields W^{ab} and $\bar W_{ab}$ and constraints for the gauge connections. For example, the first constraint in equation (2.21a) means that the left-hand side transforms as a singlet under Lorentz transformations and as a **6** under $SU(4)$ flavor transformations. In other words, $\bar W_{ab}$ is a rank-2 antisymmetric tensor. It obeys the Hermiticity condition $\bar W_{ab} = (W^{ab})^{\dagger}$.

Let us note here a crucial difference to $\mathcal{N} = 1$ superfields. In that case, the first two constraints in equations (2.21a) and (2.21b) have a trivial right-hand side. This allows to solve the constraints in this case.

The superfields W^{ab} and $\bar{W}_{ab}$ are very natural superfields. They have mass dimension 1 and their flavor-symmetry transformations are such that their bottom component in the θ, $\bar{\theta}$ expansion is the scalars fields ϕ_{ab} in the $\mathcal{N} = 4$ supermultiplet. More precisely, the scalars ϕ_{ab} are the bottom component in the $\bar{W}_{ab}$ multiplet, while the conjugate scalars $\phi^{ab} = (\phi_{ab})^\dagger$ are part of the W^{ab} multiplet. The higher components contain the fermions ψ^a_α, $\bar{\psi}_{a\dot{\alpha}} = (\psi^a_\alpha)^\dagger$ and the field strength $F_{\alpha\beta}$ and $\bar{F}_{\dot{\alpha}\dot{\beta}} = (F_{\alpha\beta})^\dagger$.

The scalar fields in $\mathcal{N} = 4$ SYM satisfy a reality condition $\phi^{ab} = (\phi_{ab})^\dagger = \frac{1}{2}\varepsilon^{abcd}\phi_{cd}$. The superfields themselves are related by a similar relation[7]

$$W^{ab} = \tfrac{1}{2}\varepsilon^{abcd}\bar{W}_{cd}. \tag{2.22}$$

The constraint on the superfield imposes proper reality constraints on the members ψ, $\bar{\psi}$, F and $\bar{F}$ of the multiplet.

3. Gauge field propagator

In this section, we derive a two-point function for the gauge fields of $\mathcal{N} = 4$ SYM in superspace. This is the relevant object for the one-loop contribution to a Wilson loop expectation value. Quantization of gauge fields in extended superspace is troublesome due to the constraints, and we start by sketching our procedure and results in terms of a simple example. Subsequently, we will lift the results to $\mathcal{N} = 4$ SYM.

3.1. Sketch for a scalar field

The first problem we have to face is that the constraints for the gauge field in superspace force it on-shell. A standard Feynman propagator takes the form $1/(p^2 - \mathrm{i}\epsilon)$, which clearly is ill-defined when $p^2 = 0$. Nevertheless, there exists a well-defined on-shell propagator which we can use for the calculation of the Wilson loop expectation value. This is the VEV of two fields in canonical QFT,

$$\Delta(x - y) = \langle 0|\phi(x)\phi(y)|0\rangle. \tag{3.1}$$

Here, we explicitly mean the VEV *without time-ordering*. This is not the same as the expectation value in a path integral which equals the *time-ordered* VEV,

$$\mathrm{i}\Delta_\mathrm{F}(x - y) = \langle 0|\mathrm{T}[\phi(x)\phi(y)]|0\rangle = \langle\phi(x)\phi(y)\rangle. \tag{3.2}$$

There is no obvious formulation for the VEV without time-ordering in the path-integral formalism, and thus we have to use the language of quantized fields.

Consider a real scalar field $\phi(x)$ and the Klein–Gordon equation $\partial^2\phi + m^2\phi = 0$ with mass m. The standard mode expansion for the field equation reads

$$\phi(\vec{x}, t) = \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3 2E(\vec{p})}(\exp(\mathrm{i}\vec{p}\cdot\vec{x} + \mathrm{i}E(\vec{p})t)a^\dagger(\vec{p}) + \exp(-\mathrm{i}\vec{p}\cdot\vec{x} - \mathrm{i}E(\vec{p})t)a(\vec{p})), \tag{3.3}$$

with the energy $E(\vec{p}) = +\sqrt{\vec{p}^{\,2} + m^2}$. The canonical commutator of two modes equals their VEV (without time-ordering) and reads

$$[a(\vec{p}), a^\dagger(\vec{q})] = (2\pi)^3\, 2E(\vec{p})\delta^3(\vec{p} - \vec{q}) = \langle 0|a(\vec{p})a^\dagger(\vec{q})|0\rangle. \tag{3.4}$$

The resulting VEV of two fields in position space reads

$$\Delta(\vec{x}, t) = \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3 2E(\vec{p})}\exp\big(-\mathrm{i}\vec{p}\cdot\vec{x} - \mathrm{i}E(\vec{p})t\big). \tag{3.5}$$

[7] This relation permits the insertion of a complex phase which has no impact on physical quantities.

All of the above relations are on-shell. In the massless case, there is a convenient and covariant formulation in terms of the unconstrained spinor variables $\lambda, \bar{\lambda}$. The corresponding mode expansion now reads[8]

$$\phi(x) = \frac{1}{8\pi^2}\int d^2\lambda\, d^2\bar{\lambda}\, \exp\left(-\frac{i}{2}\langle\lambda|x|\bar{\lambda}]\right)A(\lambda,\bar{\lambda}). \tag{3.6}$$

Here we integrate over real on-shell momenta, i.e. over all complex λ with $\bar{\lambda} = \pm\lambda^\dagger$, but later in section 5, it will be convenient to complexify the integration contours. The field $A(\lambda,\bar{\lambda})$ contains both the positive and negative energy modes $a(\vec{p})$ and $a^\dagger(\vec{p})$ for $\bar{\lambda} = \pm\lambda^\dagger$, and the integral is also over positive and negative energies. Furthermore, the field obeys the scaling $A(z\lambda, z^{-1}\lambda) = A(\lambda,\bar{\lambda})$. The corresponding VEV reads

$$\langle 0|A(\lambda,\bar{\lambda})A(\lambda',\bar{\lambda}')|0\rangle = \theta(E(\lambda,\bar{\lambda}))\int\frac{dz}{2\pi i z}\delta^2(\lambda'+z^{-1}\lambda)\delta^2(\bar{\lambda}'-z\bar{\lambda}). \tag{3.7}$$

Here $z = e^{i\alpha}$ is a pure complex phase. Furthermore, $E(\lambda,\bar{\lambda})$ refers to the energy described by the pair of spinors $\lambda, \bar{\lambda}$. It appears only as an argument to the step function θ. Consequently, only the sign of $\bar{\lambda} = \pm\lambda^\dagger$ is relevant, and thus the VEV remains manifestly Lorentz covariant. The resulting VEV in position space reads

$$\Delta(x) = \frac{1}{64\pi^4}\int_+ d^2\lambda\, d^2\bar{\lambda}\, \exp\left(-\frac{i}{2}\langle\lambda|x|\bar{\lambda}]\right). \tag{3.8}$$

The $+$ subscript of the integral means that we restrict the integration to positive energy by means of a factor $\theta(E(\lambda,\bar{\lambda}))$. It equals the above position-space two-point function for $m = 0$.

The above Gaussian integral can be performed easily, but proper attention should be paid to singular contributions in the imaginary part

$$\Delta(x) = -\frac{1}{4\pi^2}\frac{1}{x^2 - i\,\mathrm{sign}(x^0)\epsilon} = -\frac{1}{4\pi^2}\left(\frac{1}{x^2} + i\pi\,\mathrm{sign}(x^0)\delta(x^2)\right). \tag{3.9}$$

Note that this expression is not symmetric under $x \to -x$; due to the non-commutativity of quantum fields, this is not necessary. The Feynman propagator is the time-ordering of the same expression

$$\Delta_F(x) = -i(\theta(x^0)\Delta(x) + \theta(-x^0)\Delta(-x)) = \frac{i}{4\pi^2}\frac{1}{x^2 - i\epsilon} = \frac{i}{4\pi^2}\left(\frac{1}{x^2} + i\pi\delta(x^2)\right). \tag{3.10}$$

Curiously, the VEV differs from the Feynman propagator only by a distributional amount in position space. This fact will become important for the Wilson loop calculation. The situation in momentum space is quite different:

$$\begin{aligned}\Delta(p) &= -\frac{1}{4\pi^2}\int d^4x\frac{\exp(ix\cdot p)}{x^2 - i\,\mathrm{sign}(x^0)\epsilon} = 2\pi\theta(p_0)\delta(p^2),\\ \Delta_F(p) &= \frac{i}{4\pi^2}\int d^4x\frac{\exp(ix\cdot p)}{x^2 - i\epsilon} = \frac{1}{p^2 + i\epsilon}.\end{aligned} \tag{3.11}$$

Here the VEV is defined on-shell, while the Feynman propagator is clearly off-shell.

Our strategy for $\mathcal{N} = 4$ SYM is to derive the VEVs of gauge fields in the spinor formalism. This can be done on-shell while fully respecting the superspace constraints. The VEVs can be converted to position space, from which Feynman propagators follow. This will give us all the information needed to compute a Wilson loop expectation value at one loop.

[8] The factor of $\frac{1}{2}$ in the exponent has its origin in the identity $x\cdot y = x_\mu y^\mu = \frac{1}{2}x_{\alpha\dot{\alpha}}y^{\dot{\alpha}\alpha}$. Also, we are using a shorthand notation for spinor index contraction, as detailed in appendix A.

3.2. Gauge prepotentials

In the following we will compute the supersymmetric Wilson loop in full superspace to one-loop order. To this order, apart from a global color factor, there is no difference between the Abelian and non-Abelian theory. Therefore, it is sufficient to consider the linearized theory.

To solve the linearized version of constraints (2.21), we make an ansatz for the fermionic components of the gauge field (see also [37])

$$A_{a\alpha} = D_{a\beta}B^{\beta}{}_{\alpha} + D_{a\alpha}\Lambda, \quad \bar{A}^{a}_{\dot{\alpha}} = -\bar{D}^{a}_{\dot{\beta}}\bar{B}^{\dot{\beta}}{}_{\dot{\alpha}} + \bar{D}^{a}_{\dot{\alpha}}\Lambda, \tag{3.12}$$

in terms of a pair of chiral and antichiral prepotentials $B^{\alpha\beta}(x^+,\theta)$ and $\bar{B}^{\dot{\alpha}\dot{\beta}}(x^-,\bar{\theta})$ with symmetric indices as well as an explicit gauge transformation $\Lambda(x,\theta,\bar{\theta})$. The prepotentials B and $\bar{B}$ are Hermitian conjugates, $B^{\dagger} = \bar{B}$, while Λ is antiHermitian.

Constraint (2.21*c*) defines the bosonic components of the gauge field

$$A_{\alpha\dot{\alpha}} = \partial_{\beta\dot{\alpha}}B^{\beta}{}_{\alpha} - \partial_{\alpha\dot{\beta}}\bar{B}^{\dot{\beta}}{}_{\dot{\alpha}} + \partial_{\alpha\dot{\alpha}}\Lambda. \tag{3.13}$$

Constraints (2.21*a*) and (2.21*b*) imply that the prepotentials are chiral harmonic functions

$$D_{a}{}^{\alpha}D_{b\alpha}B_{\beta\gamma} = 0, \quad \bar{D}^{a\dot{\alpha}}\bar{D}^{b}_{\dot{\alpha}}\bar{B}_{\dot{\beta}\dot{\gamma}} = 0. \tag{3.14}$$

Applying further fermionic derivatives to these equations shows that B and $\bar{B}$ also obey the massless wave equation. Finally, together with (2.22), constraints (2.21*a*) and (2.21*b*) imply a relationship between the two prepotentials,

$$-\bar{D}^{a}_{\dot{\alpha}}\bar{D}^{b}_{\dot{\beta}}\bar{B}^{\dot{\alpha}\dot{\beta}} = \tfrac{1}{2}\varepsilon^{abcd}D_{c\alpha}D_{d\beta}B^{\alpha\beta}. \tag{3.15}$$

It is important to note that there is a redundancy in the definition of the prepotentials,

$$\delta B_{\alpha\beta} = \partial_{\alpha\dot{\beta}}\Gamma^{\dot{\beta}}_{\beta} + \partial_{\beta\dot{\beta}}\Gamma^{\dot{\beta}}_{\alpha}, \quad \delta\bar{B}_{\dot{\alpha}\dot{\beta}} = \partial_{\beta\dot{\alpha}}\bar{\Gamma}^{\beta}_{\dot{\beta}} + \partial_{\beta\dot{\beta}}\bar{\Gamma}^{\beta}_{\dot{\alpha}}, \quad \delta\Lambda = -\partial_{\alpha\dot{\alpha}}(\Gamma^{\alpha\dot{\alpha}} - \bar{\Gamma}^{\alpha\dot{\alpha}}), \tag{3.16}$$

where $\Gamma_{\alpha\dot{\alpha}}$ is a chiral harmonic function and $\bar{\Gamma}_{\alpha\dot{\alpha}} = (\Gamma_{\alpha\dot{\alpha}})^{\dagger}$ is its Hermitian conjugate. This transformation leaves the gauge potentials A and $\bar{A}$ invariant.

The prepotentials B and $\bar{B}$ have an interesting analog in the case of bosonic Yang–Mills theory. We refer to appendix B for more details.

3.3. On-shell momentum space

The prepotentials are harmonic functions on chiral superspace and thus obey the massless wave equation. They can be written as the Fourier transformation

$$\begin{aligned} B^{\alpha\beta}(x^+,\theta) &= \frac{1}{8\pi^2}\int d^2\lambda\, d^2\bar{\lambda}\, d^{0|4}\bar{\eta}\, \exp\left(\frac{i}{2}\langle\lambda|x^+|\bar{\lambda}] + \langle\lambda|\theta|\bar{\eta}]\right) C^{\alpha\beta}(\lambda,\bar{\lambda},\bar{\eta}),\\ \bar{B}^{\dot{\alpha}\dot{\beta}}(x^-,\bar{\theta}) &= \frac{1}{8\pi^2}\int d^2\lambda\, d^2\bar{\lambda}\, d^{0|4}\eta\, \exp\left(\frac{i}{2}\langle\lambda|x^-|\bar{\lambda}] - \langle\eta|\bar{\theta}|\bar{\lambda}]\right) \bar{C}^{\dot{\alpha}\dot{\beta}}(\lambda,\bar{\lambda},\eta), \end{aligned} \tag{3.17}$$

in terms of the on-shell momentum space fields $C^{\alpha\beta}(\lambda,\bar{\lambda},\bar{\eta})$ and $\bar{C}^{\dot{\alpha}\dot{\beta}}(\lambda,\bar{\lambda},\eta)$. We have used the shorthand notation $\langle\lambda|x|\bar{\lambda}] = \lambda^{\alpha}x_{\alpha\dot{\alpha}}\bar{\lambda}^{\dot{\alpha}}$, $\langle\lambda|\theta|\bar{\eta}] = \lambda^{\alpha}\theta^{a}_{\alpha}\bar{\eta}_a$ and $\langle\eta|\bar{\theta}|\bar{\lambda}] = \eta^{a}\bar{\theta}_{a\dot{\alpha}}\bar{\lambda}^{\dot{\alpha}}$. The on-shell Fourier transformation in equations (3.17) includes states with both positive and negative energies for $\bar{\lambda} = \pm\lambda^{\dagger}$. Reality conditions imply the following conjugation property of the modes:

$$C^{\alpha\beta}(\lambda,\bar{\lambda},\bar{\eta})^{\dagger} = \bar{C}^{\dot{\alpha}\dot{\beta}}(\lambda,-\bar{\lambda},\eta). \tag{3.18}$$

The harmonic constraints in (3.14) are satisfied because the two derivatives each pull a λ which are subsequently contracted to $\langle\lambda\lambda\rangle = 0$. Constraint (3.15) relates the two mode expansions

$$\bar\lambda_{\dot\alpha}\bar\lambda_{\dot\beta}\bar C^{\dot\alpha\dot\beta}(\lambda,\bar\lambda,\eta) = 4\int d^{0|4}\bar\eta \exp\left(\frac{1}{2}\eta\bar\eta\right)\lambda_\alpha\lambda_\beta C^{\alpha\beta}(\lambda,\bar\lambda,\bar\eta). \tag{3.19}$$

The above spinor integrals have the following scaling symmetry:

$$(\lambda,\bar\lambda,\eta,\bar\eta) \to (z\lambda, z^{-1}\bar\lambda, z\eta, z^{-1}\bar\eta). \tag{3.20}$$

Consequently, the fields C and $\bar C$ have to obey the scaling property

$$C^{\alpha\beta}(z\lambda, z^{-1}\bar\lambda, z^{-1}\bar\eta) = z^{-4}C^{\alpha\beta}(\lambda,\bar\lambda,\bar\eta), \quad \bar C^{\dot\alpha\dot\beta}(z\lambda, z^{-1}\bar\lambda, z\eta) = z^4\bar C^{\dot\alpha\dot\beta}(\lambda,\bar\lambda,\eta). \tag{3.21}$$

The reality conditions for spacetime with (3, 1) signature imply that $z = e^{i\phi}$ is a pure complex phase. Hence, the compact scaling symmetry merely leads to a factor of 2π in the integral and does not need to be 'gauge fixed' otherwise.

In terms of the fields $C^{\alpha\beta}$, $\bar C^{\dot\alpha\dot\beta}$, the redundancy of equation (3.16) becomes

$$\delta C^{\alpha\beta} = i\big(\lambda^\alpha\bar\lambda^{\dot\alpha}\tilde\Gamma^\beta_{\dot\alpha} + \lambda^\beta\bar\lambda^{\dot\alpha}\tilde\Gamma^\alpha_{\dot\alpha}\big), \quad \delta\bar C^{\dot\alpha\dot\beta} = i\big(\bar\lambda^{\dot\alpha}\lambda^\alpha\tilde{\bar\Gamma}^{\dot\beta}_\alpha + \bar\lambda^{\dot\beta}\lambda^\alpha\tilde{\bar\Gamma}^{\dot\alpha}_\alpha\big), \tag{3.22}$$

where $\tilde\Gamma$ and $\tilde{\bar\Gamma}$ are the Fourier transforms of Γ and $\bar\Gamma$, respectively. Note that the contractions of $\tilde\Gamma$ and $\tilde{\bar\Gamma}$ with $\bar\lambda$ and λ, respectively, leave two redundant d.o.f. in C and $\bar C$. Effectively, $C^{\alpha\beta}$ and $\bar C^{\dot\alpha\dot\beta}$ have only one physical component.

3.4. Light-cone gauge

In (3.16), we have seen that the prepotential carries some on-shell (chiral harmonic) redundant d.o.f. To eliminate them, we introduce a pair of reference spinors l^α, $\bar l^{\dot\alpha}$ defining a null vector $l^\alpha\bar l^{\dot\alpha}$. For a light-cone gauge, we impose that $l_\alpha B^{\alpha\beta} = 0$, $\bar l_{\dot\alpha}\bar B^{\dot\alpha\dot\beta} = 0$. These conditions are solved by

$$C^{\alpha\beta}(\lambda,\bar\lambda,\bar\eta) = \frac{l^\alpha l^\beta}{\langle\lambda l\rangle^2}C(\lambda,\bar\lambda,\bar\eta), \quad \bar C^{\dot\alpha\dot\beta}(\lambda,\bar\lambda,\eta) = \frac{\bar l^{\dot\alpha}\bar l^{\dot\beta}}{[\bar l\bar\lambda]^2}\bar C(\lambda,\bar\lambda,\eta), \tag{3.23}$$

where C and $\bar C$ are on-shell physical modes. The scaling property (3.21) translates to

$$C(z\lambda, z^{-1}\bar\lambda, z^{-1}\bar\eta) = z^{-2}C(\lambda,\bar\lambda,\bar\eta), \quad \bar C(z\lambda, z^{-1}\bar\lambda, z\eta) = z^2\bar C(\lambda,\bar\lambda,\eta). \tag{3.24}$$

Furthermore, they are related by constraint (3.19),

$$\bar C(\lambda,\bar\lambda,\eta) = 4\int d^{0|4}\bar\eta \exp\left(\frac{1}{2}\eta\bar\eta\right)C(\lambda,\bar\lambda,\bar\eta). \tag{3.25}$$

The fields $\bar C$ and C are also related by complex conjugation. As a consequence of (3.17) and $B^\dagger = \bar B$, we have

$$C^\dagger(\lambda,\bar\lambda,\bar\eta) = \bar C(-\lambda,\bar\lambda,-\eta) = \bar C(\lambda,-\bar\lambda,\eta), \tag{3.26}$$

where the last equality follows from the scaling symmetry in (3.24).

It is physically evident that this mode expansion is complete because for every light-like momentum given in terms of $\lambda,\bar\lambda$, the expansion of C in terms of holomorphic $\eta \in \mathbb{C}^{0|4}$ yields the desired 16 on-shell states of $\mathcal{N} = 4$ SYM. The conjugate field $\bar C$ is fully determined by C and does not carry additional d.o.f.

Note that the l-dependence in the above expressions is merely a gauge artifact. The variation of C w.r.t. the spinor l^α reads

$$\delta C^{\alpha\beta} = \frac{\langle l\delta l\rangle}{\langle l\lambda\rangle^3}(\lambda^\alpha l^\beta + \lambda^\beta l^\alpha)C, \tag{3.27}$$

where we decomposed δl on the basis l, λ. This corresponds to the redundancy of the gauge fields specified by (3.22) with

$$\tilde{\Gamma}^{\alpha\dot{\alpha}} = \frac{\mathrm{i} l^{\alpha} \bar{l}^{\dot{\alpha}} \langle l \delta l \rangle}{[\bar{\lambda}\bar{l}]\langle l\lambda\rangle^3} C. \tag{3.28}$$

The answer for $\tilde{\bar{\Gamma}}$ can be obtained by complex conjugation. Another way to see that the l-dependence is gauge is to compute the quantities $\bar{W}_{ab} = -D_{a\alpha} D_{b\beta} B^{\alpha\beta}$ and similarly for W^{ab} and note that they are independent of l and also invariant with respect to the linearized gauge transformations. For example,

$$\bar{W}_{ab} = -\frac{1}{8\pi^2} \int \mathrm{d}^2\lambda\, \mathrm{d}^2\bar{\lambda}\, \mathrm{d}^{0|4}\bar{\eta}\, \big(\bar{\eta}_a + 2\bar{\lambda}_{\dot{\alpha}} \bar{\theta}^{\dot{\alpha}}_a\big)\big(\bar{\eta}_b + 2\bar{\lambda}_{\dot{\beta}} \bar{\theta}^{\dot{\beta}}_b\big) \times \exp\left(\frac{\mathrm{i}}{2}\langle\lambda|x^+|\bar{\lambda}] + \langle\lambda|\theta|\bar{\eta}]\right) C(\lambda, \bar{\lambda}, \bar{\eta}). \tag{3.29}$$

The description of the on-shell states in $\mathcal{N} = 4$ in terms of the superfield C (or $\bar{C}$) should be related to the light-cone description by Mandelstam [38] and by Brink *et al* [39]. If we set B or $\bar{B}$ to zero (thus breaking the reality condition relating B to $\bar{B}$), we obtain an (anti-)self-dual theory. Actions for this theory with $\mathcal{N} = 4$ supersymmetry have been found in [40, 41].

3.5. Quantization

Conventionally, the quantization of a theory starts with the derivation of the propagator from the kinetic terms in the action. Unfortunately, it is far from trivial to write down an action for extended supersymmetric Yang–Mills theory, at least if supersymmetry is to be manifest. Nevertheless, we can construct a supersymmetric propagator, and show that it agrees with our expectations.

The major problem we have to face is that the linearized constraints for the gauge field force it on-shell. Consequently, we have expressed the solution to the constraints through the momentum space superfields C and $\bar{C}$ which are manifestly on-shell. We now lift the VEVs discussed in section 3.1 to the gauge prepotentials of $\mathcal{N} = 4$ SYM.

The Grassmann components of the fields C, $\bar{C}$ contain precisely the physical fields of $\mathcal{N} = 4$ SYM in light-cone gauge. Hence, we could use their VEVs to define the VEVs for the superfields C and $\bar{C}$. This is tedious, and instead we use a number of constraints that the VEV $\langle 0|CC|0\rangle$ must satisfy. It has to satisfy the momentum conservation $\lambda\bar{\lambda} + \lambda'\bar{\lambda}' = 0$. Moreover, it has to conserve the supersymmetric analog of momentum, $\lambda\bar{\eta} + \lambda'\bar{\eta}' = 0$. It has to have the right transformation under $(\lambda, \bar{\lambda}, \bar{\eta}) \to (z\lambda, z^{-1}\bar{\lambda}, z^{-1}\bar{\eta})$. Finally, it has to have the right mass dimension. A suitable expression, analogous to equation (3.7), which satisfies all the constraints is

$$\langle 0|C(\lambda, \bar{\lambda}, \bar{\eta})\, C(\lambda', \bar{\lambda}', \bar{\eta}')|0\rangle = \theta(E(\lambda, \bar{\lambda})) \int \frac{\mathrm{d}z}{2\pi \mathrm{i} z^3}\, \delta^2(\lambda' + z^{-1}\lambda)\, \delta^2(\bar{\lambda}' - z\bar{\lambda})\, \delta^{0|4}(\bar{\eta}' - z\bar{\eta}). \tag{3.30}$$

Here the integral is over a pure complex phase $z = \mathrm{e}^{\mathrm{i}\phi}$.

By (3.25), the VEV of a C field and a $\bar{C}$ field is

$$\langle 0|C(\lambda, \bar{\lambda}, \bar{\eta})\bar{C}(\lambda', \bar{\lambda}', \eta')|0\rangle = 4\theta(E(\lambda, \bar{\lambda})) \int \frac{\mathrm{d}z}{2\pi \mathrm{i} z^3} \delta^2(\lambda' + z^{-1}\lambda)\delta^2(\bar{\lambda}' - z\bar{\lambda}) \exp\left(\frac{1}{2} z\eta'\bar{\eta}\right). \tag{3.31}$$

By using (3.25) again, we find that the VEV of two $\bar{C}$ is, as expected, similar to that of two C,

$$\langle 0|\bar{C}(\lambda,\bar{\lambda},\eta)\bar{C}(\lambda',\bar{\lambda}',\eta')|0\rangle = \theta(E(\lambda,\bar{\lambda}))\int \frac{z\,\mathrm{d}z}{2\pi\mathrm{i}}\,\delta^2(\lambda'+z^{-1}\lambda)\delta^2(\bar{\lambda}'-z\bar{\lambda})\delta^{0|4}(\eta'+z^{-1}\eta), \tag{3.32}$$

which is a consistency check for our $\langle 0|CC|0\rangle$.

Let us test that this choice for the $\langle CC\rangle$ propagator yields the results we expect, by computing the scalar two-point functions. To do this computation, note that since $\phi_{ab}(x) = \bar{W}_{ab}(x,\theta=0,\bar{\theta}=0)$, we obtain

$$\langle 0|\phi_{ab}(x)\phi_{cd}(x')|0\rangle = \frac{1}{64\pi^4}\int \mathrm{d}^2\lambda\,\mathrm{d}^2\bar{\lambda}\mathrm{d}^{0|4}\bar{\eta}\,\mathrm{d}^2\lambda'\,\mathrm{d}^2\bar{\lambda}'\,\mathrm{d}^{0|4}\bar{\eta}'\;\bar{\eta}_a\bar{\eta}_b\bar{\eta}'_c\bar{\eta}'_d$$
$$\times\exp\left(\frac{\mathrm{i}}{2}\langle\lambda|x|\bar{\lambda}] + \frac{\mathrm{i}}{2}\langle\lambda'|x'|\bar{\lambda}']\right)\langle 0|C(\lambda,\bar{\lambda},\bar{\eta})C(\lambda',\bar{\lambda}',\bar{\eta}')|0\rangle. \tag{3.33}$$

Using usual manipulations for the delta functions, we can show that this two-point function is

$$\langle 0|\phi_{ab}(x)\phi_{cd}(x')|0\rangle = \frac{-1}{4\pi^2}\frac{\varepsilon_{abcd}}{(x-x')^2} + \text{singular support}, \tag{3.34}$$

which is the expected result (the missing distributional terms are given in equation (3.9)). This computation also allows us to fix the normalization of the CC two-point function.

4. Wilson loop expectation value

We now turn to the calculation of the one-loop expectation value of a null polygonal Wilson loop $\mathcal{W}[C]$ in full superspace,

$$\frac{g^2N}{64\pi^2}M^{(1)}[C] = \frac{1}{N}\oint_C\oint_C{}' \frac{1}{2}\langle \mathrm{Tr}\,AA'\rangle. \tag{4.1}$$

In the following, we will not explicitly write down the factors of g^2N, since they can easily be restored when needed. Here C is the contour of a null polygon in full $\mathcal{N}=4$ superspace (see [42]), and A and A' denote one copy of the gauge connection for each of the two integrals. At this perturbative level, one needs only a two-point correlation function which we obtained in the previous section. Interaction vertices are not needed. Furthermore, the color algebra can be performed to reduce the computation to the Abelian case.

4.1. Chiral decomposition

As a first step we write the gauge connection as a differential form on superspace and substitute the prepotential ansatz discussed in section 3.2,

$$\begin{aligned}A &= \tfrac{1}{2}E^{\dot{\alpha}\alpha}A_{\alpha\dot{\alpha}} + E^{a\alpha}A_{a\alpha} + E^{\dot{\alpha}}_a\bar{A}^a_{\dot{\alpha}}\\ &= \tfrac{1}{2}\left(\mathrm{d}x^{\dot{\alpha}\alpha} - 2\mathrm{i}\mathrm{d}\theta^{a\alpha}\bar{\theta}^{\dot{\alpha}}_a - 2\mathrm{i}\mathrm{d}\bar{\theta}^{\dot{\alpha}}_a\theta^{a\alpha}\right)\left(-\partial_{\alpha\dot{\beta}}\bar{B}^{\dot{\beta}}{}_{\dot{\alpha}} + \partial_{\beta\dot{\alpha}}B^{\beta}{}_{\alpha} + \partial_{\alpha\dot{\alpha}}\Lambda\right)\\ &\quad + \mathrm{d}\theta^{a\alpha}\left(D_{a\beta}B^{\beta}{}_{\alpha} + D_{a\alpha}\Lambda\right) - \mathrm{d}\bar{\theta}^{\dot{\alpha}}_a\left(-\bar{D}^a_{\dot{\beta}}\bar{B}^{\dot{\beta}}{}_{\dot{\alpha}} + \bar{D}^a_{\dot{\alpha}}\Lambda\right).\end{aligned} \tag{4.2}$$

We use the relations[9]

$$\mathrm{d}(x^{\pm})^{\dot{\alpha}\alpha} = \mathrm{d}x^{\dot{\alpha}\alpha} \pm 2\mathrm{i}\mathrm{d}\theta^{a\alpha}\bar{\theta}^{\dot{\alpha}}_a \mp 2\mathrm{i}\mathrm{d}\bar{\theta}^{\dot{\alpha}}_a\theta^{a\alpha}, \tag{4.3}$$

[9] We use the notation $\partial^{\pm}_{\alpha\dot{\alpha}} \equiv \sigma^{\mu}_{\alpha\dot{\alpha}}\partial/\partial(x^{\pm})^{\mu}$.

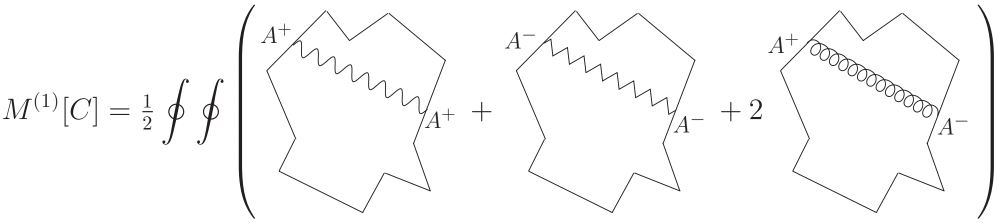

Figure 1. The one-loop expectation value of a Wilson loop, including chiral, antichiral and mixed-chirality contributions.

$$D_{a\alpha} B^{\alpha}{}_{\beta}(x^+,\theta) = \left(\frac{\partial}{\partial\theta^{a\alpha}} + 2\mathrm{i}\bar\theta^{\dot\alpha}_a \partial^+_{\alpha\dot\alpha}\right) B^{\alpha}{}_{\beta}(x^+,\theta), \tag{4.4}$$

$$\bar D^a_{\dot\alpha}\bar B^{\dot\alpha}{}_{\dot\beta}(x^-,\bar\theta) = \left(-\frac{\partial}{\partial\bar\theta^{\dot\alpha}_a} - 2\mathrm{i}\theta^{a\alpha}\partial^-_{\alpha\dot\alpha}\right)\bar B^{\dot\alpha}{}_{\dot\beta}(x^-,\bar\theta), \tag{4.5}$$

to simplify the connection

$$\begin{aligned} A &= A^+ + A^- + \mathrm{d}\Lambda, \\ A^+ &= \frac{1}{2}\mathrm{d}(x^+)^{\dot\alpha\alpha}\partial^+_{\beta\dot\alpha}B^{\beta}{}_{\alpha} + \mathrm{d}\theta^{a\alpha}\frac{\partial}{\partial\theta^{a\beta}}B^{\beta}{}_{\alpha}, \end{aligned} \tag{4.6}$$

$$A^- = -\frac{1}{2}\mathrm{d}(x^-)^{\dot\alpha\alpha}\partial^-_{\alpha\dot\beta}\bar B^{\dot\beta}{}_{\dot\alpha} - \mathrm{d}\bar\theta^{\dot\alpha}_a\frac{\partial}{\partial\bar\theta^{\dot\beta}_a}\bar B^{\dot\beta}{}_{\dot\alpha}. \tag{4.7}$$

So we see that the connection nicely splits into a connection on the chiral and antichiral parts of the full superspace and a non-chiral gauge transformation which has no impact on closed Wilson loops.

For the one-loop Wilson loop expectation value, this implies three terms, as illustrated in figure 1,

$$M^{(1)}[C] = M^{(1)}_{++}[C] + M^{(1)}_{--}[C] + 2M^{(1)}_{+-}[C], \qquad \frac{1}{64\pi^2}M^{(1)}_{\pm\pm'}[C] = \oint_C\oint_C{}' \tfrac{1}{2}\langle A^{\pm}A'^{\pm'}\rangle. \tag{4.8}$$

The three types of contributions above have different forms which will not mix. In particular, they can be distinguished by a charge counting the number of θ minus the number of $\bar\theta$.

The former two terms in the above equation are fully chiral or antichiral, respectively; they depend only on the projections of the Wilson loop onto the chiral or antichiral subspaces of superspace. The chiral part of the result, by construction, agrees with the expectation value of the supersymmetric Wilson loop proposed in [22, 23]. It is going to be a finite rational function. The antichiral part is (almost[10]) the complex conjugate of the chiral part. The latter term in the above equation is mixed chiral; it depends non-trivially on all superspace coordinates. The bosonic truncation of this part, by construction, agrees with the expectation value of a Wilson loop in ordinary spacetime, see [15, 16]. It is going to be a divergent function of transcendentality 2 (Li_2, $\log^2$).

[10] The imaginary part of the Feynman propagator causes some subtle distributional discrepancy due to unitarity.

4.2. Use of correlators

Conventionally, Wilson loop expectation values $\langle W_C\rangle$ are evaluated in the path integral. In particular, the two-point correlation function $\langle AA'\rangle$ translates to a Feynman propagator Δ_{F}. The Feynman propagator almost obeys the equation of motion (e.o.m.) of the corresponding field. Importantly, however, the e.o.m. are violated at coincident points where a delta distribution remains. A Feynman propagator is off-shell. This is a mostly negligible effect in position space, where Wilson loop expectation values are ordinarily computed. For our supersymmetric Wilson loop, it puts us in a slightly inconvenient position: on the one hand, the gauge connection has to be constrained in such a way that the e.o.m. are implied. The fields must obey the e.o.m. On the other hand, Feynman propagators are intrinsically off-shell. More concretely, the field C defined in (3.17) exists only for $p^2 = 0$, whereas the Feynman propagator is of the form $1/p^2$. In the following we shall explain how to resolve the apparent clash.

First of all, there is nothing that prevents us from performing the calculation in position space. For illustration purposes, we shall use the example of a scalar field ϕ instead of the full-fledged gauge connection on superspace. We can compute the two-point correlator of two fields (3.1),

$$\Delta(x-y) = \langle 0|\phi(x)\phi(y)|0\rangle = -\frac{1}{4\pi^2}\left(\frac{1}{x^2} + \mathrm{i}\pi\,\mathrm{sign}(x^0)\delta(x^2)\right). \tag{4.9}$$

The corresponding Feynman propagator (3.2) can be derived from the two-point correlator by a simple manipulation (3.10)

$$\begin{aligned}\mathrm{i}\Delta_{\mathrm{F}}(x-y) &= \langle 0|\mathrm{T}[\phi(x)\phi(y)]|0\rangle = \theta(x^0)\Delta(x) + \theta(-x^0)\Delta(-x)\\ &= -\frac{1}{4\pi^2}\left(\frac{1}{x^2} + \mathrm{i}\pi\delta(x^2)\right).\end{aligned} \tag{4.10}$$

This construction extends without further ado to superspace, and can be applied to the calculation of the Wilson loop expectation value $\langle W\rangle$.

There is another option at our disposal: if we blindly replace the Feynman propagator $\mathrm{i}\Delta_{\mathrm{F}}$ by the two-point correlator Δ, we actually compute $\langle 0|W|0\rangle$ which is different from $\langle W\rangle$. The difference between the two is computed via the difference

$$\mathrm{i}\Delta_{\mathrm{F}}(x) - \Delta(x) = -\frac{\mathrm{i}}{2\pi}\theta(-x^0)\delta(x^2). \tag{4.11}$$

This difference is localized to the light cone, and it is purely imaginary. It is similar to the cut discontinuity of the Feynman propagator,

$$\mathrm{disc}\,\mathrm{i}\Delta_{\mathrm{F}}(x) = -\frac{\mathrm{i}}{2\pi}\delta(x^2), \tag{4.12}$$

which yields the cut discontinuity of the ordinary Wilson loop expectation value $\langle W\rangle$. The latter is well known to be a simpler function (usually of one degree of transcendentality less). One can convince oneself that the same applies to the difference[11].

We will be satisfied with computing the most complicated part (highest transcendentality) of the Wilson loop expectation value $\langle W\rangle$. Consequently, we can instead compute $\langle 0|W|0\rangle$ by replacing the Feynman propagators $\mathrm{i}\Delta_{\mathrm{F}}$ by the two-point correlators Δ. Then all the correlators are perfectly on-shell, and the constraints on the superspace connection fully apply. Alternatively, we could decide to compute the discontinuity $\mathrm{disc}\langle W\rangle$. The cut of the Feynman propagator $\mathrm{disc}\,\mathrm{i}\Delta_{\mathrm{F}}$ is another perfectly on-shell quantity. Eventually $\langle W\rangle$ is recovered from a dispersion integral on $\mathrm{disc}\langle W\rangle$.

[11] The result of $\langle 0|W|0\rangle$ depends on the choice of operator ordering in W. The totally symmetrized ordering actually yields precisely $\mathrm{disc}\,\mathrm{i}\Delta_{\mathrm{F}}(x)$; hence, $\langle 0|W|0\rangle = \mathrm{Re}\langle W\rangle$ in this case.

A final option may be to Fourier transform the obtained Feynman propagators from position space to momentum space[12]. Here one would have to understand in how far the constraints on the superspace connection apply and can be used for simplifications.

In this paper, we shall mainly adopt the calculation of $\langle 0|W|0\rangle$. We will then use the mode expansion (3.17) of the on-shell fields in terms of the spinor-helicity field C. This will allow us to shortcut the calculation substantially. For a position space calculation, see appendix C.

4.3. Vertex correlators

The shape of the Wilson loop is a null polygon in superspace [42], i.e. a sequence of points $(x_j, \theta_j, \bar\theta_j)$ which are joined by null lines.

For the null line that joins the vertices j and $j+1$, we define λ_j, $\bar\lambda_j$ by $x^{\dot\alpha\alpha}_{j,j+1} = \lambda^\alpha_j \bar\lambda^{\dot\alpha}_j$, where $x_{j,j+1}$ is the superspace interval as defined in equation (2.7). The null line can then be parametrized as follows:

$$x^{\dot\alpha\alpha} = x^{\dot\alpha\alpha}_j + \tau\lambda^\alpha_j\bar\lambda^{\dot\alpha}_j + 2\mathrm{i}\lambda^\alpha_j\sigma^a\bar\theta^{\dot\alpha}_{j,a} - 2\mathrm{i}\theta^{\alpha a}_j\bar\sigma_a\bar\lambda^{\dot\alpha}_j, \quad \theta^{a\alpha} = \theta^{a\alpha}_j + \lambda^\alpha_j\sigma^a, \quad \bar\theta^{\dot\alpha}_a = \bar\theta^{\dot\alpha}_{j,a} + \bar\sigma_a\bar\lambda^{\dot\alpha}_j. \tag{4.13}$$

Here τ is a bosonic coordinate, and $\sigma, \bar\sigma$ are four additional complex fermionic coordinates. The null line is 'fat'; it is a 1|8-dimensional subspace of superspace. The Wilson line a 1|0-dimensional curve on the null line. The restrictions on the gauge field curvature (2.21) imply that the precise choice of curve does not matter [43, 44]. A Wilson line only depends on the start and end points $(x_j, \theta_j, \bar\theta_j)$ and $(x_{j+1}, \theta_{j+1}, \bar\theta_{j+1})$. We can thus pick any $\sigma(\tau), \bar\sigma(\tau)$ that interpolates between the vertices j and $j+1$. This implies $\sigma(0) = 0$, $\sigma(1) = \bar\eta_i$, $\bar\sigma(0) = 0$, $\bar\sigma(1) = \eta_i$.

Correspondingly, the gauge connection $A = A^+ + A^- + \mathrm{d}\Lambda$ (4.6) is a total derivative when restricted to the null line (4.13)[13]

$$A^+_j = \mathrm{d}G^+_j, \quad A^-_j = \mathrm{d}G^-_j. \tag{4.14}$$

Using the definition of A^+ and A^- in terms of B and $\bar B$ (3.12), the mode expansion (3.17) and the light-cone gauge condition (3.23), a quick computation shows that $G^\pm$ has a solution in the closed form,

$$G^+_j(x^+, \theta) = \frac{1}{8\pi^2}\int \mathrm{d}^2\lambda\, \mathrm{d}^2\bar\lambda\, \mathrm{d}^{0|4}\bar\eta \exp\left(\frac{\mathrm{i}}{2}\langle\lambda|x^+|\bar\lambda] + \langle\lambda|\theta|\bar\eta]\right)\frac{\langle jl\rangle}{\langle\lambda l\rangle\langle\lambda j\rangle}C(\lambda, \bar\lambda, \bar\eta), \tag{4.15}$$

$$G^-_j(x^-, \bar\theta) = \frac{1}{8\pi^2}\int \mathrm{d}^2\lambda\, \mathrm{d}^2\bar\lambda\, \mathrm{d}^{0|4}\eta \exp\left(\frac{\mathrm{i}}{2}\langle\lambda|x^-|\bar\lambda] - \langle\eta|\bar\theta|\bar\lambda]\right)\frac{[j\bar l]}{[\bar\lambda\bar l][\bar\lambda j]}\bar C(\lambda, \bar\lambda, \eta). \tag{4.16}$$

The Wilson loop integral can now be written as a sum of potential shifts over the edges of the polygon

$$\oint A = \sum_{j=1}^n \Delta G_j, \quad \Delta G_j = G_j(x_{j+1}, \theta_{j+1}, \bar\theta_{j+1}) - G_j(x_j, \theta_j, \bar\theta_j). \tag{4.17}$$

[12] Fourier transforms of full superspace are cumbersome due to superspace torsion: the fermionic momenta anticommute onto the bosonic momentum, and momentum space would be non-commutative. However, the prepotentials B are chiral and a flat chiral momentum space does exist.

[13] Obviously, this is a classical statement which depends very much on the classical e.o.m. to hold. In our case, we can rely on the linearized classical e.o.m. because the two-point correlator is perfectly on-shell (and even the Feynman propagator is on-shell except for coincident points whose contributions are minute).

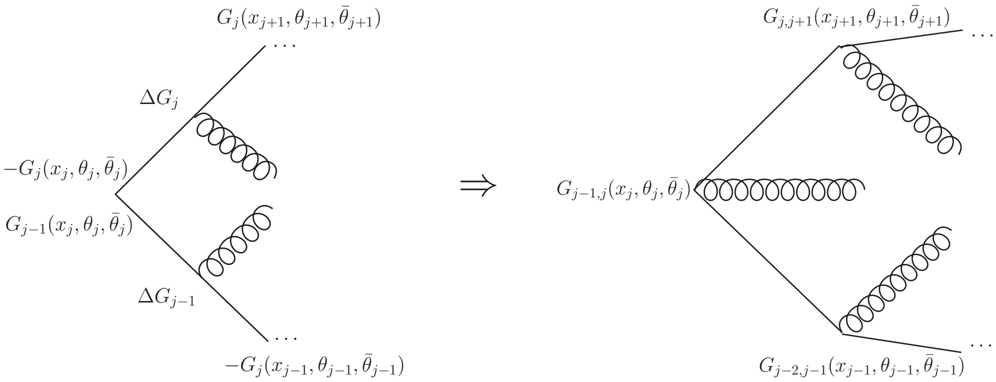

Figure 2. Rearrange the sum of potential shifts over the edges of the polygon, to a sum over shifts at the vertices.

Now there is an interesting rearrangement of the sum

$$\oint A = \sum_{j=1}^{n} G_{j-1,j}, \qquad G_{j-1,j} = G_{j-1}(x_j,\theta_j,\bar\theta_j) - G_j(x_j,\theta_j,\bar\theta_j), \tag{4.18}$$

which expresses the Wilson loop as a sum over potential shifts at the vertices, see figure 2. The latter read

$$\begin{aligned} G^+_{j-1,j} &= \frac{1}{8\pi^2}\int d^2\lambda\, d^2\bar\lambda\, d^{0|4}\bar\eta \exp\left(\frac{i}{2}\langle\lambda|x^+_j|\bar\lambda] + \langle\lambda|\theta_j|\bar\eta]\right)\frac{\langle j{-}1\,j\rangle}{\langle j{-}1\lambda\rangle\langle\lambda j\rangle}C(\lambda,\bar\lambda,\bar\eta),\\ G^-_{j-1,j} &= \frac{1}{8\pi^2}\int d^2\lambda\, d^2\bar\lambda\, d^{0|4}\eta \exp\left(\frac{i}{2}\langle\lambda|x^-_j|\bar\lambda] - \langle\eta|\bar\theta_j|\bar\lambda]\right)\frac{[j{-}1\,j]}{[j{-}1\bar\lambda][\bar\lambda j]}\bar C(\lambda,\bar\lambda,\eta). \end{aligned} \tag{4.19}$$

At first sight it may be surprising to see that the dependence on the light-cone gauge reference vector l has dropped out from $G_{j-1,j}$. In fact, the reason is simply that $G_{j-1,j}$ is localized at vertex j, and changes of the gauge must cancel between the contributions G_{j-1} and G_j. It is evident that the early cancellation of gauge artifacts will substantially simplify the subsequent calculation.

For the Wilson loop expectation value (4.1), we thus have two equivalent representations,

$$\frac{1}{64\pi^2}M^{(1)}_n = \oint\oint' \frac{1}{2}\langle AA'\rangle = \sum_{j,k=1}^{n}\frac{1}{2}\langle\Delta G_j\Delta G_k\rangle = \sum_{j,k=1}^{n}\frac{1}{2}\langle G_{j-1,j}G_{k-1,k}\rangle. \tag{4.20}$$

The former uses a sum over edge correlators, the latter a sum over vertex correlators; the latter will be more convenient to use.

Note that along the lines of the discussion in section 4.2, we shall replace the expectation value in (4.20) by a VEV. This allows us to perform the calculation using on-shell fields in the first place. Secondly, according to section 4.1, the expectation values split into three terms of different chirality. The resulting one-loop expectation thus reads (see figure 3)

$$\frac{1}{64\pi^2}M^{(1)}_n = \sum_{j,k=1}^{n}\left(\frac{1}{2}\langle 0|G^+_{j-1,j}G^+_{k-1,k}|0\rangle + \frac{1}{2}\langle 0|G^-_{j-1,j}G^-_{k-1,k}|0\rangle + \langle 0|G^+_{j-1,j}G^-_{k-1,k}|0\rangle\right). \tag{4.21}$$

In the following, we shall consider the chiral and the mixed-chiral contributions by substituting the vertex gauge potential shifts $G_{j-1,j}$, evaluating the correlators and performing the integrals.

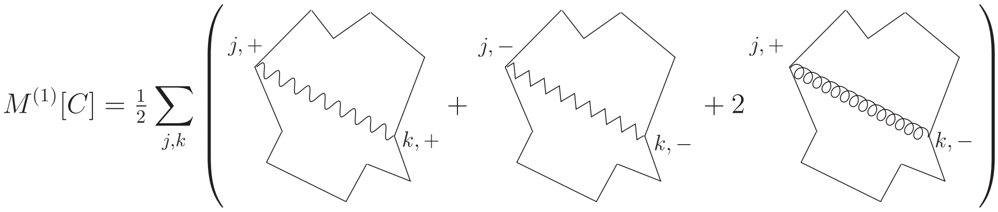

Figure 3. The one-loop expectation value of a Wilson loop, including sums of chiral, antichiral and mixed-chirality vertex correlators.

Note that the vertex correlators also play an important role in twistor space calculations. As we shall see in section 5.1, in twistor space, each vertex corresponds to an edge connecting two adjacent ambitwistors.

4.4. Chiral correlator

In this section, we compute the expectation values of the chiral–chiral (or equivalently antichiral–antichiral) vertex shifts of equation (4.21). The remaining mixed-chiral expectation values are computed in the next section.

Using the two-point function of two C fields, we obtain the following result for the two-point function of two G^+ fields:

$$\langle 0|G^+_{j-1,j}G^+_{k-1,k}|0\rangle = \frac{1}{64\pi^4}\int_+ \mathrm{d}^2\lambda\,\mathrm{d}^2\bar\lambda\,\mathrm{d}^{0|4}\bar\eta \\ \times\exp\left(\frac{\mathrm{i}}{2}\langle\lambda|x^+_{k,j}|\bar\lambda] + \langle\lambda|\theta_{k,j}|\bar\eta]\right)\frac{\langle j-1\,j\rangle}{\langle j-1\,\lambda\rangle\langle j\lambda\rangle}\frac{\langle k-1\,k\rangle}{\langle k-1\,\lambda\rangle\langle k\lambda\rangle}. \tag{4.22}$$

Then, we multiply the numerator and the denominator of the integrand by $[\bar\lambda\bar\rho]^4$ and make a change of variable $\bar\eta = \bar\zeta[\bar\lambda\bar\rho]$ to obtain

$$\langle 0|G^+_{j-1,j}(x^+_j,\theta_j)G^+_{k-1,k}(x^+_k,\theta_k)|0\rangle = \frac{1}{64\pi^4}\int_+ \mathrm{d}^2\lambda\,\mathrm{d}^2\bar\lambda\,\mathrm{d}^{0|4}\bar\zeta \\ \times\exp\left(\frac{\mathrm{i}}{2}\langle\lambda|x^+_{k,j}|\bar\lambda] - \langle\lambda|\theta_{k,j}|\bar\zeta][\bar\rho\bar\lambda]\right)\frac{\langle j-1\,j\rangle}{\langle j-1\,\lambda\rangle\langle j\lambda\rangle[\bar\lambda\bar\rho]^2}\frac{\langle k-1\,k\rangle}{\langle k-1\,\lambda\rangle\langle k\lambda\rangle[\bar\lambda\bar\rho]^2}. \tag{4.23}$$

Now consider the following differential operators:

$$\mathcal{D}_\ell = -\mathrm{i}\langle\ell|\sigma^\mu|\bar\rho]\frac{\partial}{\partial x^{+\,\mu}_{k,j}}. \tag{4.24}$$

These differential operators have been designed to cancel the λ and $\bar\lambda$ dependence in the denominator of the integrand in equation (4.24). Since

$$\mathcal{D}_\ell\left(\frac{\mathrm{i}}{2}\langle\lambda|x^+_{k,j}|\bar\lambda]\right) = \langle\ell\lambda\rangle[\bar\rho\bar\lambda], \tag{4.25}$$

we have that

$$\mathcal{D}_{j-1}\mathcal{D}_j\mathcal{D}_{k-1}\mathcal{D}_k\langle 0|G^+_{j-1,j}G^+_{k-1,k}|0\rangle \\ = \langle j-1\,j\rangle\langle k-1\,k\rangle\int \mathrm{d}^{0|4}\bar\zeta\left(-\frac{1}{4\pi^2}\right)\frac{1}{(x^+_{k,j}+2\mathrm{i}\theta_{k,j}|\bar\zeta][\bar\rho|)^2}. \tag{4.26}$$

The integral over $\bar{\zeta}$ can be done as follows:

$$\int \mathrm{d}^{0|4}\bar{\zeta}\,\frac{1}{\left(x^+_{k,j}+[\bar{\rho}|\bar{\sigma}\theta_{k,j}|\bar{\zeta}]\right)^2} = \int \mathrm{d}^{0|4}\bar{\zeta}\,\exp(\mathrm{i}[\bar{\rho}|\bar{\sigma}^{\mu}\theta_{k,j}|\bar{\zeta}]\partial_\mu)\frac{1}{\left(x^+_{k,j}\right)^2}$$
$$= 2^4 4!\,\frac{\delta^{0|4}(\theta_{k,j}|x^+_{k,j}|\bar{\rho}])}{\left(\left(x^+_{k,j}\right)^2\right)^5}, \tag{4.27}$$

where for the first equality we have used a translation operator applied to $1/\left(x^+_{k,j}\right)^2$ and for the second equality, we have expanded the exponential.

Now we want to find another expression which gives the same result when acted upon by the product $\mathcal{D}_{j-1}\mathcal{D}_j\mathcal{D}_{k-1}\mathcal{D}_k$ of differential operators. This seems to be very hard, but consider the action on $1/\left(x^+_{k,j}\right)^2$. It is straightforward to show that

$$\mathcal{D}_{j-1}\mathcal{D}_j\mathcal{D}_{k-1}\mathcal{D}_k\frac{1}{\left(x^+_{k,j}\right)^2} = 2^4 4!\,\frac{\langle j-1|x^+_{k,j}|\bar{\rho}]\langle j|x^+_{k,j}|\bar{\rho}]\langle k-1|x^+_{k,j}|\bar{\rho}]\langle k|x^+_{k,j}|\bar{\rho}]}{\left(\left(x^+_{k,j}\right)^2\right)^5}. \tag{4.28}$$

If we use the fact that $\mathcal{D}_\ell\langle\lambda|x^+_{k,j}|\bar{\rho}] = 0$ and $\mathcal{D}_\ell\theta_{k,j}|x^+_{k,j}|\bar{\rho}] = 0$ for any λ and ℓ, we get that

$$\mathcal{D}_{j-1}\mathcal{D}_j\mathcal{D}_{k-1}\mathcal{D}_k\left(-\frac{1}{4\pi^2}\frac{\langle j-1\,j\rangle\langle k-1\,k\rangle\delta^{0|4}(\theta_{k,j}|x^+_{k,j}|\bar{\rho}])}{\left(x^+_{k,j}\right)^2\langle j-1|x^+_{k,j}|\bar{\rho}]\langle j|x^+_{k,j}|\bar{\rho}]\langle k-1|x^+_{k,j}|\bar{\rho}]\langle k|x^+_{k,j}|\bar{\rho}]}\right)$$
$$= \mathcal{D}_{j-1}\mathcal{D}_j\mathcal{D}_{k-1}\mathcal{D}_k\langle 0|G^+_{j-1,j}G^+_{k-1,k}|0\rangle. \tag{4.29}$$

In conclusion, up to terms which vanish under the action of $\mathcal{D}_{j-1}\mathcal{D}_j\mathcal{D}_{k-1}\mathcal{D}_k$, we have

$$\langle 0|G^+_{j-1,j}\,G^+_{k-1,k}|0\rangle = -\frac{1}{4\pi^2}\frac{\langle j-1\,j\rangle\langle k-1\,k\rangle\delta^{0|4}(\theta_{k,j}|x^+_{k,j}|\bar{\rho}])}{\left(x^+_{k,j}\right)^2\langle j-1|x^+_{k,j}|\bar{\rho}]\langle j|x^+_{k,j}|\bar{\rho}]\langle k-1|x^+_{k,j}|\bar{\rho}]\langle k|x^+_{k,j}|\bar{\rho}]}. \tag{4.30}$$

The right-hand side of equation (4.30) is the spacetime form of the R-invariant of the points (x^+_{j-1},θ_{j-1}), (x^+_j,θ_j), (x^+_{k-1},θ_{k-1}), (x^+_k,θ_k) and a reference spinor $\bar{\rho}$. The equivalence of this spacetime form of the R-invariant and the twistor form is explicitly shown in appendix D (a proof can also be found in [45]).

One may be surprised by the appearance of $\bar{\rho}$ on the right-hand side, when there is no such dependence on the left-hand side of equation (4.30). However, the dependence on $\bar{\rho}$ can be interpreted as an integration constant. Indeed, using identities between R-invariants, one can show that the difference between the expressions on the right-hand side of equation (4.30) for two different values of $\bar{\rho}$ is annihilated by $\mathcal{D}_{j-1}\mathcal{D}_j\mathcal{D}_{k-1}\mathcal{D}_k$.[14]

If we now sum up the contributions of all the chiral–chiral vertex correlators, with the same reference spinor $\bar{\rho}$, we find the NMHV scattering amplitude in the form obtained by Mason and Skinner in [22]. In section 5, we will see that the analogous computation in twistor space is more straightforward.

The antichiral correlator between the vertices j and k, $\langle G^-_{j-1,j}G^-_{k-1,k}\rangle$, is given by the conjugate R-invariant, which depends on a conjugate reference spinor ρ.

[14] The difference is a linear combination of four R-invariants, each depending on only three of the four points $j-1$, j, $k-1$, k.

4.5. Mixed chirality correlator

Using the two-point function of a C and a $\bar{C}$ field, we find, after performing some trivial integrations, that

$$\langle 0|G^{+}_{j-1,j}G^{-}_{k-1,k}|0\rangle = -\frac{1}{256\pi^4}\int_{+} \mathrm{d}^2\lambda\, \mathrm{d}^2\bar{\lambda}\, \exp\left(-\frac{\mathrm{i}}{2}\langle\lambda|x^{+-}_{j,k}|\bar{\lambda}]\right)\frac{\langle j-1 j\rangle[k-1k]}{\langle j-1\lambda\rangle\langle\lambda j\rangle[k-1\bar{\lambda}][\bar{\lambda}k]}, \tag{4.31}$$

where $x^{+-}_{j,k} = x^{-}_{k} - x^{+}_{j} + 4\mathrm{i}\theta_j\bar{\theta}_k$ (2.8).

It is not obvious how to compute these integrals directly, but we can use the fact that the answer satisfies differential equations with simple source terms. The solution to these differential equations is not unique, but there is a discrete symmetry that fixes the coefficient of the homogeneous solution.

It is convenient to use (4.31) and find differential operators with respect to the components of $x^{+-}_{j,k}$ in the basis $(\lambda_{j-1}, \lambda_j; \bar{\lambda}_{k-1}, \bar{\lambda}_k)$,

$$a = \langle j{-}1|x^{+-}_{j,k}|k{-}1], \quad b = \langle j|x^{+-}_{j,k}|k{-}1], \quad c = \langle j{-}1|x^{+-}_{j,k}|k], \quad d = \langle j|x^{+-}_{j,k}|k]; \tag{4.32}$$

then

$$\left(x^{+-}_{j,k}\right)^2 = \frac{1}{2}\frac{ad-bc}{\langle j{-}1 j\rangle[k{-}1k]}. \tag{4.33}$$

We also have that

$$\langle\lambda|x^{+-}_{j,k}|\bar{\lambda}] = \frac{\langle\lambda j\rangle[\bar{\lambda}k]a}{\langle j{-}1 j\rangle[k{-}1k]} + \frac{\langle j{-}1\lambda\rangle[\bar{\lambda}k]b}{\langle j{-}1 j\rangle[k{-}1k]} + \frac{\langle\lambda j\rangle[k{-}1\bar{\lambda}]c}{\langle j{-}1 j\rangle[k{-}1k]} + \frac{\langle j{-}1\lambda\rangle[k{-}1\bar{\lambda}]d}{\langle j{-}1 j\rangle[k{-}1k]}. \tag{4.34}$$

Integral (4.31) only depends on $x^{+-}_{j,k}$ through $\exp(-\frac{\mathrm{i}}{2}\langle\lambda|x^{+-}_{j,k}|\bar{\lambda}])$, and differentiating with respect to a gives a factor $-\frac{\mathrm{i}}{2}\langle\lambda j\rangle[\bar{\lambda}k]/\langle j-1j\rangle[k-1k]$, etc. Therefore, the second-order differential operator with respect to a, d or b, c removes all brackets in the denominator

$$\begin{aligned}\frac{\partial^2}{\partial a\partial d}\langle 0|G^{+}_{j-1,j}G^{-}_{k-1,k}|0\rangle &= \frac{\partial^2}{\partial b\partial c}\langle 0|G^{+}_{j-1,j}G^{-}_{k-1,k}|0\rangle \\ &= \frac{1}{1024\pi^4}\frac{1}{\langle j{-}1 j\rangle[k{-}1k]}\int_{+}\mathrm{d}^2\lambda\,\mathrm{d}^2\bar{\lambda}\,\exp\left(\frac{\mathrm{i}}{2}\langle\lambda|x^{+-}_{j,k}|\bar{\lambda}]\right).\end{aligned} \tag{4.35}$$

The differential operators reduce the integral to a simpler one, which is nothing but the momentum representation of the scalar propagator,

$$\int_{+}\mathrm{d}^2\lambda\,\mathrm{d}^2\bar{\lambda}\,\exp\left(\frac{\mathrm{i}}{2}\langle\lambda|x|\bar{\lambda}]\right) = -16\pi^2\frac{1}{x^2}. \tag{4.36}$$

Therefore, we have

$$\frac{\partial^2}{\partial a\partial d}\langle 0|G^{+}_{j-1,j}G^{-}_{k-1,k}|0\rangle = \frac{\partial^2}{\partial b\partial c}\langle 0|G^{+}_{j-1,j}G^{-}_{k-1,k}|0\rangle = -\frac{1}{64\pi^2}\frac{1}{ad-bc}. \tag{4.37}$$

A solution to these two differential equations is easily found by integration in terms of dilogarithms and logarithms,

$$\frac{\partial^2}{\partial a\partial d}\left(-\mathrm{Li}_2\frac{bc}{ad} + \frac{1}{2}\log(ad)\log\frac{bc}{ad}\right) = \frac{\partial^2}{\partial b\partial c}\left(-\mathrm{Li}_2\frac{bc}{ad} + \frac{1}{2}\log(ad)\log\frac{bc}{ad}\right) = \frac{1}{bc-ad}. \tag{4.38}$$

Besides an additive constant which is not very interesting, there is an ambiguity in solving the equations in the class of transcendentality-2 functions. It corresponds to adding a factor of $\log(ad)\log(bc)$ times a rational number. Such a term is annihilated by both second-order

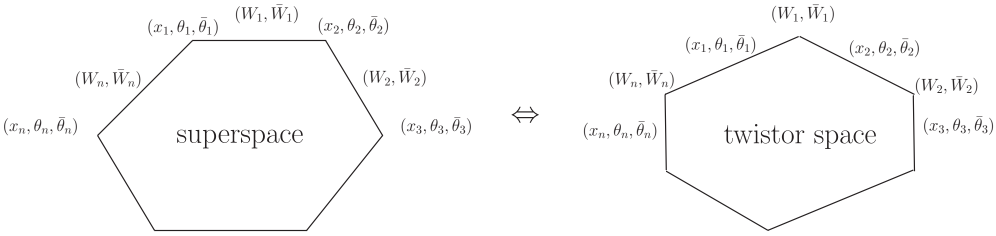

Figure 4. A null polygon in superspace and the dual polygon in twistor space.

differential operators we considered. To fix the coefficient, it suffices to demand that the expression is antisymmetric with respect to interchanges of λ_{j-1} and λ_j or $\bar{\lambda}_{k-1}$ and $\bar{\lambda}_k$. This obvious symmetry of integral (4.31) should be reflected in the integrand (up to shifts by constants which we neglect).

In conclusion, the resulting integral reads

$$64\pi^2\langle 0|G^+_{j-1,j}G^-_{k-1,k}|0\rangle = -\,\mathrm{Li}_2\left(\frac{\langle j-1|x^{+-}_{j,k}|k]\langle j|x^{+-}_{j,k}|k-1]}{\langle j-1|x^{+-}_{j,k}|k-1]\langle j|x^{+-}_{j,k}|k]}\right)$$
$$+\frac{1}{2}\log\left(\langle j-1|x^{+-}_{j,k}|k-1]\langle j|x^{+-}_{j,k}|k]\right)\log\left(\frac{\langle j-1|x^{+-}_{j,k}|k]\langle j|x^{+-}_{j,k}|k-1]}{\langle j-1|x^{+-}_{j,k}|k-1]\langle j|x^{+-}_{j,k}|k]}\right). \quad (4.39)$$

We should note that if the points j and k become too close, then the answer in equation (4.39) becomes divergent. This is how the UV divergences of the Wilson loop manifest themselves. In section 6, we will discuss some ways to regularize these divergences.

It ought to be mentioned that the above expression is not invariant under rescaling of the spinor variables. It is however reassuring to observe that in the sum over all vertices, this dependence drops out. This cancellation depends crucially on the correct choice of coefficient for the homogeneous solution of the above differential equations.

5. Twistor space calculation

The above results for the vertex correlators have convenient expressions in terms of twistor variables. Here we present our twistor conventions, translate our above results and show how the calculations can be cut short if performed directly in twistor space.

5.1. Ambitwistors

The Wilson loop is a sequence of null lines. In section 4.3, we specified these null lines through the polygon vertices. A useful alternative description of a null line is through an ambitwistor $(W, \bar{W})$ [46]. Consequently, the Wilson loop contour is also specified through a sequence of ambitwistors, see figure 4. We will now specify these twistor variables for the polygon and spell out their relations. See [42] for more details of the construction.

The twistor equations $\mu_{\dot{\alpha}} = \frac{1}{4}\lambda^\alpha x^+_{\alpha\dot{\alpha}}$, $\chi^a = \lambda^\alpha\theta^a_\alpha$ and $\bar{\mu}_\alpha = \frac{1}{4}x^-_{\alpha\dot{\alpha}}\bar{\lambda}^{\dot{\alpha}}$, $\bar{\chi}_a = \bar{\theta}_{\dot{\alpha}a}\bar{\lambda}^{\dot{\alpha}}$ for $(x, \theta, \bar{\theta})$ define a null line. They are solved precisely by the explicit parametrization of null lines given in (4.13). It then makes sense to collect the quantities $\mu_{\dot{\alpha}}$, χ^a, $\bar{\mu}_\alpha$, $\bar{\chi}_a$ in the twistor variables $W = (-\mathrm{i}\lambda^\alpha, \mu_{\dot{\alpha}}, \chi^a)$ and $\bar{W} = (\bar{\mu}_\alpha, \mathrm{i}\bar{\lambda}^{\dot{\alpha}}, \bar{\chi}_a)$ which transform nicely as projective vectors under the superconformal group $\mathrm{PSL}(2, 2|4)$.

For the null segment connecting the vertices j and $j+1$, we shall use the ambitwistor $(W_j, \bar{W}_j)$. More explicitly, we define λ_j, $\bar{\lambda}_j$ by $x_{j,j+1} = \lambda_j \bar{\lambda}_j$, where $x_{j,j+1}$ is a superspace interval as defined in equation (2.7). Also, we set

$$\begin{aligned} \mu_j &:= \tfrac{1}{4}\langle j|x_j^+ = \tfrac{1}{4}\langle j|x_{j+1}^+, & \bar{\mu}_j &:= \tfrac{1}{4}x_j^-|j] = \tfrac{1}{4}x_{j+1}^-|j], \\ \chi_j &:= \langle j|\theta_j = \langle j|\theta_{j+1}, & \bar{\chi}_j &:= \bar{\theta}_j|j] = \bar{\theta}_{j+1}|j]; \end{aligned} \tag{5.1}$$

then the twistors and dual twistors have the components

$$W_i = \big(-\mathrm{i}\lambda_i^\alpha, \mu_{i\dot{\alpha}}, \chi_i^a\big), \qquad \bar{W}_i = \big(\bar{\mu}_{i\alpha}, \mathrm{i}\bar{\lambda}_i^{\dot{\alpha}}, \bar{\chi}_{ia}\big). \tag{5.2}$$

The scalar product $\langle j, k] := W_j \cdot \bar{W}_k$ between a twistor and dual twistor is defined by

$$\langle j, k] = -\mathrm{i}\lambda_j^\alpha \bar{\mu}_{k\alpha} + \mathrm{i}\mu_{j\dot{\alpha}}\bar{\lambda}_k^{\dot{\alpha}} + \chi_j^a \bar{\chi}_{ka} = -\frac{\mathrm{i}}{4}\langle j|(x_k^- - x_j^+ + 4i\theta_j\bar{\theta}_k)|k] = -\frac{\mathrm{i}}{4}\langle j|x_{j,k}^{+-}|k]. \tag{5.3}$$

The relation $x^+ - x^- = 4\mathrm{i}\theta\bar{\theta}$ along with the relations in (5.1) implies the following identities:

$$\langle j, j-1] = \langle j, j] = \langle j, j+1] = 0. \tag{5.4}$$

In general, the answers become simpler in twistor language. For instance, the mixed-chiral vertex correlator (4.39) has the following simple expression in terms of twistor products:

$$64\pi^2 \langle 0|G_{j-1,j}^+ G_{k-1,k}^-|0\rangle = -\operatorname{Li}_2 X_{j,k} + \tfrac{1}{2}\log\langle j-1, k-1]\langle j, k]\log X_{j,k}, \tag{5.5}$$

where we have defined the twistor cross-ratios

$$X_{j,k} = \frac{\langle j-1, k]\langle j, k-1]}{\langle j-1, k-1]\langle j, k]}, \tag{5.6}$$

which are invariant under rescaling of any of the involved twistors as well as under superconformal transformations.

5.2. Correlators

Next we transform the on-shell momentum space fields $C(\lambda, \bar{\lambda}, \bar{\eta})$ and $\bar{C}(\lambda, \bar{\lambda}, \eta)$ to twistor space. There are several reasons to do this. First of all, we hope to obtain the tree-level NMHV amplitude, which is most naturally expressed in twistor space. Moreover, the one-loop amplitudes also have a simple form in twistor space. Another reason to study the transformation to twistor space is the fact that the superconformal symmetry becomes more obvious in this language.

An immediate drawback of the twistor transformation is that it is hard to define properly in Minkowski signature. Typically, one Wick rotates to the $++--$ signature or complexifies spacetime altogether. The resulting expressions remain meaningful after this transformation. Unfortunately, integration contours are not obvious anymore and would have to be specified in order to make sense of most integrals. We will not elaborate on the choice (or existence) of contours in this paper.

We use the following definitions for the twistor transforms of the mode expansions C and $\bar{C}$[15]:

$$C(\lambda, \mu, \chi) \equiv \int \mathrm{d}^2\bar{\lambda}\, \mathrm{d}^{0|4}\bar{\eta}\, \exp(-2\mathrm{i}[\mu\bar{\lambda}] + \chi\bar{\eta}) C(\lambda, \bar{\lambda}, \bar{\eta}), \tag{5.7}$$

$$\bar{C}(\bar{\lambda}, \bar{\mu}, \bar{\chi}) \equiv \int \mathrm{d}^2\lambda\, \mathrm{d}^{0|4}\eta\, \exp(-2\mathrm{i}\langle\lambda\bar{\mu}\rangle - \eta\bar{\chi})\, \bar{C}(\lambda, \bar{\lambda}, \eta). \tag{5.8}$$

[15] The dimension of λ (used to represent null momenta) is not the same as the dimension of λ_j (used to represent null distances). Due to the projective nature of twistors, this difference stays without consequences.

Relation (3.25) translates to a relation between the twistor fields,

$$\bar{C}(\bar{W}) = \frac{1}{(2\pi)^2}\int d^{4|4}W \exp(2W\cdot\bar{W})C(W). \tag{5.9}$$

Prepotentials (3.17) in light-cone gauge (3.23) also find a simple expression in terms of the twistor fields,

$$\begin{aligned} B^{\alpha\beta}(x^+,\theta) &= \frac{1}{8\pi^2}\int \frac{\langle\lambda d\lambda\rangle l^\alpha l^\beta}{\langle\lambda l\rangle^2} C\left(\lambda, \frac{1}{4}\lambda x^+, \lambda\theta\right), \\ \bar{B}^{\dot\alpha\dot\beta}(x^-,\bar\theta) &= \frac{1}{8\pi^2}\int \frac{[\bar\lambda d\bar\lambda]\bar l^{\dot\alpha}\bar l^{\dot\beta}}{[\bar l\bar\lambda]^2}\bar C\left(\bar\lambda, \frac{1}{4}x^-\bar\lambda, \bar\theta\bar\lambda\right). \end{aligned} \tag{5.10}$$

The above expressions are integrals over a contour in $\mathbb{CP}^1$ which are the twistor duals of the points (x^+,θ), $(x^-,\bar\theta)$ in chiral or antichiral superspace. As described in more detail in [42], a point in full superspace corresponds in complexified ambitwistor space to a $\mathbb{CP}^1\times\mathbb{CP}^1$. Each of the $\mathbb{CP}^1$ factors can be seen as the twistor (or conjugate twistor) associated with the points in chiral (or antichiral space).

Finally, we need to transform the two-point correlator (3.30) to twistor space. Here, the main complication is the restriction to positive energies in integrals $\int_+$, which makes sense only in Minkowski signature, but not in split signature or complexified spacetime. Simply dropping the step function is not an option because in integrals the negative energy contributions typically cancel most of the positive energy contributions, and the result would almost vanish[16].

We cast the step function to the form of a Fourier integral

$$\theta(x) = \frac{1}{2\pi i}\int \frac{dt}{t-i\epsilon}\exp(ixt), \tag{5.11}$$

which can be taken to a different signature up to a suitable choice of integration contour. Furthermore, the energy $E(\lambda,\bar\lambda)$ is not a convenient expression in twistor space. As we are only interested in distinguishing the positive from the negative light cone, we can safely replace the energy E by a light-cone energy given by $E_{\text{l.c.}} = -2\langle\lambda\rho\rangle[\bar\rho\bar\lambda]$ where the spinors ρ, $\bar\rho$ describe a reference null direction. In other words, we replace

$$\theta(E(\lambda,\bar\lambda)) \to \frac{1}{2\pi i}\int\frac{dt}{t}\exp\big(-2it\langle\lambda\rho\rangle[\bar\rho\bar\lambda]\big), \tag{5.12}$$

and obtain for the twistor space the correlation function

$$\langle 0|C(\lambda,\mu,\chi)C(\lambda',\mu',\chi')|0\rangle = -\frac{1}{4}\int\frac{dt}{t}\frac{dz}{z}\delta^2(\lambda+z\lambda')\delta^2(\mu+z\mu'+t\langle\lambda\rho\rangle\bar\rho)\delta^{0|4}(\chi+z\chi'). \tag{5.13}$$

By rescaling the integration variables, we end up with a neat twistor space expression for the chiral and antichiral correlators[17],

$$\begin{aligned} \langle 0|C(W)C(W')|0\rangle &= -\frac{1}{4}\int\frac{ds}{s}\frac{dt}{t}\delta^{4|4}(sW+tW'+(W\cdot\bar W_\star)W_\star), \\ \langle 0|\bar C(\bar W)\bar C(\bar W')|0\rangle &= -\frac{1}{4}\int\frac{ds}{s}\frac{dt}{t}\delta^{4|4}(s\bar W+t\bar W'+(W_\star\cdot\bar W)\bar W_\star), \end{aligned} \tag{5.14}$$

where $W_\star = (0,\bar\rho,0)$ and $\bar W_\star = (\rho,0,0)$ are the reference twistors. Remarkably, this is the propagator of the twistor field in the axial gauge, as shown by Mason and Skinner in [22]. It has support when the twistors W, W' and $W_\star$ lie on a common projective line.

[16] According to the discussion in section 4.2, dropping the step function amounts to computing the discontinuity on the expectation value. For Wilson loops, the discontinuity usually has one degree of transcendentality less.
[17] It is tempting to scale away $\langle\lambda\rho\rangle \simeq W\cdot\bar W_\star$ as well, but such a rescaling would obscure the conjugation relation (5.9) between C and $\bar C$, and may have other undesired side effects.

The mixed-chiral correlator in twistor space reads

$$\langle 0|C(W)\bar{C}(\bar{W}')|0\rangle = -\frac{1}{16\pi^2}\int \frac{\mathrm{d}s}{s}\frac{\mathrm{d}t}{t}\exp(sW\cdot\bar{W}' + t(W\cdot\bar{W}_\star)(W_\star\cdot\bar{W}')). \tag{5.15}$$

Corresponding to the above observation, this expression might serve as the mixed-chiral propagator in an ambitwistor theory.

5.3. Vertex correlators

Here we will compute the vertex correlators directly in twistor space. First we transform the shift of gauge potential at a vertex (4.19),

$$\begin{aligned} G^+_{j-1,j} &= \frac{1}{8\pi^2}\int \frac{\langle\lambda \mathrm{d}\lambda\rangle\langle j{-}1\, j\rangle}{\langle j{-}1\,\lambda\rangle\langle\lambda j\rangle} C\left(\lambda, \frac{1}{4}\lambda x^+_j, \lambda\theta_j\right), \\ G^-_{j-1,j} &= -\frac{1}{8\pi^2}\int \frac{[\bar\lambda \mathrm{d}\bar\lambda][j{-}1\, j]}{[j{-}1\,\bar\lambda][\bar\lambda j]} \bar{C}\left(\bar\lambda, \frac{1}{4}x^-_j\bar\lambda, \bar\theta_j\bar\lambda\right). \end{aligned} \tag{5.16}$$

We expand $\lambda, \bar\lambda$ as $\lambda = \lambda_{j-1} + u\lambda_j$, $\bar\lambda = \bar\lambda_{j-1} + v\bar\lambda_j$ and use identities (5.1) to find the following twistor space representation:

$$G^+_{j-1,j} = \frac{1}{8\pi^2}\int \frac{\mathrm{d}u}{u} C(W_{j-1} + uW_j), \quad G^-_{j-1,j} = -\frac{1}{8\pi^2}\int \frac{\mathrm{d}v}{v}\bar{C}(\bar{W}_{j-1} + v\bar{W}_j). \tag{5.17}$$

Now it is straightforward to compute the chiral correlator between the two vertices j and k which reads after some rescaling of integration variables

$$\langle 0|G^+_{j-1,j}G^+_{k-1,k}|0\rangle = -\frac{1}{256\pi^4}\int \frac{\mathrm{d}\alpha}{\alpha}\frac{\mathrm{d}\beta}{\beta}\frac{\mathrm{d}\gamma}{\gamma}\frac{\mathrm{d}\delta}{\delta}\delta^{4|4}(\alpha W_{j-1} + \beta W_j + \gamma W_{k-1} + \delta W_k + W_\star). \tag{5.18}$$

The antichiral correlator is simply the conjugate expression. We recognize this as the correlator between the two edges j and k in twistor space. This is expected since each vertex in spacetime corresponds to a line in twistor space, and $W_\star$ we have here corresponds to the reference twistor in the axial gauge form of the propagator in twistor space.

Now we proceed to the mixed-chirality correlator between two vertices. It reads simply

$$\langle 0|G^+_{j-1,j}G^-_{k-1,k}|0\rangle = \frac{1}{1024\pi^6}\int \frac{\mathrm{d}u}{u}\frac{\mathrm{d}v}{v}\frac{\mathrm{d}s}{s}\frac{\mathrm{d}t}{t}\exp(sW_u\cdot\bar{W}_v + t(W_u\cdot\bar{W}_\star)(W_\star\cdot\bar{W}_v)), \tag{5.19}$$

where $W_u := W_{j-1} + uW_j$ and $\bar{W}_v := \bar{W}_{k-1} + v\bar{W}_k$. It would be desirable to show that this multiple integral evaluates to (4.39). We have not made serious attempts in this direction, but it appears that a careful consideration of integration contours may be required to prove the equivalence.

6. Regularizations

From now on, we will consider only the mixed-chirality contributions since the purely chiral contributions are rational, finite and equal to the well-known counterparts in the chiral Wilson loop.

Now that we have the vertex correlator, we need to sum over all pairs of vertices as in equation (4.21),

$$\frac{1}{64\pi^2}M^{(1)}_{n,+-} = \sum_{j,k=1}^{n}\langle G^+_{j-1,j}G^-_{k-1,k}\rangle. \tag{6.1}$$

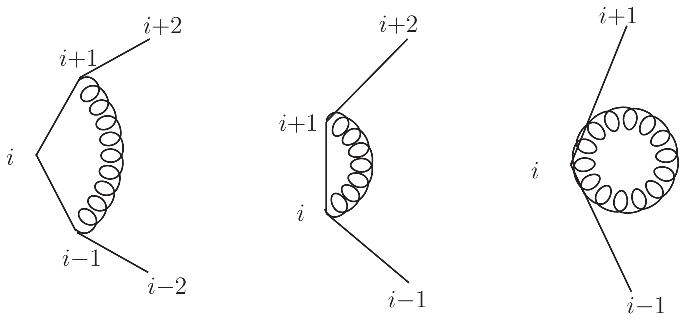

Figure 5. Diagrams for divergent mixed-chirality correlators.

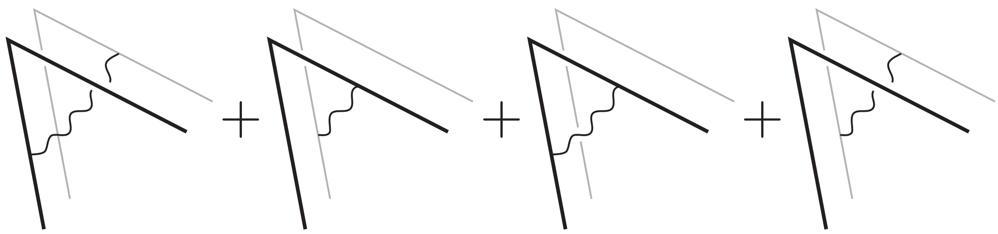

Figure 6. The one-loop expectation value of two Wilson loops. The initial Wilson loop contour C is represented by a solid line, while the contour of its displaced copy C' is represented by a gray line. There are four kinds of contributions to the one-loop expectation value of $\langle\mathcal{W}[C]\mathcal{W}[C']\rangle$, but when dividing by $\langle\mathcal{W}[C]\rangle\langle\mathcal{W}[C']\rangle$, the contribution of the last two diagrams is canceled.

However, it is easy to see that the Li_2 term diverges for $|k-j|<2$, so does the log log term for $|k-j|<3$, see figure 5. In these cases, either a regularization or a finite quantity to be extracted from the full answer is needed, and there are various ways to do it as we discuss now.

6.1. Framing

One way to regularize the one-loop result is to frame the Wilson loop. By shifting each vertex of the null polygon C by any vector[18], which preserves the null condition, we have a shifted null polygon, C', and we consider the ratio

$$\frac{\langle\mathcal{W}[C]\mathcal{W}[C']\rangle}{\langle\mathcal{W}[C]\rangle\langle\mathcal{W}[C']\rangle}. \tag{6.2}$$

At one loop, it is equivalent to the sum

$$M^{(1)}_{\text{framing}}[C,C'] := \tfrac{1}{2}M^{(1)}[C,C'] - \tfrac{1}{2}M^{(1)}[C] - \tfrac{1}{2}M^{(1)}[C'], \tag{6.3}$$

which is given by (one half) the sum of correlators between edges (or vertices) of C and edges (or vertices) of C' (see figure 6),

$$\begin{aligned}\frac{1}{64\pi^2}M^{(1)}_{\text{framing}}[C,C'] &= \frac{1}{2}\oint_C\oint_{C'}(\langle A^+A^-\rangle + \langle A^-A^+\rangle)\\ &= \frac{1}{2}\sum_{j\in C,k\in C'}(\langle G^+_{j-1,j}G^-_{k-1,k}\rangle + \langle G^-_{j-1,j}G^+_{k-1,k}\rangle).\end{aligned} \tag{6.4}$$

[18] In order to obtain the right branch cut structure, the safest option is to take the shift to be space-like.

Note that this is symmetric under the exchange of C and C'. The contributions of chiral–antichiral and antichiral–chiral vertex–vertex correlators in equation (6.4) differ by quantities which vanish when the contours C and C' become coincident. If we discard such vanishing terms in the following, we can use

$$\frac{1}{64\pi^2} M^{(1)}_{\text{framing}}[C, C'] = \sum_{j\in C, k\in C'} \langle G^+_{j-1,j} G^-_{k-1,k}\rangle. \tag{6.5}$$

Since the contour C is specified by a sequence of momentum ambitwistors $W_k, \bar{W}_k$, the contour C' can be specified by the shifted momentum ambitwistors $W'_k, \bar{W}'_k$. A particular choice is to shift all twistors (conjugates) along the same direction of a reference twistor W_* (conjugate $\bar{W}_*$), with $\langle *, *] \neq 0$,

$$W'_k = W_k + \mathrm{i}\epsilon \frac{\langle k, *]}{\langle *, *]} W_*, \qquad \bar{W}'_k = \bar{W}_k - \mathrm{i}\epsilon \frac{\langle *, k]}{\langle *, *]} \bar{W}_*, \tag{6.6}$$

for which, up to $O(\epsilon^2)$ terms, indeed we have $W'_k\bar{W}'_k = W'_k\bar{W}'_{k\pm1} = 0$.

The correlators between well-separated points have a finite limit as the framing goes away ($\epsilon \to 0$). The divergent terms are regularized simply by replacing

$$\langle j, k] \to \mathrm{i}\epsilon \frac{\langle j, *]\langle *, k]}{\langle *, *]} := \epsilon \langle j, k]^* \tag{6.7}$$

for $k = j, j\pm1$. Then, the superconformal cross-ratios defined in equation (5.6) are regularized as follows:

$$X_{j,j\pm2} \to \epsilon X^*_{j,j\pm2}, \quad X_{j,j\pm1} \to \epsilon^{-1} X^*_{j,j\pm1}, \quad X_{j,j} \to 1. \tag{6.8}$$

Explicitly, the regularized one-loop expectation value using W_*–$\bar{W}_*$ framing is given by[19]

$$\begin{aligned} M^{(1)}_{n,*} = \sum_{j=1}^{n} \Bigg\{ &-\log^2\epsilon + \log\epsilon \log \frac{\langle j+1, j-1]\langle j-1, j+1]}{\langle j+1, j]^*\langle j, j+1]^*} \\ &+ \sum_{k=j+3}^{j-3} \left(-\mathrm{Li}_2 X_{j,k} + \frac{1}{2}\log\big(\langle j-1, k-1]\langle j, k]\big) \log X_{j,k} \right) \\ &+ \sum_{k=j\pm2} \frac{1}{2} \log\big(\langle j-1, k-1]\langle j, k]\big) \log X^*_{j,k} \\ &+ \sum_{k=j\pm1} \frac{1}{2} \log\big(\langle j, k-1]^{(*)}\langle j-1, k]^{(*)}\big) \log X^*_{j,k} \Bigg\} + \mathcal{O}(\epsilon), \end{aligned} \tag{6.9}$$

where we have neglected terms with $k = j$ since they simply give constants like $\zeta(2)$ which we are not careful about. We have checked that the weight in each of the twistors and conjugates vanishes. Roughly speaking, the W_*–$\bar{W}_*$ framing can be viewed as an axial regularization, which breaks superconformal symmetry explicitly by the axial directions, W_* and $\bar{W}_*$.

6.2. Super-Poincaré

Instead of the reference twistors W_* and $\bar{W}_*$, we can try to use the matrix corresponding to the infinity twistor I for regularization purposes. The antisymmetric matrix I projects any twistor (or conjugate twistor) to its λ (or $\bar{\lambda}$) component; thus

$$W I W' = \langle\lambda\lambda'\rangle, \qquad \bar{W} I \bar{W}' = [\bar{\lambda}\bar{\lambda}']. \tag{6.10}$$

The matrix I breaks superconformal symmetry down to super-Poincaré symmetry.

[19] Just as for the symbol described in [47], from here on we shall not be careful with the signs of the arguments of logarithm functions. This amounts to a choice of branch cuts.

A motivation for introducing the spinor brackets for regularization is that they arise naturally in the dimensional reduction scheme which preserves super-Poincaré symmetry. By fully supersymmetrizing the bosonic result [33] using twistor/spinor brackets, we can propose a super-Poincaré invariant expression for the regularized one-loop expectation value. At the moment we have no first-principles derivation for the following expression, it remains a guess[20]:

$$\begin{aligned} M^{(1)}_{n,I} = &-\sum_{j=1}^{n}\left(\frac{2}{\epsilon^2}+\frac{2}{\epsilon}\log\frac{\langle j-1, j+1]\langle j+1, j-1]}{\mu^2\langle j, j+1\rangle[j, j+1]}\right) \\ &-\sum_{j=1}^{n}\sum_{k=j+3}^{j-3}\mathrm{Li}_2\,X_{j,k}-\frac{1}{2}\sum_{j=1}^{n}\left(\log^2\frac{\langle j, j+1\rangle}{[j, j+1]}-\log^2\frac{\langle j-1, j\rangle}{[j, j+1]}-\log^2\frac{[j-1, j-2]}{\langle j, j-1\rangle}\right) \\ &-\frac{1}{2}\sum_{j=1}^{n}\left(\log^2\frac{\langle j-1, j+1]}{\mu[j, j+1]}+\log^2\frac{\mu[j-1, j-2]}{\langle j, j-2]}+\log^2\frac{\langle j, j+2]}{\mu\langle j, j+1\rangle}+\log^2\frac{\mu\langle j, j-1\rangle}{\langle j, j-2]}\right) \\ &-\frac{1}{2}\sum_{j=1}^{n}\left(\sum_{k=j+2}^{j-3}\log^2\frac{\langle j-1, k]}{\langle j, k]}+\sum_{k=j+3}^{j-2}\log^2\frac{\langle j, k]}{\langle j, k-1]}-\sum_{k=j+2}^{j-2}\log^2\frac{\langle j-1, k-1]}{\langle j, k]}\right) \\ &+\frac{1}{2}\gamma\sum_{j=1}^{n}\log^2\frac{\langle j-1, j+1]\langle j, j+1\rangle[j-1, j]}{\langle j+1, j-1]\langle j-1, j\rangle[j, j+1]}. \end{aligned} \tag{6.11}$$

Note that there is some freedom in supersymmetrizing the bosonic result. Requiring proper scaling for all twistors and for all conjugate twistors yields some constraints that guided us to the above result. We note that the structure multiplied by the coefficient γ has proper twistor scaling, reduces to zero when dropping fermionic coordinates and obeys some discrete symmetries. It also does not modify the well-defined finite correlator to be obtained in section 6.3. Hence, we have no means to fix the coefficient γ, but for simplicity we will subsequently set it to zero.

We can also take the derivative of (6.11), which is essentially its polylogarithm symbol,

$$\begin{aligned} \delta M^{(1)}_{n,I} = &\sum_{j=1}^{n}\left(+\frac{2}{\epsilon}+\log\frac{[j, j+1]\langle j+1, j-1]\langle j, j+2]}{\mu^2\langle j, j+1\rangle[j, j-1][j+1, j+2]}\right)\delta\log\langle j, j+1\rangle \\ &+\sum_{j=1}^{n}\left(+\frac{2}{\epsilon}+\log\frac{\langle j, j+1\rangle\langle j-1, j+1]\langle j+2, j]}{\mu^2[j, j+1]\langle j-1, j\rangle\langle j+2, j+1\rangle}\right)\delta\log[j, j+1] \\ &+\sum_{j=1}^{n}\left(-\frac{2}{\epsilon}+\log\frac{\mu^2\langle j-1, j+2]\langle j, j+1\rangle[j+1, j+2]\langle j, j+3]}{\langle j-1, j+1]\langle j, j+2]\langle j, j+2]\langle j+1, j+3]}\right)\delta\log\langle j, j+2] \\ &+\sum_{j=1}^{n}\left(-\frac{2}{\epsilon}+\log\frac{\mu^2\langle j+1, j-2]\langle j, j-1\rangle[j-1, j-2]\langle j, j-3]}{\langle j+1, j-1]\langle j, j-2]\langle j, j-2]\langle j-1, j-3]}\right)\delta\log\langle j, j-2] \\ &+\sum_{j=1}^{n}\sum_{k=j+3}^{j-3}\log\frac{X_{j,k}X_{j+1,k+1}(1-X_{j+1,k})(1-X_{j,k+1})}{(1-X_{j,k})(1-X_{j+1,k+1})}\delta\log\langle j, k]. \end{aligned} \tag{6.12}$$

[20] We should note that the divergent part is similar to the one in framing regularization (6.9). Also, the divergent part is a bit more complicated than in the bosonic case. In particular, it depends on odd variables as well as next-to-adjacent twistors.

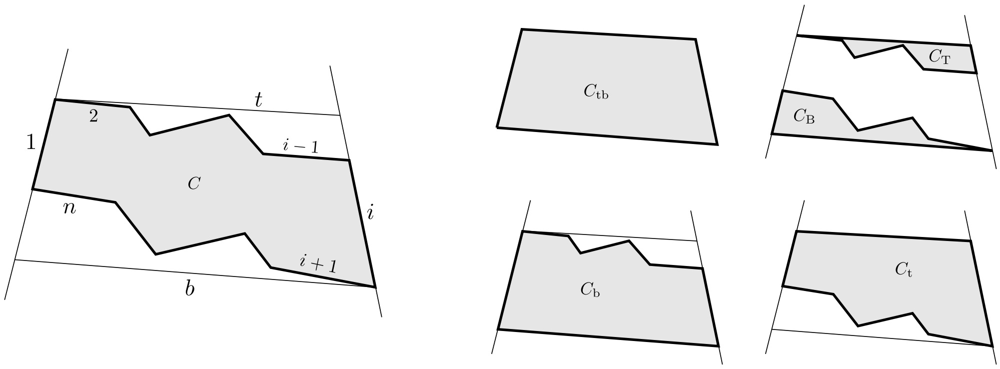

Figure 7. The boxing regularization. We have labeled the momenta of the initial polygonal contour from 1 to n. The vertices v_i are at the intersection of the edges labeled by $i-1$ and i and have the coordinates $(x_i, \theta_i, \bar{\theta}_i)$. The new sides t and b are light-like. We represent the six relevant contours $C, C_{\rm t}, C_{\rm b}, C_{\rm tb}, C_{\rm T}$ and $C_{\rm B}$.

It is straightforward to see that the result is a supersymmetrization of the regularized bosonic one-loop expectation value. For $|j-k| \geqslant 3$, by discarding all fermionic components, the combination

$$\frac{(1-X_{j+1,k})(1-X_{j,k+1})X_{j,k}X_{j+1,k+1}}{(1-X_{j,k})(1-X_{j+1,k+1})} = \frac{(x^{+-}_{j+1,k})^2(x^{+-}_{j,k+1})^2}{(x^{+-}_{j,k})^2(x^{+-}_{j+1,k+1})^2} := u^{+-}_{j,k} \tag{6.13}$$

reduces to the bosonic cross-ratio $u_{j,k} := (x^2_{j+1,k}x^2_{j,k+1})/(x^2_{j+1,k+1}x^2_{j,k})$, and $\langle j,k] \to \langle jk{-}1kk{+}1\rangle/\langle k{-}1k\rangle\langle kk{+}1\rangle$; thus, the term reduces to the derivative of the finite correlator between the edges j and k,

$$\log u_{j,k}\delta \log \frac{\langle jk{-}1kk{+}1\rangle}{\langle k{-}1k\rangle\langle kk{+}1\rangle}. \tag{6.14}$$

Terms with $k = j\pm 1, j\pm 2$ depend on I, and they also reduce to the derivative of regularized terms in the bosonic result. Note that $\langle j{\mp}1, j{\pm}1] \to \langle j{\mp}1, j\rangle[j, j{\pm}1]$.

6.3. Boxing

Finally, similar to [48] in the bosonic case, we can use the following 'boxing' procedure to extract a finite and superconformal quantity of the one-loop expectation value, as shown in figure 7. It is a prescription to compute a finite object, which is not a simple Wilson loop, and we call it the 'boxed Wilson loop'. This prescription, when applied to Wilson loops in any regularization scheme, should yield the same answer, as we will confirm below.

First we pick two edges, say 1 and i, and extend them from v_1 and v_i to two new vertices, which are then connected to v_{i+1} and v_2 by two additional null edges b and t, respectively (see figure 7). Then the boxed Wilson loop is defined as a combination of four Wilson loop expectation values,

$$\frac{\langle \mathcal{W}[C]\rangle\langle \mathcal{W}[C_{\rm tb}]\rangle}{\langle \mathcal{W}[C_{\rm t}]\rangle\langle \mathcal{W}[C_{\rm b}]\rangle}, \tag{6.15}$$

where we have specified the four polygons C, $C_{\rm tb}$, $C_{\rm t}$ and $C_{\rm b}$ by listing the twistors, including

$$W_{\rm t} = W_1 - \frac{\langle 1, i]}{\langle 2, i]}W_2, \qquad W_{\rm b} = W_i - \frac{\langle i, 1]}{\langle i{+}1, 1]}W_{i+1}, \tag{6.16}$$

and similarly for conjugate twistors[21]. At one loop, the combination reduces to the following remainder function:

$$\begin{aligned} r :&= M^{(1)}[C] + M^{(1)}[C_{\rm tb}] - M^{(1)}[C_{\rm t}] - M^{(1)}[C_{\rm b}] \\ &= M^{(1)}(1,\ldots,n) + M^{(1)}(1,{\rm t},i,{\rm b}) - M^{(1)}(1,{\rm t},i,\ldots) - M^{(1)}(1,\ldots,i,{\rm b}). \end{aligned} \tag{6.17}$$

By (4.8), the one-loop mixed-chirality expectation value is given by a double integral along the null polygonal contour C,

$$\frac{1}{64\pi^2} M^{(1)}[C] = \oint_C \oint_C \langle A^+ A^- \rangle; \tag{6.18}$$

thus, the boxed Wilson loop at one loop is given by[22]

$$r = \oint_{C_{\rm T}} \oint_{C_{\rm B}} \left(\langle A^+ A^- \rangle + \langle A^- A^+ \rangle \right) = \sum_{\substack{j\in C_{\rm T}\\ k\in C_{\rm B}}} \left(\langle G^+_{j-1,j} G^-_{k-1,k} \rangle + \langle G^-_{j-1,j} G^+_{k-1,k} \rangle \right). \tag{6.19}$$

The sum is over pairs of edges (or vertices), j of the top null polygon $C_{\rm T}$ and k of the bottom one $C_{\rm B}$. In terms of twistors, the two contours are $C_{\rm T} = \{2,\ldots,i{-}1,i,{\rm t}\}$ and $C_{\rm B} = \{i{+}1,\ldots,n,1,{\rm b}\}$, respectively.

A generic edge (or vertex) of the top polygon is well separated from one of the bottom one, yielding finite correlators for the remainder function. There are special cases when some correlators naively diverge, because e.g. the vertex v_2 lies on the null line 1. However, similar to the bosonic case shown in appendix C of [48], if we carefully take the limit when v_2 approaches line 1, we find $r = r^+ + r^-$ with

$$\begin{aligned} r^+ = \sum_{j=4}^{i} \sum_{k=i+2}^{n} &\left(-{\rm Li}_2 X_{j,k} + \frac{1}{2} \log\langle j{-}1, k{-}1] \langle j,k] \log X_{j,k} \right) \\ &+ \frac{1}{2}\log^2 \frac{\langle 3,n]}{\langle 3,1]} + \frac{1}{2}\log^2 \frac{\langle 2,{\rm b}]}{\langle 3,{\rm b}]} - \frac{1}{2}\log^2 \frac{\langle i,n]}{\langle i,1]} - \frac{1}{2}\log^2 \frac{\langle 2,n]}{\langle 3,n]} \\ &+ \frac{1}{2} \log\langle i,1]\langle {\rm t},{\rm b}] \log X'_{{\rm t},{\rm b}} + \frac{1}{2}\log\langle 2,n]\langle 3,1] \log X'_{3,{\rm b}}, \end{aligned} \tag{6.20}$$

where $X'_{i,{\rm b}} := \langle i-1,{\rm b}]\langle i,n]/\langle i,{\rm b}]\langle i-1,n]$, and similarly for its conjugate r^-. Thus, we confirm that the boxed Wilson loop (6.19) is indeed finite and superconformal, and its explicit expression agrees with that of [48], if we replace the supersymmetric products $\langle j,k]$ by the bosonic ones $\langle jk{-}1kk{+}1\rangle/\langle k-1k\rangle\langle kk+1\rangle$.

As an important consistency check, we have explicitly used regularized expectation values, the axial-framing and the super-Poincaré forms, to calculate the boxed Wilson loop. By plugging (6.9) and (6.11) into (6.17), we find the same one-loop result as above in both cases. In particular, all reference twistors W_*, $\bar W_*$ or infinity twistors I_{AB}, I^{AB}, as well as all divergent contributions, neatly cancel.

7. Yangian symmetry and anomalies

Let us now turn to the definition of the Yangian in full superspace and to the analysis of its anomalies.

[21] A light-like line in spacetime is dual to a twistor W and a conjugate twistor $\bar W$ such that $W\cdot\bar W = 0$. Two light-like lines, represented by two twistor pairs $(W,\bar W)$ and $(W',\bar W')$, intersect if and only if $W\cdot\bar W' = W'\cdot\bar W = 0$. The incidence relations in figure 7 imply that $W_{\rm t}\cdot\bar W_i = W_{\rm t}\cdot\bar W_1 = W_{\rm t}\cdot\bar W_2 = 0$, which are solved by the first equality in (6.16).

[22] Since $\oint_C + \oint_{C_{\rm tb}} - \oint_{C_{\rm t}} - \oint_{C_{\rm b}} = 0$, it is easy to show $\oint_C\oint_C + \oint_{C_{\rm tb}}\oint_{C_{\rm tb}} - \oint_{C_{\rm t}}\oint_{C_{\rm t}} - \oint_{C_{\rm b}}\oint_{C_{\rm b}} = \oint_{C_{\rm B}}\oint_{C_{\rm T}} + \oint_{C_{\rm T}}\oint_{C_{\rm B}}$.

7.1. Yangian generators in ambitwistor space

The space of functions of ambitwistor space variables $(W^A, \bar{W}_B)$ admits a representation of the generators $\mathrm{J}^A{}_B$ of the unitary superalgebra $\mathfrak{u}(2,2|4)$ by single-derivative operators,

$$\mathrm{J}^A{}_B = \sum_{i=1}^{n} \mathrm{J}^A_{i,C} = \sum_{i=1}^{n} (-1)^A W_i^A \partial_{i,B} - (-1)^{AB} \bar{W}_{i,B} \bar{\partial}_i^A, \tag{7.1}$$

where the sum is taken over the sites of the Wilson loop. The central charge C and the hypercharge B are obtained by taking the supertrace and trace of $\mathrm{J}^A{}_B$, respectively.

The level-1 generators $\widehat{\mathrm{J}}^A{}_B$ of the Yangian [29] transform in the adjoint representation under the level-0 generators,

$$[\mathrm{J}^A{}_B, \widehat{\mathrm{J}}^C{}_D] = (-1)^C \delta^C_B \widehat{\mathrm{J}}^A{}_D - (-1)^{C+(A+B)(C+D)} \delta^A_D \widehat{\mathrm{J}}^C{}_B. \tag{7.2}$$

They are represented by a bilocal formula[23]

$$\widehat{\mathrm{J}}^A{}_B = \sum_{i,j=1}^{n} \mathrm{sign}(j-i)\, \mathrm{J}^A_{i\ C} \mathrm{J}^C_{j\ B} = \sum_{i<j} \mathrm{J}^A_{i\ C} \mathrm{J}^C_{j\ B} - (i \leftrightarrow j), \tag{7.3}$$

where we use the sign function to rewrite the ordered sums on the far right-hand side of the equation above in terms of sums taken over all sites of the Wilson loop.

Yangian invariance of a function of ambitwistor variables $F(W^A, \bar{W}_B)$ is achieved when

$$jF(W^A, \bar{W}_B) = 0 \tag{7.4}$$

holds for all $j = \mathrm{J}$ or $j = \widehat{\mathrm{J}}$.

Superconformal invariance. All generators

$$\{\mathrm{P}, \mathrm{L}, \bar{\mathrm{L}}, \mathrm{K}, \mathrm{D} | \mathrm{Q}, \bar{\mathrm{Q}}, \mathrm{S}, \bar{\mathrm{S}}\} \tag{7.5}$$

of the superconformal algebra $\mathfrak{psu}(2,2|4)$ are neatly represented by $\mathrm{J}^A{}_B$ so that we can treat them all at once.

The ambitwistor brackets $\langle k, l]$ defined in section 5.1 are superconformal invariants,

$$\mathrm{J}^A{}_B \langle i, j] = 0, \qquad |i-j| \geqslant 2. \tag{7.6}$$

Since the generators $\mathrm{J}^A{}_B$ are represented by single-derivative operators on ambitwistor space, any function $F(\langle k, l])$ of finite ambitwistor brackets is a superconformal invariant, too

$$\mathrm{J}^A{}_B F(\langle k, l]) = 0. \tag{7.7}$$

It is important to note that the dual Coxeter number of $\mathfrak{psu}(2,2|4)$ is zero. This fact is very helpful during calculations where we often encounter terms proportional to $(-1)^A \delta^A_A$. Further comments about superconformal invariants can be found in appendix E.

Due to regularization (see section 6), a wider class of functions F_{reg} with additional dependences on the auxiliary twistors W_* as in the framing regularization or explicitly non-superconformally invariant combinations of the twistor data like the angled and square brackets

$$\langle a, b\rangle = W_a^A I_{AB} W_b^B, \qquad [a, b] = \bar{W}_{a,A} I^{AB} \bar{W}_{b,B} \tag{7.8}$$

in supersymmetric regularization has to be considered. These do not in general satisfy superconformal invariance. We expect therefore an anomalous remainder $\mathcal{A}$ of the invariance equations

$$\mathrm{J}^A{}_B F_{\mathrm{reg}} = \mathcal{A}. \tag{7.9}$$

This has implications for Yangian invariance.

[23] The sign factor $(-1)^A$ was included to eliminate a corresponding factor in the definition of the Yangian charges, see below.

Yangian invariance. The generators of the first level in the Yangian $\mathrm{Y}[\mathfrak{psu}(2,2|4)]$ are given by second-order derivatives. This requires any superconformally invariant function of ambitwistors $(W^A, \bar{W}_B)$ to satisfy an additional second-order differential equation

$$\widehat{\mathrm{J}}^A{}_B F(W^A, \bar{W}_B) = 0. \tag{7.10}$$

It is easily checked that a single ambitwistor bracket $\langle k, l]$ on its own is also invariant under the first-level generators of $\mathrm{Y}[\mathfrak{psu}(2,2|4)]$. However, a generic function $F(\langle k, l])$ of brackets is in general not an invariant as (7.10)[24]

$$\begin{aligned}\widehat{\mathrm{J}}^A{}_B F(\langle m, n]) = (-1)^A \sum_{i,j,k,l=1}^{n} S_{kl,ij} W_i^A \bar{W}_{l,B} \langle k, j] \partial_{k,l} \partial_{i,j} F(\langle m, n]) \\ - \delta_B^A \sum_{k,l=1}^{n} \mathrm{sign}(k-l) \langle k, l] \partial_{k,l} F(\langle m, n])\end{aligned} \tag{7.11}$$

is a non-trivial second-order partial differential equation. The trace term proportional to δ_B^A in (7.11) only appears when considering the level-1 hypercharge $\widehat{\mathrm{B}}$ of the Yangian $\mathrm{Y}[\mathfrak{u}(2,2|4)]$. This generator was shown to be an additional symmetry of the scattering amplitudes of $\mathcal{N} = 4$ SYM [49] not contained in the Yangian $\mathrm{Y}[\mathfrak{psu}(2,2|4)]$. This trace term contains a single derivative with respect to the brackets as can be seen above. Thus, ambitwistor brackets transform covariantly under $\widehat{\mathrm{B}}$,

$$\widehat{\mathrm{B}} \langle k, l] = 8\, \mathrm{sign}(l-k) \langle k, l]. \tag{7.12}$$

Furthermore, it is worth mentioning that the twistor constraints (5.4) are even invariant under the full $\mathrm{Y}[\mathfrak{u}(2,2|4)]$.

7.2. Anomaly of Yangian symmetry

It has been shown that one-loop corrections to the chiral supersymmetric Wilson loop [23, 22] break the chiral $\mathcal{N} = 4$ supersymmetry transformations [8]. Its conformal anomaly has been investigated most recently in [50].

Also the non-chiral supersymmetric n-polygonal Wilson loop $\langle \mathcal{W} \rangle$ presented in this paper suffers from UV divergences in the regions close to the cusps. These need to be regularized which in turn breaks Yangian invariance,

$$jF_n \neq 0 \tag{7.13}$$

for $j \in \mathrm{Y}[\mathfrak{psu}(2,2|4)]$. In contradistinction to the chiral super Wilson loop however, it should be possible to find a regularization for the non-chiral Wilson loop that at least preserves super-Poincaré symmetry. A very promising guess for such a regularization was given in section 6.

In the following, we treat the anomalies

$$jM_n^{(1)} = \mathcal{A}_{n,j} \tag{7.14}$$

for the non-chiral MHV one-loop expectation value in different regularizations. We investigate not only the anomalies of the symmetry generators $\mathrm{J}^A{}_B$ of the superconformal algebra $\mathfrak{psu}(2,2|4)$ but also the anomalies

$$\widehat{\mathrm{J}}^A{}_B M_n^{(1)} = \widehat{\mathcal{A}}^A_{n,B} \tag{7.15}$$

of the Yangian generators.

[24] The occurring derivative is defined by $\partial_{k,l} = \partial/\partial\langle k, l]$. The function S is a factor defined by $S_{kl,ij} = \mathrm{sign}(k-i) - \mathrm{sign}(k-j) - \mathrm{sign}(l-i) + \mathrm{sign}(l-j)$.

Naturally, it would be better to check explicitly finite, regularization-independent quantities for superconformal and Yangian invariance. An interesting class of such quantities is provided by $r = M^{(1)}[C] + M^{(1)}[C_{\mathrm{tb}}] - M^{(1)}[C_{\mathrm{t}}] - M^{(1)}[C_{\mathrm{b}}]$ in (6.17) that is obtained by the boxing procedure in section 6.3. We find that these are clearly superconformally invariant,

$$\mathrm{J}^A{}_B\, r = 0 \tag{7.16}$$

as they have no dependence on the regulators. On the other hand, this does not extend to Yangian symmetries which remain broken even when used on these finite quantities.

7.3. Vertex correlators

We begin by inspecting finite mixed correlators (4.39) $\langle 0|G^+_{j-1,j}G^-_{k-1,k}|0\rangle$ with j and k well separated. These are obviously invariant under superconformal transformations as they are functions of ambitwistor brackets alone.

How do the Yangian level-1 generators fare? When simply acting with $\widehat{\mathrm{J}}^A{}_B$ on the vertex correlators in (4.39), we find

$$\widehat{\mathrm{J}}^A{}_B\langle 0|G^+_{j-1,j}G^-_{k-1,k}|0\rangle = 64\pi^2(-1)^A\left[\frac{W^A_{j-1}\bar{W}_{k,B}}{\langle j-1,k]} - \frac{W^A_j\bar{W}_{k-1,B}}{\langle j,k-1]}\right] + \delta^A_B \log\left(\frac{\langle j-1,k]\langle j,k-1]}{\langle j-1,k-1]\langle j,k]}\right) \tag{7.17}$$

so they are not Yangian invariant on their own. Nevertheless, the anomaly is of the form $f_{j-1,k} - f_{j,k-1}$ (the trace term is slightly different, but the conclusion is the same) which naively telescopes in the sum over all vertices,

$$\sum_{j,k=1}^{n}(f_{j-1,k} - f_{j,k-1}) = \sum_{j,k=1}^{n}(f_{j,k} - f_{j,k}) = 0. \tag{7.18}$$

The trouble is that (7.17) holds only for the finite vertex correlators with $|j-k| \geqslant 3$. The divergent correlators for $|j-k| \leqslant 2$ need to be regularized. This turns out to inevitably break superconformal and Yangian invariance. Therefore, it is fair to say that the one-loop Wilson loop expectation value is perfectly superconformal and Yangian invariant except for the effects of regularization. Only the divergent correlators of nearby vertices call for regularization and break both symmetries in an analogous fashion. These anomaly terms are computed in the subsequent subsections.

It is worth mentioning that the expression in (7.17) makes no reference to the vertex which defines the ordering in the Yangian action (7.3). This is because the function is also superconformally invariant in which case the Yangian action respects cyclic symmetry [29]. However, the regularized vertex correlators for $|j-k| \leqslant 2$ break superconformal symmetry and consequently introduce dependence on the reference vertex.

It is helpful to cast $\langle 0|G^+_{j-1,j}G^-_{k-1,k}|0\rangle$ into the form of a symbol

$$\mathcal{S}\langle 0|G^+_{j-1,j}G^-_{k-1,k}|0\rangle = \sum_{\substack{i=j-1,j\\ l=k-1,k}} R_{i,l}\otimes\langle i,l]. \tag{7.19}$$

It is remarkable that there are only single brackets in the second entry. A very similar observation for the form of the symbols of scattering amplitudes has been made in [24]. The $R_{i,l}$ represent the rational functions which appear as the first entry of the symbol for a given second entry $\langle i,l]$. A generator of $\mathfrak{u}(2,2|4)$ acts like a logarithmic derivative on the last entry of a symbol, thus lowering the transcendentality by 1. As can be seen from (7.19), this is just the bracket $\langle i, l]$. However, the Yangian level-1 generators $\widehat{\mathrm{J}}^A{}_B$ generically act on both parts of the symbol, thus producing rational terms when acting on a finite correlator.

On inspection of (7.11), it is evident that the only generator acting twice on the second part of a symbol of the form (7.19) is the level-1 hypercharge generator $\widehat{\mathrm{B}}$. This explains the logarithmic terms in (7.17) proportional to the trace δ^A_B. The anomaly of $\widehat{\mathrm{B}}$ therefore suffers from additional single-logarithm contributions.

Correlators that need regularization can be inspected in the same way. Supersymmetric and axial regularization also have symbols with only one bracket in the second entry for the divergent propagators $|j-k| < 3$:

$$\mathcal{S}\langle G^+_{j-1,j} G^-_{k-1,k}\rangle_{\text{susy}} = \sum_{\substack{i=j-1,j\\ l=k-1,k}} \left(R^{(1)}_{i,l} \otimes \langle i, l] + R^{(2)}_{i,l} \otimes \langle i, l\rangle + R^{(3)}_{i,l} \otimes [i, l]\right), \tag{7.20}$$

$$\mathcal{S}\langle G^+_{j-1,j} G^-_{k-1,k}\rangle_{\text{axial}} = \sum_{\substack{i=j-1,j\\ l=k-1,k}} \left(\mathcal{R}^{(1)}_{i,l} \otimes \langle i, l] + \mathcal{R}^{(2)}_{i,l} \otimes \langle i, \bar{*}] + \mathcal{R}^{(3)}_{i,l} \otimes \langle *, l] + \mathcal{R}^{(4)}_{i,l} \otimes \langle *, \bar{*}]\right). \tag{7.21}$$

The functions $R^{(i)}$ and $\mathcal{R}^{(i)}$ in (7.20) are all rational and they differ in both schemes. The presence of non-invariant brackets $\langle\cdot,\cdot\rangle$ and $[\cdot,\cdot]$ in super-Poincaré regularization or $\langle *, \cdot]$ and $\langle\cdot, \bar{*}]$ in axial regularization in the second entries breaks superconformal invariance. Similarly, we expect further contributions to the anomalies of all $j \in \mathrm{Y}(\mathfrak{u}(2,2|4))$.

7.4. Super-Poincaré regularization

The superconformal anomaly. From the variation of $M^{(1)}_n$ given in (6.12) follows that it is only necessary to know the action of the generators of $\mathfrak{psu}(2,2|4)$ on spinor brackets. We write any superconformal generator $\mathrm{J}^A{}_B$ acting on a function

$$F = F(\langle k,\ l], \langle k, k+1\rangle, [k, k+1]) \tag{7.22}$$

as a function of derivatives with respect to brackets,

$$\mathrm{J}^A{}_B F = (-1)^A I_{BC} \sum_{i=1}^{n} \left(W^A_i W^C_{i+1} - W^A_{i+1} W^C_i\right)\partial_i F - (-1)^A I^{AC} \sum_{i=1}^{n} (\bar{W}_{i,B}\bar{W}_{i+1,C} - \bar{W}_{i+1,B}\bar{W}_{i,C})\bar{\partial}_i F, \tag{7.23}$$

where $\partial_i = \partial/\partial\langle i, i+1\rangle$, similarly for $\bar{\partial}_i$. For $F = M^{(1)}_n$ in supersymmetric regularization, the right-hand side is

$$\begin{aligned}
\mathrm{J}^A{}_B M^{(1)}_n = (-1)^A I_{BC} &\sum_i \left(\frac{W^A_i W^C_{i+1} - W^A_{i+1} W^C_i}{\langle i, i+1\rangle}\right)\\
&\times \left[\frac{2}{\epsilon} + \log\left(\frac{[i, i+1]\langle i, i+2]\langle i+1, i-1]}{\mu^2\langle i, i+1\rangle[i+1, i+2][i, i-1]}\right)\right]\\
&- I^{AC} \sum_i \left(\frac{\bar{W}_{i,B}\bar{W}_{i+1,C} - \bar{W}_{i+1,B}\bar{W}_{i,C}}{[i, i+1]}\right)\\
&\times \left[\frac{2}{\epsilon} + \log\left(\frac{\langle i, i+1\rangle\langle i-1, i+1]\langle i+2, i]}{\mu^2\langle i-1, i\rangle[i, i+1]\langle i+2, i+1\rangle}\right)\right],
\end{aligned} \tag{7.24}$$

where I_{AB} and I^{AB} are infinity (bi-)twistors. The right-hand side of (7.24) is zero for any of the Poincaré generators as well as supersymmetry and R-symmetry, thus realizing full super-Poincaré symmetry free of anomalies. We are left with the conformal anomaly of the Wilson loop.

When comparing this anomaly to the literature, e.g. [17], note that the bosonic result is often split

$$\langle \mathcal{W}[C_n]\rangle = Z_n F_n \tag{7.25}$$

into a divergent part Z_n and a finite part F_n. The divergent part Z_n is defined such that it contains the full dependence on the renormalization scale μ. Reference [17] computed the anomaly of the conformal group, when acting on $\log F_n$. This fact must be taken into account when comparing to the above anomaly of the whole answer, including the contribution of the divergent part Z_n. After taking into account these differences in conventions, we find agreement with the conformal anomaly computed in [17].

The Yangian anomaly. The calculation of the Yangian anomaly

$$\widehat{\mathrm{J}}^A{}_B M_n^{(1)} = \widehat{\mathcal{A}}^A_{n,B} \tag{7.26}$$

of $M_n^{(1)}$ can be done in a similar fashion.

As an example, we will give the form of the anomaly of the level-1 hypercharge $\widehat{\mathrm{B}}$. Its form is especially nice compared to the anomalies of the other first-level generators $\widehat{\mathrm{J}}^A{}_B$ which can be deduced using (7.2). Just as before we can find the action of $\widehat{\mathrm{B}}$ on a function F in terms of derivatives with respect to brackets. The result of acting on $F = M_n^{(1)}$ is

$$\begin{aligned}\widehat{\mathrm{B}}M_n^{(1)} = 2\sum_{j=1}^{n}\Bigg[&\frac{\langle j-1,j+2](-1)^A W_j^A \bar{W}_{j+1,A}}{\langle j-1,j+1]\langle j,j+2]} - \frac{\langle j+2,j-1](-1)^A W_{j+1}^A \bar{W}_{j,A}}{\langle j+1,j-1]\langle j+2,j]}\\ &+2\left(\frac{(-1)^A W_{j+2}^A \bar{W}_{j,A}}{\langle j+2,j]} - \frac{(-1)^A W_j^A \bar{W}_{j+2,A}}{\langle j,j+2]}\right)\Bigg] + 16\sum_{j=1}^{n-2}\log\left(\frac{\langle j+2,j]}{\langle j,j+2]}\right)\\ &+16\log\left(\frac{\langle 1,2\rangle[n-1,n]}{\langle n-1,n\rangle[1,2]}\right),\end{aligned} \tag{7.27}$$

where the regularization-dependent part of the anomaly is fully contained in the terms proportional to $W_i\bar{W}_{i+2}$ and $W_{i+2}\bar{W}_i$. The last term is a contribution from the $1, n$ boundary.

For Yangian level-1 generators, $\widehat{\mathrm{J}}^A{}_B$ invariance under cyclic shifts $i \to i+1$ needs to be checked explicitly. This is done by calculating the difference between a Yangian generator $\widehat{\mathrm{J}}^A_{1,n\ B}$ between the site 1 and site n and a Yangian generator which is shifted by one site $\widehat{\mathrm{J}}^A_{2,n+1\ B}$. For $\mathfrak{psu}(2,2|4)$, one finds

$$\widehat{\mathrm{J}}^A_{2,n+1\ B} - \widehat{\mathrm{J}}^A_{1,n\ B} = 2(-1)^{(A+C)(C+B)}\mathrm{J}^C_{1,B}\mathrm{J}^A{}_C - 2\mathrm{J}^A_{1,C}\mathrm{J}^C{}_B. \tag{7.28}$$

Superconformally as well as cyclically invariant functions will be annihilated by the right-hand side, proving the compatibility of the Yangian with cyclic shifts. In the anomalous case presented here, the right-hand side is non-vanishing which is the reason for the cyclic asymmetry of the last term in (7.27).

7.5. Axial regularization

The superconformal anomaly. Now consider framing as described in section 6. When acting with $\mathrm{J}^A{}_B$ on a function

$$F = F(\langle k,l], \langle k,\bar{*}], \langle *,k], \langle *,\bar{*}]) \tag{7.29}$$

in axial regularization, the invariance equation is no longer trivially satisfied,

$$\mathrm{J}^A{}_B F = (-1)^A \sum_{j=1}^{n}\left[W_j^A \bar{W}_{\bar{*},B}\frac{\partial F}{\partial\langle j,\bar{*}]} - W_*^A \bar{W}_{j,B}\frac{\partial F}{\partial\langle *,j]}\right]. \tag{7.30}$$

Setting $F = M^{(1)}_{n,*}$, we find

$$\mathrm{J}^A{}_B M^{(1)}_{n,*} = (-1)^A \sum_{i=1}^{n} \left[\frac{W^A_* \bar{W}_{i,B}}{\langle *, i]} - \frac{W^A_i \bar{W}_{\bar{*},B}}{\langle i, \bar{*}]} \right] \log \left(\epsilon^2 \frac{\langle i-1, i+1]^* \langle i+1, i-1]^*}{\langle i-1, i+1]\langle i+1, i-1]} \right) \tag{7.31}$$

employing notation (6.7) introduced in section 6.

This compares nicely with (7.24). In both cases, there are single-logarithmic terms weighted by rational functions depending on the symmetry-breaking brackets.

The twistors W_* and $\bar{W}_{\bar{*}}$ do not get transformed under the action of the generators of $\mathfrak{psu}(2,2|4)$. Hence, the brackets $\langle i, \bar{*}]$ and $\langle *, i]$ are not invariant. Obviously, if the auxiliary twistors W_* and $W_{\bar{*}}$ were to be transformed under superconformal transformations, we would find the expectation value (6.9) $M^{(1)}_{n,*}$ to be an invariant $\mathrm{J}'^A{}_B M^{(1)}_{n,*} = 0$.

The Yangian anomaly. In the following, we will use some additional notation to shorten the expression for the Yangian anomaly. We write[25]

$$(i\ j) \cap k := W_i \langle j, k] - W_j \langle i, k]. \tag{7.32}$$

This resembles the notation used in [20]. Similarly, for antichiral twistor variables, we use

$$k \cap (i\ j) := \bar{W}_i \langle k, j] - \bar{W}_k \langle k, i]. \tag{7.33}$$

They satisfy the relation

$$\langle (i\ j) \cap k, m] = \langle j, i \cap (k\ m)]. \tag{7.34}$$

Finally, to write the Yangian anomaly in a more compact form, we will make use of the notation

$$([i\ j]k) \cap (l\ m) = W_i \langle (j\ k) \cap l, m] - W_j \langle (i\ k) \cap l, m]. \tag{7.35}$$

When restricted to bosonic components, this quantity indicates that the points $(j\,k)\cap l$, $(i\,k)\cap l$ and $(i\,j)\cap l$ are linearly related which enables us via a Plücker identity to replace this expression by a simpler one. However, on inclusion of the fermionic directions, there are additional sign factors from the fermions that prevent us from doing so.

The Yangian anomaly can be straightforwardly calculated. It is given by

$$\begin{aligned}
\widehat{\mathrm{J}}^A{}_B M^{(1),*}_n = & \sum_{i=1}^{n-1} \left(2 \frac{\langle ([i-1\,*]i) \cap (i+2\,\bar{*}), i+1]}{\langle i-1, i+1]\langle i, i+2]\langle *, \bar{*}]} - 1 \right) \frac{(-1)^A W^A_i \bar{W}_{i+1,B}}{\langle i, i+1]^*} \\
& + \left(2 \frac{\langle i+1, (*\,i+2) \cap ([\bar{*}\,i-1]i)]}{\langle i+1, i-1]\langle i+2, i]\langle *, \bar{*}]} - 1 \right) \frac{(-1)^A W^A_{i+1} \bar{W}_{i,B}}{\langle i+1, i]^*} \\
& - \sum_{i=1}^{n} \left(\frac{(-1)^A W^A_i \bar{W}_{i+2,B}}{\langle i, i+2]} - \frac{(-1)^A W^A_{i+2} \bar{W}_{i,B}}{\langle i+2, i]} \right) \\
& + 2 \frac{(-1)^A [(n-1\,*) \cap 1]^A \bar{W}_{n,B}}{\langle n-1, 1]\langle *, n]} - 2 \frac{(-1)^A W^A_n [1 \cap (n-1\,\bar{*})]_B}{\langle 1, n-1]\langle n, \bar{*}]} \\
& + 2 \frac{(-1)^A W^A_i [n \cap (2\,\bar{*})]_B}{\langle 1, \bar{*}]\langle n, 2]} - 2 \frac{(-1)^A [(2*) \cap n]^A \bar{W}_{1,B}}{\langle 2, n]\langle *, 1]} \\
& - \frac{(-1)^A W^A_n \bar{W}_{1,B}}{\langle n, 1]^*} + \frac{(-1)^A W^A_1 \bar{W}_{n,B}}{\langle 1, n]^*} + 2\delta^A_B \sum_{j=1}^{n-2} \log \left(\frac{\langle j+2, j]}{\langle j, j+2]} \right).
\end{aligned} \tag{7.36}$$

[25] When restricted to bosonic components, this denotes the intersection point between a line (jk) and the plane $\bar{W}_k$.

Despite the fact that we could make superconformal symmetry exact by transforming the auxiliary twistors W_* and $\bar{W}_{\bar{*}}$ too, the same trick does not cure the Yangian anomaly $\widehat{\mathcal{A}}^A{}_B$. The bilocal structure of the Yangian generators distinguishes the auxiliary sites as we need to insert these into the chain $1 \to \ldots \to n \to 1$. Putting them between n and 1, the new level-1 generators $\widehat{\mathrm{J}}'^A{}_B$ are defined by $\widehat{\mathrm{J}}^A{}_B$ and additional pieces from the new sites

$$\widehat{\mathrm{J}}'^A{}_B = \widehat{\mathrm{J}}^A{}_B + \mathrm{J}^A{}_C \mathrm{J}^C_{*B} + \mathrm{J}^A{}_C \mathrm{J}^C_{\bar{*}B} - \mathrm{J}^A_{\bar{*}C}\mathrm{J}^C{}_B - \mathrm{J}^A_{*C}\mathrm{J}^C{}_B + \mathrm{J}^A_{*C}\mathrm{J}^C_{\bar{*}B} - \mathrm{J}^A_{\bar{*}C}\mathrm{J}^C_{*B}. \quad (7.37)$$

Their action on $M_n^{(1),*}$ is given by

$$\widehat{\mathrm{J}}'^A{}_B M_n^{(1),*} = \widehat{\mathcal{A}}^A_{n,B} - \mathrm{J}'^A{}_C \mathcal{A}^C_{n,B} + f^A{}_E{}^F{}_D{}^G{}_B \mathrm{J}^E_{*,F}\mathrm{J}^D_{\bar{*},G} M_n^{(1),*} \quad (7.38)$$

with $f^A{}_E{}^F{}_D{}^G{}_B = (-1)^A \delta^A_E \delta^F_D \delta^G_B - (-1)^{A+(A+G)(G+F)} \delta^A_D \delta^F_B \delta^G_E$. In particular, cyclic symmetry remains broken after the inclusion of the auxiliary points into the superconformal generators.

7.6. Boxing the Wilson loop

Let us now discuss the Yangian anomalies of the boxed Wilson loop defined in section 6.3. A related (but different in certain aspects) study of the Yangian anomalies for the boxed Wilson loop in two-dimensional kinematics was done in [51].

We saw that the above two regularized Wilson loop expectation values break parts of superconformal and Yangian symmetry. Moreover, the anomaly terms are different in both cases. This is particularly inconvenient when the aim is to construct the result from unbroken or anomalous symmetry consideration. This is, however, not very surprising because both results are divergent when the regulator is removed, $\epsilon \to 0$. In other words, the above Wilson loops are regularized but not renormalized, and therefore all answers certainly depend on the regularization scheme. It only makes sense to consider the symmetries of a regularized but not renormalized quantity within any given regularization scheme.

Let us take a look at correlators of local operators in a conformal theory. Naively they are also divergent and need to be regularized. In addition, local operators are renormalized, and when the regulator is removed, the correlation functions are not only perfectly finite, but also transform nicely under superconformal symmetry (albeit with quantum corrections to scaling dimensions).

The boxed Wilson loop introduced in section 6.3 can be regarded as such a renormalization of a Wilson loop. Quantity (6.17) and the ones obtained by choosing different reference twistors i and j do not depend on the regularization scheme; they are finite and manifestly superconformally invariant. However, when inspecting r^+ in section 6.3, we note the occurrence of brackets like

$$\langle k, t] = \langle k, 1] - \frac{\langle i, 1]}{\langle i, 2]} \langle k, 2]. \quad (7.39)$$

Their occurrence breaks Yangian invariance. This is easily seen when considering the symbols $\mathcal{S}r$ of these quantities. We find terms like

$$R_{i,j,k,l} \otimes \left(1 - \frac{\langle i, k]\langle j, l]}{\langle i, l]\langle j, k]}\right). \quad (7.40)$$

The Yangian acts twice on the second entry of the symbol leaving behind additional logarithmic terms on the right-hand side of the anomalous invariance equations of Yangian level-1 generators.

The boxed Wilson loop is finite and respects superconformal symmetry, but it does not respect Yangian symmetry. Naively this seems to imply that superconformal symmetry is exact while Yangian symmetry is broken or anomalous. However, one has to bear in mind that the boxed Wilson loop is not a simple planar Wilson loop expectation value anymore. For instance, at the one-loop level, the boxed Wilson loop is equivalent to the correlator of two Wilson loops,

$$r = \frac{\langle W[C]\rangle\langle W[C_{\mathrm{tb}}]\rangle}{\langle W[C_{\mathrm{t}}]\rangle\langle W[C_{\mathrm{b}}]\rangle} = \frac{\langle W[C_{\mathrm{T}}]W[C_{\mathrm{B}}]\rangle}{\langle W[C_{\mathrm{T}}]\rangle\langle W[C_{\mathrm{B}}]\rangle} + \mathcal{O}(g^4), \tag{7.41}$$

where C_{T} and C_{B} refer to the top and bottom polygons enclosed by the edges $(\mathrm{t}, 2, \ldots, i)$ and $(\mathrm{b}, i+1, \ldots, n, 1)$ in figure 7. In the string worldsheet picture, the simple planar Wilson loop has the topology of a disk, while the correlator has annulus topology. Yangian invariance is expected only for disk topology, because a loop surrounding the disk which represents a Yangian generator can be contracted to a point, see the discussions in [52]. Hence, it is not surprising that we find no Yangian invariance from the quantities obtained through boxing despite the fact that they are finite and superconformally invariant.

Once again from experience with local operators, we know that two-point functions of local operators do not exhibit Yangian invariance. Hence, it is not surprising that we find no Yangian invariance from the quantities obtained through boxing despite the fact that they are finite and superconformally invariant.

8. Conclusions

In conclusion, we have computed the one-loop expectation value of polygonal light-like Wilson loops in full superspace. The answer we obtained has two pieces: one rational of Grassmann weight 4 and one transcendental of transcendentality degree 2.

For the rational part, the computation in full superspace is identical to the computation in chiral superspace, in the sense that they are both computed by integrating the end points of a propagator along the sides of the Wilson loop.

However, for the transcendental part, the computation looks different. Both in the twistor [22] and spacetime [23] version of the chiral Wilson loop, the one-loop computation uses a quadratic interaction vertex, besides the integration along the sides. The extra interaction vertex gives rise to the integrand of the Wilson loop, which is the same as the integrand of the corresponding scattering amplitude. In contrast, the corresponding computation in non-chiral superspace directly yields the integrated result, and does not employ any interaction vertices. It would be interesting to see if there is a useful notion of integrand for the non-chiral Wilson loop.

We have presented several computations: in momentum space, in spacetime and in momentum twistor space. In order to regularize the divergences, we have used the framing regularization. We have also presented a guess for the finite part of the answer which preserves Poincaré supersymmetry.

Another way to deal with divergences is to construct finite quantities. Inspired by [34], we considered a finite combination of Wilson loops called the ‘boxed Wilson loop’, which depends on a choice of two reference edges.

In the chiral case, the Poincaré supersymmetry generators are $\partial/\partial\theta$ and $\theta\partial/\partial x$. The first chiral-half of Poincaré supersymmetry is not anomalous, but the second one is. However, if we use a non-chiral formalism, the generators and the answer are modified in such a way that Q and $\bar{\mathrm{Q}}$ symmetries are both exact.

If we expand the transcendental part in powers of $\theta\bar{\theta}$, the second chiral-half anomaly of the chiral result, i.e. the term at zeroth order, can be interpreted as coming from the generator acting on higher order terms, since the full result is invariant (see [24]). At zeroth order in the $\theta\bar{\theta}$ expansion, the answer is identical to the answer obtained for the chiral Wilson loop.

Finally, we have investigated the superconformal and Yangian anomaly of several regularized one-loop Wilson loop expectation values. It turned out that the result is superconformally invariant whenever it is finite. Conversely, no regularization turned out to be exactly invariant under the Yangian. In fact, it is quite common for integrable models that the Yangian is not an exact symmetry. For example, the Hamiltonian of an integrable spin chain is typically not Yangian invariant. Instead, the Yangian action converts the bulk Hamiltonian to a telescoping sum. The resulting boundary terms usually remain and break exact Yangian invariance. Gladly, this behavior turns out to be sufficient for integrability. Here the situation is very similar: the Yangian action (7.17) leaves behind some terms which telescope in a sum. Naively, we thus have Yangian invariance. Unfortunately, some (boundary) terms require regularization and spoil exact invariance. Nevertheless, the cancellation of the majority of terms is very remarkable and should be taken as a consequence of integrability of the problem.

We have computed the Yangian anomaly for different types of regularizations. In particular, we have seen that the transcendentality of the Yangian anomaly is reduced by two degrees compared to the Wilson loop expectation value. This seems to imply that it would be substantially simpler to compute in practice. It would therefore be good to be able to calculate or quantify this anomaly in more general terms. Along the lines of [24, 26], this could give easy access to yet higher loop orders.

Obviously one would like to compute this Wilson loop in full superspace to higher loops. Beyond one-loop level, one has to use interaction vertices and work with non-Abelian gauge fields. Furthermore, the causal structure of propagators and the framing regularization certainly require more care than was needed in this work. Presumably, the two-loop answer will contain the tree-level N^2MHV, the one-loop NMHV and the two-loop MHV answers, with a similar pattern for higher loops. This is in line with the recent findings that, in some sense, a measure of the difficulty of a computation is given by the NMHV level plus the loop order.

We believe that the results of this work will contribute toward understanding the perturbation theory of $\mathcal{N} = 4$ SYM in ambitwistor space. This ambitwistor theory is very elegant but it has proven hard to quantize. We hope that availability of results in a non-chiral formulation will contribute to the understanding of the quantization of this theory.

Finally, let us comment on the duality with scattering amplitudes. As we have already mentioned and as discussed in more detail in [42], there is no straightforward correspondence with scattering amplitudes. This happens because supersymmetric intervals in full superspace contain terms quadratic in the fermionic variables and therefore are not given by the differences $x_i - x_j$. This implies that a direct identification of the particle momenta with the supersymmetric intervals will violate momentum conservation.

Instead, one could attempt to identify particle momenta with the differences $x_i - x_j$ of the bosonic superspace coordinates. This satisfies momentum conservation, but the particles are not massless anymore since $x_{ij}^2 \neq 0$. If we want to take this proposal seriously, we need to explain the discrepancy in the number of d.o.f.; a massless $\mathcal{N} = 4$ multiplet containing states with helicities between 1 and -1 has $2^4 = 16$ states (corresponding to a superfield in 4 $\bar{\eta}$), while a massive multiplet has $2^8 = 256$ states (corresponding to a superfield in 4 $\bar{\eta}$ and 4 η).

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Appendix A. Conventions

Our convention for the metric signature is $+---$.

We raise and lower spinor indices with the $SL(2)$-invariant antisymmetric matrices ϵ, $\psi_\alpha = \varepsilon_{\alpha\beta}\psi^\beta$, $\psi^\alpha = \varepsilon^{\alpha\beta}\psi_\beta$, $\bar\psi_{\dot\alpha} = \varepsilon_{\dot\alpha\dot\beta}\bar\psi^{\dot\beta}$ and $\bar\psi^{\dot\alpha} = \varepsilon^{\dot\alpha\dot\beta}\bar\psi_{\dot\beta}$. We have $\varepsilon_{\alpha\beta}\varepsilon^{\beta\gamma} = \delta_\alpha^\gamma$, $\varepsilon_{\dot\alpha\dot\beta}\varepsilon^{\dot\beta\dot\gamma} = \delta_{\dot\alpha}^{\dot\gamma}$.

We use a shorthand notation for index contractions: $\langle\psi\chi\rangle = \psi^\alpha\chi_\alpha$, $[\bar\psi\bar\chi] = \bar\psi_{\dot\alpha}\bar\chi^{\dot\alpha}$. Complex conjugation changes the order of the Grassmann variables $\langle\psi\chi\rangle^* = [\bar\chi\bar\psi]$.

We also use the 2×2 matrices $\sigma^\mu_{\alpha\dot\alpha} = (\mathbf{1},\vec\sigma)_{\alpha\dot\alpha}$ and $(\bar\sigma^\mu)^{\dot\alpha\alpha} = (\mathbf{1},-\vec\sigma)^{\dot\alpha\alpha}$ where $\vec\sigma$ are the three 3-dimensional Pauli matrices. The σ^μ and $\bar\sigma^\mu$ matrices are related by

$$(\bar\sigma^\mu)^{\dot\alpha\alpha} = \varepsilon^{\alpha\beta}\varepsilon^{\dot\alpha\dot\beta}\sigma^\mu_{\beta\dot\beta}. \tag{A.1}$$

The σ^μ and $\bar\sigma^\mu$ matrices also satisfy the following relations:

$$\sigma^\mu\bar\sigma^\nu + \sigma^\nu\bar\sigma^\mu = 2\eta^{\mu\nu}\mathbf{1}, \tag{A.2}$$

$$\bar\sigma^\mu\sigma^\nu + \bar\sigma^\nu\sigma^\mu = 2\eta^{\mu\nu}\mathbf{1}. \tag{A.3}$$

Finally, some relations which are useful in calculations are

$$\sigma^\mu_{\alpha\dot\alpha}\bar\sigma_\mu^{\dot\beta\beta} = 2\delta_\alpha^\beta\delta_{\dot\alpha}^{\dot\beta}, \quad \sigma^\mu_{\alpha\dot\alpha}\bar\sigma_\nu^{\dot\alpha\alpha} = 2\delta_\nu^\mu. \tag{A.4}$$

Throughout the text, we use the notations

$$x_{\alpha\dot\alpha} = x_\mu\sigma^\mu_{\alpha\dot\alpha}, \quad x^{\dot\alpha\alpha} = x_\mu\,(\bar\sigma^\mu)^{\dot\alpha\alpha}, \quad \partial_{\alpha\dot\alpha} = \sigma^\mu_{\alpha\dot\alpha}\partial_\mu. \tag{A.5}$$

Note that with this convention, we have $\partial_{\alpha\dot\alpha}x^{\dot\beta\beta} = 2\delta_\alpha^\beta\delta_{\dot\alpha}^{\dot\beta}$.

Appendix B. Bosonic prepotentials

In this appendix, we discuss the bosonic reduction of the supersymmetric calculations presented above. Since the meaning of the different fields appearing in the supersymmetric computation may be unclear, we aim at understanding them better by considering only their lowest components in the Grassmann expansion.

The construction described below is valid for an Abelian gauge theory, or for a non-Abelian theory at the linearized level. This is sufficient for computing one-loop corrections to the Wilson loops. It is not clear how to extend this construction to make it work at the nonlinear level.

We start by imposing the Lorentz gauge condition $\partial^\mu A_\mu = 0$. This gauge condition can be solved by $A_\mu = \partial^\nu H_{\nu\mu}$ where $H_{\mu\nu}$ is an antisymmetric rank-2 tensor. The field H is known as the Hertz potential.

The 'prepotential' $H_{\mu\nu}$ has its own gauge symmetry under which the potential A is unchanged $\delta H_{\mu\nu} = \partial^\rho K_{\mu\nu\rho}$, where $K_{\mu\nu\rho}$ is a rank-3 completely antisymmetric tensor. The rank-3 antisymmetric tensor $K_{\mu\nu\rho}$ can be dualized to a vector so the gauge transformation of $H_{\mu\nu}$ can be alternatively written as $\delta H_{\mu\nu} = \varepsilon_{\mu\nu\rho\sigma}\partial^\rho K^\sigma$.

Note that we could also add a piece proportional to $\eta_{\mu\nu}H$ to $H_{\mu\nu}$. Then $H_{\mu\nu}$ will not be antisymmetric anymore and a variation of H would produce a gauge transformation of A, $\delta A_\mu = \partial_\mu \delta H$. Therefore, we can alternatively describe a $U(1)$ theory by a rank-2 tensor $H_{\mu\nu}$ of a special kind, which can be decomposed to an antisymmetric tensor with a gauge symmetry and a scalar.

Let us write the tensor $H_{\mu\nu}$ in spinor language:

$$H_{\mu\nu}\sigma^\mu_{\alpha\dot\alpha}\sigma^\nu_{\beta\dot\beta} \equiv \varepsilon_{\dot\alpha\dot\beta}H_{\alpha\beta} - \varepsilon_{\alpha\beta}\bar H_{\dot\alpha\dot\beta}. \tag{B.1}$$

If A is antiHermitian, then $H_{\mu\nu}$ is antiHermitian and $H_{\alpha\beta}$ and $\bar H_{\dot\alpha\dot\beta}$ are related by Hermitian conjugation.

If $H_{\mu\nu} = -H_{\nu\mu}$, then $H_{\alpha\beta} = H_{\beta\alpha}$ and $\bar H_{\dot\alpha\dot\beta} = \bar H_{\dot\beta\dot\alpha}$. This is the usual decomposition of an antisymmetric tensor in self-dual and anti-self-dual parts. If $H_{\mu\nu}$ is not antisymmetric but it can be decomposed to an antisymmetric tensor and a scalar, then the same spinor decomposition holds but $H_{\alpha\beta}$ and $\bar H_{\dot\alpha\dot\beta}$ are not symmetric anymore.

Now we can identify up to a factor the fields $H_{\alpha\beta}$ and $\bar H_{\dot\alpha\dot\beta}$ with the fields $B_{\alpha\beta}$ and $\bar B_{\dot\alpha\dot\beta}$ defined before, since the equation $A_\mu = \partial^\nu H_{\nu\mu}$ translates to

$$A_{\alpha\dot\alpha} = -\tfrac{1}{2}\partial_{\alpha\dot\beta}\bar H^{\dot\beta}{}_{\dot\alpha} + \tfrac{1}{2}\partial_{\beta\dot\alpha}H^{\beta}{}_{\alpha} \tag{B.2}$$

in spinor language. This is to be compared with equation (3.13). If we take B, $\bar B$, H and $\bar H$ to be symmetric, we have that

$$B_{\alpha\beta} = \tfrac{1}{2}H_{\alpha\beta}, \quad \bar B_{\dot\alpha\dot\beta} = \tfrac{1}{2}\bar H_{\dot\alpha\dot\beta}. \tag{B.3}$$

Let us now impose a light-cone gauge condition $l^\mu H_{\mu\nu} = 0$ with $l^2 = 0$ on the field $H_{\mu\nu}$, just like in the supersymmetric computation. This implies that $l^\mu A_\mu = 0$ as well. In momentum space, this constraint and the Lorentz gauge constraint read

$$l^\mu A_\mu(p) = 0, \quad \text{with } l^2 = 0, \qquad p^\mu A_\mu(p) = 0, \quad \text{for all } p. \tag{B.4}$$

The propagator in this gauge reads

$$\langle A_\mu(p)A_\nu(q)\rangle = \delta^4(p+q)\frac{-\mathrm{i}}{p^2}\left(\eta_{\mu\nu} - \frac{l_\mu p_\nu + p_\mu l_\nu}{p\cdot l} + p^2\frac{l_\mu l_\nu}{(p\cdot l)^2}\right). \tag{B.5}$$

Now we can obtain the $\langle H_{\mu\rho}(p)H_{\nu\sigma}(q)\rangle$ from $\langle A_\mu(p)A_\nu(q)\rangle = p^\rho p^\sigma \langle H_{\mu\rho}(p)H_{\nu\sigma}(q)\rangle$ and the symmetries of the fields H. This yields

$$\langle H_{\mu\rho}(p)H_{\nu\sigma}(q)\rangle = \delta^4(p+q)\frac{-\mathrm{i}(\eta_{\mu\nu}l_\rho l_\sigma - \eta_{\rho\nu}l_\mu l_\sigma - \eta_{\mu\sigma}l_\rho l_\nu + \eta_{\rho\sigma}l_\mu l_\nu)}{p^2(p\cdot l)^2}. \tag{B.6}$$

In spinor language, this reads

$$\langle H_{\alpha\beta}(p)H_{\gamma\delta}(q)\rangle = 0, \qquad \langle \bar H_{\dot\alpha\dot\beta}(p)\bar H_{\dot\gamma\dot\delta}(q)\rangle = 0, \tag{B.7}$$

$$\langle H_{\alpha\beta}(p)\bar H_{\dot\alpha\dot\beta}(q)\rangle = -2\mathrm{i}\delta^4(p+q)\frac{l_\alpha l_\beta \bar l_{\dot\alpha}\bar l_{\dot\beta}}{p^2(p\cdot l)^2}. \tag{B.8}$$

Of course, when we do the same computation in $\mathcal{N} = 4$ SYM, all the fields are on-shell and the naive computation of the right-hand side yields $\frac{1}{0}$. This infinity is regularized by putting in the Feynman $+\mathrm{i}\epsilon$ prescription for the propagator. We will use the identity

$$\frac{1}{x+0\mathrm{i}} = \mathrm{p.v.}\frac{1}{x} - \mathrm{i}\pi\delta(x). \tag{B.9}$$

We can put the fields H and $\bar{H}$ on-shell by setting

$$H_{\alpha\beta}(p) = \delta(p^2)\frac{l_\alpha l_\beta}{\langle\lambda l\rangle^2}H(\lambda,\bar{\lambda}), \quad \bar{H}_{\dot{\alpha}\dot{\beta}}(q) = \delta(q^2)\frac{\bar{l}_{\dot{\alpha}}\bar{l}_{\dot{\beta}}}{[\bar{\lambda}'\bar{l}]^2}\bar{H}(\lambda',\bar{\lambda}'), \tag{B.10}$$

where we have set $p_{\alpha\dot{\alpha}} = \lambda_\alpha\bar{\lambda}_{\dot{\alpha}}$ on the support of $\delta(p^2)$ and $q_{\alpha\dot{\alpha}} = \lambda'_\alpha\bar{\lambda}'_{\dot{\alpha}}$ on the support of $\delta(q^2)$. The fields $H(\lambda,\bar{\lambda})$ and $\bar{H}(\lambda',\bar{\lambda}')$ are, up to a factor $\frac{1}{2}$, the same as the fields C and $\bar{C}$, truncated to their lowest component in the Grassmann expansion. Note that the fact that the $\langle CC\rangle$ two-point function is of Grassmann weight 4 is consistent with the vanishing of the $\langle HH\rangle$ two-point function.

Using the on-shell version of the fields and the fact that the propagator with the Feynman $+\mathrm{i}\epsilon$ prescription reduces to $-\mathrm{i}\pi\delta(p^2)$ on-shell, we obtain

$$\delta(q^2)\langle H(\lambda,\bar{\lambda})\bar{H}(\lambda',\bar{\lambda}')\rangle\frac{[\bar{\lambda}\bar{l}]^2}{[\bar{\lambda}'\bar{l}]^2} = -8\pi\delta^4(p+q). \tag{B.11}$$

It is not hard to show that

$$\delta^4(p+q) = \frac{1}{4}\delta(q^2)\left|\frac{\lambda_1}{\lambda'_1}\right|^2\int\frac{\mathrm{d}s}{s}\delta^2(\lambda-s\lambda')\delta^2(s\bar{\lambda}+\bar{\lambda}'), \tag{B.12}$$

so we finally obtain

$$\langle H(\lambda,\bar{\lambda})\bar{H}(\lambda',\bar{\lambda}')\rangle = -2\pi\int\mathrm{d}s\,s^3\delta^2(\lambda-s\lambda')\delta^2(s\bar{\lambda}+\bar{\lambda}'). \tag{B.13}$$

One can check that the scaling constraints

$$H(z\lambda,z^{-1}\bar{\lambda}) = z^2H(\lambda,\bar{\lambda}), \quad \bar{H}(z\lambda,z^{-1}\bar{\lambda}) = z^{-2}\bar{H}(\lambda,\bar{\lambda}) \tag{B.14}$$

are satisfied by the two-point function in equation (B.13). Note that, because of the absence of Grassmann variables, the exponent of s in equation (B.13) is different from the exponent of s in equation (3.30).

Appendix C. Position space calculations

C.1. Prepotential correlators

With the propagators in equations (3.30), (3.31) and (3.32) it is possible to compute correlators of the prepotentials

$$\begin{aligned}
\langle 0|B^{\alpha\beta}(x_1^+,\theta_1)B^{\gamma\delta}(x_2^+,\theta_2)|0\rangle &= \Delta^{\alpha\beta\gamma\delta}(x_{12}^+,\theta_{12}),\\
\langle 0|B^{\alpha\beta}(x_1^+,\theta_1)\bar{B}^{\dot{\gamma}\dot{\delta}}(x_2^-,\bar{\theta}_2)|0\rangle &= \Delta^{\alpha\beta\dot{\gamma}\dot{\delta}}(x_{12}^{+-}),\\
\langle 0|\bar{B}^{\dot{\alpha}\dot{\beta}}(x_1^-,\bar{\theta}_1)\bar{B}^{\dot{\gamma}\dot{\delta}}(x_2^-,\bar{\theta}_2)|0\rangle &= \Delta^{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}}(x_{12}^-,\bar{\theta}_{12}),
\end{aligned} \tag{C.1}$$

where, after some initial trivial integrations,

$$\begin{aligned}
\Delta^{\alpha\beta\gamma\delta}(x_{12}^+,\theta_{12}) &= \frac{1}{64\pi^4}\int_+\mathrm{d}^2\lambda\,\mathrm{d}^2\bar{\lambda}\exp\left(-\frac{\mathrm{i}}{2}\langle\lambda|x_{12}^+|\bar{\lambda}]\right)\frac{l^\alpha l^\beta l^\gamma l^\delta}{\langle\lambda l\rangle^4}\delta^{0|4}(\langle\lambda|\theta_{12}\rangle),\\
\Delta^{\alpha\beta\dot{\gamma}\dot{\delta}}(x_{12}^{+-}) &= \frac{1}{256\pi^4}\int_+\mathrm{d}^2\lambda\,\mathrm{d}^2\bar{\lambda}\exp\left(-\frac{\mathrm{i}}{2}\langle\lambda|x_{12}^{+-}|\bar{\lambda}]\right)\frac{l^\alpha l^\beta\bar{l}^{\dot{\gamma}}\bar{l}^{\dot{\delta}}}{\langle\lambda l\rangle^2[\bar{l}\bar{\lambda}]^2},\\
\Delta^{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}}(x_{12}^-,\bar{\theta}_{12}) &= \frac{1}{64\pi^4}\int_+\mathrm{d}^2\lambda\,\mathrm{d}^2\bar{\lambda}\exp\left(-\frac{\mathrm{i}}{2}\langle\lambda|x_{12}^-|\bar{\lambda}]\right)\frac{\bar{l}^{\dot{\alpha}}\bar{l}^{\dot{\beta}}\bar{l}^{\dot{\gamma}}\bar{l}^{\dot{\delta}}}{[\bar{l}\bar{\lambda}]^4}\delta^{0|4}(\bar{\theta}_{12}|\bar{\lambda}]).
\end{aligned} \tag{C.2}$$

To solve the remaining integrals, we have to perform a calculation very similar to the calculation of the vertex correlators $\langle 0|G^{\pm}_{j-1,j}G^{\pm}_{k-1,k}|0\rangle$ presented in section 4.4. The remaining instances of λ and $\bar{\lambda}$ can be represented through $(\partial/\partial x)$ contracted with $\bar{l}$ and l, respectively. This leads to the derivative operators

$$\mathcal{D}^{+} = \mathrm{i}\langle l|\sigma^{\mu}|\bar{l}]\frac{\partial}{\partial x_{12}^{+\,\mu}}, \quad \mathcal{D}^{\pm} = \mathrm{i}\langle l|\sigma^{\mu}|\bar{l}]\frac{\partial}{\partial x_{12}^{+-\,\mu}}, \quad \mathcal{D}^{-} = \mathrm{i}\langle l|\sigma^{\mu}|\bar{l}]\frac{\partial}{\partial x_{12}^{-\,\mu}}, \tag{C.3}$$

for the three different correlators. Making use of the momentum representation of the scalar propagator

$$\int_{+} \mathrm{d}^2\lambda\, \mathrm{d}^2\bar{\lambda}\, \exp\left(-\frac{\mathrm{i}}{2}\langle\lambda|x|\bar{\lambda}]\right) = -16\pi^2\frac{1}{x^2}, \tag{C.4}$$

it is possible to perform the corresponding integrations and we obtain the solutions

$$\begin{aligned}
\Delta^{\alpha\beta\gamma\delta}(x_{12}^{+},\theta_{12}) &= -\frac{1}{4\pi^2}\frac{l^{\alpha}l^{\beta}l^{\gamma}l^{\delta}\delta^{0|4}(\theta_{12}|x_{12}^{+}|\bar{l}])}{\langle l|x_{12}^{+}|\bar{l}]^4(x_{12}^{+})^2},\\
\Delta^{\alpha\beta\dot{\gamma}\dot{\delta}}(x_{12}^{+-}) &= -\frac{1}{64\pi^2}\frac{l^{\alpha}l^{\beta}\bar{l}^{\dot{\gamma}}\bar{l}^{\dot{\delta}}(x_{12}^{+-})^2(\alpha+\log(x_{12}^{+-})^2)}{\langle l|x_{12}^{+-}|\bar{l}]^2} - \frac{1}{64\pi^2}\frac{l^{\{\alpha}(x_{12}^{+-})^{\beta\}\{\dot{\gamma}}\bar{l}^{\dot{\delta}\}}}{\langle l|x_{12}^{+-}|\bar{l}]},\\
\Delta^{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}}(x_{12}^{-},\bar{\theta}_{12}) &= -\frac{1}{4\pi^2}\frac{\bar{l}^{\dot{\alpha}}\bar{l}^{\dot{\beta}}\bar{l}^{\dot{\gamma}}\bar{l}^{\dot{\delta}}\delta^{0|4}(\langle l|x_{12}^{-}|\bar{\theta}_{12})}{\langle l|x_{12}^{-}|\bar{l}]^4(x_{12}^{-})^2}.
\end{aligned} \tag{C.5}$$

where α is an unspecified integration constant.

One can confirm that these propagators are harmonic functions (3.14): contractions of the second derivatives in x and θ all vanish. Furthermore, the duality constraint (3.15) is fulfilled. To see this, it is possible to evaluate the equation

$$D^{(1)}_{a\alpha}D^{(1)}_{b\beta}\Delta^{\alpha\beta\dot{\gamma}\dot{\delta}}(x_{12}^{+-}) = -\tfrac{1}{2}\varepsilon_{abcd}\bar{D}^{(1)c}{}_{\dot{\alpha}}\bar{D}^{(1)d}{}_{\dot{\beta}}\Delta^{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}}(x_{12}^{-},\bar{\theta}_{12}) \tag{C.6}$$

at $\theta_1 = 0$ using a supersymmetry translation. The left-hand side is given by

$$D^{(1)}_{a\alpha}D^{(1)}_{b\beta}\Delta^{\alpha\beta\dot{\gamma}\dot{\delta}}(x_{12}^{+-})\Big| = \frac{1}{4\pi^2}\frac{\bar{l}^{\dot{\gamma}}\bar{l}^{\dot{\delta}}\langle l|x_{12}^{-}|\bar{\theta}_{12}]_a\langle l|x_{12}^{-}|\bar{\theta}_{12}]_b}{\langle l|x_{12}^{-}|\bar{l}]^2(x_{12}^{-})^2} \tag{C.7}$$

which is easily seen to be equal to the right-hand side of (C.6) when $\theta_1 = 0$. Similar considerations work for (3.15) between chiral–chiral and mixed-chirality correlators.

C.2. Mixed edge correlator

Let us consider the mixed-chirality correlator $\langle 0|A_j^{+}A_k^{-}|0\rangle$ in the chiral decomposition (4.6) and propagator (C.1). Importantly, the mixed propagator depends only on the mixed-chirality interval $x_{jk}^{+-} = -x_j^{+} + x_k^{-} + 4\mathrm{i}\theta_j\bar{\theta}_k$. There is no explicit dependence on the fermionic coordinates; they merely enter through x_{jk}^{+-}. This fact simplifies the result somewhat

$$\begin{aligned}
\langle 0|A_j^{+}A_k^{-}|0\rangle = &-\left(-\frac{1}{2}(\mathrm{d}x_j^{+})^{\dot{\beta}\gamma} + 2\mathrm{i}(\mathrm{d}\theta_j)^{\gamma b}(\bar{\theta}_k)_b{}^{\dot{\beta}}\right)\left(\frac{1}{2}(\mathrm{d}x_k^{-})^{\dot{\gamma}\beta} - 2\mathrm{i}(\mathrm{d}\bar{\theta}_k)_b{}^{\dot{\gamma}}(\theta_j)^{\beta b}\right)\\
&\times\partial_{\delta\dot{\beta}}\partial_{\beta\dot{\delta}}\Delta^{\delta}{}_{\gamma}{}^{\dot{\delta}}{}_{\dot{\gamma}}(x_{jk}^{+-}) - 2\mathrm{i}(\mathrm{d}\theta_j)^{\gamma b}(\mathrm{d}\bar{\theta}_k)_b{}^{\dot{\gamma}}\partial_{\delta\dot{\delta}}\Delta^{\delta}{}_{\gamma}{}^{\dot{\delta}}{}_{\dot{\gamma}}(x_{jk}^{+-}).
\end{aligned} \tag{C.8}$$

Moreover, the propagator is a harmonic function and thus one can exchange the indices β and δ in the second derivative. Effectively, the propagator appears only in the combination

$$\partial_{\gamma\dot{\gamma}}\Delta^{\gamma\delta\dot{\gamma}\dot{\delta}}(x) = -\frac{1}{32\pi^2}\frac{l^{\delta}\bar{l}^{\dot{\delta}}(\alpha' + \log x^2)}{\langle l|x|\bar{l}]}, \qquad \alpha' = \alpha + 3. \tag{C.9}$$

Now consider the case where point j is restricted to a twistor and point k to a conjugate twistor. Using the parametrization in equation (4.13), we have

$$\begin{aligned}
x_j^{+}(\bar{\kappa},\sigma) &= x_j^{+} + \lambda_j\bar{\kappa}, & \theta_j(\bar{\kappa},\sigma) &= \theta_j + \lambda_j\sigma,\\
x_k^{-}(\kappa,\bar{\sigma}) &= x_k^{-} + \kappa\bar{\lambda}_k, & \bar{\theta}_k(\kappa,\bar{\sigma}) &= \bar{\theta}_k + \bar{\sigma}\bar{\lambda}_k.
\end{aligned} \tag{C.10}$$

Since the gauge connections are exact on the sides of the Wilson loop (see equation (4.14)), we can make the ansatz

$$\langle 0|A_j^+ A_k^-|0\rangle = \frac{1}{64\pi^2} d_j d_k I_{jk}^{+-}(x_{jk}^{+-}). \tag{C.11}$$

Comparing both sides, one finds the following two differential equations on the integral function I :

$$\begin{aligned}\frac{1}{64\pi^2}\lambda_j^{\delta}\bar{\lambda}_k^{\dot{\delta}}\partial_{\delta\dot{\delta}} I_{jk}^{+-}(x) &= -\lambda_j^{\alpha}\bar{\lambda}_k^{\dot{\alpha}}\partial_{\delta\dot{\delta}}\Delta^{\delta}{}_{\alpha}{}^{\dot{\delta}}{}_{\dot{\alpha}}(x),\\ \frac{1}{64\pi^2}\lambda_j^{\delta}\bar{\lambda}_k^{\dot{\delta}}\partial_{\beta\dot{\delta}}\partial_{\delta\dot{\beta}} I_{jk}^{+-}(x) &= -\lambda_j^{\gamma}\bar{\lambda}_k^{\dot{\gamma}}\partial_{\delta\dot{\beta}}\partial_{\beta\dot{\delta}}\Delta^{\delta}{}_{\gamma}{}^{\dot{\delta}}{}_{\dot{\gamma}}(x).\end{aligned} \tag{C.12}$$

Now in general we can assume that the spinors l (or $\bar{l}$) are not collinear to the λ_j (or $\bar{\lambda}_k$), and thus they form a basis for spinors. We decompose the coordinate x in this basis,

$$x = \frac{\langle l|x|\bar{l}]\lambda_j\bar{\lambda}_k - \langle j|x|\bar{l}]l\bar{\lambda}_k - \langle l|x|\bar{k}]\lambda_j\bar{l} + \langle j|x|\bar{k}]l\bar{l}}{\langle l\, j\rangle[k\bar{l}]}, \tag{C.13}$$

which implies that

$$x^2 = \frac{\langle l|x|\bar{l}]\langle j|x|\bar{k}] - \langle l|x|\bar{k}]\langle j|x|\bar{l}]}{\langle l\, j\rangle[k\bar{l}]}. \tag{C.14}$$

We rewrite the differential equations in these coordinates,

$$\frac{\partial I_{jk}^{+-}(x)}{\partial\langle l|x|\bar{l}]} = \frac{1}{\langle l|x|\bar{l}]}\log\frac{\langle l|x|\bar{l}]\langle j|x|\bar{k}] - \langle l|x|\bar{k}]\langle j|x|\bar{l}]}{\langle l\, j\rangle[k\bar{l}]} + \frac{\alpha'}{\langle l|x|\bar{l}]}, \tag{C.15}$$

$$\frac{\partial^2 I_{jk}^{+-}(x)}{\partial\langle j|x|\bar{l}]\partial\langle l|x|\bar{k}]} = \frac{1}{\langle l|x|\bar{l}]\langle j|x|\bar{k}] - \langle l|x|\bar{k}]\langle j|x|\bar{l}]}. \tag{C.16}$$

Up to functions of $(\langle l|x|\bar{k}], \langle j|x|\bar{k}])$ and of $(\langle j|x|\bar{l}], \langle j|x|\bar{k}])$, the solution reads

$$I_{jk}^{+-}(x) = \mathrm{Li}_2\left(\frac{\langle l|x|\bar{k}]\,\langle j|x|\bar{l}]}{\langle l|x|\bar{l}]\,\langle j|x|\bar{k}]}\right) + \frac{1}{2}\log^2\left(\frac{\langle\rho\, j\rangle[\bar{k}\bar{\rho}]}{\langle l|x|\bar{l}]\,\langle j|x|\bar{k}]}\right) + \alpha'\log\langle l|x|\bar{l}]\,. \tag{C.17}$$

Note that $\langle j|x|\bar{k}]$ is constant along the edges since

$$\langle j|x_{jk}^{+-}|\bar{k}] = \langle j|x_{j+1,k}^{+-}|\bar{k}] = \langle j|x_{j,k+1}^{+-}|\bar{k}] = \langle j|x_{j+1,k+1}^{+-}|\bar{k}] = 4\mathrm{i}W_j\bar{W}_k. \tag{C.18}$$

Now we can straightforwardly integrate $\langle A_j^+A_k^-\rangle$ along the two edges j and k which interpolate between the points x_j and x_{j+1} and between the points x_k and x_{k+1}. Because the gauge fields are exact differentials on these edges, we have only contributions from the boundary terms:

$$\sum_{j,k}\left(I_{jk}^{+-}(x_{j+1,k+1}^{+-}) - I_{jk}^{+-}(x_{j,k+1}^{+-}) - I_{jk}^{+-}(x_{j+1,k}^{+-}) + I_{jk}^{+-}(x_{j,k}^{+-})\right). \tag{C.19}$$

Just like for the other computations, here also we need a regularization. This can be done as in section 6.

The sum should also be independent of l and $\bar{l}$. This is not obvious since dilogarithm identities are necessary to show it. It is easy to see that the dependence on the integration constant α' cancels in the sum. We have also checked the independence of l and $\bar{l}$ at the level of the symbol and we have confirmed that the answer obtained in this way agrees with the answer obtained by the other methods described in this paper.

Appendix D. *R*-invariants

Let us show that the *R*-invariant

$$[j-1jk-1k\star] = \frac{\delta^{0|4}(\langle j-1jk-1k\rangle\chi_\star + \text{cycle})}{\langle j-1jk-1k\rangle\langle jk-1k\star\rangle\langle k-1k\star j-1\rangle\langle k\star j-1j\rangle\langle\star j-1jk-1\rangle} \tag{D.1}$$

is, up to a factor, the same as the one in equation (4.30), for $W_\star = (0, \bar\rho|0)$.

First, we need to compute $\langle aa+1b\star\rangle$. We have

$$\langle aa+1b\star\rangle = \begin{vmatrix} \lambda_a & \lambda_{a+1} & \lambda_b & 0 \\ \frac{1}{4}\langle a|x_a & \frac{1}{4}\langle a+1|x_{a+1} & \frac{1}{4}\langle b|x_b & \bar\rho \end{vmatrix} = \begin{vmatrix} \lambda_a & \lambda_{a+1} & \lambda_b & 0 \\ 0 & 0 & \frac{1}{4}\langle b|(x_b - x_{a+1}) & \bar\rho \end{vmatrix} = \frac{1}{4}\langle aa+1\rangle\langle b|x_b - x_{a+1}|\bar\rho], \tag{D.2}$$

where we have used equation (5.1) to rewrite the expressions.

Similarly, we can compute $\langle aa+1bb+1\rangle$. We find

$$\begin{aligned}\langle aa+1bb+1\rangle &= \begin{vmatrix} \lambda_a & \lambda_{a+1} & \lambda_b & \lambda_{b+1} \\ \frac{1}{4}\langle a|x_a & \frac{1}{4}\langle a+1|x_{a+1} & \frac{1}{4}\langle b|x_b & \frac{1}{4}\langle b+1|x_{b+1} \end{vmatrix} \\ &= \begin{vmatrix} \lambda_a & \lambda_{a+1} & \lambda_b & \lambda_{b+1} \\ 0 & 0 & \frac{1}{4}\langle b|(x_b - x_{a+1}) & \frac{1}{4}\langle b|(x_{b+1} - x_{a+1}) \end{vmatrix} \\ &= \frac{1}{16}\langle aa+1\rangle\langle bb+1\rangle(x_{a+1} - x_{b+1})^2. \end{aligned} \tag{D.3}$$

After some computation using $\chi_\star = 0$ and $\chi_j = \langle\lambda_j|\theta_j$, we find that

$$\delta^{0|4}(\langle j-1jk-1k\rangle\chi_\star + \text{cycle}) = \frac{1}{4^4}\langle j-1j\rangle^4\langle k-1k\rangle^4\delta^4(\theta_{j,k}|x^+_{j,k}|\bar\rho]). \tag{D.4}$$

Putting everything together, we find

$$[j-1jk-1k\star] = 16\frac{\langle j-1j\rangle\langle k-1k\rangle\delta^{0|4}(\theta_{k,j}|x^+_{k,j}|\bar\rho])}{\left(x^+_{k,j}\right)^2\langle j-1|x^+_{k,j}|\bar\rho]\langle j|x^+_{k,j}|\bar\rho]\langle k-1|x^+_{k,j}|\bar\rho]\langle k|x^+_{k,j}|\bar\rho]}, \tag{D.5}$$

which, up to a constant factor, is the same as the right-hand side of equation (4.30).

Appendix E. Invariants and cross-ratios

In this appendix, we contemplate about superconformal invariants which can be constructed from the ambitwistor variables W_j and $\bar W_j$. Finite superconformal observables should be their functions.

In the absence of fermionic d.o.f., there are essentially two types of invariants: mixed and chiral brackets

$$\langle jk] = W^a_j \bar W_{k,a}, \qquad \langle jkmn\rangle = \varepsilon_{abcd}W^a_j W^b_k W^c_m W^d_n. \tag{E.1}$$

Due to momentum ambitwistor constraints, the two types of brackets are even related $\langle jk] \sim \langle j, k-1, k, k+1\rangle$, and it suffices to consider only the chiral brackets. Superconformal invariants are constructed from these quantities, taking care that the overall twistor weights vanish.

In the full superspace, the picture is similar, but there are also important differences. First of all, the totally antisymmetric tensor ε_{abcd} is not superconformally invariant, and there is no replacement in the form of a tensor. Nevertheless, there is a generalization of the chiral bracket which exists for superalgebras[26],

$$\int \mathrm{d}^n s f(s) \delta^{4|4}(s \cdot W), \tag{E.2}$$

where $f(s)$ is some function of n variables s_k. The Grassmannian integrals (see [30, 53, 54]) yielding the N^{k}MHV tree-level scattering amplitudes are precisely of this kind. The essential feature of this integral is that it merely depends on chiral data.

On the other hand, the mixed-chiral bracket $\langle j, k]$ generalizes straightforwardly to the supersymmetric case. In (finite) loop corrections to scattering amplitudes, these brackets usually appear in cross-ratios with balanced twistor weights,

$$X_{j,k} := \frac{\langle j{-}1, k]\langle j, k-1]}{\langle j{-}1, k-1]\langle j, k]}. \tag{E.3}$$

The main difference between the bosonic and supersymmetric case is that in the presence of fermions, there exist additional conformal cross-ratios. Let us briefly consider the cases of $n = 4, 5, 6$ edges. In the bosonic case, it is well known that there exist no conformal cross-ratios for $n = 4, 5$, and there are three cross-ratios for $n = 6$. In the supersymmetric case, there are also no superconformal invariants for $n = 4$. However, for $n = 5$, there is the following superconformal invariant[27]:

$$\frac{\langle 13]\langle 35]\langle 52]\langle 24]\langle 41]}{\langle 14]\langle 42]\langle 25]\langle 53]\langle 31]} = 1 + \mathcal{O}(\chi\bar{\chi}). \tag{E.4}$$

When the fermions are discarded, there are some relations to show that this expression is exactly 1, and hence cannot be considered an independent invariant. In the presence of fermions, however, the invariant receives non-trivial corrections in $\chi\bar{\chi}$.

For $n = 6$, the picture is similar. We find $6 + 1$ independent superconformal invariants,

$$X_{k,k+3} = \frac{\langle k-1, k+3]\langle k, k+2]}{\langle k-1, k+2]\langle k, k+3]} \quad \text{and} \quad \frac{\langle 13]\langle 24]\langle 35]\langle 46]\langle 51]\langle 62]}{\langle 14]\langle 25]\langle 36]\langle 41]\langle 52]\langle 63]}. \tag{E.5}$$

In the bosonic case, these are also invariant, but there are four constraints which leave behind the well-known three independent conformal cross-ratios.

It would be interesting to investigate further the number of independent superconformal cross-ratios. In the bosonic case, some general considerations of the dimension of the (little) group and number of d.o.f. yield the answer. However, the presence of fermionic variables obscures the counting somewhat.

References

[1] Beisert N *et al* 2012 Review of AdS/CFT integrability: an overview *Lett. Math. Phys.* **99** 3 (arXiv:1012.3982)

[2] Eden B, Korchemsky G P and Sokatchev E 2011 From correlation functions to scattering amplitudes *J. High Energy Phys.* JHEP12(2011)002 (arXiv:1007.3246)

[3] Alday L F, Eden B, Korchemsky G P, Maldacena J and Sokatchev E 2011 From correlation functions to Wilson loops *J. High Energy Phys.* JHEP09(2011)123 (arXiv:1007.3243)

[4] Eden B, Heslop P, Korchemsky G P and Sokatchev E 2011 The super-correlator/super-amplitude duality: part I arXiv:1103.3714

[5] Eden B, Heslop P, Korchemsky G P and Sokatchev E 2011 The super-correlator/super-amplitude duality: part II arXiv:1103.4353

[6] Alday L F and Maldacena J M 2007 Gluon scattering amplitudes at strong coupling *J. High Energy Phys.* JHEP06(2007)064 (arXiv:0705.0303)

[26] Note that one can write $\langle 1234\rangle$ as the integral $\int \mathrm{d}^{0|4}\sigma\, \delta^{0|4}(\sigma \cdot W)$ over four fermionic variables σ_k.

[27] Note that this invariant cannot be written as a rational function of the above cross-ratios.

[7] Alday L F, Henn J M, Plefka J and Schuster T 2010 Scattering into the fifth dimension of $\mathcal{N} = 4$ super Yang–Mills *J. High Energy Phys.* JHEP01(2010)077 (arXiv:0908.0684)
[8] Belitsky A V, Korchemsky G P and Sokatchev E 2012 Are scattering amplitudes dual to super Wilson loops? *Nucl. Phys.* B **855** 333 (arXiv:1103.3008)
[9] Bern Z, Czakon M, Kosower D, Roiban R and Smirnov V A 2006 Two-loop iteration of five-point $\mathcal{N} = 4$ super-Yang–Mills amplitudes *Phys. Rev. Lett.* **97** 181601 (arXiv:hep-th/0604074)
[10] Bern Z, Dixon L J, Kosower D A, Roiban R, Spradlin M, Vergu C and Volovich A 2008 The two-loop six-gluon MHV amplitude in maximally supersymmetric Yang–Mills theory *Phys. Rev.* D **78** 045007 (arXiv:0803.1465)
[11] Kosower D A, Roiban R and Vergu C 2011 The six-point NMHV amplitude in maximally supersymmetric Yang–Mills theory *Phys. Rev.* D **83** 065018 (arXiv:1009.1376)
[12] Berkovits N and Maldacena J 2008 Fermionic T-duality, dual superconformal symmetry, and the amplitude/Wilson loop connection *J. High Energy Phys.* JHEP09(2008)062 (arXiv:0807.3196)
[13] Anastasiou C, Bern Z, Dixon L J and Kosower D A 2003 Planar amplitudes in maximally supersymmetric Yang–Mills theory *Phys. Rev. Lett.* **91** 251602 (arXiv:hep-th/0309040)
[14] Bern Z, Dixon L J and Smirnov V A 2005 Iteration of planar amplitudes in maximally supersymmetric Yang–Mills theory at three loops and beyond *Phys. Rev.* D **72** 085001 (arXiv:hep-th/0505205)
[15] Brandhuber A, Heslop P and Travaglini G 2008 MHV amplitudes in $\mathcal{N} = 4$ super Yang–Mills and Wilson loops *Nucl. Phys.* B **794** 231 (arXiv:0707.1153)
[16] Drummond J M, Henn J, Korchemsky G P and Sokatchev E 2008 On planar gluon amplitudes/Wilson loops duality *Nucl. Phys.* B **795** 52 (arXiv:0709.2368)
[17] Drummond J M, Henn J, Korchemsky G P and Sokatchev E 2010 Conformal Ward identities for Wilson loops and a test of the duality with gluon amplitudes *Nucl. Phys.* B **826** 337 (arXiv:0712.1223)
[18] Drummond J M, Henn J, Korchemsky G P and Sokatchev E 2008 The hexagon Wilson loop and the BDS ansatz for the six-gluon amplitude *Phys. Lett.* B **662** 456 (arXiv:0712.4138)
[19] Anastasiou C *et al* 2009 Two-loop polygon Wilson loops in $\mathcal{N} = 4$ SYM *J. High Energy Phys.* JHEP05(2009)115 (arXiv:0902.2245)
[20] Arkani-Hamed N, Bourjaily J L, Cachazo F, Caron-Huot S and Trnka J 2011 The all-loop integrand for scattering amplitudes in planar $\mathcal{N} = 4$ SYM *J. High Energy Phys.* JHEP01(2011)041 (arXiv:1008.2958)
[21] Arkani-Hamed N, Bourjaily J L, Cachazo F and Trnka J 2010 Local integrals for planar scattering amplitudes arXiv:1012.6032
[22] Mason L J and Skinner D 2010 The complete planar *S*-matrix of $\mathcal{N} = 4$ SYM as a Wilson loop in twistor space *J. High Energy Phys.* JHEP12(2010)018 (arXiv:1009.2225)
[23] Caron-Huot S 2011 Notes on the scattering amplitude/Wilson loop duality *J. High Energy Phys.* JHEP07(2011)058 (arXiv:1010.1167)
[24] Caron-Huot S 2011 Superconformal symmetry and two-loop amplitudes in planar $\mathcal{N} = 4$ super Yang–Mills *J. High Energy Phys.* JHEP12(2011)066 (arXiv:1105.5606)
[25] Bullimore M and Skinner D 2011 Descent equations for superamplitudes arXiv:1112.1056
[26] Caron-Huot S and He S 2011 Jumpstarting the all-loop *S*-matrix of planar $\mathcal{N} = 4$ super Yang–Mills arXiv:1112.1060
[27] Drummond J M, Henn J, Korchemsky G P and Sokatchev E 2010 Dual superconformal symmetry of scattering amplitudes in $\mathcal{N} = 4$ super-Yang–Mills theory *Nucl. Phys.* B **828** 317 (arXiv:0807.1095)
[28] Brandhuber A, Heslop P and Travaglini G 2008 A note on dual superconformal symmetry of the $\mathcal{N} = 4$ super Yang–Mills *S*-matrix *Phys. Rev.* D **78** 125005 (arXiv:0807.4097)
[29] Drummond J M, Henn J M and Plefka J 2009 Yangian symmetry of scattering amplitudes in $\mathcal{N} = 4$ super Yang–Mills theory *J. High Energy Phys.* JHEP05(2009)046 (arXiv:0902.2987)
[30] Drummond J M and Ferro L 2010 Yangians, Grassmannians and T-duality *J. High Energy Phys.* JHEP07(2010)027 (arXiv:1001.3348)
[31] Hodges A 2009 Eliminating spurious poles from gauge-theoretic amplitudes arXiv:0905.1473
[32] Mason L J and Skinner D 2009 Dual superconformal invariance, momentum twistors and Grassmannians *J. High Energy Phys.* JHEP11(2009)045 (arXiv:0909.0250)
[33] Bern Z, Dixon L J, Dunbar D C and Kosower D A 1994 One loop *n* point gauge theory amplitudes, unitarity and collinear limits *Nucl. Phys.* B **425** 217 (arXiv:hep-ph/9403226)
[34] Alday L F, Gaiotto D, Maldacena J, Sever A and Vieira P 2011 An operator product expansion for polygonal null Wilson loops *J. High Energy Phys.* JHEP04(2011)088 (arXiv:1006.2788)
[35] Sohnius M F 1978 Bianchi identities for supersymmetric gauge theories *Nucl. Phys.* B **136** 461
[36] Galperin A S, Ivanov E A, Ogievetsky V I and Sokatchev E S 2001 *Harmonic Superspace* (Cambridge: Cambridge University Press)

[37] Siegel W 2011 New superspaces/algebras for superparticles/strings arXiv:1106.1585
[38] Mandelstam S 1983 Light cone superspace and the ultraviolet finiteness of the $\mathcal{N} = 4$ model *Nucl. Phys.* B **213** 149
[39] Brink L, Lindgren O and Nilsson B E W 1983 $\mathcal{N} = 4$ Yang–Mills theory on the light cone *Nucl. Phys.* B **212** 401
[40] Parkes A 1992 A cubic action for self-dual Yang–Mills *Phys. Lett.* B **286** 265 (arXiv:hep-th/9203074)
[41] Siegel W 1992 The $\mathcal{N} = 2$ (4) string is self-dual $\mathcal{N} = 4$ Yang–Mills *Phys. Rev.* D **46** R3235 (arXiv:hep-th/9205075)
[42] Beisert N and Vergu C 2012 On the geometry of null polygons in full $\mathcal{N} = 4$ superspace arXiv:1203.0525
[43] Witten E 1978 An interpretation of classical Yang–Mills theory *Phys. Lett.* B **77** 394
[44] Witten E 1986 Twistor-like transform in ten dimensions *Nucl. Phys.* B **266** 245
[45] Bullimore M, Mason L J and Skinner D 2010 MHV diagrams in momentum twistor space *J. High Energy Phys.* JHEP12(2010)032 (arXiv:1009.1854)
[46] Mason L J and Skinner D 2006 An ambitwistor Yang–Mills Lagrangian *Phys. Lett.* B **636** 60 (arXiv:hep-th/0510262)
[47] Goncharov A B, Spradlin M, Vergu C and Volovich A 2010 Classical polylogarithms for amplitudes and Wilson loops *Phys. Rev. Lett.* **105** 151605 (arXiv:1006.5703)
[48] Gaiotto D, Maldacena J, Sever A and Vieira P 2011 Pulling the straps of polygons *J. High Energy Phys.* JHEP12(2011)011 (arXiv:1102.0062)
[49] Beisert N and Schwab B U W 2011 Bonus Yangian symmetry for the planar *S*-matrix of $\mathcal{N} = 4$ super Yang–Mills *Phys. Rev. Lett.* **106** 231602 (arXiv:1103.0646)
[50] Belitsky A V 2012 Conformal anomaly of super Wilson loop *Nucl. Phys.* B **862** 430 (arXiv:1201.6073)
[51] Drummond J M, Ferro L and Ragoucy E 2011 Yangian symmetry of light-like Wilson loops *J. High Energy Phys.* JHEP11(2011)049 (arXiv:1011.4264)
[52] Beisert N 2009 T-duality, dual conformal symmetry and integrability for strings on $AdS_5 \times S^5$ *Fortschr. Phys.* **57** 329 (arXiv:0903.0609)
[53] Arkani-Hamed N, Cachazo F, Cheung C and Kaplan J 2010 A duality for the *S* matrix *J. High Energy Phys.* JHEP03(2010)020 (arXiv:0907.5418)
[54] Korchemsky G P and Sokatchev E 2010 Superconformal invariants for scattering amplitudes in $\mathcal{N} = 4$ SYM theory *Nucl. Phys.* B **839** 377 (arXiv:1002.4625)