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Variable transformation defects

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Abstract. We investigate defects between supersymmetric Landau-Ginzburg models whose superpotentials are related by a variable transformation. It turns out that there is one natural defect, which can then be used to relate boundary conditions and defects in the different models. In particular this defect can be used to relate Grassmannian Kazama-Suzuki models and minimal models, and one can generate rational boundary conditions in the Kazama-Suzuki models from those in minimal models. The defects that appear here are closely related to the defects that are used in Khovanov-Rozansky link homology.

1. Introduction

Matrix factorisations are a beautiful mathematical subject in the sense that they are easy to define and still have a lot of interesting structures. Furthermore they can be used and applied in physics, where they describe boundary conditions and defects in $N = 2$ supersymmetric Landau-Ginzburg (LG) models (see e.g. [**17**] for an overview).

In the simplest setting, a matrix factorisation consists of two quadratic matrices p^0 and p^1 of the same size with polynomial entries whose product is the identity matrix multiplied by a given potential $W \in \mathbb{C}[x_1, \ldots, x_n]$,

$$p^0 \cdot p^1 = W \cdot \mathbb{1} \ , \quad p^1 \cdot p^0 = W \cdot \mathbb{1} \ . \tag{1.1}$$

An example of that is given by

$$x^{\ell} \cdot x^{k-\ell} = x^k \ , \tag{1.2}$$

where p^0 and p^1 are just polynomials (1×1 matrices). This example describes B-type boundary conditions in $N = 2$ minimal models [**18, 6**].

In addition to boundary conditions one can also consider B-type defects between Landau-Ginzburg models with superpotentials W and W'. They can be described by matrix factorisations of the difference $W - W'$ [**19, 20, 8**]. Defects are very important and useful objects in two-dimensional field theory: one of their most crucial properties is that they can be fused by bringing them on top of each other to produce a new defect [**23, 8**]. In such a way, defects define an interesting algebraic structure that turns out to be useful in analysing symmetries and dualities (see e.g. [**14**]), and bulk and boundary renormalisation group flows (see e.g. [**16, 3, 9, 13**]) in such models. As defects can also be fused onto boundaries, they may be used to relate or to generate boundary conditions. In particular, if we know defects between different theories, we can generate boundary conditions in one model from boundary conditions in the other model by fusion of the defect.

In this work we will analyse defects between LG models with potentials W and W' that are related by a variable transformation. If these transformations are

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non-linear, the two physical theories will be different. We will see that in such a situation there is one natural defect that acts in a simple, but non-trivial way on matrix factorisations. After analysing its properties we will apply it in a number of examples. In particular we demonstrate how it can be used to generate matrix factorisations in Kazama-Suzuki models from those in minimal models.

2. Variable transformations via defects

A B-type defect separating two $N = 2$ supersymmetric Landau-Ginzburg models with superpotentials W and W', respectively, can be described by a matrix factorisation of the difference $W - W'$ of the potentials [**8**]. To be more precise, let R and R' be polynomial algebras over $\mathbb{C}$, and $W \in R$, $W' \in R'$. A (W, W')-defect matrix factorisation is then a pair $({}_RM_{R'}, Q)$ where ${}_RM_{R'} = {}_RM^0_{R'} \oplus {}_RM^1_{R'}$ is a free, $\mathbb{Z}/2\mathbb{Z}$ graded R-R'-bimodule, and Q is an odd bimodule map,

$$Q = \begin{pmatrix} 0 & p^1 \\ p^0 & 0 \end{pmatrix} , \tag{2.1}$$

such that $Q^2 = W \cdot \mathrm{id}_M - \mathrm{id}_M \cdot W'$. As M is assumed to be free, Q can be written as a matrix with polynomial entries. A B-type boundary condition is a special defect, for which one side is trivial, e.g. $R' = \mathbb{C}$, $W' = 0$.

Morphisms between defects $({}_RM_{R'}, Q)$ and $({}_R\tilde{M}_{R'}, \tilde{Q})$ are bimodule maps $\varphi : M \to \tilde{M}$ with $\tilde{Q} \circ \varphi = \varphi \circ Q$ modulo exact maps of the form $\tilde{Q} \circ \psi + \psi \circ Q$. Matrix factorisations are considered to be equivalent if there exist two morphisms $\phi : M \to \tilde{M}$ and $\psi : \tilde{M} \to M$ such that $\phi \circ \psi$ and $\psi \circ \phi$ equal the identity map up to exact terms. Consider e.g. $({}_RM_R, Q)$ and $({}_RM_R, S \circ Q \circ S^{-1})$ for an even isomorphism $S : M \to M$. These factorisations are then equivalent with the morphisms being $\phi = S$ and $\psi = S^{-1}$. When we write the $\mathbb{Z}/2\mathbb{Z}$ gradation explicitly, the action of $S = \left(\begin{smallmatrix} s^0 & 0 \\ 0 & s^1 \end{smallmatrix}\right)$ on p^0 and p^1 amounts to similarity transformations,

$$p^0 \mapsto s^1 p^0 (s^0)^{-1} , \quad p^1 \mapsto s^0 p^1 (s^1)^{-1} . \tag{2.2}$$

One of the most interesting properties of defects is that they can be fused. Physically this means that two defects can be put on top of each other producing a new defect [**23, 8**]. Mathematically this amounts to define the tensor product [**24**] of two matrix factorisations $({}_RM_{R'}, Q)$ and $({}_{R'}\tilde{M}_{R''}, \tilde{Q})$. As a module this is simply the graded tensor product

$$M \otimes \tilde{M} = \left(M^0 \otimes_{R'} \tilde{M}^0 \oplus M^1 \otimes_{R'} \tilde{M}^1\right) \oplus \left(M^1 \otimes_{R'} \tilde{M}^0 \oplus M^0 \otimes_{R'} \tilde{M}^1\right) , \tag{2.3}$$

and the associated module map is

$$Q \hat{\otimes} \tilde{Q} := \begin{pmatrix} 0 & 0 & p^1 \otimes \mathrm{id} & \mathrm{id} \otimes \tilde{p}^1 \\ 0 & 0 & -\mathrm{id} \otimes \tilde{p}^0 & p^0 \otimes \mathrm{id} \\ p^0 \otimes \mathrm{id} & -\mathrm{id} \otimes \tilde{p}^1 & 0 & 0 \\ \mathrm{id} \otimes \tilde{p}^0 & p^1 \otimes \mathrm{id} & 0 & 0 \end{pmatrix} . \tag{2.4}$$

For $R' = R$ and $W' = W$, there is a special defect called the identity defect, which we denote by $({}_RI_R, {}_W\mathcal{I}_W)$. Fusing the identity defect onto some defect reproduces the original defect, it serves therefore as a unit object with respect to the tensor product. Its precise construction can be found in [**20, 19, 10**].

For different superpotentials $W \in R$ and $W' \in S$ there is in general no natural defect factorisation. On the other hand, if there exists a ring homomorphism

$$\phi : R \to S \ , \quad \text{such that } \phi(W) = W' \ , \tag{2.5}$$

then we can naturally map R-modules to S-modules and vice versa by extension or restriction of scalars: via the homomorphism ϕ the ring S has a natural R-S-bimodule structure, ${}_RS_S$, where the multiplication from the left is defined via the homomorphism ϕ. Given a right R-module M_R we can then map it to a right S-module by

$$\phi^* : M_R \mapsto (M_R) \otimes_R ({}_RS_S) \ , \tag{2.6}$$

which describes the extension of scalars from R to S. On the other hand, a left S-module ${}_S\tilde{M}$ has a natural R-module structure using the homomorphism ϕ. This restriction of scalars from S to R can be written as the map

$$\phi_* : {}_S\tilde{M} \mapsto ({}_RS_S) \otimes_S ({}_S\tilde{M}) \ . \tag{2.7}$$

ϕ^* and ϕ_* act also on module homomorphisms in an obvious way, so they define functors on the categories of R- and S-modules. Notice that ϕ^* maps free modules to free modules, whereas this is not guaranteed for ϕ_*. We assume in the following that the R-module ${}_RS$ is free, such that ϕ_* maps free modules to free modules.

We can apply these functors also to matrix factorisations. In particular we can apply them to the identity factorisations $({}_RI_R, {}_W\mathcal{I}_W)$ and $({}_SI_S, {}_{W'}\mathcal{I}_{W'})$ to obtain two (W,W')-defects with $W' = \phi(W)$,

$$({}_RI_S^A, {}_W\mathcal{I}_{W'}^A) = (\phi^*({}_RI_R), \phi^*({}_W\mathcal{I}_W)) \ , \quad ({}_RI_S^B, {}_W\mathcal{I}_{W'}^B) = (\phi_*({}_SI_S), \phi_*({}_{W'}\mathcal{I}_{W'})) \ . \tag{2.8}$$

We now claim that these two defects are actually equivalent. To show this we take the first defect and fuse the identity defect ${}_SI_S$ from the right, and compare it to the second defect onto which we fuse the identity defect ${}_RI_R$ from the left. As a module we obtain

$$({}_RI_S^A) \otimes_S ({}_SI_S) \cong ({}_RI_R) \otimes_R ({}_RS_S) \otimes_S ({}_SI_S) \cong ({}_RI_R) \otimes_R ({}_RI_S^B) \ . \tag{2.9}$$

Since

$$({}_W\mathcal{I}_W \otimes_R \mathrm{id}_S) \,\hat{\otimes}\, ({}_{W'}\mathcal{I}_{W'}) = ({}_W\mathcal{I}_W) \,\hat{\otimes}\, (\mathrm{id}_S \otimes_S \ {}_{W'}\mathcal{I}_{W'}) \ , \tag{2.10}$$

also the factorisations agree, so that we indeed find that these two defects are equivalent. We call them $({}_RI_S, {}_W\mathcal{I}_{W'})$.

By a similar consideration as above we see that when we fuse $({}_RI_S, {}_W\mathcal{I}_{W'})$ to the left, it acts by the functor ϕ^*, whereas it acts by the functor ϕ_* when we fuse it to defects to the right. Thus we have a very simple description for the fusion result for this defect. Analogously we can construct the defect $({}_SI_R, {}_{W'}\mathcal{I}_W)$.

Let us explicitly describe how the defect $({}_RI_S, {}_W\mathcal{I}_{W'})$ acts by fusion. First consider the (simpler) fusion to the left on a defect $({}_{R'}M_R, Q)$. For a rank $2m$ free R'-R-bimodule ${}_{R'}M_R$ we can think of Q as a $2m \times 2m$ matrix with entries Q_{ij} in $R' \otimes_{\mathbb{C}} R$. Fusing $({}_RI_S, {}_W\mathcal{I}_{W'})$ onto this defect from the right, we obtain a free R'-S-module of rank $2m$, and a matrix $\tilde{Q}$ with entries $\tilde{Q}_{ij} = (\mathrm{id} \otimes \phi)(Q_{ij})$, i.e. we just replace the variables of R by the variables of S via the map ϕ.

We now assume that ${}_RS$ is a finite rank free R-module,

$$\rho : R^{\oplus n} \xrightarrow{\sim} {}_RS \ . \tag{2.11}$$

With the help of the R-module isomorphism ρ we can then explicitly describe how the defect $({}_R I_S, {}_W\mathcal{I}_{W'})$ acts by fusion to the right on a defect $({}_S M_{S'}, Q)$. If ${}_S M_{S'}$ is free of rank $2m$, then Q can be represented as a $2m \times 2m$ matrix with entries $Q_{ij} \in S \otimes_{\mathbb{C}} S'$. After the fusion we have a R-S'-module of rank $2mn$, and each entry Q_{ij} is replaced by the $n \times n$ matrix that represents the map $\rho^{-1} \circ Q_{ij} \circ \rho$ (where we tacitly extend ρ to mean $\rho \otimes \mathrm{id}_{S'}$).

A particular situation occurs when all Q_{ij} are of the form $\phi(\tilde{Q}_{ij})$. As ρ is an R-module map, the map $\rho^{-1} \circ Q_{ij} \circ \rho$ can then be represented by the $n \times n$ matrix $\tilde{Q}_{ij} \cdot \mathbb{1}_{n\times n}$. The resulting defect is therefore a direct sum of n identical defects. As an example, consider the fusion of $({}_R I_S, {}_W\mathcal{I}_{W'})$ on $({}_S I_R, {}_{W'}\mathcal{I}_W)$. By the arguments above this fusion results in a direct sum of n identity defects,

$$({}_R I_S, {}_W\mathcal{I}_{W'}) \otimes ({}_S I_R, {}_{W'}\mathcal{I}_W) \cong ({}_R I_R, {}_W\mathcal{I}_W)^{\oplus n} \ . \tag{2.12}$$

In the special case that ϕ is a ring isomorphism, $\phi : R \to R$, and $W' = \phi(W) = W$, the construction above leads to symmetry or group-like defects $G^\phi = ({}_R M_R, (\mathrm{id} \otimes \phi)({}_W\mathcal{I}_W))$ (which have been discussed in [**14, 8**]). The fusion of such defects is particularly simple,

$$G^\phi \otimes G^\psi \cong G^{\psi\circ\phi} \ , \tag{2.13}$$

and G^ϕ is invertible with inverse $G^{\phi^{-1}}$. These defects therefore form a group.

3. Examples and applications

In this section we want to apply the formalism of the foregoing section to physically interesting examples.

Minimal models. Let us first look at the one variable case, $R = \mathbb{C}[y]$, and choose the potential to be $W(y) = y^k$. The corresponding Landau-Ginzburg model describes a minimal model at level $k-2$. Consider now the ring homomorphism

$$\phi_1 : p(y) \mapsto p(x^d) \tag{3.1}$$

that maps polynomials in R to those in $S = \mathbb{C}[x]$. The transformed potential is $W'(x) = x^{kd}$. We observe that ${}_R S$ is a free R-module of rank d,

$$\begin{array}{rrcl} \rho : & R^{\oplus d} & \to & S \\ & (p_1(y), \ldots, p_d(y)) & \mapsto & \sum_{j=1}^{d} x^{j-1} p_j(x^d) \ . \end{array} \tag{3.2}$$

Let us now look at the corresponding defect between these two minimal models. We consider the explicit construction $({}_R I_S^B, {}_W\mathcal{I}_{W'}^B)$ via ϕ_*. We start with the identity defect $({}_S I_S, {}_{W'}\mathcal{I}_{W'})$ that is given by a rank 2 matrix ${}_{W'}\mathcal{I}_{W'} = \left(\begin{smallmatrix} 0 & \imath_0 \\ \imath_1 & 0 \end{smallmatrix}\right)$ with $\imath_0 = (x - x')$ and $\imath_1 = (W(x) - W(x'))/(x - x')$. Here we denoted by x' the variable corresponding to the right S-module structure. Under the map ϕ_* acting on the left S-module structure the entry $\imath_0$ is then replaced by

$$\tilde{\imath}_0 = \begin{pmatrix} -x' & & & y \\ 1 & -x' & & \\ & \ddots & \ddots & \\ & & 1 & -x' \end{pmatrix} \xrightarrow[\text{transformation}]{\text{similarity}} \begin{pmatrix} y - x'^d & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \ . \tag{3.3}$$

We therefore explicitly see that this defect is equivalent to $({}_R I_S^A, {}_W\mathcal{I}_{W'}^A)$ that we obtain from the identity defect in y-variables by expressing one of the variables in terms of x. This defect is related to the generalised permutation boundary conditions in two minimal models [**11, 12**] by the folding trick.

We now want to apply this defect to matrix factorisations $({}_SM, Q)$ that describe boundary conditions. The elementary factorisation $x^\ell \cdot x^{kd-\ell}$ will be called $Q_\ell^{(x)}$, and correspondingly $Q_\ell^{(y)}$ refers to the y-factorisation $y^\ell \cdot y^{k-\ell}$. Fusing the defect $({}_RI_S, {}_W\mathcal{I}_{W'})$ to $(S^{\oplus 2}, Q_{rd+\ell}^{(x)})$ results in a superposition $\big(R^{\oplus d}, \big(Q_r^{(y)}\big)^{\oplus d-\ell} \oplus \big(Q_{r+1}^{(y)}\big)^{\oplus \ell}\big)$ (for $0 \le \ell \le d-1$). The factorisation $Q_0^{(y)}$ is trivial, so we see that the basic factorisation $Q_1^{(x)}$ is just mapped to the basic factorisation $Q_1^{(y)}$.

We can also consider defects in minimal models. Of particular interest are the group-like defects $G_{(y)}^n$ [**8**] that induce the map $y \mapsto \eta^n y$. Here $\eta = \exp \frac{2\pi i}{k}$ such that the potential y^k is invariant. Obviously we have $G^n \cong G^{n+k}$, and the group law is just $G_{(y)}^m \otimes G_{(y)}^n = G_{(y)}^{m+n}$. As a (W,W)-defect matrix factorisation, $G_{(y)}^n$ corresponds to $(y-\eta^n y') \cdot \frac{W(y)-W(y')}{y-\eta^n y'}$. Similarly, $G_{(x)}^n$ denotes the group-like defect corresponding to the map $x \mapsto \exp \frac{2\pi i n}{kd} x$. Given such a defect $G_{(x)}^n$ one can ask what happens to it when we sandwich it between the defects ${}_RI_S$ and ${}_SI_R$. Surprisingly the result can again be expressed in terms of group like-defects, namely

$$({}_RI_S, {}_W\mathcal{I}_{W'}) \otimes G_{(x)}^n \otimes ({}_SI_R, {}_{W'}\mathcal{I}_W) \cong \Big(G_{(y)}^n\Big)^{\oplus d} \; . \tag{3.4}$$

***SU*(3)/*U*(2) Kazama-Suzuki model.** As a more interesting example we look at a defect between an $SU(3)/U(2)$ Kazama-Suzuki model and a product of two minimal models. Consider the two variable polynomial rings $R = \mathbb{C}[y_1, y_2]$ and $S = \mathbb{C}[x_1, x_2]$, and the ring homomorphism

$$\phi : p(y_1, y_2) \mapsto p(x_1 + x_2, x_1 x_2) \; , \tag{3.5}$$

which replaces the y_i by the elementary symmetric polynomials in the x_j. The potential in x-variables is that of two minimal models,

$$W'(x_1, x_2) = x_1^k + x_2^k \qquad (k \ge 4) \; . \tag{3.6}$$

It is symmetric in x_1 and x_2 and thus it can be expressed in terms of the elementary symmetric polynomials leading to the potential $W(y_1, y_2)$ in the y-variables such that $\phi(W) = W'$. This then describes the $SU(3)/U(2)$ Kazama-Suzuki model (see e.g. [**15, 5**]).

The R-module ${}_RS$ is free of rank 2 with the explicit R-module isomorphism

$$\begin{array}{lrcl} \rho : & R \oplus R & \to & {}_RS \\ & (p_1(y_1,y_2), p_2(y_1,y_2)) & \mapsto & p_1(x_1+x_2, x_1x_2) + (x_1 - x_2) p_2(x_1+x_2, x_1x_2) \, , \end{array} \tag{3.7}$$

with inverse

$$\begin{array}{lrcl} \rho^{-1} : & {}_RS & \to & R \oplus R \\ & p(x_1,x_2) & \mapsto & \Big(p_s(x_1,x_2)\big|_{y_i}, \frac{p_a(x_1,x_2)}{x_1-x_2}\big|_{y_i} \Big) \; , \end{array} \tag{3.8}$$

where $p_{s/a}(x_1, x_2) = \frac{1}{2}(p(x_1,x_2) \pm p(x_2,x_1))$, and $|_{y_i}$ means to replace in a symmetric polynomial in x_j the elementary symmetric polynomials by the y_i.

The (W, W')-defect between the Kazama-Suzuki model (y-variables) and the minimal models (x-variables) acts on y-factorisations simply by replacing variables.

However, given an x-factorisation with matrix Q, a matrix element Q_{ij} is replaced by a 2×2 matrix,

$$Q_{ij} \mapsto \begin{pmatrix} (Q_{ij})_s & (x_1-x_2)(Q_{ij})_a \\ \frac{(Q_{ij})_a}{x_1-x_2} & (Q_{ij})_s \end{pmatrix}\Bigg|_{y_1,y_2} . \tag{3.9}$$

As an example consider the boundary condition based on the factorisation $(x_1 - \xi x_2)\cdot \frac{W'(x_1,x_2)}{x_1-\xi x_2}$ with $\xi = \exp\frac{\pi i}{k}$ (these are the so-called permutation factorisations [**2, 7**]). By the map (3.9) the factor $(x_1-\xi x_2)$ is mapped to

$$(x_1-\xi x_2) \mapsto \begin{pmatrix} \frac{1-\xi}{2}y_1 & \frac{1+\xi}{2}(y_1^2-4y_2) \\ \frac{1+\xi}{2} & \frac{1-\xi}{2}y_1 \end{pmatrix} \xrightarrow[\text{transf.}]{\text{similarity}} \begin{pmatrix} y_1^2 - 2(1+\cos\frac{\pi}{k+2})y_2 & 0 \\ 0 & 1 \end{pmatrix} . \tag{3.10}$$

This means that the linear polynomial factorisation in x is mapped to a polynomial factorisation in the y-variables. The interesting fact is now that both factorisations describe rational boundary states in the corresponding conformal field theories [**5**]. One can go further and consider the x-factorisation

$$\big((x_1-\xi x_2)(x_1-\xi^3 x_2)\big)\cdot \frac{W'(x_1,x_2)}{(x_1-\xi x_2)(x_1-\xi^3x_2)} = W'(x_1,x_2) . \tag{3.11}$$

The quadratic factor is mapped to

$$\begin{aligned} &(x_1-\xi x_2)(x_1-\xi^3x_2) \\ &\quad\mapsto \begin{pmatrix} \frac{1-\xi^4}{2}(y_1^2-2y_2)-(\xi+\xi^3)y_2 & \frac{1+\xi^4}{2}(y_1^2-4y_2)y_1 \\ \frac{1+\xi^4}{2}y_1 & \frac{1-\xi^4}{2}(y_1^2-2y_2)-(\xi+\xi^3)y_2 \end{pmatrix} \\ &\xrightarrow[\text{transf.}]{\text{similarity}} \begin{pmatrix} y_1^2-2(1+\cos\frac{\pi}{k})y_2 & 0 \\ y_1 & y_1^2-2(1+\cos\frac{3\pi}{k})y_2 \end{pmatrix} . \end{aligned} \tag{3.12}$$

Again this factorisation has been identified with a rational boundary condition in the Kazama-Suzuki model in [**5**]. This example shows that the variable transformation defect is indeed very useful to generate interesting matrix factorisations. In a subsequent publication we will show that with the help of this variable transformation defect, one also can generate rational defects in Kazama-Suzuki models which then allow to generate in principle all factorisations corresponding to rational boundary conditions in these models.

The defect considered here actually also appears in the link homology of Khovanov and Rozansky [**20**], namely the diagram on the right in figure 1 corresponds in our language to the defect $({}_{S'}I_R)\otimes({}_RI_S)$ (where $S'=\mathbb{C}[x_3,x_4]$). The diagram on the left of figure 1 simply corresponds to the identity defect in x-variables. One of the fundamental equivalences in the link homology displayed in figure 2 would read in our notation (with $S''=\mathbb{C}[x_5,x_6]$)

$$\big(({}_{S''}I_R)\otimes({}_RI_{S'})\big)\otimes\big(({}_{S'}I_R)\otimes({}_RI_S)\big) \cong \big(({}_{S''}I_R)\otimes({}_RI_S)\big)^{\oplus 2} , \tag{3.13}$$

which follows immediately from (2.12). It would be very interesting to also consider the morphisms between the defects in figure 1 that are needed to formulate the complex of defects assigned to crossings (see [**20**, figure 46]) in our framework, but we leave this for future work.

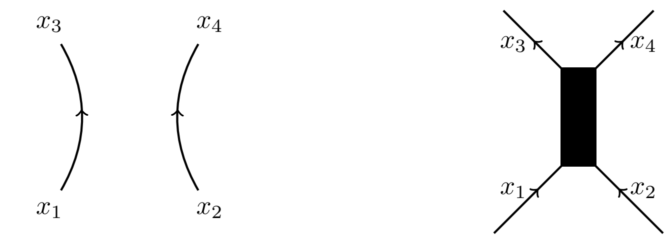

FIGURE 1. Basic building blocks that appear in the resolution of crossings [**20**, figure 9]: the identity defect ${}_{S'}I_S$ in x-variables to the left, and the basic wide-edge graph on the right corresponding to $({}_{S'}I_R) \otimes ({}_R I_S)$ (with $S = \mathbb{C}[x_1, x_2]$, $S' = \mathbb{C}[x_3, x_4]$).

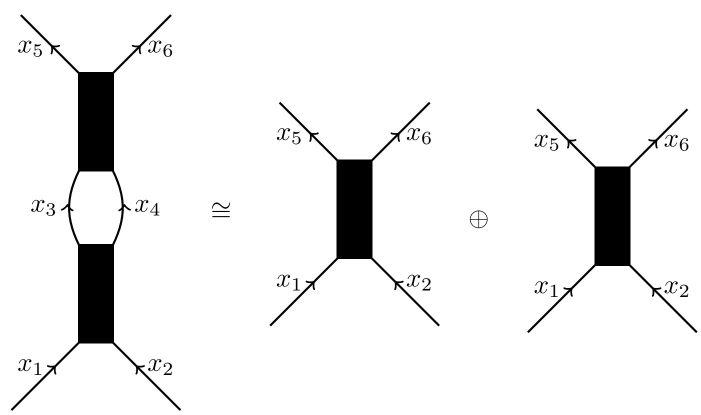

FIGURE 2. One of the fundamental diagram equivalences of [**20**, figure 35 and Prop. 30] (up to grading).

***$SU(n+1)/U(n)$* Kazama-Suzuki models.** The last example has a beautiful generalisation to a defect between a $SU(n+1)/U(n)$ Kazama-Suzuki model and n copies of minimal models. We consider the polynomial rings $R = \mathbb{C}[y_1, \ldots, y_n]$ and $S = \mathbb{C}[x_1, \ldots, x_n]$, and the potential

$$W'(x_1, \ldots, x_n) = x_1^k + \cdots + x_n^k \ . \tag{3.14}$$

The ring homomorphism is defined by

$$\phi(y_j) = \sum_{i_1 < \cdots < i_j} x_{i_1} \cdot \cdots \cdot x_{i_j} \ , \tag{3.15}$$

and it maps the y_j to the elementary symmetric polynomials in the x_i. It is an old result in invariant theory [**1**, section II.G] that ${}_R S$ is a free R-module of rank $n!$. To get an explicit R-module isomorphism between ${}_R S$ and $R^{\oplus n!}$, one needs to choose a good basis in S. The simplest choice [**1**] is to take the $n!$ polynomials given by

$$x_1^{\nu_1} x_2^{\nu_2} \cdot \cdots \cdot x_n^{\nu_n} \ , \text{ where } \nu_i \leq i - 1 \ . \tag{3.16}$$

Another possibility with some computational advantages is provided by the Schubert polynomials $X_\sigma(x_1, \ldots, x_n)$, for which there is one for each permutation σ in the symmetric group S_n. It was shown in [**21**] that any polynomial $p(x_1,\dots,x_n)$ has a unique expansion

$$p(x_1,\dots,x_n)=\sum_{\sigma\in S_n} p_\sigma(x_1,\dots,x_n)X_\sigma(x_1,\dots,x_n)\ , \tag{3.17}$$

where the p_σ are totally symmetric. The map ρ is then given by

$$\begin{aligned}\rho:\qquad\qquad R^{\oplus n!} &\to S\\ (p_\sigma(y_1,\dots,y_n))_{\sigma\in S_n} &\mapsto \sum_{\sigma\in S_n} p_\sigma(x_1+\dots+x_n,\dots,x_1\cdots x_n)X_\sigma(x_1,\dots,x_n)\,.\end{aligned} \tag{3.18}$$

4. Conclusion

We have seen that there is a natural defect between Landau-Ginzburg theories whose superpotentials are related by a variable transformation. The fusion of this defect onto other factorisations has an explicit and simple description via the functors ϕ^* and ϕ_* corresponding to extension and restriction of scalars.

The examples have shown that these defects can be used to relate boundary conditions or defects in different LG models. In particular, one can use such defects between minimal models and Grassmannian Kazama-Suzuki models to put into use the knowledge that is already available for minimal models to obtain factorisations for the Kazama-Suzuki models. In a such a way one can for example generate all factorisations corresponding to rational boundary conditions in the $SU(3)/U(2)$ model, as we will show in a subsequent publication.

In the $SU(3)/U(2)$ example it turns out that the defects discussed here are crucial to construct factorisations for rational topological defects. These finitely many elementary defects and their superpositions form a closed semi-ring that is isomorphic to the fusion semi-ring. Realising such a finite-dimensional semi-ring (in the sense that as a semi-group it is isomorphic to a direct product of finitely many copies of $\mathbb{N}_0$) in terms of defect matrix factorisations reflects the rational structure of the conformal field theory that is otherwise hard to see in the LG formulation. It would be interesting to investigate whether the existence of such an algebraic structure automatically signals an enhanced symmetry in the CFT.

Finally, we have seen that our defects generate the building blocks of Khovanov-Rozansky homology, except for the morphisms between defect building blocks. In other words, one could say that our formulation provides a physical setup of the Khovanov-Rozansky factorisations as a sequence of Kazama-Suzuki models separated by defects. By generalisation to $SU(n+1)/U(n)$ models, this is also true for the higher graphs appearing in the MOY calculus [**22**]. Our analysis can therefore be seen as a physical supplement to the recent results in [**4, 10**].

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