Spinor Fields and Symmetries of the Spacetime

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ABSTRACT

In the background of a stationary black hole, the "conserved current" of a particular spinor field always approaches the null Killing vector on the horizon. What's more, when the black hole is asymptotically flat and when the coordinate system is asymptotically static, then the same current also approaches the time Killing vector at the spatial infinity. We test these results against various black hole solutions and no exception is found. The spinor field only needs to satisfy a very general and simple constraint.

1 Introduction and Summary of Key Results

In [1] it was noticed that, for the Kerr black hole and the five dimensional Myers-Perry black hole, there exists a particular vector field which interpolates between the time Killing vector at the spatial infinity and the null Killing vector on the horizon. The existence of such a vector field can be very interesting in that it may contain important (possibly non-geometrical) information about the spacetime itself.

In this paper, we want to suggest that the existence of such a vector is a general feature of all stationary black holes. For all the examples tested, the vector field always approaches the null Killing vector on the horizon; when the black hole is asymptotically flat and when the coordinate system is asymptotically static, then the same vector field also approaches the time Killing vector at the spatial infinity. In [1], such features are interpreted as describing a possible fluid flow underlying the spacetime. Here, we want to leave the physical interpretation behind and merely demonstrate the existence of the vector field.

The vector field is constructed using an auxiliary spinor field,

$$\xi^\mu = c_\psi \bar{\psi}\gamma^\mu\psi\,, \quad \gamma^\mu = e_A{}^\mu\gamma^A\,, \tag{1}$$

where c_ψ is a constant and γ^A is defined in the vielbein basis, $e^A = e^A{}_\mu dx^\mu$. If ψ was to obey the Dirac equation, then (1) is nothing but the conserved current of the spinor field. Here we do not require ψ to be a Dirac fermion, but we will still occasionally refer to (1) as the "conserved current" for simplicity.

For a stationary and axisymmetric black hole, it is empirically known that one can always put the metric into one of the following forms [2]

$$ds^2 = -f_t\Delta(dt + f_a d\phi^a)^2 + \frac{f_r}{\Delta}dr^2 + h_i d\theta^{i2} + g_{ab}(d\phi^a - w^a dt)(d\phi^b - w^b dt)\,, \tag{2}$$

$$ds^2 = -f_t\Delta(f_a d\phi^a)^2 + \frac{f_r}{\Delta}dr^2 + h_i d\theta^{i2} + g_{ab}(d\phi^a - w^a dt)(d\phi^b - w^b dt)\,, \tag{3}$$

where $\Delta = \Delta(r)$, and the functions $f_t, f_a, f_r, h_i, g_{ab}$ and w^a only depend on r and θ^i's. The black hole horizon r_0 is located at the (largest) root of $\Delta(r_0) = 0$. For many solutions, one can explicitly choose the coordinate system to be non-rotating at the spatial infinity ($r \to +\infty$), and the spatial coordinates can be identified with those from a usual spherical coordinate system — namely the radius r, latitudinal angles θ^i ($i = 1, \cdots, [\frac{d}{2}] - 1$) and azimuthal angles ϕ^a ($a = 1, \cdots, [\frac{d+1}{2}] - 1$), where d is the dimension of the spacetime. In such cases, one has [2]

$$w^a \to \Omega^a \quad \text{as} \quad r \to r_0\,, \tag{4}$$

where Ω^a is the angular velocity of the black hole in the ϕ^a direction.

It is also empirically known that the functions f_t, f_r, h_i and the matrix (g_{ab}) can always be made *positive definite* near the black hole horizon [2]. For this reason, one can always rewrite (2) and (3) in terms of vielbeins,

$$ds_d^2 = \eta_{AB} e^A e^B\,, \quad A, B = 0, \cdots, d-1\,, \tag{5}$$

where $\eta = diag\{- + \cdots +\}$, and

$$\begin{aligned} e^0 = \sqrt{f_t\Delta}(dt + f_a d\phi^a) \quad \text{or} \quad e^0 = \sqrt{f_t\Delta}(f_a d\phi^a)\,, \quad e^1 = \sqrt{\frac{f_r}{\Delta}}dr\,, \\ e^{1+i} = \sqrt{h_i}d\theta^i \text{ (no summation over } i) : \quad i = 1, \cdots, [\frac{d}{2}] - 1\,, \\ e^{[\frac{d}{2}]+a} : \quad a = 1, \cdots, [\frac{d+1}{2}] - 1\,, \end{aligned} \tag{6}$$

where $e^{[\frac{d}{2}]+a}$'s are obtained by diagonalizing the last terms in (2) and (3). In general, (5) and (6) are only well defined near the black hole horizon, where all the vielbeins in (6) are real. But for many solutions given by (2), the functions f_t, f_r, h_i and the matrix (g_{ab}) are in fact positive definite in the whole region outside the black hole horizon [2]. So for these solutions, (5) and (6) are well defined in the whole region outside the black hole horizon.

Our main result of this paper is to demonstrate the following two results:

Result #1: *Given (2) (or (3)) and that the spinor field ψ obeys*[1]

$$(\gamma^0 + \gamma^d)\psi = 0 \quad \text{or} \quad (\gamma^0 - \gamma^d)\psi = 0\,, \tag{7}$$

then (1) always reduces to (Note one can always normalized ψ to let $\xi^t = 1$.)

$$\xi^\mu \partial_\mu = \partial_t + w^a \partial_{\phi^a}\,, \tag{8}$$

which means

$$\xi^2 = \begin{cases} -(1 + f_a w^a)^2 f_t \Delta & : \text{ for(2)}\,, \\ -(f_a w^a)^2 f_t \Delta & : \text{ for(3)}\,, \end{cases} \tag{9}$$

vanishes on the horizon. Since w^a's become constants on the horizon [2], ξ becomes nothing but the null Killing vector on the horizon.

[1] All the gamma matrices in (7) are defined in the vielbein basis. In even dimensions,

$$\gamma^d_{even} = (-1)^{\frac{d-2}{4}} \gamma^1 \cdots \gamma^{d-1}\gamma^0\,, \quad (\gamma^d_{even})^\dagger = \gamma^d_{even}\,, \quad (\gamma^d_{even})^2 = \mathbf{1}_d\,,$$

and in odd dimensions,

$$\gamma^d_{odd} = (-1)^{\frac{d-1}{4}} \gamma^1 \cdots \gamma^{d-1}\gamma^0 \propto \mathbf{1}_d\,, \quad (\gamma^d_{odd})^\dagger = -\gamma^d_{odd}\,, \quad (\gamma^d_{odd})^2 = -\mathbf{1}_d\,,$$

where $\mathbf{1}_d$ is the unit matrix in d dimensions.

Result #2 *When the black hole is asymptotically flat and when the coordinate system is asymptotically static, then*

$$w^a \to 0 \quad \text{as} \quad r \to +\infty\,. \tag{10}$$

So in this case, ξ interpolates between the time Killing vector at the spatial infinity and the null Killing vector on the horizon.

Note one can only test (10) for cases where one can explicitly find the vielbeins that are well defined in the whole region outside the black hole horizon. As a side remark, also note that given (2) (or (3)) and (8),

$$\nabla_\mu \xi^\mu = 0\,, \quad \Longrightarrow \quad \nabla_\rho \nabla_\mu \xi^\rho = R_{\mu\rho}\xi^\rho\,, \tag{11}$$

which partially justifies calling ξ the "conserved current".

In the next section, we will use black hole solutions in various spacetime dimensions to prove result #1. Then we will prove result #2 in the section that follows. After that, we will conclude with a short discussion.

2 Testing Result #1

In this section, we will show that (8) is true for the general metrics given in (2) and (3). Since both equations in (7) lead to the same result in (8), we will only use the first equation of (7) in all the following calculations.

It is difficult to do the calculations for all dimensions at once, so we will only test the result from three to eight dimensions. This should give us enough confidence in the generality of the result.

2.1 $d = 3$

The gamma matrices in the dreibein basis are chosen as[2]

$$\gamma^0 = i\sigma^3\,, \quad \gamma^1 = \sigma^1\,, \quad \gamma^2 = \sigma^2\,, \tag{12}$$

where $\sigma^{1,2,3}$ are the usual Pauli matrices. The spinor field is

$$\psi = \begin{pmatrix} \phi_{1a} + i\phi_{1b} \\ \phi_{2a} + i\phi_{2b} \end{pmatrix}\,, \tag{13}$$

[2]Note (1) is independent of the choice of the gamma matrices.

where all the functions are real. From the first equation in (7), we find

$$\phi_{2a} = \phi_{2b} = 0 \,. \tag{14}$$

In three dimensions, (2) becomes

$$ds^2 = -f_t(dt + f d\phi)^2 + f_r dr^2 + g_p(d\phi - w dt)^2 \,. \tag{15}$$

The corresponding dreibeins are

$$e^0 = \sqrt{f_t}(dt + f d\phi)\,, \quad e^1 = \sqrt{f_r} dr\,, \quad e^2 = \sqrt{g_p}(d\phi - w dt)\,. \tag{16}$$

Plug (14) into (1), we find

$$\xi^\mu = \partial_t + w\partial_\phi \,, \tag{17}$$

just as given in (8).

Similarly, (3) becomes

$$ds^2 = -f_t(f d\phi)^2 + f_r dr^2 + g_p(d\phi - w dt)^2 \,. \tag{18}$$

The corresponding dreibeins are

$$e^0 = \sqrt{f_t}(f d\phi)\,, \quad e^1 = \sqrt{f_r} dr\,, \quad e^2 = \sqrt{g_p}(d\phi - w dt)\,. \tag{19}$$

Plug (14) into (1), we also get

$$\xi^\mu \partial_\mu = \partial_t + w\partial_\phi \,, \tag{20}$$

as given in (8).

2.2 d=4

The gamma matrices in the vierbein basis are chose as

$$\gamma^0 = i\sigma^3 \otimes \mathbf{1}_2\,, \quad \gamma^j = -\sigma^2 \otimes \sigma^j\,, \quad j = 1, 2, 3\,. \tag{21}$$

The spinor field is

$$\psi = \begin{pmatrix} \phi_{1a} + i\phi_{1b} \\ \phi_{2a} + i\phi_{2b} \\ \phi_{3a} + i\phi_{3b} \\ \phi_{4a} + i\phi_{4b} \end{pmatrix}, \tag{22}$$

where all the functions are real. From the first equation in (7), we find

$$\phi_{1a} = \phi_{3b}\,, \quad \phi_{1b} = -\phi_{3a}\,, \quad \phi_{2a} = \phi_{4b}\,, \quad \phi_{2b} = -\phi_{4a}\,. \tag{23}$$

In four dimensions, (2) becomes

$$ds^2 = -f_t(dt + f d\phi)^2 + f_r dr^2 + f_y d\theta^2 + g_p(d\phi - w dt)^2 \,. \tag{24}$$

The corresponding vierbeins are

$$e^0 = \sqrt{f_t}(dt + f d\phi) \,, \quad e^1 = \sqrt{f_r} dr \,, \quad e^2 = \sqrt{f_y} d\theta \,, \quad e^3 = \sqrt{g_p}(d\phi - w dt) \,. \tag{25}$$

Plug (23) into (1), we find

$$\xi^\mu = \partial_t + w\partial_\phi \,, \tag{26}$$

just as given in (8).

Similarly, (2) becomes

$$ds^2 = -f_t(f d\phi)^2 + f_r dr^2 + f_y d\theta^2 + g_p(d\phi - w dt)^2 \,. \tag{27}$$

The corresponding vierbeins are

$$e^0 = \sqrt{f_t}(f d\phi) \,, \quad e^1 = \sqrt{f_r} dr \,, \quad e^2 = \sqrt{f_y} d\theta \,, \quad e^3 = \sqrt{g_p}(d\phi - w dt) \,. \tag{28}$$

Plug (23) into (1), we find

$$\xi^\mu = \partial_t + w\partial_\phi \,, \tag{29}$$

also as given in (8).

2.3 d=5

The gamma matrices in the fuenfbein basis is taken to be

$$\gamma^0 = i\sigma^1 \otimes \mathbf{1}_2 \,, \quad \gamma^4 = \sigma^3 \otimes \mathbf{1}_2 \,, \quad \gamma^j = -\sigma^2 \otimes \sigma^j \,, \quad j = 1, 2, 3 \,. \tag{30}$$

The spinor field is

$$\psi = \begin{pmatrix} \phi_{1a} + i\phi_{1b} \\ \phi_{2a} + i\phi_{2b} \\ \phi_{3a} + i\phi_{3b} \\ \phi_{4a} + i\phi_{4b} \end{pmatrix} \,, \tag{31}$$

where all the functions are real. From the first equation in (7), we find

$$\phi_{1a} = \phi_{3a} \,, \quad \phi_{1b} = \phi_{3b} \,, \quad \phi_{2a} = \phi_{4a} \,, \quad \phi_{2b} = \phi_{4b} \,. \tag{32}$$

In five dimensions, (2) becomes

$$ds^2 = -f_t(dt + f_1 d\phi_1 + f_2 d\phi_2)^2 + f_r dr^2 + f_y d\theta^2 + g_{22}(d\phi_2 - w_2 dt)^2$$
$$+g_{11}\Big[d\phi_1 - w_1 dt + g_{12}(d\phi_2 - w_2 dt)\Big]^2 . \tag{33}$$

The corresponding fuenfbeins are

$$e^0 = \sqrt{f_t}(dt + f_1 d\phi_1 + f_2 d\phi_2)\,, \quad e^1 = \sqrt{f_r}dr\,, \quad e^2 = \sqrt{f_y}d\theta\,,$$
$$e^3 = \sqrt{g_{11}}\Big[d\phi_1 - w_1 dt + g_{12}(d\phi_2 - w_2 dt)\Big]\,, \quad e^4 = \sqrt{g_{22}}(d\phi_2 - w_2 dt)\,. \tag{34}$$

Plug (32) into (1), we find

$$\xi^\mu = \partial_t + w_1\partial_{\phi_1} + w_2\partial_{\phi_2}\,, \tag{35}$$

just as given in (8).

Similarly, (3) becomes

$$ds^2 = -f_t(f_1 d\phi_1 + f_2 d\phi_2)^2 + f_r dr^2 + f_y d\theta^2 + g_{22}(d\phi_2 - w_2 dt)^2$$
$$+g_{11}\Big[d\phi_1 - w_1 dt + g_{12}(d\phi_2 - w_2 dt)\Big]^2 . \tag{36}$$

The corresponding fuenfbeins are

$$e^0 = \sqrt{f_t}(f_1 d\phi_1 + f_2 d\phi_2)\,, \quad e^1 = \sqrt{f_r}dr\,, \quad e^2 = \sqrt{f_y}d\theta\,,$$
$$e^3 = \sqrt{g_{11}}\Big[d\phi_1 - w_1 dt + g_{12}(d\phi_2 - w_2 dt)\Big]\,, \quad e^4 = \sqrt{g_{22}}(d\phi_2 - w_2 dt)\,. \tag{37}$$

Plug (32) into (1), we find

$$\xi^\mu = \partial_t + w_1\partial_{\phi_1} + w_2\partial_{\phi_2}\,, \tag{38}$$

also as given in (8).

2.4 d=6

The gamma matrices in the sechsbein basis are taken to be

$$\gamma^0 = i\sigma^1 \otimes \mathbf{1}_4\,, \quad \gamma^5 = \sigma^3 \otimes \mathbf{1}_4\,, \quad \gamma^j = -\sigma^2 \otimes \gamma_4^j\,, \quad j = 1,2,3,4\,, \tag{39}$$

where $\gamma_4^{1,2,3,4}$ are gamma matrices in the vierbein basis from four-dimensions, and we have replaced γ_4^0 by $\gamma_4^4 = -i\gamma_4^0$. The spinor field is

$$\psi = \begin{pmatrix} \phi_{1a} + i\phi_{1b} \\ \vdots \\ \phi_{8a} + i\phi_{8b} \end{pmatrix}\,, \tag{40}$$

where all the functions are real. From the first equation in (7), we find

$$\begin{aligned}&\phi_{1a}=-\phi_{3a}\,,\quad \phi_{1b}=-\phi_{3b}\,,\quad \phi_{2a}=-\phi_{4a}\,,\quad \phi_{2b}=-\phi_{4b}\,,\\ &\qquad\phi_{5a}=\phi_{7a}\,,\quad \phi_{5b}=\phi_{7b}\,,\quad \phi_{6a}=\phi_{8a}\,,\quad \phi_{6b}=\phi_{8b}\,.\end{aligned}\tag{41}$$

In six dimensions, (2) becomes

$$\begin{aligned}ds^2&=-f_t(dt+f_1d\phi_1+f_2d\phi_2)^2+f_rdr^2+f_{11}d\theta_1^2+f_{22}d\theta_2^2\\ &+g_{11}\Big[d\phi_1-w_1dt+g_{12}(d\phi_2-w_2dt)\Big]^2+g_{22}(d\phi_2-w_2dt)^2\,.\end{aligned}\tag{42}$$

The corresponding sechsbeins are

$$\begin{aligned}&e^0=\sqrt{f_t}(dt+f_1d\phi_1+f_2d\phi_2)\,,\quad e^1=\sqrt{f_r}dr\,,\quad e^2=\sqrt{f_{11}}d\theta_1\,,\quad e^3=\sqrt{f_{22}}d\theta_2\,,\\ &e^4=\sqrt{g_{11}}\Big[d\phi_1-w_1dt+g_{12}(d\phi_2-w_2dt)\Big]\,,\quad e^5=\sqrt{g_{22}}(d\phi_2-w_2dt)\,.\end{aligned}\tag{43}$$

Plug (41) into (1), we find

$$\xi^\mu=\partial_t+w_1\partial_{\phi_1}+w_2\partial_{\phi_2}\,,\tag{44}$$

just as given in (8).

Similarly, (3) becomes

$$\begin{aligned}ds^2&=-f_t(f_1d\phi_1+f_2d\phi_2)^2+f_rdr^2+f_{11}d\theta_1^2+f_{22}d\theta_2^2\\ &+g_{11}\Big[d\phi_1-w_1dt+g_{12}(d\phi_2-w_2dt)\Big]^2+g_{22}(d\phi_2-w_2dt)^2\,.\end{aligned}\tag{45}$$

The corresponding sechsbeins are

$$\begin{aligned}&e^0=\sqrt{f_t}(f_1d\phi_1+f_2d\phi_2)\,,\quad e^1=\sqrt{f_r}dr\,,\quad e^2=\sqrt{f_{11}}d\theta_1\,,\quad e^3=\sqrt{f_{22}}d\theta_2\,,\\ &e^4=\sqrt{g_{11}}\Big[d\phi_1-w_1dt+g_{12}(d\phi_2-w_2dt)\Big]\,,\quad e^5=\sqrt{g_{22}}(d\phi_2-w_2dt)\,.\end{aligned}\tag{46}$$

Plug (41) into (1), we find

$$\xi^\mu=\partial_t+w_1\partial_{\phi_1}+w_2\partial_{\phi_2}\,,\tag{47}$$

also as given in (8).

2.5 d=7

The gamma matrices in the siebbein basis are taken to be

$$\gamma^0=i\sigma^1\otimes\mathbf{1}_5\,,\quad \gamma^6=\sigma^3\otimes\mathbf{1}_5\,,\quad \gamma^j=-\sigma^2\otimes\gamma_5^j\,,\quad j=1,2,3,4,5\,,\tag{48}$$

where $\gamma_5^{1,2,3,4,5}$ are gamma matrices in the fuenfbein basis from five-dimensions, and we have replaced γ_5^0 by $\gamma_5^5 = -i\gamma_5^0$. The spinor field is

$$\psi = \begin{pmatrix} \phi_{1a} + i\phi_{1b} \\ \vdots \\ \phi_{8a} + i\phi_{8b} \end{pmatrix} , \tag{49}$$

where all the functions are real. From the first equation in (7), we find

$$\begin{aligned} &\phi_{1a} = \phi_{5a} \,, \quad \phi_{1b} = \phi_{5b} \,, \quad \phi_{2a} = \phi_{6a} \,, \quad \phi_{2b} = \phi_{6b} \,, \\ &\phi_{3a} = \phi_{7a} \,, \quad \phi_{3b} = \phi_{7b} \,, \quad \phi_{4a} = \phi_{8a} \,, \quad \phi_{4b} = \phi_{8b} \,. \end{aligned} \tag{50}$$

In seven dimensions, (2) becomes

$$\begin{aligned} ds^2 = &-f_t(dt + f_1 d\phi_1 + f_2 d\phi_2 + f_3 d\phi_3)^2 + f_r dr^2 + f_{11} d\theta_1^2 + f_{22} d\theta_2^2 \\ &+g_{11}\Big[d\phi_1 - w_1 dt + g_{12}(d\phi_2 - w_2 dt) + g_{13}(d\phi_3 - w_3 dt)\Big]^2 \\ &+g_{22}\Big[d\phi_2 - w_2 dt + g_{23}(d\phi_3 - w_3 dt)\Big]^2 + g_{33}(d\phi_3 - w_3 dt)^2 \,. \end{aligned} \tag{51}$$

The corresponding siebbeins are

$$\begin{aligned} &e^0 = \sqrt{f_t}(dt + f_1 d\phi_1 + f_2 d\phi_2 + f_3 d\phi_3) \,, \quad e^1 = \sqrt{f_r} dr \,, \quad e^2 = \sqrt{f_{11}} d\theta_1 \,, \\ &e^3 = \sqrt{f_{22}} d\theta_2 \,, \quad e^4 = \sqrt{g_{11}}\Big[d\phi_1 - w_1 dt + g_{12}(d\phi_2 - w_2 dt) + g_{13}(d\phi_3 - w_3 dt)\Big] \,, \\ &e^5 = \sqrt{g_{22}}\Big[d\phi_2 - w_2 dt + g_{23}(d\phi_3 - w_3 dt)\Big] \,, \quad e^6 = \sqrt{g_{33}}(d\phi_3 - w_3 dt) \,. \end{aligned} \tag{52}$$

Plug (50) into (1), we find

$$\xi^\mu = \partial_t + w_1 \partial_{\phi_1} + w_2 \partial_{\phi_2} + w_3 \partial_{\phi_3} \,, \tag{53}$$

just as given in (8).

Similarly, (3) becomes

$$\begin{aligned} ds^2 = &-f_t(f_1 d\phi_1 + f_2 d\phi_2 + f_3 d\phi_3)^2 + f_r dr^2 + f_{11} d\theta_1^2 + f_{22} d\theta_2^2 \\ &+g_{11}\Big[d\phi_1 - w_1 dt + g_{12}(d\phi_2 - w_2 dt) + g_{13}(d\phi_3 - w_3 dt)\Big]^2 \\ &+g_{22}\Big[d\phi_2 - w_2 dt + g_{23}(d\phi_3 - w_3 dt)\Big]^2 + g_{33}(d\phi_3 - w_3 dt)^2 \,. \end{aligned} \tag{54}$$

The corresponding siebbeins are

$$\begin{aligned} &e^0 = \sqrt{f_t}(f_1 d\phi_1 + f_2 d\phi_2 + f_3 d\phi_3) \,, \quad e^1 = \sqrt{f_r} dr \,, \quad e^2 = \sqrt{f_{11}} d\theta_1 \,, \\ &e^3 = \sqrt{f_{22}} d\theta_2 \,, \quad e^4 = \sqrt{g_{11}}\Big[d\phi_1 - w_1 dt + g_{12}(d\phi_2 - w_2 dt) + g_{13}(d\phi_3 - w_3 dt)\Big] \,, \end{aligned}$$

$$e^5=\sqrt{g_{22}}\Big[d\phi_2-w_2dt+g_{23}(d\phi_3-w_3dt)\Big]\,,\quad e^6=\sqrt{g_{33}}(d\phi_3-w_3dt)\,. \tag{55}$$

Plug (50) into (1), we find

$$\xi^\mu=\partial_t+w_1\partial_{\phi_1}+w_2\partial_{\phi_2}+w_3\partial_{\phi_3}\,, \tag{56}$$

also as given in (8).

2.6 d=8

The gamma matrices in the vielbein basis are taken to be

$$\gamma^0=i\sigma^1\otimes\mathbf{1}_6\,,\quad \gamma^7=\sigma^3\otimes\mathbf{1}_6\,,\quad \gamma^j=-\sigma^2\otimes\gamma_6^j\,,\quad j=1,2,3,4,5,6\,, \tag{57}$$

where $\gamma_6^{1,2,3,4,5,6}$ are gamma matrices in the sechsbein basis from six-dimensions, and we have replaced γ_6^0 by $\gamma_6^6=-i\gamma_6^0$. The spinor field is

$$\psi=\begin{pmatrix}\phi_{1a}+i\phi_{1b}\\ \vdots\\ \phi_{16a}+i\phi_{16b}\end{pmatrix}\,, \tag{58}$$

where all the functions are real. From the first equation in (7), we find

$$\begin{aligned}
\phi_{1a}&=\phi_{7b}\,,\quad \phi_{1b}=-\phi_{7a}\,,\quad \phi_{2a}=\phi_{8b}\,,\quad \phi_{2b}=-\phi_{8a}\,,\\
\phi_{3a}&=\phi_{5b}\,,\quad \phi_{3b}=-\phi_{5a}\,,\quad \phi_{4a}=\phi_{6b}\,,\quad \phi_{4b}=-\phi_{6a}\,,\\
\phi_{9a}&=-\phi_{15b}\,,\quad \phi_{9b}=\phi_{15a}\,,\quad \phi_{10a}=-\phi_{16b}\,,\quad \phi_{10b}=\phi_{16a}\,,\\
\phi_{11a}&=-\phi_{13b}\,,\quad \phi_{11b}=\phi_{13a}\,,\quad \phi_{12a}=-\phi_{14b}\,,\quad \phi_{12b}=\phi_{14a}\,.
\end{aligned} \tag{59}$$

In eight dimensions, (2) becomes

$$\begin{aligned}
ds^2=&-f_t(dt+f_1d\phi_1+f_2d\phi_2+f_3d\phi_3)^2+f_rdr^2+f_{11}d\theta_1^2+f_{22}d\theta_2^2+f_{33}d\theta_3^2\\
&+g_{11}\Big[d\phi_1-w_1dt+g_{12}(d\phi_2-w_2dt)+g_{13}(d\phi_3-w_3dt)\Big]^2\\
&+g_{22}\Big[d\phi_2-w_2dt+g_{23}(d\phi_3-w_3dt)\Big]^2+g_{33}(d\phi_3-w_3dt)^2\,.
\end{aligned} \tag{60}$$

The corresponding vielbeins are

$$\begin{aligned}
e^0&=\sqrt{f_t}(dt+f_1d\phi_1+f_2d\phi_2+f_3d\phi_3)\,,\quad e^1=\sqrt{f_r}dr\,,\\
e^2&=\sqrt{f_{11}}d\theta_1\,,\quad e^3=\sqrt{f_{22}}d\theta_2\,,\quad e^4=\sqrt{f_{33}}d\theta_3\,,\\
e^5&=\sqrt{g_{11}}\Big[d\phi_1-w_1dt+g_{12}(d\phi_2-w_2dt)+g_{13}(d\phi_3-w_3dt)\Big]\,,
\end{aligned}$$

$$e^6 = \sqrt{g_{22}}\Big[d\phi_2 - w_2 dt + g_{23}(d\phi_3 - w_3 dt)\Big]\,, \quad e^7 = \sqrt{g_{33}}(d\phi_3 - w_3 dt)\,. \tag{61}$$

Plug (50) into (1), we find

$$\xi^\mu = \partial_t + w_1\partial_{\phi_1} + w_2\partial_{\phi_2} + w_3\partial_{\phi_3}\,, \tag{62}$$

just as given in (8).

Similarly, (3) becomes

$$\begin{aligned} ds^2 = -f_t(f_1 d\phi_1 + f_2 d\phi_2 + f_3 d\phi_3)^2 + f_r dr^2 + f_{11} d\theta_1^2 + f_{22} d\theta_2^2 + f_{33} d\theta_3^2 \\ +g_{11}\Big[d\phi_1 - w_1 dt + g_{12}(d\phi_2 - w_2 dt) + g_{13}(d\phi_3 - w_3 dt)\Big]^2 \\ +g_{22}\Big[d\phi_2 - w_2 dt + g_{23}(d\phi_3 - w_3 dt)\Big]^2 + g_{33}(d\phi_3 - w_3 dt)^2\,. \end{aligned} \tag{63}$$

The corresponding vielbeins are

$$\begin{aligned} e^0 = \sqrt{f_t}(f_1 d\phi_1 + f_2 d\phi_2 + f_3 d\phi_3)\,, \quad e^1 = \sqrt{f_r}dr\,, \\ e^2 = \sqrt{f_{11}}d\theta_1\,, \quad e^3 = \sqrt{f_{22}}d\theta_2\,, \quad e^4 = \sqrt{f_{33}}d\theta_3\,, \\ e^5 = \sqrt{g_{11}}\Big[d\phi_1 - w_1 dt + g_{12}(d\phi_2 - w_2 dt) + g_{13}(d\phi_3 - w_3 dt)\Big]\,, \\ e^6 = \sqrt{g_{22}}\Big[d\phi_2 - w_2 dt + g_{23}(d\phi_3 - w_3 dt)\Big]\,, \quad e^7 = \sqrt{g_{33}}(d\phi_3 - w_3 dt)\,. \end{aligned} \tag{64}$$

Plug (50) into (1), we find

$$\xi^\mu = \partial_t + w_1\partial_{\phi_1} + w_2\partial_{\phi_2} + w_3\partial_{\phi_3}\,, \tag{65}$$

also as given in (8).

2.7 Summary of the section

To summarize this section, we have explicitly shown that (8) is true for any metric of the form (2) or (3), given that (7) is satisfied. The calculation is only done in three through eight dimensions. But we do not see any particular reason that such a pattern will break down in higher dimensions. Given the fact that (2) and (3) are quite general structures for all known stationary and axisymmetric black holes [2], one can conclude that result #1 holds for all such black holes.

3 Testing Result #2

In this section, we will use examples in different dimensions to demonstrate result #2. In this case, we need to make sure that the coordinate system is non-rotating at the spatial infinity and that the vielbeins (6) are well defined in the whole region outside the black hole horizon. So more detail of each specific solution will be needed.

3.1 $d = 3$

In three spacetime dimensions, an interesting example is the BTZ black hole [3],

$$ds^2 = -f d\hat{t}^2 + \frac{dr^2}{f} + r^2\Big(d\hat{\phi} - \frac{J}{2r^2}d\hat{t}\Big)^2 , \quad f = -m + g^2r^2 + \frac{J^2}{4r^2} , \tag{66}$$

which solves the Einstein equation with a cosmological constant,

$$R_{\mu\nu} = \frac{2\Lambda}{d-2}g_{\mu\nu} , \quad \Lambda = -\frac{(d-1)(d-2)}{2}g^2 . \tag{67}$$

The coordinates in (66) are related to the static ones by

$$\begin{aligned} d\hat{t} &= \frac{1}{\sqrt{2(m-\tilde{m})}}\Big(\frac{Jg}{\sqrt{\tilde{m}}}dt + \frac{\sqrt{\tilde{m}}}{g}d\phi\Big) , \\ d\hat{\phi} &= \frac{1}{\sqrt{2(m-\tilde{m})}}\Big(\frac{Jg}{\sqrt{\tilde{m}}}d\phi + g\sqrt{\tilde{m}}dt\Big) , \end{aligned} \tag{68}$$

where $\tilde{m} = m - \sqrt{m^2 - J^2g^2}$. Now the metric becomes

$$ds^2 = -\frac{\Delta r_0^2}{r_0^2 - r_c^2}\Big(dt + \frac{r_c}{gr_0}d\phi\Big) + \frac{dr^2}{\Delta} + \frac{r_0^2(r^2-r_c^2)^2}{r^2(r_0^2-r_c^2)}\Big[d\phi + \frac{gr_c(r^2-r_0^2)}{r_0(r^2-r_c^2)}dt\Big]^2 . \tag{69}$$

where

$$\Delta = \frac{g^2(r^2-r_0^2)(r^2-r_c^2)}{r^2} , \quad r_c = \frac{J}{2gr_0} < r_0 . \tag{70}$$

For this metric, the horizon angular velocity is zero, $\Omega = 0$.

The dreibeins can be read off (69) in a straightforward manor. From (1) and (14), we find

$$\xi^\mu\partial_\mu = \partial_t - \frac{gr_c(r^2-r_0^2)}{r_0(r^2-r_c^2)}\partial_\phi , \tag{71}$$

just as given in (8). As $r \to +\infty$,

$$\xi \to \partial_t - \frac{gr_c}{r_0}\partial_\phi . \tag{72}$$

For the BTZ black hole, one can never set $g = 0$, so ξ will not become the time Killing vector at the spatial infinity. On the other hand, one sees that $\xi = \partial_t$ on the horizon when $r = r_0$. This is a peculiar feature of the BTZ black hole.

3.2 d=4

In four dimensions, we consider the rotating solution in $U(1)^4$ gauged supergravity with four charges pairwise equal [8]. The metric is given by (5) with the vierbeins,

$$e^0 = \sqrt{\frac{R}{H(r^2+y^2)}}\Big[d\hat{t} - \frac{a^2-y^2}{a(1-g^2a^2)}d\hat{\phi}\Big] ,$$

$$e^1 = \sqrt{\frac{H(r^2+y^2)}{R}}dr\,, \quad e^2 = \sqrt{\frac{H(r^2+y^2)}{Y}}dy\,,$$
$$e^3 = \sqrt{\frac{Y}{H(r^2+y^2)}}\frac{r_1r_2+a^2}{a(1-g^2a^2)}\left[d\hat{\phi} - \frac{a(1-g^2a^2)}{r_1r_2+a^2}d\hat{t}\right]\,, \tag{73}$$

where $r_1 = r + 2ms_1^2$, $r_2 = r + 2ms_2^2$ and

$$R = r^2 + a^2 - 2mr + g^2 r_1 r_2 [r_1 r_2 + a^2]\,,$$
$$Y = (1 - g^2y^2)(a^2 - y^2)\,, \quad H = \frac{r_1r_2 + y^2}{r^2+y^2}\,. \tag{74}$$

The coordinates are related to the static ones by

$$d\hat{t} = dt\,, \quad d\hat{\phi} = d\phi - g^2 a dt\,. \tag{75}$$

The horizon is located at $R(r_0) = 0$, and the angular velocity is

$$\Omega = \frac{a(1+g^2r_{10}r_{20})}{r_{10}r_{20}+a^2}\,, \tag{76}$$

where $r_{10} = r_1(r_0)$ and $r_{20} = r_2(r_0)$.

From (1) and (23), we find

$$\xi^\mu \partial_\mu = \partial_t + \frac{a(1+g^2r_1r_2)}{r_1r_2+a^2}\partial_\phi\,, \tag{77}$$

just as given in (8). We see that $\xi \to \partial_t + g^2 a \partial_\phi$ as $r \to +\infty$. The solution is asymptotically flat when $g = 0$. In this case, $\xi \to \partial_t$ as $r \to +\infty$.

It will also be interesting to consider the single-charge and two-charge rotating black hole in the gauged supergravity [5, 6], and the four-charge black hole in the ungauged supergravity [7, 8]. But for these solutions, we have not been able to put the metrics into desired the form. So the corresponding calculation is not done.

3.3 d=5

In five dimensions, we consider the rotating solution in $U(1)^3$ gauged supergravity with two of the charges equal [10]. The metric is given by (5) with the fuenfbeins,

$$e^0 = \frac{h_3^{1/6} r\sqrt{X}}{h_1^{2/3}\sqrt{r_3}}\sigma_t\,, \quad e^1 = \frac{h_1^{1/3}h_3^{1/6}}{\sqrt{X}}dr\,, \quad e^2 = \frac{h_1^{1/3}h_3^{1/6}}{\sqrt{1-g^2y^2}}d\theta\,,$$
$$e^3 = \frac{(a^2-b^2)h_3^{1/6}\sqrt{1-g^2y^2}\cos\theta\sin\theta}{h_1^{2/3}y}\sigma_a\,, \quad e^4 = \frac{(abh_3 + 2mc_3s_3y^2)\sigma_t + h_1 r_3 \sigma_b}{h_1^{2/3}h_3^{1/3}y\sqrt{r_3}}\,, \tag{78}$$

where

$$\sigma_t = \frac{1-g^2y^2}{Z_aZ_b}dt - \frac{a\sin^2\theta}{Z_a}d\phi_1 - \frac{b\cos^2\theta}{Z_b}d\phi_2\,,$$

$$
\begin{aligned}
\sigma_a &= \frac{1+g^2 r_1}{Z_a Z_b} dt - \frac{a(a^2+r_1)}{Z_a(a^2-b^2)} d\phi_1 + \frac{b(b^2+r_1)}{Z_b(a^2-b^2)} d\phi_2 \,, \\
\sigma_b &= \frac{g^2 ab(1-g^2y^2)}{Z_a Z_b} dt - \frac{b\sin^2\theta}{Z_a} d\phi_1 - \frac{a\cos^2\theta}{Z_b} d\phi_2 \,, \\
y^2 = a^2\cos^2\theta + b^2\sin^2\theta \,, &\quad Z_a = 1-g^2a^2 \,, \quad Z_b = 1-g^2b^2 \,, \\
r_1 = r_2 + 2ms_1^2 \,, \quad r_3 = r_2 + 2ms_3^2 \,, &\quad r_2 = r^2 - \frac{2}{3} m(2s_1^2+s_3^2) \,, \\
h_1 = r_1 + y^2 \,, \quad h_3 = r_3 + y^2 \,, &\quad c_1 = \sqrt{1+s_1^2} \,, \quad c_3 = \sqrt{1+s_3^2} \,, \\
X &= r^{-2}\Big[(r_2+a^2)(r_2+b^2) + g^2(r_1+a^2)(r_1+b^2)r_3 \\
&\qquad -2m(r_2 - 2abc_3s_3 - a^2s_3^2 - b^2s_3^2)\Big] \,. \qquad (79)
\end{aligned}
$$

The gauge fields are

$$
A_1 = A_2 = \frac{2mc_1s_1}{h_1}\sigma_t \,, \quad A_3 = \frac{2m[c_3s_3\sigma_t - (s_1^2-s_3^2)\sigma_b]}{h_3} \,. \qquad (80)
$$

The horizon is located at $X(r_0)=0$, and the angular velocities are

$$
\begin{aligned}
\Omega_a &= \frac{b(ab+2ms_3c_3) + a[1+g^2(r_{10}+b^2)]r_{30}}{ab(ab+2ms_3c_3) + (r_{10}+a^2+b^2)r_{30}} \,, \\
\Omega_b &= \frac{a(ab+2ms_3c_3) + b[1+g^2(r_{10}+a^2)]r_{30}}{ab(ab+2ms_3c_3) + (r_{10}+a^2+b^2)r_{30}} \,, \qquad (81)
\end{aligned}
$$

where $r_{10} = r_1(r=r_0)$ and $r_{30} = r_3(r=r_0)$.

From (1) and (32), we find

$$
\xi^\mu \partial_\mu = \partial_t + w_1 \partial_{\phi_1} + w_2 \partial_{\phi_2} \,, \qquad (82)
$$

with

$$
\begin{aligned}
w_1 &= \frac{b(ab+2ms_3c_3) + a[1+g^2(r_1+b^2)]r_3}{ab(ab+2ms_3c_3) + (r_1+a^2+b^2)r_3} \,, \\
w_2 &= \frac{a(ab+2ms_3c_3) + b[1+g^2(r_1+a^2)]r_3}{ab(ab+2ms_3c_3) + (r_1+a^2+b^2)r_3} \,. \qquad (83)
\end{aligned}
$$

It is obvious that the result agrees with (8). In the limit $r \to +\infty$, we find

$$
w_1 = g^2 a + \mathcal{O}(\frac{1}{r}) \,, \quad w_2 = g^2 b + \mathcal{O}(\frac{1}{r}) \,. \qquad (84)
$$

In the case $g=0$, the solution is asymptotically flat, and we have $\xi = \partial_t$ as $r \to +\infty$.

It will also be interesting to consider the equal rotation solution in $U(1)^3$ gauged supergravity with arbitrary charges [9] and the Cvetič-Youm solution [11] in ungauged supergravity. But for these solutions, we have not been able to put the metrics into the desired form. So the corresponding calculation is not done.

3.4 d=6

In six dimensions, we consider the single-charge two-rotation solution in $SU(2)$ gauged supergravity found in [12]. The metric is given in (5) with the sechsbeins,

$$e^0 = \sqrt{\frac{R}{H^{3/2}U}}\mathcal{A}\,, \quad e^1 = \sqrt{\frac{H^{1/2}U}{R}}dr\,,$$
$$e^2 = \sqrt{\frac{H^{1/2}(r^2+y^2)(y^2-z^2)}{Y}}dy\,, \quad e^3 = \sqrt{\frac{H^{1/2}(r^2+z^2)(z^2-y^2)}{Z}}dz\,,$$
$$e^4 = \sqrt{\frac{H^{1/2}Y}{(r^2+y^2)(y^2-z^2)}}\mathcal{A}_Y\,, \quad e^5 = \sqrt{\frac{H^{1/2}Z}{(r^2+z^2)(z^2-y^2)}}\mathcal{A}_Z\,, \tag{85}$$

where (Note $\hat{\phi}_1 = \phi_1 - g^2 at$ and $\hat{\phi}_2 = \phi_2 - g^2 bt$)

$$\mathcal{A}_Y = dt - (r^2+a^2)(a^2-z^2)\frac{d\hat{\phi}_1}{\epsilon_1} - (r^2+b^2)(b^2-z^2)\frac{d\hat{\phi}_2}{\epsilon_2} - \frac{qr\widetilde{\mathcal{A}}}{HU}\,,$$
$$\mathcal{A}_Z = dt - (r^2+a^2)(a^2-y^2)\frac{d\hat{\phi}_1}{\epsilon_1} - (r^2+b^2)(b^2-y^2)\frac{d\hat{\phi}_2}{\epsilon_2} - \frac{qr\widetilde{\mathcal{A}}}{HU}\,,$$
$$\mathcal{A} = dt - (a^2-y^2)(a^2-z^2)\frac{d\hat{\phi}_1}{\epsilon_1} - (b^2-y^2)(b^2-z^2)\frac{d\hat{\phi}_2}{\epsilon_2}\,,$$
$$R = (r^2+a^2)(r^2+b^2) + g^2[r(r^2+a^2)+q][r(r^2+b^2)+q] - 2mr\,. \tag{86}$$

More detail of the solution can be found in [12, 13].

From (1) and (41), we find that

$$\xi^\mu\partial_\mu = \partial_t + w_1\partial_{\phi_1} + w_2\partial_{\phi_2}\,, \tag{87}$$

with

$$w_1 = \frac{a[g^2qr + (b^2+r^2)(1+g^2r^2)]}{qr + (a^2+r^2)(b^2+r^2)}\,,$$
$$w_2 = \frac{b[g^2qr + (a^2+r^2)(1+g^2r^2)]}{qr + (a^2+r^2)(b^2+r^2)}\,. \tag{88}$$

Note $\Omega_a = w_1(r_0)$ and $\Omega_b = w_2(r_0)$ are just the two angular velocities of the black hole. It is obvious that the result agrees with (8). In the limit $r \to +\infty$, we find

$$w_1 = g^2 a + \mathcal{O}(\frac{1}{r})\,, \quad w_2 = g^2 b + \mathcal{O}(\frac{1}{r})\,. \tag{89}$$

In the case $g=0$, the solution is asymptotically flat, and we have $\xi = \partial_t$ as $r \to +\infty$.

3.5 d=7

In seven dimensions, we consider the single-charge three-rotation solution in $SO(5)$ gauged supergravity found in [14]. The metric is given in (5) with the siebbeins,

$$e^0 = \sqrt{\frac{R}{H^{8/5}U}}\mathcal{A}\,, \quad e^1 = \sqrt{\frac{H^{2/5}U}{R}}dr\,,$$

$$e^2 = \sqrt{\frac{H^{2/5}(r^2+y^2)(y^2-z^2)}{Y}}dy\,, \quad e^3 = \sqrt{\frac{H^{2/5}(r^2+z^2)(z^2-y^2)}{Z}}dz\,,$$
$$e^4 = \sqrt{\frac{H^{2/5}Y}{(r^2+y^2)(y^2-z^2)}}\mathcal{A}_Y\,, \quad e^5 = \sqrt{\frac{H^{2/5}Z}{(r^2+z^2)(z^2-y^2)}}\mathcal{A}_Z\,, \quad e^6 = \frac{a_1a_2a_3}{ryz}\mathcal{A}_7\,, \tag{90}$$

where (Note $\hat{\phi}_i = \phi_i - g^2 a_i t$, $i = 1, 2, 3$.)

$$\mathcal{A}_Y = dt - \sum_{i=1}^{3}\frac{(\hat{r}^2+a_i^2)\gamma_i}{a_i^2-y^2}\frac{d\hat{\phi}_i}{\epsilon_i} - \frac{q}{HU}\mathcal{A}\,, \quad \mathcal{A}_Z = dt - \sum_{i=1}^{3}\frac{(\hat{r}^2+a_i^2)\gamma_i}{a_i^2-z^2}\frac{d\hat{\phi}_i}{\epsilon_i} - \frac{q}{HU}\mathcal{A}\,,$$
$$\mathcal{A}_7 = dt - \sum_{i=1}^{3}\frac{(\hat{r}^2+a_i^2)\gamma_i}{a_i^2}\frac{d\hat{\phi}_i}{\epsilon_i} - \frac{q}{HU}\Big(1+\frac{gy^2z^2}{a_1a_2a_3}\Big)\mathcal{A}\,, \quad \mathcal{A} = d\hat{t} - \sum_{i=1}^{3}\gamma_i\frac{d\hat{\phi}_i}{\epsilon_i}\,,$$
$$R = \frac{1+g^2r^2}{r^2}\prod_{i=1}^{3}(r^2+a_i^2) + qg^2(2r^2+a_1^2+a_2^2+a_3^2) - \frac{2qga_1a_2a_3}{r^2} + \frac{q^2g^2}{r^2} - 2m\,. \tag{91}$$

More detail of the solution can be found in [13, 14].

From (1) and (50), we find

$$\xi^\mu\partial_\mu = \partial_t + w_1\partial_{\phi_1} + w_2\partial_{\phi_2} + w_3\partial_{\phi_3}\,, \tag{92}$$

with

$$w_1 = \frac{a_1(r^2+a_2^2)(r^2+a_3^2)(1+g^2r^2) - gq(a_2a_3 - a_1gr^2)}{(r^2+a_1^2)(r^2+a_2^2)(r^2+a_3^2) - q(a_1a_2a_3g - r^2)}\,,$$
$$w_2 = \frac{a_2(r^2+a_1^2)(r^2+a_3^2)(1+g^2r^2) - gq(a_1a_3 - a_2gr^2)}{(r^2+a_1^2)(r^2+a_2^2)(r^2+a_3^2) - q(a_1a_2a_3g - r^2)}\,,$$
$$w_3 = \frac{a_3(r^2+a_1^2)(r^2+a_2^2)(1+g^2r^2) - gq(a_1a_2 - a_3gr^2)}{(r^2+a_1^2)(r^2+a_2^2)(r^2+a_3^2) - q(a_1a_2a_3g - r^2)}\,. \tag{93}$$

Note $\Omega_1 = w_1(r_0)$, $\Omega_2 = w_2(r_0)$ and $\Omega_3 = w_3(r_0)$ are the three angular velocities of the black hole. It is obvious that the result agrees with (8). In the limit $r \to +\infty$, we find

$$w_1 = g^2a_1 + \mathcal{O}(\frac{1}{r})\,, \quad w_2 = g^2a_2 + \mathcal{O}(\frac{1}{r})\,, \quad w_3 = g^2a_3 + \mathcal{O}(\frac{1}{r})\,. \tag{94}$$

In the case $g = 0$, the solution is asymptotically flat, and we have $\xi = \partial_t$ as $r \to +\infty$.

3.6 d=8

In arbitrary dimensions, there is the Kerr-NUT-AdS solution found in [15]. Still we are not able to discuss the general case of arbitrary dimensions. Here we will only consider the case of eight dimensions (lower dimension ones are already covered in the previous examples). Since the metrics obtained in [15] are already of the form (2), we do not need to repeat the calculation done in the last section. By comparing equation (13) in [15] with our metric (60), we find that

$$w_\alpha = \frac{a_\alpha(1+g^2r^2)}{r^2+a_\alpha^2}\,, \quad \alpha = 1,2,3\,. \tag{95}$$

In the limit $r \to +\infty$, we find

$$w_\alpha = g^2 a_\alpha + \mathcal{O}(\frac{1}{r})\,, \quad \alpha = 1,2,3\,. \tag{96}$$

In the case $g=0$, the solution is asymptotically flat, and we have $\xi = \partial_t$ as $r \to +\infty$.

It is straightforward to generalize the calculation to higher dimensions, and we expect that the basic properties of the result stay the same.

3.7 Summary of the section

To summarize this section, we have shown that, when the black hole solutions are asymptotically flat and when the coordinate system is asymptotically static, then the vector field (8) approaches the time Killing vector at the spatial infinity. Because of technical reasons, we have not been able to do the calculation for several interesting examples. But it is quite likely that result #2 will hold for all stationary black holes that have a well defined vielbein expression outside the black hole horizon.

4 Summary

In this paper, we have constructed a vector field by using the "conserved current" of a particular spinor field. We have shown that, in the background of a stationary black hole, the vector field always approaches the null Killing vector on horizon. When the black hole is asymptotically flat and when the coordinate system is asymptotically static, the same vector field also becomes the time Killing vector at the spatial infinity. The required constraint on the spinor field is simple and universal (valid for any spacetime dimensions).

It is still not clear as to the physical nature of the vector field or the corresponding spinor field. Our original motivation for studying the vector field was to construct a possible fluid flow underlying the spacetime [1]. For asymptotically flat black hole solutions, the behavior of the vector field fits very well with our intuitive picture about the speculated fluid that may underly our spacetime. One can imagine that the fluid is dragged by the black hole horizon (Hence the same velocity on the horizon[3]). Then the angular velocity steadily

[3]Here we simply let $U=\xi$, where U^μ is the velocity of the fluid. We no longer assume $U^2=-1$ as was done in [1].

decreases until it vanishes at the spatial infinity. However, such a picture is still highly hypothetical, and one should be open minded towards other possible explanations.

Another interesting possibility is to treat (8) as sort of generalized angular velocity function. This may be offer some insight towards peculiar objects such as the BTZ black hole [3, 16]. In particular, one may say that the black hole now has a finite angular velocity at the spatial infinity, even though the horizon angular velocity is zero. Again, much more work is needed before one can take such a possibility seriously.

Regardless of what the physical interpretation may be, it is unexpected and also quite amazing that something like (7) and (8) can exist. Given the remarkable features summarized in **Result #1** and **Result #2**, it will be very interesting to see possible applications of the vector field (8), or the corresponding spinor field (7), or both.

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