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RESEARCH ARTICLE

Spinor fields and symmetries of the spacetime

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Abstract We show that in the background of a stationary and axisymmetric black hole, there is a particular spinor field whose "conserved current" interpolates between the null Killing vector on the horizon and the time Killing vector at the spatial infinity. The spinor field only needs to satisfy a very general and simple constraint.

Keywords Killing vector · Black hole solutions · Spinor field

1 Introduction

In [1] it was noticed that, for the Kerr black hole and the five dimensional Myers–Perry black hole, there exists a particular vector field which interpolates between the time Killing vector at the spatial infinity and the null Killing vector on the horizon. The existence of such a vector field can be very interesting in that it may contain important (possibly non-geometrical) information about the spacetime itself.

In this paper, we want to suggest that the existence of such a vector is a general feature of all stationary and axisymmetric black holes. In [1], the particular property of the vector field is interpreted as describing a possible fluid flow underlying the spacetime. Here, we want to leave the physical interpretation behind and merely demonstrate the existence of the vector field.

We clarify our notations in the next section. Then we present and prove our main result in Sect. 3. A short summary is at the end.

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2 Notations

The vector field is constructed using a spinor field,

$$\xi^\mu = -\bar{\psi}\gamma^\mu\psi = -\eta_{AB}\,\breve{g}^{\mu\nu}\,e^B{}_\nu\,\bar{\psi}\gamma^A\psi, \quad A, B = 0, 1, \ldots, d-1, \tag{1}$$

where $\eta = diag\{- + \cdots +\}$, $\breve{g}$ stands for the background metric, and the gamma matrices γ^A are in the vielbein basis, $e^A = e^A{}_\mu dx^\mu$. Note we will always use Capital Latin letters (such as) as internal indices and Greek letters (such as) as indices for real coordinates. If ψ was to obey the Dirac equation, then (1) is nothing but the conserved current of the spinor field. Here we do not require ψ to be a Dirac fermion, but we will still refer to (1) as the "conserved current" for simplicity.

The background we consider is that of a stationary and axisymmetric black hole. It is empirically known that one can always put the metric into one of the following forms [2]

$$ds_I^2 = -f_t\Delta(dt + f_a d\phi^a)^2 + \frac{f_r}{\Delta}dr^2 + h_i d\theta^{i2} + g_{ab}(d\phi^a - w^a dt)(d\phi^b - w^b dt), \tag{2}$$

$$ds_{II}^2 = -f_t\Delta(f_a d\phi^a)^2 + \frac{f_r}{\Delta}dr^2 + h_i d\theta^{i2} + g_{ab}(d\phi^a - w^a dt)(d\phi^b - w^b dt), \tag{3}$$

where the coordinates are r the radial direction, t the asymptotic time, θ^i ($i = 1, \ldots, [\frac{d}{2}] - 1$) the longitudinal angles and ϕ^a ($a = 1, \ldots, [\frac{d+1}{2}] - 1$) the azimuthal angles. Note we will always use the beginning Latin letters (such as "$a, b, \ldots$") to label the azimuthal angles (e.g. ϕ^a) and the middle Latin letters (such as "$i, j, \ldots$") to label the longitudinal angles (e.g. θ^i). All the functions in (2) and (3) depend on r and θ^i, except for Δ which only depends on r. The (outer) horizon is located at the (largest) root of $\Delta(r_0) = 0$. All other functions f_t, f_a, f_r, h_i, g_{ab} and w^a are non-divergent on the horizon. Usually the first law of black hole thermodynamics is most conveniently studied in a coordinate system that is non-rotating at the spatial infinity [4,5], otherwise additional boundary terms may have to be included into the action [6]. In any case, $\Omega^a = w^a(r_0)$ is the angular velocity of the horizon in the ϕ^a direction.

There is some ambiguity in the values of f_a, which we fix by massaging both metrics (2) and (3) into the ADM form [3],

$$ds^2 = -\tilde{f}_t\Delta dt^2 + \frac{f_r}{\Delta}dr^2 + h_i d\theta^{i2} + \tilde{g}_{ab}(d\phi^a - \tilde{w}^a dt)(d\phi^b - \tilde{w}^b dt), \tag{4}$$

$$\tilde{g}_{ab} = g_{ab} - f_a f_b f_t\Delta, \quad \tilde{g}^{ab} = g^{ab} + \frac{f^a f^b f_t\Delta}{1 - f^2 f_t\Delta}, \quad f^a = g^{ab} f_b,$$

$$\tilde{w}^a = w^a + \frac{k f^a f_t\Delta}{1 - f^2 f_t\Delta}, \quad \tilde{f}_t = \frac{k^2 f_t}{1 - f^2 f_t\Delta}, \quad f^2 = f_a f^a, \tag{5}$$

where $k = 1 + f_a w^a$ for (2) and $k = f_a w^a$ for (3). If we want the coordinate system to be non-rotating (i.e., no cross terms like $d\phi^a dt$) at the spatial infinity, then

$$\tilde{w}^a(r, \theta^i) \longrightarrow 0, \quad \text{as} \quad r \longrightarrow +\infty. \tag{6}$$

Since $\tilde{g}_{ab}, \tilde{w}^a$ and $\tilde{f}_t$ differ from g_{ab}, w^a and f_t only by a term proportional to Δ, $\tilde{w}^a(r_0) = w^a(r_0) = \Omega^a$ is still the angular velocity of the horizon in the ϕ^a direction. For later convenience, let's use $\breve{g}$ to denote the full metric (4) and write down all the non-vanishing (inverse) metric elements,

$$\breve{g}_{rr} = \frac{f_r}{\Delta}, \quad \breve{g}_{ij} = h_i \delta_{ij}, \quad \breve{g}_{ab} = \tilde{g}_{ab}, \quad \breve{g}_{at} = -\tilde{g}_{ab}\tilde{w}^b, \quad \breve{g}_{tt} = -\tilde{f}_t\Delta + \tilde{g}_{ab}\tilde{w}^a\tilde{w}^b,$$
$$\breve{g}^{rr} = \frac{\Delta}{f_r}, \quad \breve{g}^{ij} = \frac{\delta_{ij}}{h_i}, \quad \breve{g}^{ab} = \tilde{g}^{ab} - \frac{\tilde{w}^a\tilde{w}^b}{\tilde{f}_t\Delta}, \quad \breve{g}^{at} = -\frac{\tilde{w}^a}{\tilde{f}_t\Delta}, \quad \breve{g}^{tt} = -\frac{1}{\tilde{f}_t\Delta}. \tag{7}$$

We have collected several examples in the "Appendix" to illustrate the general properties described here.

For a well defined black hole spacetime, the functions $\tilde{f}_t$, f_r, h_i and the matrix $(\tilde{g}_{ab})$ should be positive definite outside the black hole horizon. So one can rewrite the metric (4) in terms of the vielbeins,

$$ds^2 = \breve{g}_{\mu\nu}dx^\mu dx^\nu = \eta_{AB}e^A e^B, \quad A, B = 0, \ldots, d-1, \tag{8}$$

$$\begin{aligned} e^0 &= \sqrt{\tilde{f}_t\Delta}\, dt, \quad e^1 = \sqrt{\frac{f_r}{\Delta}}\, dr, \\ e^{1+i} &= \sqrt{h_i}\, d\theta^i \text{ (no summation over } i\text{)}: \quad i = 1, \ldots, \left[\frac{d}{2}\right] - 1, \\ e^{[\frac{d}{2}]+a} &: \quad a = 1, \ldots, \left[\frac{d+1}{2}\right] - 1, \end{aligned} \tag{9}$$

where $e^{[\frac{d}{2}]+a}$'s are obtained by diagonalizing the last term in (4).

3 Key result and the proof

Our main result of the paper is the following:

Key result: *Given the vielbeins* (9) *and the Hermitian gamma matrices,*

$$(\gamma^A)^\dagger = \gamma^0\gamma^A\gamma^0 = \gamma_A = \begin{cases} \gamma^A & : A = 1, \ldots, d-1; \\ -\gamma^0 & : A = 0, \end{cases} \tag{10}$$

and that the spinor field ψ obeys[1]

$$\gamma^0\psi = c\gamma^d\psi, \tag{12}$$

with $c \neq 0$ being some real constant, then (1) always reduces to

$$\xi = \xi^\mu\partial_\mu = \partial_t + \tilde{w}^a\partial_{\phi^a}, \tag{13}$$

with an appropriate normalization of ψ. As a result

$$\xi^2 = -(\xi^t)^2\tilde{f}_t\Delta + \tilde{g}_{ab}(\xi^a - \tilde{w}^a\xi^t)(\xi^b - \tilde{w}^b\xi^t) = -\tilde{f}_t\Delta, \tag{14}$$

which vanishes on the horizon. Since $\tilde{w}^a(r_0) = \Omega^a$, ξ becomes nothing but the null Killing vector on the horizon. One the other hand, because of (6), ξ also becomes the time Killing vector at the spatial infinity.

Proof From (11), one can check that

$$\begin{aligned}&(\gamma^d)^\dagger = (-1)^d\gamma^d, \quad (\gamma^d)^2 = (-1)^d\mathbf{1}_d, \Longrightarrow (\gamma^d)^\dagger\gamma^d = \mathbf{1}_d,\\ &\gamma^A\gamma^d = (-1)^{d-1}\gamma^d\gamma^A, \quad \forall\, A = 0, 1, \ldots, d-1,\end{aligned} \tag{15}$$

where $\mathbf{1}_d$ is the unit matrix in d dimensions. Using (10) and (12), one can also check that

$$\begin{aligned}\bar{\psi}\gamma^A\psi &= \psi^\dagger\gamma^0\gamma^A\psi = -\psi^\dagger\gamma^A\gamma^0\psi = -c\,\psi^\dagger\gamma^A\gamma^d\psi\\ &= (-1)^d c\,\psi^\dagger\gamma^d\gamma^A\psi = c\,\psi^\dagger(\gamma^d)^\dagger\gamma^A\psi = \psi^\dagger(\gamma^0)^\dagger\gamma^A\psi\\ &= -\bar{\psi}\gamma^A\psi = 0, \quad \forall\, A = 1, \ldots, d-1.\end{aligned} \tag{16}$$

So the only non-vanishing spinor bilinear is $\bar{\psi}\gamma^0\psi = -\psi^\dagger\psi$. Plugging into (1), we find

$$\begin{aligned}\xi^\mu &= -\eta_{AB}\,\breve{g}^{\mu\nu}\,e^B{}_\nu\,\bar{\psi}\gamma^A\psi = -\eta_{00}\,\breve{g}^{\mu\nu}\,e^0{}_\nu\,\bar{\psi}\gamma^0\psi\\ &= -\eta_{00}\,\breve{g}^{\mu t}\,e^0_t\,\bar{\psi}\gamma^0\psi = -(\psi^\dagger\psi)\sqrt{\tilde{f}_t\Delta}\;\breve{g}^{t\mu},\end{aligned} \tag{17}$$

$$\Longrightarrow\; \xi = \xi^\mu\partial_\mu = \frac{\psi^\dagger\psi}{\sqrt{\tilde{f}_t\Delta}}(\partial_t + \tilde{w}^a\partial_{\phi^a}), \tag{18}$$

where we have used (7) in the second line. One obtains (13) by setting $(\psi^\dagger\psi)^2 = \tilde{f}_t\Delta$. □

[1] All the gamma matrices in (12) are in the vielbein basis, and we define

$$\begin{aligned}\gamma^d &= (-1)^{\frac{d-2}{4}}\gamma^1\ldots\gamma^{d-1}\gamma^0 \quad : \quad \text{for } d \text{ even},\\ \gamma^d &= (-1)^{\frac{d-1}{4}}\gamma^1\ldots\gamma^{d-1}\gamma^0 \quad : \quad \text{for } d \text{ odd}.\end{aligned} \tag{11}$$

A few comments are in order,

- Both conditions (10) and (12) are necessary in leading to the key result (18). The real constant c in (12) only appears in the intermediate steps and it cancels out in the end. So its true value is not essential for our discussion. However, by applying another γ^0 to the left of both sides of (12), one can see that $c = \pm 1$. By using the redefinition $\gamma^0 \psi \to \psi$ when necessary, one can always choose to have $c = +1$.[2]
- In going from (18) to (13), we have normalized ψ so that $\xi^t = 1$. A different normalization of ψ amounts to multiplying (14) by an extra function (say $\mathcal{N}$). The vector field ξ is still null on the horizon as long as that $\mathcal{N}$ is nonsingular. On the other hand, the normalization in (13) certainly makes it easier to see the key properties of ξ, which approaches $\partial_t + \Omega^a \partial_{\phi^a}$ on the horizon and ∂_t at the spatial infinity;
- The result (18) depends on the choice of the vielbeins (9). One can see this by noting that (17) only depends on e^0. So if one keeps the gamma matrices γ^A (and hence the spinor field ψ) fixed, then any Lorentz transformation $e^A \to \Lambda^A{}_B e^B$ that changes e^0 can also lead to a different result for ξ.

As a side remark, note given (7) and (13),

$$\breve{\nabla}_\mu \xi^\mu = 0, \implies \breve{\nabla}_\rho \breve{\nabla}_\mu \xi^\rho = \breve{R}_{\mu\rho} \xi^\rho, \tag{19}$$

where everything with a " $\breve{}$ " is defined with the full metric $\breve{g}$ in (7). This partially justifies calling ξ the "conserved current".

Also note that, although the dependance of (18) on the choice of vielbeins appears as a limitation to our construction, it is so for a good reason. Technically, this is related to the fact that the Hermitian condition (10) is not invariant under a Lorentz transformation like $\gamma^A \to S\gamma^A S^{-1}$, because S is not unitary ($S^\dagger \neq S^{-1}$) in general. Physically, one of our motivation for the present construction is the emergent picture of the spacetime. In particular, a possible interpretation for the vector field in (1) is to relate it to some fluid flow coexisting with the black hole spacetime. In this case, not only is it natural to use an asymptotically static coordinate system, but also that the orientation of the vielbeins (9) could be related to structures of the underlying material and thus may have distinguished physical meanings. As such, it is sensible to work with a particular choice of the vielbeins.

As an interesting comparison, let's see what happens if we repeat the calculation for metrics (2) and (3). Because only e^0 matters, we will only write e^0 out explicitly in the following. For (2), the natural choice is

$$e^0 = \sqrt{f_t \Delta}\,(dt + f_a d\phi^a). \tag{20}$$

[2] I thank the referee for pointing out the definitive value of c.

Then from (17) and (7),

$$\xi^{\mu} = -(\psi^{\dagger}\psi)\sqrt{f_t\Delta}\,(\breve{g}^{t\mu} + f_a\breve{g}^{a\mu}),$$
$$\Longrightarrow \quad \xi^{\mu}\partial_{\mu} = \frac{\psi^{\dagger}\psi}{k\sqrt{f_t\Delta}}(\partial_t + w^a\partial_{\phi^a}). \tag{21}$$

Similarly for (3), the natural choice is

$$e^0 = \sqrt{f_t\Delta}\; f_a d\phi^a. \tag{22}$$

So from (17) and (7),

$$\xi^{\mu} = -(\psi^{\dagger}\psi)\sqrt{f_t\Delta}\; f_a\breve{g}^{a\mu},$$
$$\Longrightarrow \xi^{\mu}\partial_{\mu} = \frac{\psi^{\dagger}\psi}{k\sqrt{f_t\Delta}}(\partial_t + w^a\partial_{\phi^a}). \tag{23}$$

In deriving these results, we note (2), (3) and (4) share the same form of the full metric (7). Now if we set $\psi^{\dagger}\psi = k\sqrt{f_t\Delta}$, then both (21) and (23) lead to

$$\xi = \partial_t + w^a\partial_{\phi^a}. \tag{24}$$

Comparing with (13), we see that $\tilde{w}^a$ is replaced by w^a.

Although both $\tilde{w}^a$ and w^a approach the angular velocity Ω^a on the horizon, only $\tilde{w}^a$ is guaranteed to vanish at the spatial infinity. This difference also makes it more desirable to use (4) instead of (2) or (3) in the calculations.

4 Summary

To conclude, we have constructed a vector field (1) by using the "conserved current" of a particular spinor field. We have shown that, in the background of a stationary and axisymmetric black hole, the vector field always approaches the null Killing vector on horizon and the time Killing vector at the spatial infinity. The required constraint on the spinor field is simple and universal (valid for any spacetime dimensions).

It is still not clear as to the true physical nature of the vector field or the corresponding spinor field. Our original motivation for studying the vector field was to construct a possible fluid flow underlying the spacetime [1]. In fact, the behavior of the vector field fits very well with our intuitive picture about the speculated fluid that may coexist with our spacetime. It looks like that the fluid is dragged by the black hole horizon (Hence the same angular velocity on the horizon), and then the angular velocity steadily decreases until it vanishes at the spatial infinity. However, such a picture is still highly hypothetical, and one should be open minded towards other possible explanations.

Regardless of what the physical interpretation of (1) may be, it is unexpected and also quite amazing that a conspiracy like (10), (12) and (13) can exist. Given the remarkable features summarized in the Key result, it will be very interesting to see possible applications of the vector field (13) or the corresponding spinor field (12), or both.

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Appendix A: Some examples

In this section, we use some explicit examples to illustrate our Key result given in the main context. Our main purpose here is to show two things: (i) one can always put a black hole metric into the form of (4),[3] and (ii) the functions $\tilde{w}^a$ do behave as described around (6). Our strategy is to explicitly do the calculation for the general metric (4) in a given dimension, and then to apply the result to a particular black hole solution. Whenever possible, we will use examples from the complicated solutions in supergravity theories (e.g. [9–18]). These solutions are given in particular spacetime dimensions. For an example in arbitrary spacetime dimensions, one can use the general Kerr-AdS solution [19].

We will only carry out the calculation in three through five dimensions. For more examples, one is referred to [2]. To simplify our notations, we will drop the "tilde" from all the functions in (4) from now on.

A.1 $d = 3$

In three dimensions, (4) becomes

$$ds^2 = -f_t \Delta dt^2 + \frac{f_r}{\Delta} dr^2 + g_{11}(d\phi^1 - w^1 dt)^2. \tag{25}$$

The corresponding dreibeins are

$$e^0 = \sqrt{f_t \Delta} dt, \quad e^1 = \sqrt{\frac{f_r}{\Delta}} dr, \quad e^2 = \sqrt{g_{11}}(d\phi^1 - w^1 dt). \tag{26}$$

The gamma matrices in the dreibein basis are chosen as[4]

$$\gamma^0 = i\sigma^3, \quad \gamma^1 = \sigma^1, \quad \gamma^2 = \sigma^2, \tag{27}$$

where $\sigma^{1,2,3}$ are the usual Pauli matrices. The spinor field is

$$\psi = \begin{pmatrix} \psi_{1a} + i\psi_{1b} \\ \psi_{2a} + i\psi_{2b} \end{pmatrix}, \tag{28}$$

[3] It is trivial that one can cast a metric into the ADM form, but it is non-trivial that for the first two terms in (4), dt^2 always comes with a factor Δ and dr^2 always comes with a factor Δ^{-1}.

[4] Note one can use any set of Hermitian gamma matrices.

where all the functions are real. From (12),

$$\psi_{2a} = \psi_{2b} = 0. \tag{29}$$

Plugging (29) into (1) and normalizing ψ appropriately, we find

$$\xi = \partial_t + w^1 \partial_{\phi^1}, \tag{30}$$

just as given in (13).

For a concrete example, let's look at the BTZ black hole [7,8]. It has the exceptional feature that, even without counterterms to the action, its first law of thermodynamics works not only in an asymptotically static coordinate system, but also in one that is rotating at the spatial infinity. For this reason, we will discuss it in more detail. The metric is

$$\begin{aligned} ds^2 &= -\Delta d\hat{t}^{\,2} + \frac{d\hat{r}^2}{\Delta} + \hat{r}^2 (d\hat{\phi} - w d\hat{t}\,)^2, \\ \Delta &= -8M + \frac{\hat{r}^2}{\ell^2} + \frac{16a^2}{\hat{r}^2}, \quad w = \frac{4a}{\hat{r}^2}. \end{aligned} \tag{31}$$

It solves the equations of motion $R_{\mu\nu} = -\frac{2}{\ell^2} g_{\mu\nu}$. The horizon is located at $\hat{r}_0$ with $\Delta(\hat{r}_0) = 0$. The mass, temperature, entropy, angular velocity and the angular momentum are

$$E = M, \quad T = \frac{\Delta'(\hat{r}_0)}{4\pi}, \quad S = \frac{\pi \hat{r}_0}{2}, \quad \Omega = w(\hat{r}_0), \quad J = a, \tag{32}$$

respectively. One can check that the first law of thermodynamics is satisfied,

$$dE = TdS + \Omega dJ. \tag{33}$$

The coordinate system of (31) is *rotating* at the spatial infinity. One can switch to an asymptotically static coordinate system by a coordinate transformation,

$$\begin{aligned} \hat{t} &= t + \frac{\tilde{M}\ell^2}{J}\phi, \quad \hat{\phi} = \phi + \frac{\tilde{M}}{J}t, \quad \hat{r} = \frac{2Jr}{r_0\sqrt{\tilde{M}}}, \\ \tilde{M} &= M - \frac{r_0^2}{8\ell^2}, \quad r_0 = 2\sqrt{2}\ell\Big(M^2 - \frac{J^2}{\ell^2}\Big)^{1/4}. \end{aligned} \tag{34}$$

The metric becomes

$$ds^2 = -\Delta dt^2 + \frac{dr^2}{\Delta f_p} + r^2 f_p\, d\phi^2, \quad \Delta = \frac{r^2 - r_0^2}{\ell^2}, \quad f_p = 1 - \frac{\tilde{M}^2\ell^2 r_0^2}{J^2 r^2}. \tag{35}$$

In the new coordinate system, the horizon is located at r_0. The mass, temperature, entropy, angular velocity and the angular momentum are

$$E = \frac{r_0^4 \tilde{M}}{32 J^2 \ell^2}, \quad T = \frac{r_0^2 \sqrt{\tilde{M}}}{4\pi J \ell^2}, \quad S = \frac{\pi r_0^2 \sqrt{\tilde{M}}}{4J}, \quad \Omega = J = 0, \tag{36}$$

respectively. Again, the first law of thermodynamics (33) is satisfied.

Both (31) and (35) are of the form (25). Using (30), we find for (31),

$$\xi = \partial_{\hat{t}} + \frac{4a}{\hat{r}^2} \partial_{\hat{\phi}}, \tag{37}$$

and for (35),

$$\xi = \partial_t. \tag{38}$$

Our Key result at (13) only covers the case (38), where the coordinate system is non-rotating at the spatial infinity. But it is interesting to note that that (37) also fits the descriptions in our Key result.

A.2 $d = 4$

In four dimensions, (4) becomes

$$ds^2 = -f_t \Delta dt^2 + \frac{f_r}{\Delta} r^2 + h_1 (d\theta^1)^2 + g_{11}(d\phi^1 - w^1 dt)^2. \tag{39}$$

The corresponding vierbeins are

$$e^0 = \sqrt{f_t \Delta}\, dt, \quad e^1 = \sqrt{\frac{f_r}{\Delta}}\, dr, \quad e^2 = \sqrt{h_1}\, d\theta, \quad e^3 = \sqrt{g_{11}}(d\phi^1 - w^1 dt). \tag{40}$$

The gamma matrices in the vierbein basis are chose as

$$\gamma^0 = i\sigma^3 \otimes \mathbf{1}_2, \quad \gamma^j = -\sigma^2 \otimes \sigma^j, \quad j = 1, 2, 3. \tag{41}$$

The spinor field is

$$\psi = \begin{pmatrix} \psi_{1a} + i\psi_{1b} \\ \psi_{2a} + i\psi_{2b} \\ \psi_{3a} + i\psi_{3b} \\ \psi_{4a} + i\psi_{4b} \end{pmatrix}, \tag{42}$$

where all the functions are real. From (12), we find

$$\psi_{1a} = \psi_{3b}, \quad \psi_{1b} = -\psi_{3a}, \quad \psi_{2a} = \psi_{4b}, \quad \psi_{2b} = -\psi_{4a}. \tag{43}$$

Plugging (43) into (1) and normalizing ψ appropriately, we find

$$\xi = \partial_t + w^1 \partial_{\phi^1}, \tag{44}$$

just as given in (13).

For a concrete example, we consider the rotating solution in U(1)^4 gauged supergravity with four charges pairwise equal [11],

$$\begin{aligned} ds^2 &= -\frac{\Delta}{W}(d\hat{t} - a\sin^2\theta d\hat{\phi})^2 + W\Big(\frac{dr^2}{\Delta} + \frac{d\theta^2}{\Delta_\theta}\Big) \\ &\quad + \frac{\Delta_\theta \sin^2\theta}{W}\Big[a d\hat{t} - (r_1 r_2 + a^2) d\hat{\phi}\Big]^2, \end{aligned} \tag{45}$$

where $r_1 = r + 2ms_1^2$, $r_2 = r + 2ms_2^2$ and

$$\begin{aligned} \Delta &= r^2 + a^2 - 2mr + g^2 r_1 r_2 [r_1 r_2 + a^2], \\ \Delta_\theta &= 1 - g^2 a^2 \cos^2\theta, \quad W = r_1 r_2 + a^2\cos^2\theta. \end{aligned} \tag{46}$$

More details can be found in the original paper. The coordinates in (45) are not asymptotically static. They are related to the static ones by

$$d\hat{t} = dt, \quad d\hat{\phi} = \frac{d\phi - g^2 a dt}{1 - g^2 a^2}. \tag{47}$$

Now the metric (45) becomes

$$\begin{aligned} ds^2 &= -\frac{W\Delta_\theta\Delta}{Y}dt^2 + W\Big(\frac{dr^2}{\Delta} + \frac{d\theta^2}{\Delta_\theta}\Big) + \frac{Y\sin^2\theta}{(1-g^2a^2)^2 W}(d\phi - w dt)^2, \\ w &= a\Delta_\theta \frac{2m(1+s_1^2+s_2^2)r + 4m^2 s_1^2 s_2^2}{Y}, \quad Y = (r_1r_2 + a^2)^2\Delta_\theta - a^2\sin^2\theta\Delta. \end{aligned} \tag{48}$$

The horizon is located at $\Delta(r_0) = 0$, and the angular velocity is

$$\Omega = w(r_0) = a\frac{1 + g^2 r_{10} r_{20}}{r_{10} r_{20} + a^2}, \tag{49}$$

where $r_{10} = r_1(r_0)$ and $r_{20} = r_2(r_0)$. At the spatial infinity, we have

$$w \sim \frac{1}{r^3} \longrightarrow 0 \quad \text{as} \quad r \longrightarrow +\infty, \tag{50}$$

just as asserted in (6). The metric (48) is of the form (39), and so one can use (44) to find $\xi = \partial_t + w\partial_\phi$. Using the properties of w as described above, we note ξ behaves exactly as described in our Key result.

A.3 $d = 5$

In five dimensions, (2) becomes

$$ds^2 = -f_t \Delta (dt + f_1 d\phi^1 + f_2 d\phi^2)^2 + \frac{f_r}{\Delta} r^2 + h_1 (d\theta^1)^2 \\ + g_{11}\Big[d\phi^1 - w^1 dt + g_{12}(d\phi^2 - w^2 dt)\Big]^2 + g_{22}(d\phi^2 - w^2 dt)^2. \quad (51)$$

The corresponding fuenfbeins are

$$e^0 = \sqrt{f_t \Delta}\,(dt + f_1 d\phi^1 + f_2 d\phi^2), \quad e^1 = \sqrt{\frac{f_r}{\Delta}}\,dr, \quad e^2 = \sqrt{h_1}\,d\theta, \\ e^3 = \sqrt{g_{11}}\Big[d\phi^1 - w^1 dt + g_{12}(d\phi^2 - w^2 dt)\Big], \quad e^4 = \sqrt{g_{22}}(d\phi^2 - w^2 dt). \quad (52)$$

The gamma matrices in the fuenfbein basis are taken to be

$$\gamma^0 = i\sigma^1 \otimes \mathbf{1}_2, \quad \gamma^4 = \sigma^3 \otimes \mathbf{1}_2, \quad \gamma^j = -\sigma^2 \otimes \sigma^j, \quad j = 1, 2, 3. \quad (53)$$

The spinor field is

$$\psi = \begin{pmatrix} \psi_{1a} + i\psi_{1b} \\ \psi_{2a} + i\psi_{2b} \\ \psi_{3a} + i\psi_{3b} \\ \psi_{4a} + i\psi_{4b} \end{pmatrix}, \quad (54)$$

where all the functions are real. From the first equation in (12), we find

$$\psi_{1a} = \psi_{3a}, \quad \psi_{1b} = \psi_{3b}, \quad \psi_{2a} = \psi_{4a}, \quad \psi_{2b} = \psi_{4b}. \quad (55)$$

Plugging (55) into (1) and normalizing ψ appropriately, we find

$$\xi = \partial_t + w^1 \partial_{\phi_1} + w^2 \partial_{\phi_2}, \quad (56)$$

just as given in (13).

For a concrete example, we consider the rotating solution in $U(1)^3$ gauged supergravity with two of the charges equal [13],

$$ds^2 = H_1^{2/3} H_3^{1/3} \Bigg\{ (x^2 - y^2) \left(\frac{dx^2}{X} - \frac{dy^2}{Y} \right) - \frac{x^2 X (d\tau + y^2 d\sigma)^2}{(x^2 - y^2) f H_1^2} \\ + \frac{y^2 Y \left[d\tau + (x^2 + 2ms_1^2) d\sigma \right]^2}{(x^2 - y^2)(\gamma + y^2) H_1^2}$$

$$-U\left(d\tau + y^2 d\sigma + \frac{(x^2-y^2) f H_1 \left[abd\sigma + (\gamma + y^2)d\chi\right]}{ab(x^2-y^2)H_3 - 2ms_3c_3(\gamma+y^2)}\right)^2 \Bigg\}\,, \tag{57}$$

where more detail can be found in the original paper. The coordinates in (57) are not asymptotically static, and they are related to the non-rotating ones by

$$\begin{aligned}
x^2 &= r^2 - \frac{2m}{3}(2s_1^2 + s_3^2) - \gamma, \quad y^2 = -a^2\cos^2\theta - b^2\sin^2\theta - \gamma,\\
\tau &= \frac{(1+g^2\gamma)t}{\Xi_a\Xi_b} - \frac{\tilde{a}^2 a\,\phi^1}{\Xi_a(a^2-b^2)} + \frac{\tilde{b}^2 b\,\phi^2}{\Xi_b(a^2-b^2)},\\
\sigma &= \frac{g^2 t}{\Xi_a\Xi_b} - \frac{a\,\phi^1}{\Xi_a(a^2-b^2)} + \frac{b\,\phi^2}{\Xi_b(a^2-b^2)},\\
\chi &= \frac{g^4 ab\,t}{\Xi_a\Xi_b} - \frac{b\,\phi^1}{\Xi_a(a^2-b^2)} + \frac{a\,\phi^2}{\Xi_b(a^2-b^2)}.
\end{aligned} \tag{58}$$

Now the metric is rather messy and it is unenlightening to write it out explicitly. We only need to know that the metric can be cast into the form of (51). So the vector field (1) now takes the form of (56). What's important for us are the details of the functions w^1 and w^2, which we find to be

$$\begin{aligned}
w^1 &= w^1_{(0)} + \frac{g^2r^2\,\Xi_a V_1\Delta}{(1-g^2\tilde{y}^2)V}, \quad w^1_{(0)} = \frac{b(ab+2mc_3s_3) + a(1+g^2(r_{22}+b^2))r_{44}}{ab(ab+2mc_3s_3) + (r_{22}+a^2+b^2)r_{44}},\\
w^2 &= w^2_{(0)} + \frac{g^2r^2\,\Xi_b V_2\Delta}{(1-g^2\tilde{y}^2)V}, \quad w^2_{(0)} = \frac{a(ab+2mc_3s_3) + b(1+g^2(r_{22}+b^2))r_{44}}{ab(ab+2mc_3s_3) + (r_{22}+a^2+b^2)r_{44}},\\
V_1 &= a\,\Xi_b(r_{22}+b^2)(r_{44}+\tilde{y}^2) + 2m(a^2-b^2)(bc_3s_3 - a(s_1^2-s_3^2))\cos^2\theta,\\
V_2 &= b\,\Xi_a(r_{22}+a^2)(r_{44}+\tilde{y}^2) - 2m(a^2-b^2)(ac_3s_3 - b(s_1^2-s_3^2))\sin^2\theta,\\
V &= \left[(r_{44}+a^2)(r_{44}+b^2) + 2mr_{44}(s_1^2-s_3^2) + 2mabc_3s_3\right]\\
&\quad\times\Bigg[(\Xi_a(r_{22}+a^2) - 2ms_1^2)(\Xi_b(r_{22}+b^2) - 2ms_1^2)\\
&\quad -2m(r_{22}-\gamma)(1+g^2\gamma) - 2mg^2\tilde{a}^2\tilde{b}^2 + 4m^2s_1^2 - \frac{r^2\,\Xi_a\Xi_b\Delta}{1-g^2\tilde{y}^2}\Bigg],\\
\Delta &= \frac{(r_{42}+a^2)(r_{42}+b^2) + g^2(r_{22}+a^2)(r_{22}+b^2)r_{44} - 2m(r_{42}-\gamma)}{r^2},\\
r_{22} &= r^2 + \frac{2m(s_1^2-s_3^2)}{3}, \quad r_{42} = r_{22} - 2ms_1^2, \quad r_{44} = r_{42} + 2ms_3^2,\\
\tilde{y}^2 &= a^2\cos^2\theta + b^2\sin^2\theta, \quad \Xi_a = 1-g^2a^2, \quad \Xi_b = 1-g^2b^2,\\
\gamma &= 2abc_3s_3 + (a^2+b^2)s_3^2, \quad \tilde{a}^2 = a^2+\gamma, \quad \tilde{b}^2 = b^2+\gamma^2.
\end{aligned} \tag{59}$$

The horizon is located at $\Delta(r_0) = 0$, and so the angular velocities are

$$\Omega^1 = w^1(r_0) = w^1_{(0)}(r_0), \quad \Omega^2 = w^2(r_0) = w^2_{(0)}(r_0), \tag{60}$$

which agree with that found in [13]. At the spatial infinity ($r \to +\infty$), we find

$$
\begin{aligned}
w^1_{(0)} &\to g^2 a, \quad \frac{g^2 r^2 \Xi_a V_1 \Delta}{(1-g^2\tilde{y}^2)V} \to -g^2 a \implies w^1 \to 0,\\
w^2_{(0)} &\to g^2 b, \quad \frac{g^2 r^2 \Xi_b V_2 \Delta}{(1-g^2\tilde{y}^2)V} \to -g^2 b \implies w^2 \to 0,
\end{aligned} \tag{61}
$$

just as asserted in (6). Using these properties, we note that (56) behaves just as described in our Key result.

References

1. Mei, J.: Towards a possible fluid flow underlying the Kerr spacetime. [arXiv:1104.3728 [hep-th]]
2. Mei, J.: The entropy for general extremal black holes. JHEP **1004**, 005 (2010). [arXiv:1002.1349 [hep-th]]
3. Arnowitt, R.L., Deser, S., Misner, C.W.: The dynamics of general relativity. gr-qc/0405109
4. Gibbons, G.W., Perry, M.J., Pope, C.N.: The first law of thermodynamics for Kerr-anti-de Sitter black holes. Class. Quant. Gravit. **22**, 1503–1526 (2005). [hep-th/0408217]
5. Gibbons, G.W., Perry, M.J., Pope, C.N.: AdS/CFT Casimir energy for rotating black holes. Phys. Rev. Lett. **95**, 231601 (2005). [hep-th/0507034]
6. Papadimitriou, I., Skenderis, K.: Thermodynamics of asymptotically locally AdS spacetimes. JHEP **0508**, 004 (2005). [hep-th/0505190]
7. Banados, M., Teitelboim, C., Zanelli, J.: The Black hole in three-dimensional space-time. Phys. Rev. Lett. **69**, 1849 (1992). [arXiv:hep-th/9204099]
8. Carlip, S.: Conformal field theory, $(2+1)$-dimensional gravity, and the BTZ black hole. Class. Quant. Gravit. **22**, R85 (2005). [arXiv:gr-qc/0503022]
9. Cvetič, M., Youm, D.: Entropy of non-extreme charged rotating black holes in string theory. Phys. Rev. D **54**, 2612 (1996). [arXiv:hep-th/9603147]
10. Cvetič, M., Youm, D.: General rotating five dimensional black holes of toroidally compactified heterotic string. Nucl. Phys. B **476**, 118 (1996). [arXiv:hep-th/9603100]
11. Chong, Z.W., Cvetič, M., Lü, H., Pope, C.N.: Charged rotating black holes in four-dimensional gauged and ungauged supergravities. Nucl. Phys. B **717**, 246 (2005). [arXiv:hep-th/0411045]
12. Cvetič, M., Lü, H., Pope, C.N.: Charged rotating black holes in five dimensional $U(1)^3$ gauged $N=2$ supergravity. Phys. Rev. D **70**, 081502 (2004). [arXiv:hep-th/0407058]
13. Mei, J., Pope, C.N.: New rotating non-extremal black holes in D = 5 maximal gauged supergravity. Phys. Lett. B **658**, 64 (2007). [arXiv:0709.0559 [hep-th]]
14. Chow, D.D.K.: Equal charge black holes and seven dimensional gauged supergravity. Class. Quant. Gravit. **25**, 175010 (2008). [arXiv:0711.1975 [hep-th]]
15. Chow, D.D.K.: Charged rotating black holes in six-dimensional gauged supergravity. Class. Quant. Gravit. **27**, 065004 (2010). [arXiv:0808.2728 [hep-th]]
16. Chow, D.D.K.: Two-charge rotating black holes in four-dimensional gauged supergravity. [arXiv:1012.1851 [hep-th]]
17. Chow, D.D.K.: Single-charge rotating black holes in four-dimensional gauged supergravity. Class. Quant. Gravit. **28**, 032001 (2011). [arXiv:1011.2202 [hep-th]]
18. Wu, S.-Q.: General nonextremal rotating charged ads black holes in five-dimensional $U(1)^3$ gauged supergravity: a simple construction method. Phys. Lett. B **707**, 286 (2012). [arXiv:1108.4159 [hep-th]]
19. Chen, W., Lü, H., Pope, C.N.: General Kerr-NUT-AdS metrics in all dimensions. Class. Quant. Gravit. **23**, 5323 (2006). [arXiv:hep-th/0604125]