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Conformal Symmetries of the Einstein-Hilbert Action on Horizons of Stationary and Axisymmetric Black Holes

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ABSTRACT

We suggest a way to study possible conformal symmetries on black hole horizons. We do this by carrying out a Kaluza-Klein like reduction of the Einstein-Hilbert action along the ignorable coordinates of stationary and axisymmetric black holes. Rigid diffeomorphism invariance of the m-ignorable coordinates then becomes a global $SL(m,R)$ gauge symmetry of the reduced action. Related to each non-vanishing angular velocity there is a particular $SL(2,R)$ subgroup, which can be extended to the Witt algebra on the black hole horizons. The classical Einstein-Hilbert action thus has k-copies of infinite dimensional conformal symmetries on a given black hole horizon, with k being the number of non-vanishing angular velocities of the black hole.

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1 Introduction

It is a long standing problem to find a statistical explanation of the black hole entropy. One intriguing possibility is that the black hole entropy may have a sort of "universal" explanation, which is largely determined by some 2D conformal filed theory but depends little on the detail of the possible UV completion of quantum gravity. Discussions of such an idea can be found in, e.g. [1, 2].

There have been some evidence in support of this possibility. Soon after the original calculation of the entropy for certain black holes in string theory [3], Strominger showed that any black holes having an AdS_3 factor in their near horizon region can have their entropies calculated in a common way [4], by using the fact that quantum gravity on AdS_3 must be described by a 2D conformal field theory (CFT) [5]. Loosely related to this, it has also been suggested that, with appropriate boundary conditions imposed, quantum gravity on the horizon of black holes may also be described by a 2D conformal field theory [6, 7, 8, 9, 10]. This later argument, however, is marred by the ambiguity on the possible boundary conditions that one can impose near the black hole horizons.

More recently, the development of the Kerr/CFT correspondence [11, 2] brings more support to a possible "universal" explanation of the black hole entropy. The near horizon limit of the extremal Kerr (NHEK) metric [12] at fixed polar angles are quotients of warped AdS_3. This indicates that one may use the same techniques of [5] to discuss the asymptotic symmetry group, much like in the case of BTZ black holes [4], which are quotients of AdS_3. Indeed, for an extremal Kerr black hole with the angular momentum J, appropriate boundary conditions can be found and a copy of the Virasoro algebra can be identified. The putative CFT at the NHEK boundary was shown to have a central charge $c_L = 12J$

and temperature $T_L = \frac{1}{2\pi}$ [11]. Cardy's formula then reproduces exactly the Bekenstein-Hawking entropy. Afterwards, the calculation was generalized to black holes in higher dimensions and also in more complicated settings (for a sample of the early references, see [13, 14, 15, 16, 17, 18, 19]). Black holes in more than four dimensions can have multiple rotations. It was found in [16] that corresponding to each non-zero rotation there is an independent copy of the Virasoro algebra, and each copy of the Virasoro algebra appears to be equally good in reproducing the Bekenstein-Hawking entropy. For general treatments, it has also been shown that the method works for all extremal stationary and axisymmetric black holes, in the context of Einstein gravity [20].

As a drawback, the success of the Kerr/CFT correspondence is limited to extremal black holes. Although it is possible to discuss physics slightly away from the extremal limit (see, e.g. [21]), it will be more desirable to study the case of non-extremal black holes directly. The investigation of the hidden conformal symmetry of Kerr black hole is one such attempt [22]. In steady of looking at the symmetry structure of gravitational fluctuations directly, the authors of [22] studied the dynamics of a massless scalar field probing the background of a Kerr black hole. They found that the wave equation in the so called "near region" enjoys an enhanced $SL(2,R)_L \times SL(2,R)_R$ symmetry. By assuming that there is a putative dual 2D CFT having a ground state sharing this same $SL(2,R)_L \times SL(2,R)_R$ symmetry, the authors of [22] were able to infer for the temperatures $T_{L,R}$, which together with the central charges $c_{L,R}$ extrapolated from the Kerr/CFT calculation, reproduce the Bekenstein-Hawking entropy exactly. Further evidence of the existence of a dual 2D CFT was also provided by matching the low-energy scalar-Kerr scattering amplitude with correlators of a 2D CFT at the same temperatures. For further works one can consult [23] and references therein.

Still, the situation with non-extremal black holes is far from being satisfactory. In order to achieve the same level of success as is in the case of Kerr/CFT correspondence for extremal black holes, one will need a way to identify the full conformal symmetries of the putative dual 2D conformal field theory. In this paper, we want to report some partial results that may finally help us achieve this goal.

We will show that on the horizon of a stationary and axisymmetric black hole with k non-vanishing angular velocities, the Einstein-Hilbert action itself enjoys k-copies of infinite dimensional conformal symmetries. Note the similarity between this result and that from [16] mentioned above. Our result holds for any stationary and axisymmetric black holes in any spacetime dimensions. But since we will limit our calculation to pure Einstein gravity plus a (possibly zero) cosmological constant, the black holes should also be solutions to such a system.

Our starting point is the simple fact that stationary and axisymmetric black holes all have ignorable coordinates and that their metrics share a common structure [20]. It is then natural to seek a Kaluza-Klein like reduction of the action on the ignorable coordinates. The usual experience with Kaluza-Klein reduction suggests that it may be easier to study some of the symmetries in the system (see, e.g. [24, 25]). On the other hand, since we presume the existence of the classical black hole solutions, what we do here is not much than explicitly writing out the classical action in terms of functions that are known to be independent on the ignorable coordinates. As such, we will not expect any inconsistency that may arise in the usual Kaluza-Klein reduction of a dynamical system. Rather, the reduced action allows us to study the classical equations of motion in a much greater detail. In the case of pure gravity plus a cosmological constant, this allows us to re-derive the first law of black hole thermodynamics in a straightforward manor. In fact, the derivation echoes with [26] and partially explains why it is sensible to calculate the mass of a black hole by integrating the first law of thermodynamics.

After the reduction, we find that the rigid diffeomorphism invariance of the ignorable coordinates become a global $SL(m,R)$ gauge symmetry of the reduced action, with m being the number of the ignorable coordinates. As the key result of this paper, we will show that corresponding to each non-vanishing angular momentum there is a particular $SL(2,R)$ subgroup, which can be extended to the full Witt algebra on the black hole horizons. This means that the classical Einstein-Hilbert action, when restricted to the horizons of stationary and axisymmetric black holes, enjoys a copy of the infinite dimensional conformal symmetry for each non-vanishing angular velocity.

The plan of the paper is as follows. In section 2, we derive a scheme of Kaluza-Klein like reduction that will make it easier to deal with the special case of stationary and axisymmetric black holes. In section 3, we write down the reduced action for stationary and axisymmetric black holes. As an application, we re-derive the first law for black holes in terms of the new language. In section 4, we prove the classical conformal invariance of the reduced action on the black hole horizons. A short summary is in section 5.

2 A Kaluza-Klein Reduction of the Einstein-Hilbert Action

Consider the action in an D-dimensional spacetime Σ with a boundary $\partial\Sigma$,

$$S = \int_\Sigma d^D x \sqrt{|g|}\, (R - 2\Lambda) + \int_{\partial\Sigma} (d^{D-1}x)_\mu n^\mu \sqrt{|g|}\; K\,, \tag{1}$$

where n^μ is the unit normal vector of $\partial\Sigma$ (suppose the boundary is defined with some function $\Delta = 0$, then $n_\mu = \partial_\mu \Delta / \sqrt{g^{\varrho\sigma}\partial_\varrho \Delta \partial_\sigma \Delta}$), and K is the extrinsic curvature,

$$K = g^{\mu\nu} K_{\mu\nu}\,, \quad K_{\mu\nu} = \nabla_\mu n_\nu + \nabla_\nu n_\mu\,. \tag{2}$$

The inclusion of the Gibbons-Hawking-York boundary term is necessary for a well defined variation principle. When the metric is varied (note $\delta g^{\mu\nu} = 0$ on $\partial\Sigma$),

$$\delta S = \int_\Sigma d^D x \sqrt{|g|}\, \Big(R_{\mu\nu} - \frac{R - 2\Lambda}{2} g_{\mu\nu}\Big)\delta g^{\mu\nu}\,, \tag{3}$$

from which one can derive the equations of motion

$$R_{\mu\nu} = \frac{2\Lambda}{D-2} g_{\mu\nu}\,. \tag{4}$$

Now consider the metric of a $(D = m + n)$-dimensional spacetime,[1]

$$ds^2 = \widetilde{G}_{\mu\nu} dx^\mu dx^\nu = H_{IJ} dx^I dx^J + G_{AB} dy^A dy^B\,, \tag{5}$$

where both H_{IJ} and G_{AB} depend only on the x-coordinates. We use capital letters from the beginning of the alphabet $(A, B, C, \cdots \in \{1, \cdots, m\})$ to label the y-coordinates, and those from the middle of the alphabet $(I, J, K, \cdots \in \{1, \cdots, n\})$ to label the x-coordinates. The reason for considering such a metric will become clear in the next section. Now because both G_{AB} and H_{IJ} depend only on the x-coordinates, one can formally treat G_{AB} as some matter fields living in the curved background H_{IJ}. It is then interesting to look at the action for both G_{AB} and H_{IJ} from this new perspective. For this purpose, let's write down the metric elements explicitly,

$$\begin{aligned} & \widetilde{G}_{IJ} = H_{IJ}\,, \quad \widetilde{G}_{AB} = G_{AB}\,, \quad \widetilde{G}_{IA} = 0\,, \\ \Longrightarrow \quad & \widetilde{G}^{IJ} = H^{IJ}\,, \quad \widetilde{G}^{AB} = G^{AB}\,, \quad \widetilde{G}^{IA} = 0\,. \end{aligned} \tag{6}$$

From now on, indices $A, B, C, \cdots$ will be raised or lowered using the metric G, and indices $I, J, K, \cdots$ will be raised or lowered using the metric H. We will always write out the indices

[1] Do not confuse the number n with the normal vector n^μ of the boundary $\partial\Sigma$.

$A, B, C, \cdots$ explicitly, but will sometimes hide the $I, J, K, \cdots$ indices, in places where their presence is obvious. The elements of the original affine connection are

$$\begin{aligned}
\widetilde{\Gamma}^I_{JK} &= \Gamma^I_{JK}\,, \quad \widetilde{\Gamma}^A_{IJ} = \widetilde{\Gamma}^I_{AJ} = \widetilde{\Gamma}^A_{BC} = 0\,, \\
\widetilde{\Gamma}^I_{AB} &= -\frac{1}{2}\partial^I G_{AB}\,, \quad \widetilde{\Gamma}^A_{IB} = \frac{1}{2}G^{AC}\partial_I G_{BC}\,,
\end{aligned} \tag{7}$$

the elements of the original Ricci tensor are

$$\begin{aligned}
\widetilde{R}_{IJ} &= R_{IJ} - \nabla_I \nabla_J \ln\sqrt{|G|} + \frac{1}{4}\partial_I G_{AB}\partial_J G^{AB}\,, \quad \widetilde{R}_{IA} = 0\,, \\
\widetilde{R}_{AB} &= -\frac{1}{2}\nabla^2 G_{AB} - \frac{1}{2}\partial \ln\sqrt{|G|}\;\partial G_{AB} + \frac{1}{2}G^{CD}\partial G_{AC}\partial G_{BD}\,,
\end{aligned} \tag{8}$$

and the original Ricci scalar is

$$\begin{aligned}
\widetilde{R} &= R - (\partial \ln\sqrt{|G|})^2 - 2\nabla^2 \ln\sqrt{|G|} + \frac{1}{4}\partial G_{AB}\partial G^{AB} \\
&= R + (\partial \ln\sqrt{|G|})^2 + \frac{1}{4}\partial G_{AB}\partial G^{AB} - \frac{2}{\sqrt{|G|}}\nabla^2\sqrt{|G|}\,.
\end{aligned} \tag{9}$$

We will only consider the case when the boundary $\partial\Sigma$ is in the x-directions. Then $\widetilde{n}_A = 0$, $\widetilde{n}_I = n_I$ and

$$\begin{aligned}
&\widetilde{K}_{IJ} = K_{IJ} = \nabla_I\, n_J + \nabla_J\, n_I\,, \quad \widetilde{K}_{AB} = -2\widetilde{\Gamma}^I_{AB} n_I = n_I \partial^I G_{AB}\,, \\
&\qquad \Longrightarrow \quad \widetilde{K} = \widetilde{H}^{IJ}\widetilde{K}_{IJ} + \widetilde{G}^{AB}\widetilde{K}_{AB} = K + 2n^I\partial_I \ln\sqrt{|G|}\,.
\end{aligned} \tag{10}$$

Using these results in the original action (1), we find

$$\begin{aligned}
S &= \int_\Sigma d^n x\sqrt{|H|}\sqrt{|G|}\,\Big\{R - 2\Lambda + (\partial\ln\sqrt{|G|})^2 + \frac{1}{4}\partial G_{AB}\partial G^{AB} - \frac{2}{\sqrt{|G|}}\nabla^2\sqrt{|G|}\Big\} \\
&\quad + \int_{\partial\Sigma} (d^{D-1}x)_I\; n^I\sqrt{|H|}\sqrt{|G|}\;\Big\{K + 2n^J\partial_J \ln\sqrt{|G|}\Big\}\,, \\
&= \int_\Sigma d^n x\sqrt{|H|}\sqrt{|G|}\,\Big\{R - 2\Lambda + (\partial\ln\sqrt{|G|})^2 + \frac{1}{4}\partial G_{AB}\partial G^{AB}\Big\} \\
&\quad + \int_{\partial\Sigma} (d^{D-1}x)_I\; n^I\sqrt{|H|}\sqrt{|G|}\;K\,,
\end{aligned} \tag{11}$$

where we have divided out the volume of the y-coordinate space from the action, and Σ is redefined as the space spanned by the x-coordinate. Equations of motion from (11) is consistent with $\widetilde{R}_{\mu\nu} = \frac{2\Lambda}{D-2}\widetilde{G}_{\mu\nu}$. When varying H_{IJ}, it is important to note that

$$\begin{aligned}
\sqrt{|G|}\;\delta R &= \sqrt{|G|}\;\Big(R_{IJ} - \nabla_I\nabla_J + H_{IJ}\nabla^2\Big)\delta H^{IJ} \\
&= \sqrt{|G|}\;\Big\{R_{IJ} - \frac{\nabla_I\nabla_J\sqrt{|G|}}{\sqrt{|G|}} + H_{IJ}\frac{\nabla^2\sqrt{|G|}}{\sqrt{|G|}}\Big\}\delta H^{IJ}\,, \\
&\quad +\text{boundary terms (to be cancelled by the boundary action)}\,.
\end{aligned} \tag{12}$$

By tracing over $\widetilde{R}_{AB} = \frac{2\Lambda}{D-2}\widetilde{G}_{AB}$, we also find

$$(\partial \ln \sqrt{|G|})^2 + \nabla^2 \ln \sqrt{|G|} = \frac{\nabla^2\sqrt{|G|}}{\sqrt{|G|}} = -\frac{2m\Lambda}{D-2}\,. \tag{13}$$

It is obvious that (11) has a rigid $SL(m,R)$ symmetry: the action is invariant under the transformation,

$$G_{AB} \quad\longrightarrow\quad (\mathcal{V}\cdot G\cdot \mathcal{V}^T)_{AB}\,,\quad |\mathcal{V}| = 1\,. \tag{14}$$

This symmetry is due to the freedom in redefining the y-coordinates,

$$dy^A \quad\longrightarrow\quad (dy\cdot\mathcal{V}^{-1})^A\,. \tag{15}$$

As such, the same symmetry should continue to exist even when there are additional matter fields. Of course, the matter fields should transform appropriately to keep the physical objects invariant. For example, a vector field should transform as

$$\mathcal{A}_I \quad\longrightarrow\quad \mathcal{A}_I\,,\quad \mathcal{A}_A = \quad\longrightarrow\quad (\mathcal{V}\mathcal{A})_A\,, \tag{16}$$

which leaves $\mathcal{A} = dx^I\mathcal{A}_I + dy^A\mathcal{A}_A$ invariant.

3 First Law for Stationary and Axisymmetric Black Holes

It is well known that any metric can be cast into the ADM form,

$$ds^2 = -N^2dt^2 + g_{ij}(dx^i - N^i dt)(dx^j - N^j dt)\,. \tag{17}$$

For a stationary and axisymmetric black hole, the metric elements are further constrained, and the metrics can always be put into the following form [20],

$$ds^2 = f\Big[-\frac{\Delta}{v^2}dt^2 + \frac{dr^2}{\Delta}\Big] + h_{ij}d\theta^i d\theta^j + g_{ab}(d\phi^a - w^a dt)(d\phi^b - w^b dt)\,, \tag{18}$$

where $\Delta = \Delta(r)$, and the functions f, v, h_{ij}, g_{ab} and w^a depend only on the r and θ-coordinates. In principle, one can identify the coordinates as the asymptotic time t, the radial coordinate r, the latitudinal angles θ^i ($i = 1,\cdots,[\frac{D}{2}]-1$) and the azimuthal angles ϕ^a ($a = 1,\cdots,[\frac{D+1}{2}]-1$), where D is the total dimension of the spacetime. The black hole horizon r_0 is located at the (largest) root of $\Delta(r_0) = 0$. Near the black hole horizon, $f, v^2, (h_{ij})$ and (g_{ab}) are all positive definite. The fact that black holes are intrinsically regular on the horizon puts extra constraints on the functions,

$$v(r,\theta^i) \;=\; v_0(r) + v_1(r,\theta^i)\Delta + \mathcal{O}(\Delta^2)\,,$$

$$w^a(r,\theta^i) \;\; = \;\; w_0^a(r) + w_1^a(r,\theta^i)\Delta + \mathcal{O}(\Delta^2)\,, \tag{19}$$

which means that any dependence of v and w^a on θ^i can only begin at the order Δ. What's more, $v_0(r_0) \neq 0$ and $w_0^a(r_0) = \Omega^a$ is the angular velocity of the black hole in the ϕ^a direction. One can also choose the coordinate system to be non-rotating at the spatial infinity ($r \to +\infty$), which means that[2]

$$w^a(r,\theta^i) \quad \longrightarrow \quad 0 \quad \text{as} \quad r \to +\infty\,. \tag{20}$$

The inverse of (18) is

$$(\partial_S)^2 = \frac{\Delta}{f}\partial_r^2 + h^{ij}\partial_{\theta^i}\partial_{\theta^j} + g^{ab}\partial_{\phi^a}\partial_{\phi^b} - \frac{v^2}{f\Delta}(\partial_t + w^a\partial_{\phi^a})(\partial_t + w^b\partial_{\phi^b})\,. \tag{21}$$

It is obvious that (18) is a special case of (5). Comparing (18) with (5), we see that r and θ^i's belong to the x-coordinates and are labelled by the I,J,K indices, while t and ϕ^a's belong to the y-coordinates and are labelled by the A,B,C indices. Also $n = [\frac{D}{2}]$ and $m = [\frac{D+1}{2}]$. The non-vanishing elements of the metric are

$$\begin{aligned} H_{rr} = \frac{f}{\Delta}\,, \quad H_{ij} = h_{ij}\,, \quad G_{at} = -w_a\,, \quad G_{ab} = g_{ab}\,, \quad G_{tt} = -\frac{1}{\varrho} + w^2\,, \\ H^{rr} = \frac{\Delta}{f}\,, \quad H^{ij} = h^{ij}\,, \quad G^{at} = -\varrho w^a\,, \quad G^{ab} = g^{ab} - \varrho w^a w^b\,, \quad G^{tt} = -\varrho\,, \end{aligned} \tag{22}$$

where $\varrho = \frac{v^2}{f\Delta}$, $w_a = g_{ab}w^b$ and $w^2 = w_a w^a$. For the determinants, we have $\sqrt{H} = \sqrt{fh/\Delta}$ and $\sqrt{|G|} = \sqrt{g/\varrho}$, with h being the determinant of h_{ij} and g the determinant of g_{ab}. Note $H > 0$ outside the black hole horizon. In the following, we will still denote H_{rr} and H_{ij} collectively as H_{IJ}, $I,J \in \{r,i\}$. The action (11) can now be written as

$$\begin{aligned} S \;&=\; \int_\Sigma (d^{n-1}\theta) dr\, \mathcal{L} + \int_{\partial\Sigma} (d^{n-1}\theta\ dr)_I\, n^I \sqrt{Hg/\varrho}\, K\,, \\ \mathcal{L} \;&=\; \sqrt{Hg/\varrho}\Big\{R - 2\Lambda + (\partial \ln \sqrt{g/\varrho}\,)^2 - (\partial \ln \sqrt{\varrho}\,)^2 \\ &\qquad + \frac{1}{4}\partial g_{ab}\partial g^{ab} + \frac{\varrho}{2} g_{ab}\partial w^a \partial w^b\Big\}\,. \end{aligned} \tag{23}$$

Note this action is completely regular on the black hole horizons ($\Delta \to 0$). This is reasonable because black holes are intrinsically regular on the horizons. As mentioned before, one can formally treat (23) as a field theory of g_{ab}, ϱ and w^a, defined in the curved background H_{IJ}. So correspondingly, one can derive a new set of equations of motion,

$$-\frac{\nabla(\sqrt{g/\varrho}\;\partial g_{ab})}{2\sqrt{g/\varrho}} + \frac{1}{2}g^{cd}\partial g_{ac}\partial g_{bd} - \frac{\varrho}{2}g_{ac}g_{bd}\partial w^c\partial w^d \;\; = \;\; \frac{2\Lambda}{D-2}g_{ab}\,, \tag{24}$$

[2] As a side remark, note if we use (18) in the construction of [27], we will get a vector field that interpolates the null Killing vector on the horizon and the time Killing vector at the spatial infinity.

$$\frac{\nabla(\sqrt{g/\varrho}\,\partial\ln\sqrt{\varrho})}{\sqrt{g/\varrho}}+\frac{\varrho}{2}g_{ab}\partial w^a\partial w^b \;=\; \frac{2\Lambda}{D-2}\,, \tag{25}$$

$$\nabla\Big(\sqrt{g/\varrho}\;\varrho g_{ab}\partial w^b\Big) \;=\; 0\,, \tag{26}$$

which are equivalent to $\widetilde{R}_{AB}=\frac{2\Lambda}{D-2}\widetilde{G}_{AB},\ A,B\in\{t,a\}$. By tracing over (24) and then using (25), we find (note $\delta^a_a=m-1$)

$$\frac{\nabla^2\sqrt{g/\varrho}}{\sqrt{g/\varrho}}=-\frac{2m\Lambda}{D-2}\,, \tag{27}$$

thus recovering (13). Also, we can vary H_{IJ} to obtain

$$\begin{aligned}\frac{2\Lambda}{D-2}H_{IJ} \;&=\; R_{IJ}-\nabla_I\nabla_J\ln\sqrt{g/\varrho}-\partial_I\ln\sqrt{\varrho}\,\partial_J\ln\sqrt{\varrho}\\ &\quad+\frac{1}{4}\partial_I g_{ab}\partial_J g^{ab}+\frac{\varrho}{2}g_{ab}\partial_I w^a\partial_J w^b\,,\end{aligned} \tag{28}$$

which is equivalent to $\widetilde{R}_{IJ}=\frac{2\Lambda}{D-2}\widetilde{G}_{IJ},\ I,J\in\{r,i\}$.

As an application of the new formalism, let's re-derive the first law of black hole thermodynamics in terms of the new language. To facilitate our discussion, we firstly recall some basic formulae of the covariant phase space method, for which we follow [28, 29].

Consider the general action,

$$S=\int_{\mathcal{M}}\mathbf{L}\,,\quad \mathbf{L}=\mathcal{L}(\Phi^a,\partial_\mu\Phi^a,\partial_\mu\partial_\nu\Phi^a,\cdots)*\mathbf{1}\,, \tag{29}$$

where Φ denotes all possible fields collectively. Through out the paper, we will use a bold faced letter (e.g. $\mathbf{L}$) to denote a differential form.[3] For an arbitrary variation of the fields,

$$\delta\mathbf{L}=(\delta\Phi^a)E_a*\mathbf{1}+d\mathbf{\Theta}_\delta\,, \tag{33}$$

where all the terms involving a derivative on $\delta\Phi^a$ have been moved into $d\mathbf{\Theta}_\delta$. The Euler-Lagrange equations are just $E_a=0$. For the special case of a general diffeomorphism

[3] We will use the notation

$$(d^{D-p}x)_{\mu_1\cdots\mu_p}\equiv\frac{1}{p!(D-p)!}\varepsilon_{\mu_1\cdots\mu_p\nu_1\cdots\nu_{D-p}}dx^{\nu_1}\wedge\cdots\wedge dx^{\nu_{D-p}}\,,\quad |\varepsilon_{\ldots}|=1\,, \tag{30}$$

with which the Hodge-$*$ dual of a p-form $\mathbf{w}_p=\frac{1}{p!}w_{\mu_1\cdots\mu_p}dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_p}$ can be written as

$$*\,\mathbf{w}_p=\sqrt{|g|}\;(d^{D-p}x)_{\mu_1\cdots\mu_p}w^{\mu_1\cdots\mu_p}\,,\quad\Longrightarrow\quad *\mathbf{1}=\sqrt{|g|}\;d^Dx\,. \tag{31}$$

For the exterior and interior products, one has

$$\begin{aligned}&d*\mathbf{w}_p=\sqrt{|g|}\;(d^{D-p+1}x)_{\mu_1\cdots\mu_{p-1}}\nabla_{\mu_p}w^{\mu_1\cdots\mu_p}\,,\\ &i_\xi(d^{D-p}x)_{\mu_1\cdots\mu_p}=(d^{D-p-1}x)_{\mu_1\cdots\mu_p\mu}(p+1)\xi^\mu\,.\end{aligned} \tag{32}$$

$(\delta = \pounds_\xi = d \cdot i_\xi + i_\xi \cdot d)$,

$$\begin{aligned} \pounds_\xi \mathbf{L} = d(i_\xi \mathbf{L}) = (\pounds_\xi \Phi^a) E_a * \mathbf{1} + d\mathbf{\Theta}_\xi \,, \quad \mathbf{J}_\xi = \mathbf{\Theta}_\xi - i_\xi \mathbf{L} \,, \\ \Longrightarrow \quad d\mathbf{J}_\xi = -(\pounds_\xi \Phi^a) E_a * \mathbf{1} \approx 0 \,, \quad \Longrightarrow \quad \mathbf{J}_\xi \approx d\mathbf{Q}_\xi \,, \end{aligned} \tag{34}$$

where " $\approx$ " means equal after using the equations of motion $E_a = 0$. Now lets evolve a classical solution to a nearby one (We will focus on the particular operation $\bar\delta$ that only changes free parameters, such as mass and angular momenta, in the solution),

$$\bar\delta \mathbf{J}_\xi = \bar\delta \mathbf{\Theta}_\xi - \bar\delta (i_\xi \mathbf{L}) = \bar\delta \mathbf{\Theta}_\xi - i_\xi \cdot d\mathbf{\Theta}_{\bar\delta} = \mathbf{w}(\bar\delta, \pounds_\xi) + d(i_\xi \mathbf{\Theta}_{\bar\delta}) \,, \quad \mathbf{w}(\delta, \pounds_\xi) \equiv \delta \mathbf{Q}_\xi - \pounds_\xi \mathbf{Q}_\delta \,. \tag{35}$$

Since $\bar\delta$ only goes through classical solutions, one has $\mathbf{J}_\xi = d\mathbf{Q}_\xi$ all the time. Hence

$$\bar\delta \mathbf{J}_\xi = d\bar\delta \mathbf{Q}_\xi \,, \quad \Longrightarrow \quad \mathbf{w}(\bar\delta, \pounds_\xi) = d\mathbf{k}(\bar\delta, \pounds_\xi) \,, \quad \mathbf{k}(\bar\delta, \pounds_\xi) \equiv \bar\delta \mathbf{Q}_\xi - i_\xi \mathbf{\Theta}_{\bar\delta} \,. \tag{36}$$

In the case when ξ is a Killing vector of some classical solution,

$$\pounds_\xi = 0 \quad \Longrightarrow \quad \mathbf{w}(\bar\delta, \pounds_\xi) = 0 \,, \quad \Longrightarrow \quad 0 = \int_V \mathbf{w}(\bar\delta, \pounds_\xi) = \oint_{\partial V} \mathbf{k}(\bar\delta, \pounds_\xi) \,, \tag{37}$$

where V is a cauchy surface. Since in this paper we are mainly interested in stationary and axisymmetric black holes (18), we can take V to be the space outside the horizon(s). As a result, ∂V has two disconnect pieces: one at the spatial infinity and one at the (outer) horizon,

$$\oint_{\partial V} = \int_{+\infty} - \int_{Horizon} \,. \tag{38}$$

Usually one defines the charge corresponding to $\pounds_\xi$ through an integral at the spatial infinity,

$$\bar\delta H_\xi = \int_{+\infty} \mathbf{k}(\bar\delta, \pounds_\xi) = \int_{+\infty} (\bar\delta \mathbf{Q}_\xi - i_\xi \mathbf{\Theta}_{\bar\delta}) \,. \tag{39}$$

But because of (37) and (38), this is equivalent to defining

$$\bar\delta H_\xi = \int_{horizon} \mathbf{k}(\bar\delta, \pounds_\xi) = \int_{horizon} (\bar\delta \mathbf{Q}_\xi - i_\xi \mathbf{\Theta}_{\bar\delta}) \,. \tag{40}$$

It is this second definition that we want to use in the following.

Now consider Einstein gravity plus a cosmological constant,

$$\mathbf{L} = \Big(\frac{\widetilde{R} - 2\Lambda}{16\pi} \Big) * \mathbf{1} \,, \tag{41}$$

where we use $\widetilde{G}_{\mu\nu}$ to denote the full metric (5), with (18) being a special case. Note we have introduced the factor $\frac{1}{16\pi}$ into the Lagrangian density, just to be consistent with the usual convention of defining charges in general relativity. We will keep this factor only until the end of this section, and starting from the next section we will go back and use (1) again. For an arbitrary variation of the fields,

$$
\begin{aligned}
\delta\mathbf{L} = \frac{1}{16\pi}\Big\{\frac{\tilde{h}}{2}(\widetilde{R}-2\Lambda) + (-\widetilde{R}^{\mu\nu} + \widetilde{\nabla}^\mu\widetilde{\nabla}^\nu - \widetilde{\nabla}^2\widetilde{G}^{\mu\nu})\tilde{h}_{\mu\nu}\Big\} * \mathbf{1}\,,& \\
\Longrightarrow \quad E^{\mu\nu} = \frac{1}{16\pi}\Big[\frac{1}{2}\widetilde{G}^{\mu\nu}(\widetilde{R}-2\Lambda) - \widetilde{R}^{\mu\nu}\Big]\,,& \\
\mathbf{\Theta}_\delta = \sqrt{-\widetilde{G}}\;(d^{D-1}x)_\mu\Big(\frac{\widetilde{\nabla}_\nu\tilde{h}^{\mu\nu} - \widetilde{\nabla}^\mu\tilde{h}}{16\pi}\Big)\,,& \qquad (42)
\end{aligned}
$$

where $\tilde{h}_{\mu\nu} \equiv \delta\widetilde{G}_{\mu\nu}$. (Do not confuse it with the metric elements h_{ij} in (18).) For a diffeomorphism, one has from (34)

$$
\begin{aligned}
\mathbf{J}_\xi &= \mathbf{\Theta}_\xi - i_\xi\mathbf{L} = \sqrt{-\widetilde{G}}\;(d^{D-1}x)_\mu\Big\{\frac{-\widetilde{\nabla}_\nu\xi^{\mu\nu} + 2\widetilde{R}^{\mu\nu}\xi_\nu}{16\pi} - \Big(\frac{\widetilde{R}-2\Lambda}{16\pi}\Big)\xi^\mu\Big\} \\
&= \sqrt{-\widetilde{G}}\;(d^{D-1}x)_\mu\Big(\frac{-\widetilde{\nabla}_\nu\xi^{\mu\nu}}{16\pi}\Big) = d\mathbf{Q}_\xi\,, \\
\Longrightarrow\quad & \mathbf{Q}_\xi = \sqrt{-\widetilde{G}}\;(d^{D-2}x)_{\mu\nu}\Big(\frac{-\xi^{\mu\nu}}{16\pi}\Big)\,, \quad \xi^{\mu\nu} = \widetilde{\nabla}^\mu\xi^\nu - \widetilde{\nabla}^\nu\xi^\mu\,. \qquad (43)
\end{aligned}
$$

The metric (18) has the Killing vectors $\hat{k} = \partial_t$ and $\hat{k}_a = \partial_{\phi^a}$. The elements relevant for the integral (40) are

$$
\begin{aligned}
\hat{k}^{tr} &= \widetilde{G}^{t\mu}\widetilde{G}^{rr}(\partial_\mu\hat{k}_r - \partial_r\hat{k}_\mu) = -\widetilde{G}^{t\mu}\widetilde{G}^{rr}\partial_r\widetilde{G}_{t\mu} = -\frac{\Delta}{f}\varrho\Big[\partial_r(\frac{1}{\varrho} - w^2) + w^a\partial_r w_a\Big] \\
&= \frac{v^2}{f^2}\Big(\frac{1}{\varrho}\partial_r\ln\varrho + w_a\partial_r w^a\Big) = \frac{v^2}{f^2}\Big(w_a\partial_r w^a - \frac{f\Delta'}{v^2} + \frac{f\Delta}{v^2}\partial_r\ln\frac{v^2}{f}\Big)\,, \\
&\longrightarrow \frac{v^2}{f^2}\Big(w_a\partial_r w^a - \frac{f\Delta'}{v^2}\Big)\,, \qquad (44) \\
\hat{k}_a^{tr} &= \widetilde{G}^{t\mu}\widetilde{G}^{rr}\Big[\partial_\mu(\hat{k}_a)_r - \partial_r(\hat{k}_a)_\mu\Big] = -\widetilde{G}^{t\mu}\widetilde{G}^{rr}\partial_r\widetilde{G}_{a\mu} \\
&= -\frac{\Delta}{f}\varrho\Big[\partial_r w_a - w^b\partial_r g_{ab}\Big] = -\frac{v^2}{f^2}g_{ab}\partial_r w^b\,, \qquad (45)
\end{aligned}
$$

where " $\longrightarrow$ " means equal in the limit $\Delta \to 0$. Similarly using (42), one has for $i_\xi\mathbf{\Theta}_{\bar{\delta}} = \sqrt{-\widetilde{G}}(d^{D-2}x)_{\mu\nu}(i_\xi\mathbf{\Theta}_{\bar{\delta}})^{\mu\nu}$,

$$
\begin{aligned}
(i_\xi\mathbf{\Theta}_{\bar{\delta}})^{\mu\nu} &= \xi^\nu\Big(\frac{\widetilde{\nabla}_\rho\bar{h}^{\mu\rho} - \widetilde{\nabla}^\mu\bar{h}}{16\pi}\Big) - \xi^\mu\Big(\frac{\widetilde{\nabla}_\rho\bar{h}^{\nu\rho} - \widetilde{\nabla}^\nu\bar{h}}{16\pi}\Big)\,, \\
(i_{\hat{k}}\mathbf{\Theta}_{\bar{\delta}})^{tr} &= -\frac{1}{16\pi}\Big(\widetilde{\nabla}_\mu\bar{h}^{r\mu} - \widetilde{\nabla}^r\bar{h}\Big) \\
&= -\frac{1}{16\pi}\Big(\partial_r\bar{h}^{rr} + \tilde{\Gamma}^r_{\mu\nu}\bar{h}^{\mu\nu} + \tilde{\Gamma}^\mu_{\mu r}\bar{h}^{rr} - \widetilde{G}^{rr}\partial_r\bar{h}\Big) \\
&= -\frac{1}{16\pi}\Big(\partial_r\bar{h}^{rr} + \widetilde{G}^{rr}\partial_r\widetilde{G}_{rr}\bar{h}^{rr} - \frac{1}{2}\widetilde{G}^{rr}\partial_r\widetilde{G}_{\mu\nu}\bar{h}^{\mu\nu} + \bar{h}^{rr}\partial_r\ln\sqrt{-\widetilde{G}} \\
&\qquad -2\widetilde{G}^{rr}\partial_r\bar{\delta}\ln\sqrt{-\widetilde{G}}\;\Big)
\end{aligned}
$$

$$
\begin{aligned}
&\longrightarrow \quad -\frac{1}{16\pi}\Big(\partial_r\bar{h}^{rr}+\widetilde{G}^{rr}\partial_r\widetilde{G}_{rr}\bar{h}^{rr}+\frac{1}{2}\widetilde{G}^{rr}\partial_r\widetilde{G}^{\mu\nu}\bar{h}_{\mu\nu}+\bar{h}^{rr}\partial_r\ln\sqrt{-\widetilde{G}}\ \Big)\\
&\longrightarrow \quad -\frac{1}{16\pi}\Big\{\partial_r\Big(\frac{\Delta^2}{f^2}\bar{\delta}\frac{f}{\Delta}\Big)+\frac{\Delta^2}{f^2}\bar{\delta}\frac{f}{\Delta}\partial_r\ln\frac{f}{\Delta}+\frac{\Delta}{2f}\Big[\partial_r\frac{\Delta}{f}\bar{\delta}\frac{f}{\Delta}+\partial_r\varrho\bar{\delta}(\frac{1}{\varrho}-w^2)\\
&\qquad\qquad +2\partial_r(\varrho w^a)\bar{\delta}w_a+\partial_r(g^{ab}-\varrho w^a w^b)\bar{\delta}g_{ab}\Big]\Big\}\\
&\longrightarrow \quad -\frac{1}{16\pi}\Big[\frac{v^2}{f^2}g_{ab}\partial_r w^a\bar{\delta}w^b-\frac{v}{f}\bar{\delta}\Big(\frac{\Delta'}{v}\Big)\Big]\,,
\end{aligned}
\tag{46}
$$

where $\bar{h}_{\mu\nu}\equiv\bar{\delta}\widetilde{G}_{\mu\nu}$. (Do not confuse it with the metric elements h_{ij} in (18).) Note although we have kept Δ explicit (at where it is necessary) to show that none of the expressions diverge in the limit $\Delta\to 0$, it should be understood that the operation $\bar{\delta}$ always comes *after* taking the limit $r\to r_0$. For this reason, $\bar{\delta}\Delta=0$ holds all the time. Plugging the results back into (40), we find

$$
\begin{aligned}
\bar{\delta}E &= \bar{\delta}H_{\hat{k}}=\int_{r=r_0}(d^{D-2}x)_{\mu\nu}\Big\{\bar{\delta}\Big(\sqrt{-\widetilde{G}}\ \frac{-\hat{k}^{\mu\nu}}{16\pi}\Big)-\sqrt{-\widetilde{G}}\ (i_{\hat{k}}\mathbf{\Theta}_{\bar{\delta}})^{\mu\nu}\Big\}\\
&= \int_{r=r_0}(d^{D-2}x)_{tr}2\Big\{\bar{\delta}\Big(-\frac{\sqrt{hg}}{16\pi}\frac{v}{f}w_a\partial_r w^a+\frac{\sqrt{hg}}{16\pi}\frac{\Delta'}{v}\Big)\\
&\qquad\qquad +\frac{\sqrt{hg}}{16\pi}\frac{v}{f}g_{ab}\partial_r w^a\bar{\delta}w^b-\frac{\sqrt{hg}}{16\pi}\bar{\delta}\Big(\frac{\Delta'}{v}\Big)\Big\}\\
&= \int_{r=r_0}(d^{D-2}x)_{tr}2\Big\{w^a\bar{\delta}\Big(-\frac{\sqrt{hg}}{16\pi}\frac{v}{f}g_{ab}\partial_r w^b\Big)+\frac{\Delta'}{16\pi v}\bar{\delta}\sqrt{hg}\ \Big\}\\
&= T\bar{\delta}S+\Omega^a\bar{\delta}J_a\,,
\end{aligned}
\tag{47}
$$

where we have used $\sqrt{-\widetilde{G}}=\sqrt{hg}\dfrac{f}{v}$ and in the last step the definitions

$$
\begin{aligned}
T &= \frac{\kappa}{2\pi}=\frac{\Delta'}{4\pi v}\Big|_{r=r_0}\,,\qquad \Omega^a=w^a(r_0)\,,\\
J_a &= -H_{\hat{k}_a}=\int_{r=r_0}\sqrt{-\widetilde{G}}\ (d^{D-2}x)_{\mu\nu}\Big(\frac{\hat{k}_a^{\mu\nu}}{16\pi}\Big)\\
&= \int_{r=r_0}(d^{D-2}x)_{tr}2\Big(-\frac{\sqrt{hg}}{16\pi}\frac{v}{f}g_{ab}\partial_r w^b\Big)\,,\\
S &= \frac{1}{4}\int_{r=r_0}(d^{D-2}x)_{tr}2\sqrt{hg}=\frac{\mathcal{A}_{rea}}{4}\,,
\end{aligned}
\tag{48}
$$

where κ is the surface gravity on the horizon.

Note the above calculation is not a true "derivation" of the first law because the $\bar{\delta}$-integrability of (40) is not *a priori* obvious. As such, the above calculation, together with the observation that one can integrate the first law to recover the black hole masses [26], can be better interpreted as showing that (40) is $\bar{\delta}$-integrable for stationary and axisymmetric black holes, in the context of Einstein gravity plus a cosmological constant.

As a side remark, note [28, 29] already involved deriving the first law of thermodynamics from the general calculus of the covariant phase space method. What's new here is that (i) we are using an operation $\bar{\delta}$ that is directly related to the usual test of the first law of black hole thermodynamics, and (ii) all the quantities are now defined at the black hole horizon, without any reference to the spatial infinity (But because of (37) and (38), the results must be the same).

We want to emphasize that the above calculation becomes possible only because our formalism has made the dependence on the function $\Delta(r)$ explicit, which holds key informations of the metric (18) as it approaches the black hole horizon.

4 The Conformal Symmetries on the Horizon

As was mentioned before, the action (11) has a rigid $SL(m, R)$ symmetry, which should be inherited by the particular case (23). In this section, we want to focus on the particular $SL(2, R)$ generators like the following,

$$L_0 = \frac{1}{2}\begin{pmatrix} -1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix}, \quad L_+ = \begin{pmatrix} 0 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}, \quad L_- = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ -1 & \cdots & 0 \end{pmatrix}, \tag{49}$$

where all the matrices are m-dimensional, and all the implicit elements are zero. The transformation of the metric elements G_{AB} will be given by

$$\hat{\delta} G \equiv -(L \cdot G + G \cdot L^T)\,. \tag{50}$$

In order to see the results explicitly, let's distinguish the coordinate ϕ^1 from the rest of the azimuthal angles. We will simply denote ϕ^1 as ϕ, and will also use ϕ as the corresponding super/sub-script, e.g. $w^1 = w^\phi$ and $g_{11} = g_{\phi\phi}$. We will label all other azimuthal angles using indices with a tilde, $\phi^{\tilde{a}}$ ($\tilde{a} = 2, \cdots, m-1$). Accordingly,

$$\begin{aligned} (G_{AB}) &= \begin{pmatrix} g_{\phi\phi} & g_{\tilde{a}\phi} & -w_\phi \\ g_{\tilde{b}\phi} & g_{\tilde{a}\tilde{b}} & -w_{\tilde{b}} \\ -w_\phi & -w_{\tilde{a}} & -\frac{1}{\varrho} + w^2 \end{pmatrix}, \\ (G^{AB}) &= \begin{pmatrix} g^{\phi\phi} - \varrho w^\phi w^\phi & g^{\tilde{a}\phi} - \varrho w^{\tilde{a}} w^\phi & -\varrho w^\phi \\ g^{\tilde{b}\phi} - \varrho w^{\tilde{b}} w^\phi & g^{\tilde{a}\tilde{b}} - \varrho w^{\tilde{a}} w^{\tilde{b}} & -\varrho w^{\tilde{b}} \\ -\varrho w^\phi & -\varrho w^{\tilde{a}} & -\varrho \end{pmatrix}. \end{aligned} \tag{51}$$

Note both the indices $\{\phi, \tilde{a}\}$ are still raised and lowered using the matrix

$$(g_{ab}) = \begin{pmatrix} g_{\phi\phi} & g_{\tilde{a}\phi} \\ g_{\tilde{b}\phi} & g_{\tilde{a}\tilde{b}} \end{pmatrix}, \quad (g^{ab}) = \begin{pmatrix} g^{\phi\phi} & g^{\tilde{a}\phi} \\ g^{\tilde{b}\phi} & g^{\tilde{a}\tilde{b}} \end{pmatrix}. \tag{52}$$

As such, we will try to convert our results back to using the untilded indices (which take values form $\{\phi, 2, \cdots, m-1\}$) whenever it is possible.

Our following construction will also rely on the assumption that $w^\phi = w^1 \neq 0$. But the choice on ϕ^1 is only a matter of convenience. One can do the same for any other azimuthal angles, as long as the corresponding angular velocity is non-zero. Of course, one should accordingly relocate the the non-vanishing matrix elements in (49).

Now using (50), we find for the symmetric transformations,

$$\begin{aligned}
\hat{\delta}_0 g_{\phi\phi} &= g_{\phi\phi}\,, & \hat{\delta}_0 g^{\phi\phi} &= -g^{\phi\phi}\,, \\
\hat{\delta}_0 g_{\tilde{a}\phi} &= \frac{1}{2} g_{\tilde{a}\phi}\,, & \hat{\delta}_0 g^{\tilde{a}\phi} &= -\frac{1}{2} g^{\tilde{a}\phi}\,, \\
\hat{\delta}_0 g_{\tilde{a}\tilde{b}} &= 0\,, & \hat{\delta}_0 g^{\tilde{a}\tilde{b}} &= 0\,, \\
\hat{\delta}_0 w^\phi &= -w^\phi\,, & \hat{\delta}_0 w_\phi &= 0\,, \\
\hat{\delta}_0 w^{\tilde{a}} &= -\frac{1}{2} w^{\tilde{a}}\,, & \hat{\delta}_0 w_{\tilde{a}} &= -\frac{1}{2} w_{\tilde{a}}\,, \quad \hat{\delta}_0 \varrho = \varrho\,,
\end{aligned} \tag{53}$$

$$- - - - - - - - - - - - - - - - -$$

$$\begin{aligned}
\hat{\delta}_+ g_{\phi\phi} &= 2 w_\phi\,, & \hat{\delta}_+ g^{\phi\phi} &= -2 g^{\phi\phi} w^\phi\,, \\
\hat{\delta}_+ g_{\tilde{a}\phi} &= w_{\tilde{a}}\,, & \hat{\delta}_+ g^{\tilde{a}\phi} &= -(g^{\tilde{a}\phi} w^\phi + g^{\phi\phi} w^{\tilde{a}})\,, \\
\hat{\delta}_+ g_{\tilde{a}\tilde{b}} &= 0\,, & \hat{\delta}_+ g^{\tilde{a}\tilde{b}} &= -(g^{\tilde{a}\phi} w^{\tilde{b}} + g^{\tilde{b}\phi} w^{\tilde{a}})\,, \\
\hat{\delta}_+ w^\phi &= -(w^\phi w^\phi + g^{\phi\phi}/\varrho)\,, & \hat{\delta}_+ w_\phi &= -\frac{1}{\varrho} + w^2\,, \\
\hat{\delta}_+ w^{\tilde{a}} &= -(w^{\tilde{a}} w^\phi + g^{\tilde{a}\phi}/\varrho)\,, & \hat{\delta}_+ w_{\tilde{a}} &= 0\,, \quad \hat{\delta}_+ \varrho = 2 \varrho w^\phi\,,
\end{aligned} \tag{54}$$

$$- - - - - - - - - - - - - - - - -$$

$$\begin{aligned}
\hat{\delta}_- g_{\phi\phi} &= 0\,, & \hat{\delta}_- g^{\phi\phi} &= 0\,, \\
\hat{\delta}_- g_{\tilde{a}\phi} &= 0\,, & \hat{\delta}_- g^{\tilde{a}\phi} &= 0\,, \\
\hat{\delta}_- g_{\tilde{a}\tilde{b}} &= 0\,, & \hat{\delta}_- g^{\tilde{a}\tilde{b}} &= 0\,, \\
\hat{\delta}_- w^\phi &= -1\,, & \hat{\delta}_- w_\phi &= -g_{\phi\phi}\,, \\
\hat{\delta}_- w^{\tilde{a}} &= 0\,, & \hat{\delta}_- w_{\tilde{a}} &= -g_{\tilde{a}\phi}\,, \quad \hat{\delta}_- \varrho = 0\,,
\end{aligned} \tag{55}$$

It is easy to check that

$$[\hat{\delta}_\pm\,, \hat{\delta}_0] = \pm \hat{\delta}_\pm\,, \quad [\hat{\delta}_+\,, \hat{\delta}_-] = 2 \hat{\delta}_0\,. \tag{56}$$

For later convenience, lets define

$$\begin{aligned}
\pi^{Iab} &= \frac{\delta S}{\delta(\partial_I g_{ab})} = \sqrt{Hg/\varrho}\,\Big(g^{ab}\partial^I \ln\sqrt{g/\varrho}\, + \frac{1}{2}\partial^I g^{ab}\Big)\,,\\
\pi^I_a &= \frac{\delta S}{\delta(\partial_I w^a)} = \sqrt{Hg/\varrho}\,\Big(\varrho g_{ab}\partial^I w^b\Big)\,,\\
\pi^I_\varrho &= \frac{\delta S}{\delta(\partial_I \varrho)} = \sqrt{Hg/\varrho}\,\Big(-\frac{1}{\varrho}\partial^I \ln\sqrt{g}\Big)\,.
\end{aligned} \tag{57}$$

The Noether currents corresponding to (53), (54) and (55) are

$$\begin{aligned}
J^I_0 &= \pi^{Iab}\hat{\delta}_0 g_{ab} + \pi^I_a\hat{\delta}_0 w^a + \pi^I_\varrho\hat{\delta}_0\varrho\,,\\
&= \sqrt{Hg/\varrho}\,\Big(\frac{1}{2}g_{\phi a}\partial^I g^{a\phi} - \partial^I\ln\sqrt{\varrho} - \frac{\varrho}{2}w^a g_{ab}\partial^I w^b - \frac{\varrho}{2}w^\phi g_{\phi a}\partial^I w^a\Big)\,,\\
J^I_+ &= \pi^{Iab}\hat{\delta}_+ g_{ab} + \pi^I_a\hat{\delta}_+ w^a + \pi^I_\varrho\hat{\delta}_+\varrho\,,\\
&= \sqrt{Hg/\varrho}\,\Big(-2w^\phi\partial^I\ln\sqrt{\varrho} + w_a\partial^I g^{a\phi} - \partial^I w^\phi - \varrho w^\phi w_a\partial^I w^a\Big)\,,\\
J^I_- &= \pi^{Iab}\hat{\delta}_- g_{ab} + \pi^I_a\hat{\delta}_- w^a + \pi^I_\varrho\hat{\delta}_-\varrho = \sqrt{Hg/\varrho}\,\Big(-\varrho g_{\phi a}\partial^I w^a\Big)\,.
\end{aligned} \tag{58}$$

By using the equations of motion (24), (25) and (26), one can check that all these currents are exactly conserved.

There is an interesting connection between these currents and the charges defined in (47) and (48). Using the detail of the metric elements (22) and the relations $\varrho = \frac{v^2}{f\Delta}$ and $\sqrt{Hg/\varrho}\, = \sqrt{hg}\frac{f}{v}$, one can find that

$$J^r_- = \sqrt{hg}\frac{v}{f}\Big(-g_{\phi a}\partial_r w^a\Big)\,. \tag{59}$$

It is obvious that J^r_- is just the integrand of the angular momentum J_ϕ in (48).[4] For the energy E, it is easier to look at the asymptotically flat case ($\Lambda = 0$). In this case, it is possible to define the energy as a Komar integral,

$$E \sim -\int_{+\infty} *d\hat{k} = -\int_{Horizon} *d\hat{k} = \int_{Horizon} (d^{D-2}x)_{tr}\sqrt{hg}\frac{f}{v}2(-\hat{k}^{tr})\,, \tag{60}$$

where $\hat{k}^{tr}$ has been given in (44), and in the second step we have used $\widetilde{R}_{\mu\nu} \sim \Lambda\widetilde{G}_{\mu\nu} = 0$ and the relation $\widetilde{\nabla}_\nu\widetilde{\nabla}^\mu\xi^\nu = \widetilde{R}^\mu{}_\nu\xi^\nu$ which is valid for any Killing vector ξ. Now notice that for each azimuthal angle ϕ^a, it is possible to construct a copy of the currents (58). Using (27), we see that the following current (from summing over the J^I_0 corresponding to each azimuthal angles and then subtract out a trivial piece) is also conserved when $\Lambda = 0$,

$$J^I = \frac{2}{m}\sum_{\phi=1}^{m-1} J^I_0 + \frac{2}{m}\sqrt{Hg/\varrho}\partial^I\ln\sqrt{g/\varrho}$$

[4]The extra factor $\frac{1}{16\pi}$ comes from the difference between (1) and (41).

$$\begin{aligned} &= -\sqrt{hg}\frac{f}{v}\Big(\varrho w_a \partial^I w^a + 2\partial^I \ln\sqrt{\varrho}\Big)\,, \\ \Longrightarrow \quad J^r &\longrightarrow -\sqrt{hg}\frac{f}{v}\Big(\frac{v^2}{f^2} w_a \partial_r w^a - \frac{2\Delta'}{f}\Big)\,, \end{aligned} \tag{61}$$

where " $\longrightarrow$ " means equal in the limit $\Delta \to 0$. By comparing with (44), we see that J^r is just the integrand of (60), up to a normalization constant. Despite the fact that the connections found in this paragraph is very interesting, they will have nothing to do with our following discussions.

Given the above $SL(2,R)$ symmetry (56), it is natural to ask if one can extend it to the infinite dimensional Witt algebra,

$$\Big[\hat\delta_{\mathbf{m}}\,,\,\hat\delta_{\mathbf{n}}\Big] = (\mathbf{m}-\mathbf{n})\hat\delta_{\mathbf{m}+\mathbf{n}}\,, \quad \mathbf{m},\mathbf{n} = 0, \pm 1, \pm 2, \cdots . \tag{62}$$

In particular, we want to see if we can construct operators that satisfy (62) approximately near the black hole horizons, where $\Delta \to 0$ (i.e. $\rho \to +\infty$). Technically, given $\hat\delta_{0,\pm}$, one only needs to figure out $\hat\delta_2$ and $\hat\delta_{-2}$ to obtain the full algebra: all other operators can then be constructed by iterating the following relations,

$$\hat\delta_{\mathbf{m}+1} = \frac{1}{\mathbf{m}-1}\Big[\hat\delta_{\mathbf{m}}\,,\,\hat\delta_+\Big]\,, \quad \hat\delta_{-\mathbf{m}-1} = \frac{1}{-\mathbf{m}+1}\Big[\hat\delta_{-\mathbf{m}}\,,\,\hat\delta_-\Big]\,, \quad \mathbf{m} \geq 2\,. \tag{63}$$

We will want all the new transformations $\hat\delta_{\mathbf{m}}$ ($\mathbf{m} = \pm 2, \pm 3, \cdots$) to be regular and non-trivial on the horizon, just as $\hat\delta_0$ and $\hat\delta_\pm$ in (53), (54) and (55).

To generalize (53), (54) and (55) to infinite dimensions, let's start with

$$\Big[\hat\delta_2\,,\,\hat\delta_0\Big] = 2\hat\delta_2\,, \quad \Big[\hat\delta_{-2}\,,\,\hat\delta_0\Big] = -2\hat\delta_{-2}\,. \tag{64}$$

In combination with (53), we find

$$\begin{aligned} \hat\delta_0\hat\delta_2 g_{\phi\phi} &= -\hat\delta_2 g_{\phi\phi}\,, & \hat\delta_0\hat\delta_{-2} g_{\phi\phi} &= 3\hat\delta_{-2} g_{\phi\phi}\,, \\ \hat\delta_0\hat\delta_2 g_{\tilde a\phi} &= -\frac{3}{2}\hat\delta_2 g_{\tilde a\phi}\,, & \hat\delta_0\hat\delta_{-2} g_{\tilde a\phi} &= \frac{5}{2}\hat\delta_{-2} g_{\tilde a\phi}\,, \\ \hat\delta_0\hat\delta_2 g_{\tilde a\tilde b} &= -2\hat\delta_2 g_{\tilde a\tilde b}\,, & \hat\delta_0\hat\delta_{-2} g_{\tilde a\tilde b} &= 2\hat\delta_{-2} g_{\tilde a\tilde b}\,, \\ \hat\delta_0\hat\delta_2 w^\phi &= -3\hat\delta_2 w^\phi\,, & \hat\delta_0\hat\delta_{-2} w^\phi &= \hat\delta_{-2} w^\phi\,, \\ \hat\delta_0\hat\delta_2 w^{\tilde a} &= -\frac{5}{2}\hat\delta_2 w^{\tilde a}\,, & \hat\delta_0\hat\delta_{-2} w^{\tilde a} &= \frac{3}{2}\hat\delta_{-2} w^{\tilde a}\,, \\ \hat\delta_0\hat\delta_2 \varrho &= -\hat\delta_2 \varrho\,, & \hat\delta_0\hat\delta_{-2} \varrho &= 3\hat\delta_{-2} \varrho\,. \end{aligned} \tag{65}$$

Keeping in mind that $\hat\delta_{\pm 2}$ should be regular and non-trivial on the horizon, and also guided by (65), we try the following ansatz

$$\hat\delta_2 g_{\phi\phi} \quad = \quad u_1 g_{\tilde a\tilde b} w^{\tilde a} w^{\tilde b} + u_2 g_{\tilde a\phi} w^{\tilde a} w^\phi + u_3 g_{\phi\phi} w^\phi w^\phi\,,$$

$$\begin{aligned}
\hat{\delta}_2 g_{\tilde{a}\phi} &= u_4 g_{\tilde{a}\tilde{b}} w^{\tilde{b}} w^{\phi} + u_5 g_{\tilde{a}\phi} w^{\phi} w^{\phi} , \quad \hat{\delta}_2 g_{\tilde{a}\tilde{b}} = 0 , \\
\hat{\delta}_2 w^{\phi} &= u_6 w^{\phi} w^{\phi} w^{\phi} , \quad \hat{\delta}_2 w^{\tilde{a}} = u_7 w^{\tilde{a}} w^{\phi} w^{\phi} ,
\end{aligned} \tag{66}$$

$$\begin{aligned}
\hat{\delta}_{-2} g_{\phi\phi} &= \frac{v_1 g_{\tilde{a}\tilde{b}} w^{\tilde{a}} w^{\tilde{b}} + v_2 g_{\tilde{a}\phi} w^{\tilde{a}} w^{\phi} + v_3 g_{\phi\phi} w^{\phi} w^{\phi}}{w^{\phi} w^{\phi} w^{\phi} w^{\phi}} , \\
\hat{\delta}_{-2} g_{\tilde{a}\phi} &= \frac{v_4 g_{\tilde{a}\tilde{b}} w^{\tilde{b}} + v_5 g_{\tilde{a}\phi} w^{\phi}}{w^{\phi} w^{\phi} w^{\phi}} , \quad \hat{\delta}_{-2} g_{\tilde{a}\tilde{b}} = 0 , \\
\hat{\delta}_{-2} w^{\phi} &= v_6 / w^{\phi} , \quad \hat{\delta}_{-2} w^{\tilde{a}} = v_7 w^{\tilde{a}} / (w^{\phi} w^{\phi}) ,
\end{aligned} \tag{67}$$

where $u_1 , \cdots , u_7$ and $v_1 , \cdots , v_7$ are constants. Note g/ϱ is invariant under (53), (54) and (55). Here we further assume that g/ϱ is neutral under all the transformations. This requirement fully determines the structure of $\hat{\delta}_{\mathbf{m}} \varrho$:

$$\delta_{\mathbf{m}} \varrho = \varrho g^{ab} \delta_{\mathbf{m}} g_{ab} , \quad \forall\, \mathbf{m} = 0, \pm 1, \pm 2, \cdots . \tag{68}$$

Now since (Here "$\approx$" means equal at the leading order in $\varrho \to +\infty$)

$$\left[\hat{\delta}_2 , \hat{\delta}_- \right] \approx 3 \hat{\delta}_+ , \quad \left[\hat{\delta}_{-2} , \hat{\delta}_+ \right] \approx -3 \hat{\delta}_- , \quad \left[\hat{\delta}_2 , \hat{\delta}_{-2} \right] \approx 4 \hat{\delta}_0 , \tag{69}$$

we find $v_1 = -u_1$, and

$$\begin{aligned}
& u_2 = 6 , \quad u_3 = 3 , \quad u_4 = 3 , \quad u_5 = \frac{3}{2} , \quad u_6 = -1 , \quad u_7 = -\frac{3}{2} , \\
& v_2 = 2 , \quad v_3 = -1 , \quad v_4 = 1 , \quad v_5 = -\frac{1}{2} , \quad v_6 = -1 , \quad v_7 = \frac{1}{2} .
\end{aligned} \tag{70}$$

The currents corresponding to (66) and (67) are

$$\begin{aligned}
J^I_2 &= \pi^{Iab} \hat{\delta}_2 g_{ab} + \pi^I_a \hat{\delta}_2 w^a + \pi^I_{\varrho} \hat{\delta}_2 \varrho , \\
&= \sqrt{Hg/\varrho} \,\Big\{ u_1 g_{\tilde{a}\tilde{b}} w^{\tilde{a}} w^{\tilde{b}} \Big(\frac{1}{2} \partial^I g^{\phi\phi} - g^{\phi\phi} \partial^I \ln \sqrt{\varrho} \Big) - 3 \partial^I \ln \sqrt{\varrho}\, w^{\phi} w^{\phi} \\
&\qquad -3 \partial^I g_{ab} g^{a\phi} w^b w^{\phi} + \frac{3}{2} \partial^I g_{a\phi} g^{a\phi} w^{\phi} w^{\phi} \\
&\qquad -\frac{3}{2} \varrho g_{ab} w^a w^{\phi} w^{\phi} \partial^I w^b + \frac{1}{2} \varrho g_{a\phi} w^{\phi} w^{\phi} w^{\phi} \partial^I w^a \Big\} ,
\end{aligned} \tag{71}$$

$$\begin{aligned}
J^I_{-2} &= \pi^{Iab} \hat{\delta}_{-2} g_{ab} + \pi^I_a \hat{\delta}_{-2} w^a + \pi^I_{\varrho} \hat{\delta}_{-2} \varrho , \\
&= \frac{\sqrt{Hg/\varrho}}{w^{\phi} w^{\phi} w^{\phi} w^{\phi}} \Big\{ - u_1 g_{\tilde{a}\tilde{b}} w^{\tilde{a}} w^{\tilde{b}} \Big(\frac{1}{2} \partial^I g^{\phi\phi} - g^{\phi\phi} \partial^I \ln \sqrt{\varrho} \Big) + \partial^I \ln \sqrt{\varrho}\, w^{\phi} w^{\phi} \\
&\qquad -\partial^I g_{ab} g^{a\phi} w^b w^{\phi} + \frac{3}{2} \partial^I g_{a\phi} g^{a\phi} w^{\phi} w^{\phi} \\
&\qquad +\frac{1}{2} \varrho g_{ab} w^a w^{\phi} w^{\phi} \partial^I w^b - \frac{3}{2} \varrho g_{a\phi} w^{\phi} w^{\phi} w^{\phi} \partial^I w^a \Big\} .
\end{aligned} \tag{72}$$

With the help of the equations of motion (24), (25) and (26), and also the properties (19) and (22), the total divergence of the currents can be found as

$$\partial_I J^I_2 = \sqrt{Hg/\varrho} \,\Big(- 6 \partial \ln \sqrt{\varrho}\, w^{\phi} \partial w^{\phi} \Big) + \mathcal{O}(\frac{1}{\varrho}) + u_1 \text{ term} ,$$

$$
\begin{aligned}
&= \sqrt{hg}\,\Big(3\frac{\Delta'}{v}w^{\phi}\partial_r w^{\phi}\Big) + \mathcal{O}(\Delta) + u_1\ \text{term}\,,\\
\partial_I J^I_{-2} &= \sqrt{Hg/\varrho}\,\Big(-2\frac{\partial\ln\sqrt{\varrho}\,\partial w^{\phi}}{w^{\phi}w^{\phi}w^{\phi}}\Big) + \mathcal{O}(\frac{1}{\varrho}) + u_1\ \text{term}\,,\\
&= \sqrt{hg}\,\Big(\frac{\Delta'}{v}\cdot\frac{\partial_r w^{\phi}}{w^{\phi}w^{\phi}w^{\phi}}\Big) + \mathcal{O}(\Delta) + u_1\ \text{term}\,, \qquad (73)
\end{aligned}
$$

where all the u_1 terms have components sharing the following factor,

$$
\partial\ln\sqrt{\varrho}\,\partial(g_{\tilde a\tilde b}g^{\phi\phi}) = -\frac{\Delta'}{2f}\partial_r(g_{\tilde a\tilde b}g^{\phi\phi}) + h^{ij}\partial_i\ln\sqrt{\varrho}\,\partial_j(g_{\tilde a\tilde b}g^{\phi\phi}) + \mathcal{O}(\Delta)\,. \qquad (74)
$$

It is obvious that $\hat\delta_{\pm 2}$ are exact symmetries of the action (23) only when both Δ' and u_1 are zero. We are free to take $u_1 = 0$ because it is just a undetermined parameter. On the other hand, Δ' is related to the black hole temperature (48), and so it is non-zero in general. So it appears that the extended symmetries $\hat\delta_{\pm 2}$ are explicitly broken by the finite black hole temperature.

This problem can be fixed by introducing sub-leading terms into (66) and (67),

$$
\begin{aligned}
&\hat\delta_2 g_{\phi\phi} = 6g_{\tilde a\phi}w^{\tilde a}w^{\phi} + 3g_{\phi\phi}w^{\phi}w^{\phi} + \frac{6}{\varrho}\Big(g_{\phi\phi}g^{\phi\phi}-1\Big)\,,\\
&\hat\delta_2 g_{\tilde a\phi} = 3g_{\tilde a\tilde b}w^{\tilde b}w^{\phi} + \frac{3}{2}g_{\tilde a\phi}w^{\phi}w^{\phi} + 3g_{\tilde a\phi}g^{\phi\phi}/\varrho\,, \quad \hat\delta_2 g_{\tilde a\tilde b} = 0\,,\\
&\hat\delta_2 w^{\phi} = -w^{\phi}w^{\phi}w^{\phi} - 3g^{\phi\phi}w^{\phi}/\varrho\,, \quad \hat\delta_2 w^{\tilde a} = -\frac{3}{2}w^{\tilde a}w^{\phi}w^{\phi} - 3g^{\tilde a\phi}w^{\phi}/\varrho\,, \qquad (75)\\
&\hat\delta_{-2} g_{\phi\phi} = \frac{2g_{\tilde a\phi}w^{\tilde a}w^{\phi} - g_{\phi\phi}w^{\phi}w^{\phi} - 6(g_{\phi\phi}g^{\phi\phi}-1)/\varrho}{w^{\phi}w^{\phi}w^{\phi}w^{\phi}}\,,\\
&\hat\delta_{-2} g_{\tilde a\phi} = \frac{g_{\tilde a\tilde b}w^{\tilde b}w^{\phi} - \frac{1}{2}g_{\tilde a\phi}w^{\phi}w^{\phi} - 3g_{\tilde a\phi}g^{\phi\phi}/\varrho}{w^{\phi}w^{\phi}w^{\phi}w^{\phi}}\,, \quad \hat\delta_{-2} g_{\tilde a\tilde b} = 0\,,\\
&\hat\delta_{-2} w^{\phi} = \frac{-w^{\phi}w^{\phi} - g^{\phi\phi}/\varrho}{w^{\phi}w^{\phi}w^{\phi}}\,, \quad \hat\delta_{-2} w^{\tilde a} = \frac{\frac{1}{2}w^{\tilde a}w^{\phi} - g^{\tilde a\phi}/\varrho}{w^{\phi}w^{\phi}w^{\phi}}\,, \qquad (76)
\end{aligned}
$$

where the terms containing $1/\varrho \propto \Delta$ are of the sub-leading order. The coefficients for each sub-leading terms are determined by requiring that (75) and (76) satisfy (69) up to the sub-leading order $\mathcal{O}(\frac{1}{\varrho})$, and also that the currents are conserved up to $\mathcal{O}(1)$,

$$
\begin{aligned}
J^I_2 &= \sqrt{Hg/\varrho}\,\Big\{J^I_{\varrho} - 3w^{\phi}\partial^I w^{\phi} - 3\partial^I\ln\sqrt{\varrho}\,w^{\phi}w^{\phi}\\
&\qquad -3\partial^I g_{ab}g^{a\phi}w^b w^{\phi} + \frac{3}{2}\partial^I g_{a\phi}g^{a\phi}w^{\phi}w^{\phi}\\
&\qquad -\frac{3}{2}\varrho g_{ab}w^a w^{\phi}w^{\phi}\partial^I w^b + \frac{1}{2}\varrho g_{a\phi}w^{\phi}w^{\phi}w^{\phi}\partial^I w^a\Big\}\,, \qquad (77)\\
J^I_{-2} &= \frac{\sqrt{Hg/\varrho}}{w^{\phi}w^{\phi}w^{\phi}w^{\phi}}\Big\{-J^I_{\varrho} - w^{\phi}\partial^I w^{\phi} + \partial^I\ln\sqrt{\varrho}\,w^{\phi}w^{\phi}\\
&\qquad -\partial^I g_{ab}g^{a\phi}w^b w^{\phi} + \frac{3}{2}\partial^I g_{a\phi}g^{a\phi}w^{\phi}w^{\phi}
\end{aligned}
$$

$$+\frac{1}{2}\varrho g_{ab}w^a w^\phi w^\phi \partial^I w^b - \frac{3}{2}\varrho g_{a\phi}w^\phi w^\phi w^\phi \partial^I w^a \Big\}\,, \qquad (78)$$

$$J^I_\varrho \;=\; \frac{3}{\varrho}\Big(g^{\phi\phi}g_{a\phi}\partial^I g^{a\phi} - \partial^I g^{\phi\phi}\Big)\,. \qquad (79)$$

The first two terms in both (77) and (78) are of the sub-leading order and vanish on the black hole horizons, but the contributions from the $w^\phi \partial^I w^\phi$ terms cancel the Δ' terms in (73) exactly, while the contribution from J^I_ϱ is still negligible at the leading order.

So with (75) and (76), we have $\hat{\delta}_{\pm 2}$ acting as exact symmetries of the action (23) on the black hole horizons. By using (63), we can obtain an infinite dimensional conformal symmetry obeying the Witt algebra (62). For each azimuthal angle ϕ^a with a non-vanishing angular velocity, we will have an independent copy of the Witt algebra. So classically, the action (23) has k-copies of infinite dimensional conformal symmetries on the black hole horizon, with k being the number of non-vanishing angular velocities. Since for a given classical solution there is no essential difference between the reduced action (23) and the original action (1), the same conclusion holds for the original action (1).

Note the conformal symmetries are fully determined by the structure of the action (23) and the properties of the background H_{IJ}, but are independent of the values of g_{ab}, w^a and ϱ. (For $\varrho = \frac{v^2}{f^2}\cdot\frac{f}{\Delta}$, it is the factor $\frac{v^2}{f^2}$ that should be treated as an independent degrees of freedom, because the factor $\frac{f}{\Delta}$ is fixed in the background.) One may entertain with the idea of treating (23) as a field theory of g_{ab}, w^a and ϱ living in the fixed background H_{IJ}, with the black hole being the classical solution. Further, one can ask if the fluctuations of the fields g_{ab}, w^a and ϱ can fully describe the microstates of the black hole. We shall leave these to future works.

5 Summary

In this paper, we have carried out a Kaluza-Klein like reduction of the Einstein-Hilbert action along the ignorable coordinates of stationary and axisymmetric black holes. The reduced action enables us to study the classical equations of motion in a much greater detail. In the case of pure gravity plus a cosmological constant, this allows us to re-derive the first law of black hole thermodynamics in a straightforward manor.

The reduced action has a global $SL(m, R)$ gauge symmetry, with m being the number of ignorable coordinates. Related to each angular momentum there is a particular $SL(2, R)$ subgroup. We show that this $SL(2, R)$ can be extended to the full Witt algebra on the black hole horizons. The extended transformations are exact symmetries of the actions (23) on the horizon. For a black hole with k non-vanishing angular velocities, the action (23) then has k-copies of infinite dimensional conformal symmetries on the horizon.

Our key motivation of the present work was to search a way that can help us identify the conformal symmetries of the putative 2D CFT dual to a non-extremal black hole, as suggested by the studies of hidden conformal symmetries of black holes [22]. However, so far we have not been able to abstract any physical information from the conformal symmetries found in this work. One may try to reinterpret the extended symmetries (66), (67) and (63) as approximate diffeomorphisms of the original action (1) near the horizons, and then use the usual covariant phase space method (see, e.g. [10] for the latest) to see if the Witt algebra (62) can be promoted to a Virasoro algebra at the quantum level. This procedure is still under investigation.

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