

THE EINSTEIN–BOLTZMANN SYSTEM AND POSITIVITY

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Received 17 March 2012

Accepted 10 June 2012

Published 21 March 2013

Communicated by Philippe G. LeFloch

Abstract. The Einstein–Boltzmann (EB) system is studied, with particular attention to the non-negativity of the solution of the Boltzmann equation. A new parametrization of post-collisional momenta in general relativity is introduced and then used to simplify the conditions on the collision cross-section given by Bancel and Choquet-Bruhat. The non-negativity of solutions of the Boltzmann equation on a given curved spacetime has been studied by Bichteler and Tadmon. By examining to what extent the results of these authors apply in the framework of Bancel and Choquet-Bruhat, the non-negativity problem for the EB system is resolved for a certain class of scattering kernels. It is emphasized that it is a challenge to extend the existing theory of the Cauchy problem for the EB system so as to include scattering kernels which are physically well-motivated.

Keywords: The Einstein–Boltzmann system; general relativity; kinetic theory.

Mathematics Subject Classification 2010: 82C40, 83C55

1. Introduction

To understand how the gravitational field and matter interact in general relativity it is necessary to specify a suitable matter model. One frequently used type of model comes from kinetic theory, where matter is considered as a collection of particles described statistically. The Vlasov and Boltzmann equations are the equations describing this type of model and coupling to the gravitational field gives the Einstein–Vlasov and the Einstein–Boltzmann (EB) systems respectively. We refer to [1, 16] and their references for basic information about the Einstein–Vlasov system and the Einstein equations coupled to many different matter models. This paper considers only the EB system. Compared to the Einstein–Vlasov system, the EB system has not been studied much in the decades since local existence was proven by Bancel and Choquet-Bruhat [2, 3]. Recently several further results

have been obtained for the EB system. Mucha [11, 12] showed that the initial regularity conditions given in [2, 3] can be weakened. Noutchegueme and Dongo [13] and Noutchegueme and Takou [15] proved some global existence results for certain classes of spacetimes with symmetries. The relativistic Boltzmann equation on a given curved spacetime was studied in [4, 14, 20]. As for the nonrelativistic or special relativistic Boltzmann equations, one can find plenty of references, and we only refer to [5, 6, 8].

The EB system is a system of equations where the Einstein equations and the Boltzmann equation are coupled to each other. In harmonic coordinates, the Einstein equations take the form of the system of quasilinear wave equations

$$-\frac{1}{2}g^{\gamma\delta}\frac{\partial^2 g^{\alpha\beta}}{\partial x^\gamma\partial x^\delta} + H^{\alpha\beta} = T^{\alpha\beta} - \frac{1}{2}(g^{\gamma\delta}T_{\gamma\delta})g^{\alpha\beta}, \quad (1.1)$$

for a Lorentzian metric $g_{\alpha\beta}$, where $H^{\alpha\beta}$ is a rational function of $g_{\alpha\beta}$ and $\partial_\gamma g_{\alpha\beta}$ with denominator a power of the determinant $|g|$ and $T^{\alpha\beta}$ is the stress-energy tensor of matter described by the Boltzmann equation. The Boltzmann equation describes the dynamics of a relativistic gas, the particles of which interact through binary collisions. The unknown is the distribution function f defined on the phase space, which is the tangent bundle $T(M)$ of a four-dimensional manifold M with a metric $g_{\alpha\beta}$. If we assume that all the particles considered have the same mass, then momentum is confined to a submanifold $P(M) \subset T(M)$, which is defined by $g_{\alpha\beta}p^\alpha p^\beta = -1$, when the mass is normalized to unity. This submanifold is called the mass hyperboloid. As a consequence of this assumption the distribution function turns out to be a function of seven variables. The Boltzmann equation takes the form

$$p^\alpha \frac{\partial f}{\partial x^\alpha} - \Gamma_{\alpha\beta}^i p^\alpha p^\beta \frac{\partial f}{\partial p^i} = Q(f, f), \quad (1.2)$$

$$Q(f, f) = \int_{\mathbb{R}^3} \int_{S^2} S(x, p, q, \Omega) (f(p')f(q') - f(p)f(q)) d\Omega \frac{|g|^{\frac{1}{2}}}{-q_0} dq, \quad (1.3)$$

where $\Gamma_{\alpha\beta}^i$ are the Christoffel symbols of the metric $g_{\alpha\beta}$ and the collision operator Q will be studied in detail in Sec. 2. For simplicity we abbreviate $f(x, p')$ as $f(p')$, $f(x, q')$ as $f(q')$ and so on. The stress-energy tensor $T^{\alpha\beta}$ is defined as

$$T^{\alpha\beta} = \int_{\mathbb{R}^3} f(x, p) p^\alpha p^\beta \frac{|g|^{\frac{1}{2}}}{-p_0} dp, \quad (1.4)$$

where the integration over \mathbb{R}^3 is understood as integration on the mass hyperboloid at x . Equations (1.1)–(1.4) form the EB system [7, 17].

In this paper, we are interested in the strong solutions of the EB system constructed in [3] by Bancel and Choquet-Bruhat. In Sec. 2, the Boltzmann equation in a curved spacetime is studied and an alternative formulation using an orthonormal frame is explained. (Its relevance is discussed in Sec. 4.) Returning to the equations written in a coordinate frame as in [3] we introduce a new parametrization of

the post-collisional momenta. Unlike in the nonrelativistic case, the post-collisional momenta can usefully be parametrized in many different ways in the relativistic case [9, 18, 19]. The parametrization in Sec. 2.1 can be thought of as a generalization of that in [9]. With this parametrization, we consider the original result of Bancel and Choquet-Bruhat, and simplify some conditions in their paper. The distribution function in the Boltzmann equation should be non-negative because of its interpretation as a number density. In mathematical terms this means that a solution f of the Boltzmann equation arising from non-negative initial data should itself be non-negative. This important property was not proved in [3]. A central aim of this paper is to clarify this question, taking into the account the available results in the literature. This non-negativity problem for the EB system is considered in Sec. 3. In the case of the Boltzmann equation on a given curved spacetime, there are two results known [4, 20] concerning the non-negativity problem. If a solution of the EB system is given, in particular a four-dimensional manifold and a metric as a part of a solution, a result on non-negativity of solutions of the Boltzmann equation on suitable curved spacetimes implies a non-negativity result for the EB system. Thus in order to settle this question for the solutions obtained by Bancel and Choquet-Bruhat in [3] it suffices to examine whether the conditions assumed in [4, 20] hold in the framework of [3].

In Sec. 3.1, a result of Tadmon [20] is considered. Although [20] cites [3] it does not discuss the applicability of the results obtained to the solutions of that paper. The conditions in [20] are quite complicated, but it is shown in Sec. 3.1 that they are satisfied in the framework of Bancel and Choquet-Bruhat. In Sec. 3.2 it is pointed out that the assumptions assuring non-negativity of solutions of the Boltzmann equation in the work of Bichteler [4] do not follow from those of [3]. It is shown, however, that non-negativity of the distribution function in the solutions of [3] can be proved using a method simpler than that of Tadmon.

Section 4 contains some further general considerations on the EB system. The topics discussed include the advantages of a formulation using orthonormal frames, the issue of identifying physically relevant scattering kernels and how the non-negativity result can be made global.

1.1. Notation

In this part, we collect the notation which is used in this paper. Greek indices run from 0 to 3, while Latin indices run from 1 to 3. The spatial variable x denotes a four-vector, while the momentum variable p denotes a three-dimensional vector, i.e.

$$x = (x^0, x^1, x^2, x^3), \quad p = (p^1, p^2, p^3).$$

The metric $g_{\alpha\beta}$ has signature $(-, +, +, +)$, g denotes its determinant and the Minkowski metric is denoted by $\eta_{\alpha\beta}$. Throughout the paper, the speed of light c and mass of the particles are assumed to be unity and hence the momentum p^α

lies on a hypersurface defined by the equation $g_{\alpha\beta}p^\alpha p^\beta = -1$, which is called the mass shell. Due to the mass shell condition, p^0 and p_0 are represented by p^i as follows:

$$\begin{aligned} p^0 &= \frac{1}{-g_{00}}(g_{0i}p^i + \sqrt{(g_{0i}p^i)^2 - g_{00}(g_{ij}p^i p^j + 1)}), \\ p_0 &= -\sqrt{(g_{0i}p^i)^2 - g_{00}(g_{ij}p^i p^j + 1)}. \end{aligned} \tag{1.5}$$

In some places, we use the Euclidean norm in \mathbb{R}^d , i.e.

$$|p|^2 = \sum_{i=1}^d (p^i)^2 \quad \text{for } p \in \mathbb{R}^d,$$

and \cdot will denote the usual inner product in \mathbb{R}^d , i.e.

$$p \cdot q = \sum_{i=1}^d p^i q^i \quad \text{for } p, q \in \mathbb{R}^d.$$

We also collect the function spaces which are used in this paper and which were originally introduced in [3]. The spaces are basically Sobolev spaces weighted in the momentum variables. Two kinds of weight functions are used, $(p^0)^{N/2+|\hat{k}|}$ and e^{p^0} , where N is a positive integer and $|\hat{k}|$ is the number of derivatives which are taken with respect to momentum variables. For simplicity, we write $h_k(p) = (p^0)^{N/2+|\hat{k}|}$ and $h(p) = e^{p^0}$, which can be understood as the case $N = \infty$.

- (1) Let ω_0 be a domain of \mathbb{R}^3 , i.e. $x^0 = 0$ in \mathbb{R}^4 , and $\hat{\omega}_0 = \omega_0 \times \mathbb{R}^3$.
- (2) Let $H_\mu(\omega_0)$ and $H_{\mu,N}(\hat{\omega}_0)$ be the Sobolev spaces whose norms are defined as

$$\begin{aligned} \|u\|_{H_\mu(\omega_0)}^2 &= \sum_{|k| \leq \mu} \|D_x^k u\|_{L^2(\omega_0)}^2, \\ \|f\|_{H_{\mu,N}(\hat{\omega}_0)}^2 &= \sum_{|k| \leq \mu} \|h_k(p) D_{x,p}^k f\|_{L^2(\hat{\omega}_0)}^2, \end{aligned}$$

where k is a multi-index such that $D_{x,p}^k = D_x^{\bar{k}} D_p^{\hat{k}}$ in the latter case.

- (3) Let Ω be a domain of \mathbb{R}^4 and $\hat{\Omega} = \Omega \times \mathbb{R}^3$.
- (4) Let $H_\mu(\Omega)$ and $H_{\mu,N}(\hat{\Omega})$ be the Sobolev spaces whose norms are defined as

$$\begin{aligned} \|u\|_{H_\mu(\Omega)}^2 &= \sum_{|k| \leq \mu} \|D_x^k u\|_{L^2(\Omega)}^2, \\ \|f\|_{H_{\mu,N}(\hat{\Omega})}^2 &= \sum_{|k| \leq \mu} \|h_k(p) D_{x,p}^k f\|_{L^2(\hat{\Omega})}^2, \end{aligned}$$

where k is a multi-index such that $D_{x,p}^k = D_x^{\bar{k}} D_p^{\hat{k}}$ in the latter case.

- (5) $g_{\alpha\beta}(0)$, $\partial_0 g_{\alpha\beta}(0)$ and $f(0)$ will denote the initial data for $g_{\alpha\beta}$ and f respectively.

2. The Boltzmann Equation in a Curved Spacetime

In this section, we focus on the Boltzmann part of the EB system. We assume that a four-dimensional manifold and a metric are given, and consider the Boltzmann equation in this spacetime. Let $x^\alpha \in \mathbb{R}^4$, $p^\alpha \in \mathbb{R}^4$ and p^α satisfy $g_{\alpha\beta}p^\alpha p^\beta = -1$. For a given metric $g_{\alpha\beta}$, the Boltzmann equation reads

$$p^\alpha \frac{\partial f}{\partial x^\alpha} - \Gamma_{\alpha\beta}^i p^\alpha p^\beta \frac{\partial f}{\partial p^i} = Q(f, f),$$

$$Q(f, f)(x, p) = \int_{\mathbb{R}^3} \int_{S^2} S(x, p, q, \Omega) (f(p')f(q') - f(p)f(q)) d\Omega \frac{|g|^{\frac{1}{2}}}{-q_0} dq.$$

Here $\Gamma_{\alpha\beta}^i$ are the Christoffel symbols defined by

$$\Gamma_{\beta\gamma}^\alpha = \frac{1}{2} g^{\alpha\delta} (\partial_\gamma g_{\delta\beta} + \partial_\beta g_{\delta\gamma} - \partial_\delta g_{\beta\gamma})$$

and S is called the collision cross-section and is a non-negative function,

$$S = \lambda \varrho \sigma \quad \text{with } \sigma = \sigma(\varrho, \theta). \tag{2.1}$$

Ω is a point of S^2 , thought of as the unit sphere in \mathbb{R}^3 with standard measure $d\Omega$ and polar angle θ . The quantities σ and θ are called the scattering kernel and the scattering angle respectively. The scalar quantities λ and ϱ are defined by

$$\lambda = \sqrt{-(p_\alpha + q_\alpha)(p^\alpha + q^\alpha)}, \quad \varrho = \sqrt{(p_\alpha - q_\alpha)(p^\alpha - q^\alpha)}. \tag{2.2}$$

The variables p'^α , q'^α , p^α and q^α denote the momenta of two colliding particles, i.e. post-collisional momenta p'^α and q'^α , and pre-collisional momenta p^α and q^α , and they satisfy

$$p'^\alpha + q'^\alpha = p^\alpha + q^\alpha,$$

which expresses energy-momentum conservation. The collision operator Q can be written as $Q = Q_+ - Q_-$ where

$$Q_+ = \iint S(x, p, q, \Omega) f(p') f(q') d\Omega \frac{|g|^{\frac{1}{2}}}{-q_0} dq,$$

$$Q_- = f(p) \iint S(x, p, q, \Omega) f(q) d\Omega \frac{|g|^{\frac{1}{2}}}{-q_0} dq.$$

Q_+ and Q_- are called the gain term and the loss term respectively. Note that Q_+ and Q_- are non-negative.

In the case $g_{\alpha\beta} = \eta_{\alpha\beta}$, the above Boltzmann equation reduces to the well-known special relativistic Boltzmann equation. In this case, the Christoffel symbols vanish, $|g| = 1$ and $-q_0 = q^0$, so we obtain

$$\frac{p^\alpha}{p^0} \frac{\partial f}{\partial x^\alpha} = \iint v_\phi \sigma(\varrho, \theta) (f(p') f(q') - f(p) f(q)) d\Omega dq,$$

where v_ϕ is called the Møller velocity,

$$v_\phi = \frac{\lambda q}{p^0 q^0}.$$

The general relativistic EB system can be written in an alternative way which makes it look more similar to the special relativistic case and which is more convenient for some purposes. The idea is to introduce an orthonormal frame e_μ^α on spacetime with dual coframe θ_α^μ . Here the indices from the beginning of the Greek alphabet like α are coordinate indices as before whereas the indices like μ from the middle of the Greek alphabet label the vectors of the frame. In formulating the EB system the metric is described by its components in a coordinate frame as before but the coordinates (x^α, p^α) used to parametrize the mass shell are replaced by coordinates (x^α, v^α) where $v^\alpha = \theta_\alpha^\nu p^\nu$. By abuse of notation we will write $f(x^\alpha, v^\alpha)$ for the representation of the distribution function in the new coordinates. An advantage of the new coordinates is that the collision term no longer contains any explicit dependence on the metric and is identical in form to what it is in special relativity. On the other hand the transport part of the Boltzmann equation becomes more complicated and the Christoffel symbols are replaced by the components of the Levi-Civita connection in the chosen frame. Thus we obtain the equation

$$v^\mu e_\mu^\alpha \frac{\partial f}{\partial x^\alpha} - \eta^{k\sigma} g_{\alpha\beta} e_\sigma^\alpha e_\lambda^\beta (\partial_\delta e_\mu^\beta + \Gamma_{\delta\epsilon}^\beta e_\mu^\epsilon) v^\lambda v^\mu \frac{\partial f}{\partial v^k} = Q(f, f). \quad (2.3)$$

If the EB system is to define a closed system for the quantities $g_{\alpha\beta}(x^\gamma)$ and $f(x^\gamma, v^\alpha)$ then the orthonormal frame must be fixed in terms of the metric and the coordinates by some condition. This can be done in such a way that it depends only on the components $g_{\alpha\beta}$ and does not contain any direct dependence on the coordinates. One way of doing this is to use the Gram-Schmidt process. This algorithm for producing an orthonormal frame from a general frame is best known in the case of positive definite metrics but it works just as well for pseudo-Riemannian metrics provided the starting frame is non-degenerate in the sense that the pull-back of the metric to the subspace spanned by any non-empty subset of the vectors of the frame is a non-degenerate quadratic form. This process can be applied to the coordinate frame $\partial/\partial x^\alpha$ to get the frame e_μ^α . Then the components e_μ^α are algebraic functions of the components $g_{\alpha\beta}$. In the case of the first two elements of the basis, for instance, we get

$$e_0^\mu = \delta_0^\mu (-g_{00})^{-\frac{1}{2}}, \quad (2.4)$$

$$e_1^\mu = (-g_{00} \delta_1^\mu + g_{01} \delta_0^\mu) ((g_{00})^2 g_{11} - g_{00} (g_{01})^2)^{-\frac{1}{2}}. \quad (2.5)$$

It then follows that the components v^α are also algebraic functions of the components $g_{\alpha\beta}$. The coefficients in the transport part of the Boltzmann equation are algebraic functions of the components $g_{\alpha\beta}$ and their first-order partial derivatives.

2.1. The post-collisional momenta

A parametrization of the post-collisional momenta p'^μ and q'^μ in general relativity has been obtained in [3] using an orthonormal frame $\{e^\mu_\alpha\}$ associated to a point of the mass shell. Note that this is a different type of object from the orthonormal frame introduced above. In the frame considered in [3] $p^\mu = e^\mu_\alpha v^\alpha$, $q^\mu = e^\mu_\alpha w^\alpha$ and $v^\alpha + w^\alpha = (\lambda, 0, 0, 0)$. The post-collisional momenta in this frame are represented by

$$v'^\alpha = \frac{1}{2}(\lambda, \varrho\Omega), \quad w'^\alpha = \frac{1}{2}(\lambda, -\varrho\Omega),$$

where $\Omega \in S^2$. Consequently, p'^μ is given by

$$p'^\mu = \frac{1}{2}(e^\mu_0\lambda + e^\mu_1\varrho \cos \theta + \varrho \sin \theta(e^\mu_2 \cos \varphi + e^\mu_3 \sin \varphi)),$$

where $\Omega = (\cos \theta, \sin \theta \cos \varphi, \sin \theta \sin \varphi)$, and a similar expression is obtained for q'^μ . It is a natural approach to choose an orthonormal frame and then use formulae from special relativity, but in some points it has an inconvenient aspect concerning the orthonormal frame. In this paper, we consider the Sobolev spaces of order greater than five. Hence, high-order derivatives of some quantities are taken, and we need to estimate derivatives of the orthonormal frame chosen. If we follow the result of [3], then derivatives of the post-collisional momenta are estimated as follows:

$$|D^k_{x,p} p'| \leq (p^0)^{-|\hat{k}|} \sum \frac{(p^0 q^0)^{|r|+|\hat{s}|+1/2}}{\lambda^{2|r_1|} \varrho^{2|r_2|} |p \times q|^{|\hat{s}|}} |g^{(|\hat{k}|)}|,$$

where k is a multi-index such that $D^k_{x,p} = D_x^{\bar{k}} D_p^{\hat{k}}$, the sum is over all the possible multi-indices r_1, r_2, s satisfying $|r + s| \leq |k|$ and $r_1 + r_2 = r$ and the term $g^{(|\hat{k}|)}$ will be defined later. We can see that singularities appear in the denominator, which are certainly due to the choice of the orthonormal frame. This problem might be removed by taking another orthonormal frame at the singular points, but it would be quite complicated to find such an orthonormal frame that behaves well at those singularities. We instead use a different approach to parametrize the post-collisional momenta, where the above problem does not arise.

We introduce a new parametrization of post-collisional momenta. Suppose that p^α and q^α are given, and consider the following four-vectors:

$$n^\alpha = p^\alpha + q^\alpha \quad \text{and} \quad t^\alpha = (n_i \omega^i, -n_0 \omega) \quad \text{for } \omega \in S^2. \tag{2.6}$$

Note that t^α is orthogonal to n^α . Then p'^α and q'^α can be parametrized by

$$\begin{aligned} p'^\alpha &= p^\alpha - \frac{t_\beta(p^\beta - q^\beta)}{t_\beta t^\beta} t^\alpha = p^\alpha + 2 \frac{t_\beta q^\beta}{t_\beta t^\beta} t^\alpha, \\ q'^\alpha &= q^\alpha + \frac{t_\beta(p^\beta - q^\beta)}{t_\beta t^\beta} t^\alpha = q^\alpha - 2 \frac{t_\beta q^\beta}{t_\beta t^\beta} t^\alpha, \end{aligned} \tag{2.7}$$

where we used $t_\alpha n^\alpha = 0$. It can be easily shown that they satisfy

$$p'^\alpha + q'^\alpha = p^\alpha + q^\alpha \quad \text{and} \quad p'_\alpha p'^\alpha = q'_\alpha q'^\alpha = -1.$$

We can use another parametrization for p'^α and q'^α ,

$$p'^\alpha = \frac{p^\alpha + q^\alpha}{2} + \frac{\varrho}{2} \frac{t^\alpha}{\sqrt{t_\beta t^\beta}}, \quad q'^\alpha = \frac{p^\alpha + q^\alpha}{2} - \frac{\varrho}{2} \frac{t^\alpha}{\sqrt{t_\beta t^\beta}}, \tag{2.8}$$

where ϱ and t^α are defined by (2.2) and (2.6) respectively. We can see that the parametrizations (2.7) and (2.8) are very similar to those of the nonrelativistic case. In the nonrelativistic case, post-collisional momenta $p' \in \mathbb{R}^3$ and $q' \in \mathbb{R}^3$ are represented in two ways:

$$p' = p - ((p - q) \cdot \omega)\omega, \quad q' = q + ((p - q) \cdot \omega)\omega \quad \text{for } \omega \in S^2$$

or

$$p' = \frac{p + q}{2} + \frac{|p - q|}{2}\omega, \quad q' = \frac{p + q}{2} - \frac{|p - q|}{2}\omega \quad \text{for } \omega \in S^2.$$

Hence, (2.7) and (2.8) can be thought of as natural generalizations from the non-relativistic case.

We remark that a well-known parametrization of p'^α and q'^α in the special relativistic case by Glassey and Strauss [9] is exactly the same as (2.7). In the special relativistic case, $g_{\alpha\beta} = \eta_{\alpha\beta}$, we have

$$t_\alpha t^\alpha = -(n \cdot \omega)^2 + (n^0)^2 = (p^0 + q^0)^2 - ((p + q) \cdot \omega)^2$$

and by direct calculations,

$$\begin{aligned} t_\alpha q^\alpha &= -q^0 n \cdot \omega + n^0 \omega \cdot q = -q^0(p \cdot \omega) - q^0(q \cdot \omega) + p^0(\omega \cdot q) + q^0(\omega \cdot q) \\ &= -p^0 q^0(p/p^0 \cdot \omega) + p^0 q^0(q/q^0 \cdot \omega) = -p^0 q^0((\hat{p} - \hat{q}) \cdot \omega). \end{aligned}$$

Defining $\hat{p} = p/p^0$ and $\hat{q} = q/q^0$ we obtain the following expression:

$$p' = p - \frac{2p^0 q^0 ((\hat{p} - \hat{q}) \cdot \omega)(p^0 + q^0)}{(p^0 + q^0)^2 - ((p + q) \cdot \omega)^2} \omega.$$

This is the parametrization given in [9].

We close this subsection with a simple lemma, which shows that p^0 and $-p_0$ are equivalent in a curved spacetime when its metric is close to the Minkowski metric.

Lemma 2.1. *Consider a momentum p^α satisfying $g_{\alpha\beta} p^\alpha p^\beta = -1$. Suppose that there exists a small ε such that the metric $g_{\alpha\beta}$ satisfies*

$$|g_{\alpha\beta} - \eta_{\alpha\beta}| \leq \varepsilon \quad \text{and} \quad (1 - \varepsilon) \sum_{i=1}^3 (X^i)^2 \leq g_{ij} X^i X^j \leq (1 + \varepsilon) \sum_{i=1}^3 (X^i)^2$$

for any three-dimensional vector X . Then we have

$$|p| \leq (1 + \varepsilon_1) \min\{p^0, -p_0\} \quad \text{and} \quad (1 - \varepsilon_1)p^0 \leq -p_0 \leq (1 + \varepsilon_1)p^0$$

for some small $\varepsilon_1 > 0$.

Proof. By solving the equation $g_{\alpha\beta}p^\alpha p^\beta = -1$, p^0 and p_0 can be expressed as functions of $g_{\alpha\beta}$ and p^i as in (1.5). Hence, by direct calculations,

$$|p|^2 \leq (1 + \varepsilon)g_{ij}p^i p^j \leq (1 + \varepsilon)((g_{0i}p^i)^2 - g_{00}(g_{ij}p^i p^j + 1)) = (1 + \varepsilon)(p_0)^2, \\ -p_0 = -g_{00}p^0 - g_{0i}p^i \leq (1 + \varepsilon)p^0 + \varepsilon|p|,$$

which imply that $|p| \leq Cp_0$ and $|p| \leq Cp^0$ for some constant $C \approx 1$. From the identity $p_0 = g_{00}p^0 + g_{0i}p^i$ and the conditions on $g_{\alpha\beta}$, we get the desired equivalence between p^0 and $-p_0$. □

2.2. Derivatives of the post-collisional momenta

In this paper, we use (2.7) for p'^α and q'^α . Since the parametrization (2.7) depends on x, p and q , derivatives of the post-collisional momenta are not trivial to compute, but we can see that derivatives of p'^α and q'^α depend only on t^α . To calculate derivatives of p'^α and q'^α , we use the notation $g^{(m)}$ of Bancel and Choquet-Bruhat from [3].

A function h is said to be a $g^{(m)}$ function, if h is any linear combination of products, $\prod_j D^k g_{\mu\nu}$, with $\sum_j |k_j| \leq m$ and coefficients in the algebra of bounded functions on $\Omega \times \mathbb{R}^3 \times \mathbb{R}^3$ generated by $p^\alpha/p^0, p^\alpha/p_0, q^\alpha/q^0$ and q^α/q_0 .

Lemma 2.2. *Let t^α be a four-vector defined by (2.6) for some $\omega \in S^2$. Then, for $D^k = D_x^{\bar{k}} D_p^{\hat{k}}$ with multi-index $k = \bar{k} + \hat{k}$, we have*

$$D^k \left[2 \frac{t_\beta q^\beta}{t_\beta t^\beta} t^\alpha \right] = (p^0)^{-|\hat{k}|} \sum_{i=0}^{2+2|k|} \frac{(p^0)^i (q^0)^{3+2|k|-i}}{(t_\alpha t^\alpha)^{1+|k|}} g_i^{(|\bar{k}|)},$$

where $g_i^{(|\bar{k}|)}$ are some $g^{(|\bar{k}|)}$ functions.

Proof. We first note that p -derivatives of p^α and p_α are $g^{(0)}$ functions as follows,

$$\frac{\partial p^0}{\partial p^k} = -\frac{p_k}{p_0} = -\frac{p^\alpha}{p_0} g_{\alpha k}, \quad \frac{\partial p^i}{\partial p^k} = \delta_k^i, \\ \frac{\partial p_\alpha}{\partial p^k} = \frac{\partial p^0}{\partial p^k} g_{0\alpha} + \frac{\partial p^i}{\partial p^k} g_{i\alpha} = -\frac{p^\beta}{p_0} g_{\beta k} g_{0\alpha} + g_{k\alpha},$$

while their x^α -derivatives are $p^0 g^{(1)}$ functions,

$$\frac{\partial p^0}{\partial x^\alpha} = -\frac{p^\beta p^\gamma}{2p_0} \partial_\alpha g_{\beta\gamma}, \quad \frac{\partial p_\beta}{\partial x^\alpha} = \frac{\partial p^0}{\partial x^\alpha} g_{0\beta} + p^\gamma \partial_\alpha g_{\beta\gamma} = -\frac{p^\gamma p^\delta}{2p_0} \partial_\alpha g_{\gamma\delta} g_{0\beta} + p^\gamma \partial_\alpha g_{\beta\gamma}.$$

Hence, we can deduce that

$$\frac{\partial g^{(m)}}{\partial p^k} = \frac{g^{(m)}}{p^0} \quad \text{and} \quad \frac{\partial g^{(m)}}{\partial x^\alpha} = g^{(m+1)}.$$

To prove the lemma, we use induction on $|k|$. Since $n^\alpha = p^\alpha + q^\alpha$, the four-vector t^α is of the form

$$t^\alpha = (n_i \omega^i, -n_0 \omega) = n^\beta (g_{i\beta} \omega^i, -g_{0\beta} \omega) = p^0 g_0^{(0)} + q^0 g_1^{(0)}$$

for some $g_0^{(0)}$ and $g_1^{(0)}$, which are $g^{(0)}$ functions. Hence, we obtain for $|k| = 0$,

$$2 \frac{t_\beta q^\beta}{t_\beta t^\beta} t^\alpha = \sum_{i=0}^2 \frac{(p^0)^i (q^0)^{3-i}}{t_\alpha t^\alpha} g_i^{(0)}.$$

We use $t_\alpha t^\alpha = \sum_{i=0}^2 (p^0)^i (q^0)^{2-i} g_i^{(0)}$ to show that for $|k| + 1$.

$$\begin{aligned} D_{x^\alpha} D^k \left[2 \frac{t_\beta q^\beta}{t_\beta t^\beta} t^\gamma \right] &= D_{x^\alpha} \left[(p^0)^{-|\hat{k}|} \sum_{i=0}^{2+2|k|} \frac{(p^0)^i (q^0)^{3+2|k|-i}}{(t_\alpha t^\alpha)^{1+|k|}} g_i^{(|\bar{k}|)} \right] \\ &= (p^0)^{-|\hat{k}|} \sum_{i=0}^{2+2|k|} \frac{(p^0)^i (q^0)^{3+2|k|-i}}{(t_\beta t^\beta)^{1+|k|}} g_i^{(|\bar{k}|+1)} \\ &\quad + (p^0)^{-|\hat{k}|} \sum_{i=0}^{2+2|k|} \frac{(p^0)^i (q^0)^{3+2|k|-i}}{(t_\beta t^\beta)^{2+|k|}} D_{x^\alpha} [t_\beta t^\beta] g_i^{(|\bar{k}|)}. \end{aligned}$$

Since $D_{x^\alpha} [t_\beta t^\beta] = \sum_{i=0}^2 (p^0)^i (q^0)^{2-i} g_i^{(1)}$, we obtain

$$D_{x^\alpha} D^k \left[2 \frac{t_\beta q^\beta}{t_\beta t^\beta} t^\gamma \right] = (p^0)^{-|\hat{k}|} \sum_{i=0}^{4+2|k|} \frac{(p^0)^i (q^0)^{5+2|k|-i}}{(t_\beta t^\beta)^{2+|k|}} g_i^{(|\bar{k}|+1)}.$$

Similarly,

$$\begin{aligned} D_{p^j} D^k \left[2 \frac{t_\beta q^\beta}{t_\beta t^\beta} t^\gamma \right] &= D_{p^j} \left[(p^0)^{-|\hat{k}|} \sum_{i=0}^{2+2|k|} \frac{(p^0)^i (q^0)^{3+2|k|-i}}{(t_\alpha t^\alpha)^{1+|k|}} g_i^{(|\bar{k}|)} \right] \\ &= (p^0)^{-|\hat{k}|-1} \sum_{i=0}^{2+2|k|} \frac{(p^0)^i (q^0)^{3+2|k|-i}}{(t_\beta t^\beta)^{1+|k|}} g_i^{(|\bar{k}|)} \\ &\quad + (p^0)^{-|\hat{k}|} \sum_{i=0}^{2+2|k|} \frac{(p^0)^i (q^0)^{3+2|k|-i}}{(t_\beta t^\beta)^{2+|k|}} D_{p^j} [t_\beta t^\beta] g_i^{(|\bar{k}|)}. \end{aligned}$$

Since $D_{p^j} [t_\beta t^\beta] = (p^0)^{-1} \sum_{i=0}^2 (p^0)^i (q^0)^{2-i} g_i^{(0)}$, we obtain

$$D_{x^\alpha} D^k \left[2 \frac{t_\beta q^\beta}{t_\beta t^\beta} t^\gamma \right] = (p^0)^{-|\hat{k}|-1} \sum_{i=0}^{4+2|k|} \frac{(p^0)^i (q^0)^{5+2|k|-i}}{(t_\beta t^\beta)^{2+|k|}} g_i^{(|\bar{k}|)}$$

and this completes the proof of the lemma. □

To estimate derivatives of the post-collisional momenta, we need to obtain a lower bound for the quantity $t_\alpha t^\alpha$, which appears in the denominator in the expressions coming from Lemma 2.2. Note that $t_\alpha t^\alpha$ is non-negative, because $t_\alpha n^\alpha = 0$ and $n^\alpha = p^\alpha + q^\alpha$ is a timelike vector.

Lemma 2.3. *Let t^α be a four-vector defined by (2.6) for some $\omega \in S^2$. Suppose that there exists a small ε such that the metric $g_{\alpha\beta}$ satisfies*

$$|g_{\alpha\beta} - \eta_{\alpha\beta}| \leq \varepsilon \quad \text{and} \quad (1 - \varepsilon) \sum_{i=1}^3 (X^i)^2 \leq g_{ij} X^i X^j \leq (1 + \varepsilon) \sum_{i=1}^3 (X^i)^2$$

for any three-dimensional vector X , then we have the following lower bound.

$$t_\alpha t^\alpha \geq 2(g_{0i}\omega^i)^2 - g_{00}(g_{ij}\omega^i\omega^j) \left(2 + \frac{(p_0)^2 + (q_0)^2}{3p_0q_0} \right).$$

Proof. The proof is a direct calculation. Since $t^\alpha = (n_i\omega^i, -n_0\omega)$, we have

$$\begin{aligned} t_\alpha t^\alpha &= (n_0)^2(g_{ij}\omega^i\omega^j) - 2(n_i\omega^i)n_0(g_{0i}\omega^i) + g_{00}(n_i\omega^i)^2 \\ &= (n_0)^2(g_{ij}\omega^i\omega^j) + g_{00}((g_{ij}n^i\omega^j)^2 + 2(g_{ij}n^i\omega^j)n^0(g_{0i}\omega^i) + (n^0)^2(g_{0i}\omega^i)^2) \\ &\quad - 2((g_{ij}n^i\omega^j)n_0(g_{0i}\omega^i) + n^0n_0(g_{0i}\omega^i)^2), \end{aligned} \tag{2.9}$$

where we used $n_i\omega^i = g_{ij}n^i\omega^j + n^0(g_{0i}\omega^i)$. There are six terms on the right-hand side of (2.9). The first two of these terms can be estimated as follows using the Cauchy–Schwarz inequality:

$$(n_0)^2(g_{ij}\omega^i\omega^j) + g_{00}(g_{ij}n^i\omega^j)^2 \geq (n_0)^2(g_{ij}\omega^i\omega^j) + g_{00}(g_{ij}n^in^j)(g_{ij}\omega^i\omega^j).$$

We continue the estimation as follows:

$$\begin{aligned} &(n_0)^2(g_{ij}\omega^i\omega^j) + g_{00}(g_{ij}n^i\omega^j)^2 \\ &\geq (g_{ij}\omega^i\omega^j)((p_0)^2 + (q_0)^2 + 2p_0q_0 + g_{00}(g_{ij}p^ip^j) + g_{00}(g_{ij}q^iq^j) + 2g_{00}(g_{ij}p^iq^j)) \\ &= (g_{ij}\omega^i\omega^j)((g_{0i}p^i)^2 + (g_{0i}q^i)^2 - 2g_{00} + 2p_0q_0 + 2g_{00}(g_{ij}p^iq^j)), \end{aligned} \tag{2.10}$$

using

$$p_0 = -\sqrt{(g_{0i}p^i)^2 - g_{00}((g_{ij}p^ip^j) + 1)}$$

and the analogous formula for q_0 . The fourth and the sixth terms on the right-hand side of (2.9) are calculated as follows:

$$\begin{aligned} &(g_{0i}\omega^i)^2(g_{00}(n^0)^2 - 2n^0n_0) \\ &= (g_{0i}\omega^i)^2(g_{00}(p^0)^2 + g_{00}(q^0)^2 + 2g_{00}p^0q^0 - 2p^0p_0 - 2p^0q_0 - 2q^0p_0 - 2q^0q_0) \\ &= (g_{0i}\omega^i)^2(-g_{00}(p^0)^2 - g_{00}(q^0)^2 - 2g_{00}p^0q^0 - 2p^0(g_{0i}n^i) - 2q^0(g_{0i}n^i)), \end{aligned}$$

where we used $p_0 = g_{00}p^0 + (g_{0i}p^i)$ and the analogous formula for q_0 . To proceed with the calculation, we use the formula

$$p^0 = \frac{1}{-g_{00}}(g_{0i}p^i + \sqrt{(g_{0i}p^i)^2 - g_{00}(g_{ij}p^i p^j + 1)})$$

and the corresponding formula for q^0 . After a long calculation we obtain

$$\begin{aligned} &(g_{0i}\omega^i)^2(g_{00}(n^0)^2 - 2n^0n_0) \\ &= (g_{0i}\omega^i)^2((g_{ij}p^i p^j) + (g_{ij}q^i q^j) + 2) \\ &\quad + 2(g_{0i}\omega^i)^2 \frac{1}{-g_{00}} \sqrt{(g_{0i}p^i)^2 - g_{00}((g_{ij}p^i p^j) + 1)} \sqrt{(g_{0i}q^i)^2 - g_{00}((g_{ij}q^i q^j) + 1)} \\ &\quad - 2(g_{0i}\omega^i)^2 \frac{1}{-g_{00}}(g_{0i}p^i)(g_{0i}q^i). \end{aligned} \tag{2.11}$$

The third and the fifth terms on the right-hand side of (2.9) are estimated as follows:

$$\begin{aligned} &(g_{0i}\omega^i)(g_{ij}n^i \omega^j)(2g_{00}n^0 - 2n_0) \\ &= -2(g_{0i}\omega^i)(g_{ij}n^i \omega^j)(g_{0i}n^i) \geq -2|g_{0i}\omega^i| \sqrt{g_{ij}n^i n^j} \sqrt{g_{ij}\omega^i \omega^j} |g_{0i}n^i| \\ &\geq -(g_{0i}\omega^i)^2(g_{ij}n^i n^j) - (g_{ij}\omega^i \omega^j)(g_{0i}n^i)^2, \end{aligned} \tag{2.12}$$

where we used the Cauchy–Schwarz inequality. Hence all terms on the right-hand side of (2.9) have been estimated by (2.10)–(2.12). Moreover, since $n^\alpha = p^\alpha + q^\alpha$, the first two terms on the right-hand side of (2.10) are partially cancelled by the second term on the right-hand side of (2.12), and similarly the first two terms on the right-hand side of (2.11) are partially cancelled by the first term on the right-hand side of (2.12) as follows:

$$(g_{ij}\omega^i \omega^j)((g_{0i}p^i)^2 + (g_{0i}q^i)^2 - (g_{0i}n^i)^2) = -2(g_{ij}\omega^i \omega^j)(g_{0i}p^i)(g_{0i}q^i) \tag{2.13}$$

and

$$\begin{aligned} &(g_{0i}\omega^i)^2((g_{ij}p^i p^j) + (g_{ij}q^i q^j) - (g_{ij}n^i n^j)) \\ &= -2(g_{0i}\omega^i)^2(g_{ij}p^i q^j) \geq -2(g_{0i}\omega^i)^2 \sqrt{g_{ij}p^i p^j} \sqrt{g_{ij}q^i q^j}. \end{aligned} \tag{2.14}$$

As a consequence, (2.9) is estimated by (2.13)–(2.14) and the other terms of (2.10)–(2.11), i.e.

$$\begin{aligned} t_\alpha t^\alpha &\geq (g_{ij}\omega^i \omega^j)(-2g_{00} + 2p_0q_0 + 2g_{00}(g_{ij}p^i q^j) - 2(g_{0i}p^i)(g_{0i}q^i)) \\ &\quad + (g_{0i}\omega^i)^2(-2\sqrt{g_{ij}p^i p^j} \sqrt{g_{ij}q^i q^j} + 2) \\ &\quad + 2(g_{0i}\omega^i)^2 \frac{1}{-g_{00}} \sqrt{(g_{0i}p^i)^2 - g_{00}((g_{ij}p^i p^j) + 1)} \sqrt{(g_{0i}q^i)^2 - g_{00}((g_{ij}q^i q^j) + 1)} \\ &\quad - 2(g_{0i}\omega^i)^2 \frac{1}{-g_{00}}(g_{0i}p^i)(g_{0i}q^i) \end{aligned}$$

and we rewrite it as follows.

$$\begin{aligned}
 t_\alpha t^\alpha &\geq -2(g_{ij}\omega^i\omega^j)g_{00} + 2(g_{0i}\omega^i)^2 \\
 &\quad + 2(g_{ij}\omega^i\omega^j)(p_0q_0 + g_{00}(g_{ij}p^ip^j) - (g_{0i}p^i)(g_{0i}q^i)) \\
 &\quad + 2(g_{0i}\omega^i)^2 \frac{1}{-g_{00}} \sqrt{(g_{0i}p^i)^2 - g_{00}((g_{ij}p^ip^j) + 1)} \sqrt{(g_{0i}q^i)^2 - g_{00}((g_{ij}q^iq^j) + 1)} \\
 &\quad - 2(g_{0i}\omega^i)^2 \frac{1}{-g_{00}} (-g_{00} \sqrt{g_{ij}p^ip^j} \sqrt{g_{ij}q^iq^j} + (g_{0i}p^i)(g_{0i}q^i)). \tag{2.15}
 \end{aligned}$$

Note that the first line of (2.15) is strictly positive, and the sum of the third and the fourth lines is non-negative because of the following calculation:

$$\begin{aligned}
 &(-g_{00} \sqrt{g_{ij}p^ip^j} \sqrt{g_{ij}q^iq^j} + (g_{0i}p^i)(g_{0i}q^i))^2 \\
 &= (g_{00})^2 (g_{ij}p^ip^j)(g_{ij}q^iq^j) + (g_{0i}p^i)^2 (g_{0i}q^i)^2 \\
 &\quad - 2g_{00}(g_{0i}p^i)(g_{0i}q^i) \sqrt{g_{ij}p^ip^j} \sqrt{g_{ij}q^iq^j} \\
 &\leq (g_{00})^2 (g_{ij}p^ip^j)(g_{ij}q^iq^j) + (g_{0i}p^i)^2 (g_{0i}q^i)^2 \\
 &\quad - g_{00}((g_{0i}p^i)^2 (g_{ij}q^iq^j) + (g_{0i}q^i)^2 (g_{ij}p^ip^j)).
 \end{aligned}$$

We now have only the second line of (2.15), which is explicitly calculated as follows. Note that the coefficient $g_{ij}\omega^i\omega^j$ is strictly positive, and we have

$$\begin{aligned}
 &p_0q_0 + g_{00}(g_{ij}p^ip^j) - (g_{0i}p^i)(g_{0i}q^i) \\
 &\geq p_0q_0 - (-g_{00} \sqrt{g_{ij}p^ip^j} \sqrt{g_{ij}q^iq^j} + |g_{0i}p^i||g_{0i}q^i|) \\
 &= \frac{(p_0q_0)^2 - (-g_{00} \sqrt{g_{ij}p^ip^j} \sqrt{g_{ij}q^iq^j} + |g_{0i}p^i||g_{0i}q^i|)^2}{p_0q_0 + (-g_{00} \sqrt{g_{ij}p^ip^j} \sqrt{g_{ij}q^iq^j} + |g_{0i}p^i||g_{0i}q^i|)} \\
 &\geq \frac{1}{3p_0q_0} ((p_0q_0)^2 - (-g_{00} \sqrt{g_{ij}p^ip^j} \sqrt{g_{ij}q^iq^j} + |g_{0i}p^i||g_{0i}q^i|)^2), \tag{2.16}
 \end{aligned}$$

where we used Lemma 2.1. Note that

$$\begin{aligned}
 (p_0q_0)^2 &= (g_{0i}p^i)^2 (g_{0i}q^i)^2 \\
 &\quad - g_{00}(g_{0i}p^i)^2 (g_{ij}q^iq^j) - g_{00}(g_{0i}p^i)^2 - g_{00}(g_{0i}q^i)^2 (g_{ij}p^ip^j) - g_{00}(g_{0i}q^i)^2 \\
 &\quad + (g_{00})^2 ((g_{ij}p^ip^j)(g_{ij}q^iq^j) + g_{ij}p^ip^j + g_{ij}q^iq^j + 1)
 \end{aligned}$$

and

$$\begin{aligned}
 &(-g_{00} \sqrt{g_{ij}p^ip^j} \sqrt{g_{ij}q^iq^j} + |g_{0i}p^i||g_{0i}q^i|)^2 \\
 &= (g_{00})^2 (g_{ij}p^ip^j)(g_{ij}q^iq^j) + (g_{0i}p^i)^2 (g_{0i}q^i)^2 \\
 &\quad - 2g_{00}|g_{0i}p^i||g_{0i}q^i| \sqrt{g_{ij}p^ip^j} \sqrt{g_{ij}q^iq^j}
 \end{aligned}$$

$$\begin{aligned} &\leq (g_{00})^2(g_{ij}p^i p^j)(g_{ij}q^i q^j) + (g_{0i}p^i)^2(g_{0i}q^i)^2 \\ &\quad - g_{00}(g_{0i}p^i)^2(g_{ij}q^i q^j) - g_{00}(g_{0i}q^i)^2(g_{ij}p^i p^j). \end{aligned}$$

Then, (2.16) is estimated as follows:

$$\begin{aligned} &p_0 q_0 + g_{00}(g_{ij}p^i q^j) - (g_{0i}p^i)(g_{0i}q^i) \\ &\geq \frac{1}{3p_0 q_0}(-g_{00}(g_{0i}p^i)^2 - g_{00}(g_{0i}q^i)^2 + (g_{00})^2(g_{ij}p^i p^j + g_{ij}q^i q^j + 1)) \\ &\geq \frac{-g_{00}}{6p_0 q_0}((p_0)^2 + (q_0)^2), \end{aligned} \tag{2.17}$$

where we used the explicit formula (1.5) for p_0 and q_0 . We apply the estimate (2.17) to (2.15), and obtain the desired lower bound. \square

Remark 2.4. In the special relativistic case, we have a more refined result,

$$t_\alpha t^\alpha \geq 2 + |p \times \omega|^2 + |q \times \omega|^2 + \frac{1 + |p|^2 + |q|^2}{\sqrt{1 + |p|^2} \sqrt{1 + |q|^2}},$$

which was obtained in [9]. This inequality implies that $t_\alpha t^\alpha \geq p^0/q^0$ in the special relativistic case, and Lemma 2.3 shows that a similar inequality holds in a curved spacetime, i.e.

$$t_\alpha t^\alpha \geq C p_0/q_0 \quad \text{or} \quad t_\alpha t^\alpha \geq C p^0/q^0 \tag{2.18}$$

for some positive constant C .

We apply the inequality (2.18) to the result of Lemma 2.2, and obtain the following estimate. Under the same assumptions as in Lemma 2.2, we have

$$\begin{aligned} \left| D^k \left[2 \frac{t_\beta q^\beta}{t_\beta t^\beta} t^\alpha \right] \right| &\leq (p^0)^{-1-2|k|} \sum_{i=0}^{2+2|k|} (p^0)^{i+|\bar{k}|} (q^0)^{4+3|k|-i} |g_i^{(|\bar{k}|)}| \\ &\leq C (p^0)^{1+|\bar{k}|} (q^0)^{4+3|k|} |g^{(|\bar{k}|)}| \end{aligned}$$

for some $g^{(|\bar{k}|)}$ and a positive constant C . Since p'^α and q'^α are parametrized by (2.7) and it can be easily shown that $D^k p^\alpha = (p^0)^{1-|k|} g^{(|\bar{k}|)}$, we obtain the following result on derivatives of the post-collisional momenta.

Lemma 2.5. *Let t^α be a four-vector defined by (2.6) for some $\omega \in S^2$, and p'^α and q'^α be two post-collisional momenta parametrized by (2.7). Suppose that there exists a small ε such that the metric $g_{\alpha\beta}$ satisfies*

$$|g_{\alpha\beta} - \eta_{\alpha\beta}| \leq \varepsilon \quad \text{and} \quad (1 - \varepsilon) \sum_{i=1}^3 (X^i)^2 \leq g_{ij} X^i X^j \leq (1 + \varepsilon) \sum_{i=1}^3 (X^i)^2$$

for any three-dimensional vector X . Then, derivatives of p'^α and q'^α can be estimated as follows:

$$|D^k p'^\alpha| + |D^k q'^\alpha| \leq C (p^0)^{1+|\bar{k}|} (q^0)^{4+3|k|} |g^{(|\bar{k}|)}|$$

for some $g^{(|\bar{k}|)}$ and $D^k = D_x^{\bar{k}} D_p^{\hat{k}}$ with multi-index $k = \bar{k} + \hat{k}$.

2.3. The $\mu - N$ regularity of the collision operator

In [3], the authors introduced a regularity property of Q called $\mu - N$ regularity, and showed that the Cauchy problem for the EB system is well-posed when Q satisfies it. The collision operator Q is said to satisfy $\mu - N$ regularity if there exists a constant C satisfying for each $\hat{\omega}_t$,

$$\left\| \frac{1}{p^0} Q(f, f)(t) \right\|_{H_{\mu, N}(\hat{\omega}_t)} \leq C \|f(t)\|_{H_{\mu, N}(\hat{\omega}_t)}^2.$$

To obtain $\mu - N$ regularity, the collision cross-section S should satisfy suitable conditions, and by following the calculations of [3], we can see that the following inequality also holds under the same conditions on S :

$$\left\| \frac{1}{p^0} Q(f, g)(t) \right\|_{H_{\mu, N}(\hat{\omega}_t)} \leq C \|f(t)\|_{H_{\mu, N}(\hat{\omega}_t)} \|g(t)\|_{H_{\mu, N}(\hat{\omega}_t)}, \tag{2.19}$$

where $Q(f, g)$ is defined by

$$Q(f, g) = \int_{\mathbb{R}^3} \int_{S^2} S(x, p, q, \Omega) (f(p')g(q') - f(p)g(q)) d\Omega \frac{|g|^{\frac{1}{2}}}{-q_0} dq.$$

In the previous section, we obtained a new estimate of $D^k p'^\alpha$ and $D^k q'^\alpha$ in Lemma 2.5. This leads to a corresponding new set of conditions on S which imply the $\mu - N$ regularity of Q . Note that high-order derivatives $D_{x,p}^k [f(x, p')]$ are linear combinations of the following quantities:

$$(D^i f)(x, p') (D^{j_1} p') \cdots (D^{j_i} p') \quad \text{with } j_1 + \cdots + j_i = |k|, \tag{2.20}$$

where D^i denotes some differential operator $D_{x,p}^r$ satisfying $|r| = i$. Lemma 2.5 gives the estimate

$$|D_{x,p}^k [f(x, p')]| \leq C \sum |(D^i f)(x, p')| (p^0)^{i+|\bar{k}|} (q^0)^{4i+3|\bar{k}|} |g^{(|\bar{k}|)}|,$$

where the sum is over all the possible i satisfying (2.20). Consider the following quantity which arises from estimation of the gain term Q_+ . Let the multi-indices k, r and s satisfy $k + r + s = l$ with $|l| \leq \mu$:

$$\begin{aligned} & \left(\frac{1}{p^0} \iint (D_{x,p}^k S) D_{x,p}^r [f(p')] D_{x,p}^s [f(q')] d\Omega \frac{|g|^{\frac{1}{2}}}{-q_0} dq \right)^2 \\ & \leq C \sum \left(\iint |D_{x,p}^k S| |D^i f(p')| (p^0)^{i+|\bar{r}|} (q^0)^{4i+3|\bar{r}|} |g^{(|\bar{r}|)}| \right. \\ & \quad \left. \times |D^j f(q')| (p^0)^{j+|\bar{s}|} (q^0)^{4j+3|\bar{s}|} |g^{(|\bar{s}|)}| d\Omega \frac{1}{p_0 q_0} dq \right)^2 \\ & \leq C \sum \iint h_l^{-2}(p) h_i^2(p') |D^i f(p')|^2 |g^{(|\bar{r}|)}|^2 h_j^2(q') |D^j f(q')|^2 |g^{(|\bar{s}|)}|^2 d\Omega \frac{p'_0 q'_0}{p_0 q_0} dq \\ & \quad \times \iint |D_{x,p}^k S|^2 h_l^2(p) h_i^{-2}(p') h_j^{-2}(q') (p^0)^{2(i+j)+2|\bar{r}+\bar{s}|} (q^0)^{8(i+j)+6|\bar{r}+\bar{s}|} \frac{d\Omega dq}{p'_0 q'_0 p_0 q_0}, \end{aligned}$$

and if the second integral is bounded, then we multiply the first integral by the weight function $h_i^2(p)$ and integrate it over x and p with $\frac{p'_0 q'_0}{p_0 q_0} dp dq = dp' dq'$, to obtain the inequality (2.19) when $\mu \geq 5$. Hence, the boundedness of the second integral should be the condition on S for the $\mu - N$ regularity of Q . The second integral is estimated as follows. In the case where N is finite,

$$\begin{aligned} & \iint |D_{x,p}^k S|^2 h_i^2(p) h_i^{-2}(p') h_j^{-2}(q') (p^0)^{2(i+j)+2|\overline{r+s}|} (q^0)^{8(i+j)+6|\overline{r+s}|} \frac{d\Omega dq}{p'_0 q'_0 p_0 q_0} \\ & \leq C \iint |D_{x,p}^k S|^2 \frac{(p^0)^{N+2|\hat{l}|}}{(p^0)^{N+2\hat{i}} (q^0)^{N+2\hat{j}}} (p^0)^{2|r+s|+2|\overline{r+s}|-2} (q^0)^{14|r+s|-1} d\Omega dq \\ & \leq C \iint |D_{x,p}^k S|^2 (p^0)^{2|\hat{l}|} (p^0)^{2|r+s|+2|\overline{r+s}|-2} (q^0)^{14|r+s|-1} d\Omega dq \\ & \leq C \iint |D_{x,p}^k S|^2 (p^0)^{2|\hat{k}|} (p^0)^{4|r+s|-2} (q^0)^{14|r+s|-1} d\Omega dq, \end{aligned} \tag{2.21}$$

where we used $-p_0 \leq Cp'_0 q'_0$ and $p^0 \leq Cp^0 q^0$. For the case $N = \infty$,

$$\begin{aligned} & \iint |D_{x,p}^k S|^2 h^2(p) h^{-2}(p') h^{-2}(q') (p^0)^{2(i+j)+2|\overline{r+s}|} (q^0)^{8(i+j)+6|\overline{r+s}|} \frac{d\Omega dq}{p'_0 q'_0 p_0 q_0} \\ & \leq C \iint |D_{x,p}^k S|^2 h^{-2}(q) (p^0)^{4|r+s|-2} (q^0)^{14|r+s|-1} d\Omega dq, \end{aligned} \tag{2.22}$$

where we used $p'^0 + q'^0 = p^0 + q^0$ and $-p_0 \leq Cp'_0 q'_0$. The boundedness of the integrals (2.21) and (2.22) are the conditions on S for the gain term. The arguments for the loss term are much simpler, and it is easily shown that the condition for the gain term implies that for the loss term. Consequently, we obtain the following lemma.

Lemma 2.6 ([3]). *Let $\mu \geq 5$, and suppose that the collision cross-section S satisfies the following conditions for multi-indices $k + r = l$ satisfying $|l| \leq \mu$:*

$$\begin{aligned} & \iint |D_{x,p}^k S|^2 (q^0)^{14|r|-1} d\Omega dq \leq C(p^0)^{2-4|r|-2|\hat{k}|} \quad \text{for } N < \infty, \\ & \iint |D_{x,p}^k S|^2 e^{-2q^0} (q^0)^{14|r|-1} d\Omega dq \leq C(p^0)^{2-4|r|} \quad \text{for } N = \infty. \end{aligned}$$

Then, the collision operator Q satisfies $\mu - N$ regularity.

Remark 2.7. Note that the condition on S for finite N is stronger than that for the case $N = \infty$, and that the former condition implies the latter one:

$$\begin{aligned} & \iint |D_{x,p}^k S|^2 e^{-2q^0} (q^0)^{14|r|-1} d\Omega dq \\ & \leq \iint |D_{x,p}^k S|^2 (q^0)^{14|r|-1} d\Omega dq \leq C(p^0)^{2-4|r|-2|\hat{k}|} \leq C(p^0)^{2-4|r|}. \end{aligned}$$

In other words, if the collision cross-section S satisfies the condition for finite N , then it also satisfies the condition for the case $N = \infty$.

3. The Non-negativity Problem for the Einstein–Boltzmann System

We now consider the EB system and the question of the non-negativity of its solutions for non-negative initial data. The Cauchy problem for the EB system was first studied by Bancel and Choquet-Bruhat [2, 3]. We apply Lemma 2.6 to the results of [2, 3] to obtain the following existence theorem.

Theorem 3.1 ([2, 3]). *Suppose that Cauchy data are prescribed as*

$$g_{\alpha\beta}(0) \in H_{\mu+1}(\omega_0), \quad \partial_0 g_{\alpha\beta}(0) \in H_\mu(\omega_0), \quad f(0) \in H_{\mu,N}(\hat{\omega}_0),$$

such that

$$|g_{\alpha\beta}(0) - \eta_{\alpha\beta}| \leq \varepsilon - \delta, \quad \delta > 0.$$

If $\mu \geq 5$, $N \geq 6$, and the collision cross-section S satisfies the conditions of Lemma 2.6, then there exist a domain Ω in \mathbb{R}^4 , which admits ω_0 as a Cauchy surface, and a function f on $\hat{\Omega}$ such that

- (1) $g_{\alpha\beta} \in H_{\mu+1}(\Omega)$ and $f \in H_{\mu,N}(\hat{\Omega})$.
- (2) $g_{\alpha\beta}$ satisfies $|g_{\alpha\beta} - \eta_{\alpha\beta}| \leq \varepsilon$ on Ω .
- (3) $g_{\alpha\beta}$ and f satisfy the EB system.
- (4) $g_{\alpha\beta}$ and f induce the prescribed Cauchy data on ω and $\hat{\omega}$ respectively.

This solution is unique in Ω and depends continuously on the Cauchy data.

In this section we consider the non-negativity of the distribution function constructed in the above theorem. Note that we already have a solution $(f, g_{\alpha\beta}, \Omega)$ of the EB system, while the non-negativity problem concerns only the distribution function f . Hence, it is enough to consider the Boltzmann equation on a given curved spacetime and show non-negativity in this case. Conditions for the non-negativity of solutions of the Boltzmann equation on a given spacetime have been given by Bichteler [4] and Tadmon [20]. We will investigate to what extent the arguments for non-negativity given in [4, 20] apply to the solutions of the EB system constructed in [2, 3].

Theorem 3.1 holds for any weighted spaces with $N \geq 6$, but we will only consider the case $N = \infty$, i.e. the exponentially bounded case. First, any functions that have polynomial decay rates can be approximated by functions having exponential decay rates. Hence, non-negativity of the case $N = \infty$ implies the corresponding result in the case of finite N by suitable approximations. Second, as we checked in Remark 2.7, a collision cross-section which satisfies the condition for the case of finite N also satisfies the condition for the case $N = \infty$. So, any property that holds under the condition for the case $N = \infty$ holds under the condition for the case of finite N .

3.1. The result of Tadmon

The original method for the non-negativity problem introduced by Lu and Zhang [10] has recently been applied to the general relativistic case by Tadmon [20]. A

curved spacetime was assumed to be given, and mild solutions of the Boltzmann equation were considered, which are defined as follows. Consider the vector field $(p^\alpha, -\Gamma_{\beta\gamma}^\alpha p^\beta p^\gamma)$ as in [4], and parametrize its integral curves by $x^0 = s$ on the mass hyperboloid. Let $(X(s), P(s)) \in \mathbb{R}^3 \times \mathbb{R}^3$ denote an integral curve with $X^i(0) = x^i$ and $P(0) = p$ parametrized by s . This has the physical interpretation of a particle path. Along this curve, the Boltzmann equation is written as follows:

$$f(t, X(t), P(t)) = f(0) + \int_0^t K(f)(s, X(s), P(s)) ds, \tag{3.1}$$

where $f(0)$ is an initial datum evaluated at (x^i, p) and K denotes

$$K(f)(x, p) = \frac{1}{p^0} Q(f, f)(x, p).$$

Mild solutions are now defined as follows: *a function f is called a mild solution of the Boltzmann equation with measurable initial value $f(0)$, if f is measurable, $K(f)$ is $L^1_{loc}(\mathbb{R}_+)$ along the integral curves, and (3.1) holds.*

For the rest of this section, x will denote three-dimensional vector consisting of the spatial components x^i where

$$x^\alpha = (x^0, x^i) = (t, x) \in \mathbb{R}_+ \times \mathbb{R}^3$$

and the distribution function will be written as $f(t, x, p)$ instead of $f(x, p)$. The main theorem of [20] can be stated as follows.

Theorem 3.2. *Let f be a mild solution of the Boltzmann equation with a non-negative initial datum $f(0)$. Assume the following conditions:*

- (i) $\frac{S(t, x, p, q, \Omega)}{p^0 q_0} = \frac{S(t, x, q, p, \Omega)}{q^0 p_0}$,
- (ii) $\frac{S(t, x, p', q', \Omega)}{p'^0 q'_0} = \frac{\partial(p, q)}{\partial(p', q')} \frac{S(t, x, p, q, \Omega)}{p^0 q_0}$,
- (iii) $\text{ess sup}_{x,p} \iint_{\mathbb{R}^3 \times S^2} \frac{S q^0}{(p^0)^2 q_0} |f(t, x, q)| |g|^{\frac{1}{2}} d\Omega dq =: \alpha(t) \in L^1_{loc}(\mathbb{R}_+)$,
- (iv) $\text{ess sup}_{y,v} \frac{|l(t, y, v)| J(y, v)}{v^0} =: \beta(t) \in L^1_{loc}(\mathbb{R}_+)$,

where l is defined by

$$l(s, X(s), P(s)) := \partial_s [P^0(s) J^{-1}(X(s), P(s))]$$

with J , the Jacobian for

$$dx dp = \left| \frac{\partial(x, p)}{\partial(X(s), P(s))} \right| dX(s) dP(s) =: J(X(s), P(s)) dX(s) dP(s).$$

Then, f is non-negative.

Remark 3.3. The conditions on $\alpha(t)$ and $\beta(t)$ should be replaced by $\alpha, \beta \in L^1_{\text{loc}}(I)$ for some finite interval I , because the solutions in Theorem 3.1 are local in time. Hence, it is enough to show the boundedness of α and β on I .

In this part, we will show that, with an extra assumption, the solution of Theorem 3.1 satisfies the conditions (i)–(iv). Since, as explained below, the extra condition can be arranged by making a coordinate change, non-negativity for the EB case is ensured. We remark that the strong solutions in Theorem 3.1 satisfy the definition of mild solutions.

Non-negativity of f . We consider the conditions (i)–(iv) separately in the following.

(i)–(ii) The collision cross-section S is given by (2.1), which implies

$$S(t, x, p, q, \Omega) = S(t, x, q, p, \Omega) = S(t, x, p', q', \Omega)$$

and it is well known that the Jacobian in (ii) is given by

$$\frac{\partial(p, q)}{\partial(p', q')} = \frac{p_0 q_0}{p'_0 q'_0}.$$

Hence, the conditions (i)–(ii) can be written as $p_0/p^0 = q_0/q^0 = p'_0/p'^0$ for any p and q . However, this leads to

$$g_{0i} = 0, \tag{3.2}$$

which is not assumed in Theorem 3.1 and is the extra condition referred to above.

(iii) For the metric $g_{\alpha\beta}$ given in Theorem 3.1, its determinant $|g|$ is bounded, and q_0 and q^0 are equivalent. Therefore, (iii) is estimated as follows:

$$\begin{aligned} \alpha(t) &\leq C \frac{1}{(p^0)^2} \iint S |f(t, x, q)| d\Omega dq \\ &\leq C \frac{1}{(p^0)^2} \left(\iint S^2 e^{-2q^0} d\Omega dq \right)^{\frac{1}{2}} \left(\iint e^{2q^0} |f(t, x, q)|^2 d\Omega dq \right)^{\frac{1}{2}}, \end{aligned}$$

where e^{q^0} is the weight function for $N = \infty$ case. Since $f \in H_{\mu, N}(\hat{\Omega})$ with $\mu \geq 5$, the second integral above is bounded. The condition on S given in Lemma 2.6 implies the boundedness of the first integral, and this shows that the condition (iii) holds in Theorem 3.1.

(iv) As for the last condition, we recall that the integral curve $(X(s), P(s))$ is defined as follows. Let $X(s)$ and $P(s)$ denote for each x and p ,

$$X(s) = X(s, 0, x, p), \quad P(s) = P(s, 0, x, p)$$

and they satisfy

$$\begin{cases} \partial_s X^i(s) = \frac{P^i(s)}{P^0(s)}, & X^i(0) = x^i, \\ \partial_s P^i(s) = -\Gamma^i_{\alpha\beta} \frac{P^\alpha(s) P^\beta(s)}{P^0(s)}, & P^i(0) = p^i, \end{cases} \tag{3.3}$$

where the Christoffel symbols are evaluated at $X^\alpha(s) = (s, X(s))$. We first consider the quantity

$$\begin{aligned} \partial_s[P^0(s)] &= (\partial_{x^\alpha}P^0)(s)(\partial_sX^\alpha(s)) + (\partial_{p^i}P^0)(s)(\partial_sP^i(s)) \\ &= -\frac{P^\beta(s)P^\gamma(s)}{2P_0(s)}(\partial_\alpha g_{\beta\gamma})\frac{P^\alpha(s)}{P^0(s)} + \frac{P_i(s)}{P_0(s)}\Gamma_{\alpha\beta}^i\frac{P^\alpha(s)P^\beta(s)}{P^0(s)}, \end{aligned}$$

where we used $\partial_{x^\alpha}p^0 = -p^\beta p^\gamma \partial_\alpha g_{\beta\gamma} / (2p_0)$ and $\partial_{p^k}p^0 = -p_k/p_0$ as in the proof of Lemma 2.2 with Eq. (3.3). Since $g_{\alpha\beta} \in H_{\mu+1}(\Omega)$ with $\mu \geq 5$, $\partial_\alpha g_{\beta\gamma}$ and $\Gamma_{\alpha\beta}^i$ are bounded. Lemma 2.1 then gives the estimate

$$\partial_s[P^0(s)] \leq CP^0(s). \tag{3.4}$$

Consider now $\partial(X(s), P(s))/\partial(x, p)$. We need the following calculations. Differentiate the first equation of (3.3) with respect to x^j to obtain

$$\begin{aligned} \partial_s\partial_{x^j}X^i(s) &= \frac{\partial_{x^j}P^i(s)}{P^0(s)} - \frac{P^i(s)}{(P^0(s))^2}\partial_{x^j}[P^0(s)] \\ &= \frac{\partial_{x^j}P^i(s)}{P^0(s)} + \frac{P^i(s)}{(P^0(s))^2} \\ &\quad \times \left(\frac{P^\beta(s)P^\gamma(s)}{2P_0(s)}(\partial_\alpha g_{\beta\gamma})(\partial_{x^j}X^\alpha(s)) + \frac{P_i(s)}{P_0(s)}(\partial_{x^j}P^i(s)) \right) \end{aligned}$$

and as a consequence we obtain the estimate

$$\partial_s|\partial_x X(s)| \leq C|\partial_x X(s)| + C\frac{1}{P^0(s)}|\partial_x P(s)|. \tag{3.5}$$

Similarly we have

$$\partial_s|\partial_p X(s)| \leq C|\partial_p X(s)| + C\frac{1}{P^0(s)}|\partial_p P(s)|. \tag{3.6}$$

Differentiate the second equation of (3.3) with respect to x^j to obtain

$$\begin{aligned} \partial_s\partial_{x^j}P^i(s) &= -(\partial_\gamma\Gamma_{\alpha\beta}^i)(\partial_{x^j}X^\gamma(s))\frac{P^\alpha(s)P^\beta(s)}{P^0(s)} \\ &\quad - 2\Gamma_{\alpha\beta}^i\frac{P^\beta(s)}{P^0(s)}\partial_{x^j}[P^\alpha(s)] + \Gamma_{\alpha\beta}^i\frac{P^\alpha(s)P^\beta(s)}{(P^0(s))^2}\partial_{x^j}[P^0(s)]. \end{aligned}$$

By similar arguments, we have

$$\partial_s|\partial_x P(s)| \leq CP^0(s)|\partial_x X(s)| + C|\partial_x P(s)| \tag{3.7}$$

and similarly again

$$\partial_s|\partial_p P(s)| \leq CP^0(s)|\partial_p X(s)| + C|\partial_p P(s)|. \tag{3.8}$$

Due to (3.4), the inequalities (3.5)–(3.8) can be combined as follows:

$$\begin{aligned} \partial_s[|P^0(s)\partial_x X(s)| + |\partial_x P(s)|] &\leq C(|P^0(s)\partial_x X(s)| + |\partial_x P(s)|), \\ \partial_s[|P^0(s)\partial_p X(s)| + |\partial_p P(s)|] &\leq C(|P^0(s)\partial_p X(s)| + |\partial_p P(s)|). \end{aligned}$$

We now use Grönwall’s lemma with the conditions $X(0) = x$ and $P(0) = p$ to obtain

$$\begin{aligned} |P^0(s)\partial_x X(s)| + |\partial_x P(s)| &\leq C(|P^0(0)\partial_x X(0)| + |\partial_x P(0)|) \leq Cp^0, \\ |P^0(s)\partial_p X(s)| + |\partial_p P(s)| &\leq C(|P^0(0)\partial_p X(0)| + |\partial_p P(0)|) \leq C, \end{aligned}$$

which imply the estimates

$$\begin{aligned} |\partial_x X(s)| &\leq C \frac{p^0}{P^0(s)}, & |\partial_p X(s)| &\leq C \frac{1}{P^0(s)}, \\ |\partial_x P(s)| &\leq Cp^0, & |\partial_p P(s)| &\leq C. \end{aligned} \tag{3.9}$$

If we apply (3.9) to (3.5)–(3.8) again, we can see that s -derivatives of the above quantities satisfy the same estimates:

$$\begin{aligned} \partial_s |\partial_x X(s)| &\leq C \frac{p^0}{P^0(s)}, & \partial_s |\partial_p X(s)| &\leq C \frac{1}{P^0(s)}, \\ \partial_s |\partial_x P(s)| &\leq Cp^0, & \partial_s |\partial_p P(s)| &\leq C. \end{aligned} \tag{3.10}$$

On the other hand, we can consider the following integral curves. Let $Y(s)$ and $V(s)$ denote for each y and v ,

$$Y(s) = Y(s, t, y, v), \quad V(s) = V(s, t, y, v)$$

and they satisfy

$$\begin{aligned} \partial_s Y^i(s) &= \frac{V^i(s)}{V^0(s)}, & Y^i(t) &= y^i, \\ \partial_s V^i(s) &= -\Gamma_{\alpha\beta}^i \frac{V^\alpha(s)V^\beta(s)}{V^0(s)}, & V^i(t) &= v^i, \end{aligned}$$

where the Christoffel symbols are evaluated at $Y^\alpha(s) = (s, Y(s))$. Then, by the same arguments as above, we obtain

$$\begin{aligned} \partial_s [|V^0(s)\partial_y Y(s)| + |\partial_y V(s)|] &\leq C(|V^0(s)\partial_y Y(s)| + |\partial_y V(s)|), \\ \partial_s [|V^0(s)\partial_v Y(s)| + |\partial_v V(s)|] &\leq C(|V^0(s)\partial_v Y(s)| + |\partial_v V(s)|) \end{aligned}$$

and use Grönwall’s lemma with $Y(t) = y$ and $V(t) = v$,

$$\begin{aligned} |V^0(s)\partial_y Y(s)| + |\partial_y V(s)| &\leq C(|V^0(t)\partial_y Y(t)| + |\partial_y V(t)|) \leq Cv^0, \\ |V^0(s)\partial_v Y(s)| + |\partial_v V(s)| &\leq C(|V^0(t)\partial_v Y(t)| + |\partial_v V(t)|) \leq C \end{aligned}$$

and as a consequence the following estimates are obtained.

$$\begin{aligned} |\partial_y Y(s)| &\leq C \frac{v^0}{V^0(s)}, & |\partial_v Y(s)| &\leq C \frac{1}{V^0(s)}, \\ |\partial_y V(s)| &\leq Cv^0, & |\partial_v V(s)| &\leq C. \end{aligned} \tag{3.11}$$

We now consider the quantity

$$l(s, X(s), P(s)) = \partial_s [P^0(s)]J^{-1}(X(s), P(s)) + P^0(s)\partial_s [J^{-1}(X(s), P(s))],$$

from which we have for $y = X(t)$ and $v = P(t)$,

$$l(t, y, v)J(y, v) = \partial_s[P^0(s)]_{s=t} + v^0 J(y, v)\partial_s[J^{-1}(X(s), P(s))]_{s=t}. \tag{3.12}$$

By applying (3.4) to the first quantity above, we obtain

$$|\partial_s[P^0(s)]_{s=t}| \leq Cv^0. \tag{3.13}$$

Note that $J(y, v)$ is written as

$$J(y, v) = J(X(s), P(s))_{s=t} = \left| \frac{\partial(x, p)}{\partial(X(s), P(s))} \right|_{s=t} = \left| \frac{\partial(Y(s), V(s))}{\partial(y, v)} \right|_{s=0},$$

where we used

$$x = Y(0, t, X(t), P(t)) = Y(0, t, y, v) = Y(0) \quad \text{and similarly } p = V(0).$$

Thanks to the multilinearity of determinant, $J(y, v)$ is estimated by (3.11) as follows:

$$J(y, v) \leq C \left(\frac{v^0}{V^0(s)} \right)_{s=0}^3 = C \left(\frac{v^0}{p^0} \right)^3. \tag{3.14}$$

On the other hand, as for $J^{-1}(X(s), P(s))$, we have the same estimates for $\partial_x X(s)$ and $\partial_s \partial_x X(s)$, $\partial_p X(s)$ and $\partial_s \partial_p X(s)$ and so on, as in (3.9) and (3.10). Thanks to the multilinearity again, the following estimate is obtained:

$$|\partial_s[J^{-1}(X(s), P(s))]|_{s=t} \leq C \left(\frac{p^0}{P^0(s)} \right)_{s=t}^3 = C \left(\frac{p^0}{v^0} \right)^3. \tag{3.15}$$

We combine (3.13)–(3.15) and (3.12) to obtain

$$|l(t, y, v)|J(y, v) \leq Cv^0.$$

This gives boundedness of β on I , and therefore it is proved that the condition (iv) holds in Theorem 3.1.

It remains to discuss the extra condition $g_{0i} = 0$ which is required to ensure that the conditions of Tadmon’s theorem are satisfied. We are concerned here with a solution on a local region, where the components of the metric in a certain coordinate system are close to those of the Minkowski metric. Let the original coordinates be (t, x^i) . Now choose new coordinates (t, \tilde{x}^i) with the properties that \tilde{x}^i agrees with x^i for $t = 0$ and the x^i are constant along the integral curves of the unit normal vector field to the family of hypersurfaces of constant t . Then the components of the metric in the new coordinate system satisfy the desired condition. The new coordinates might not be defined on exactly the same region as the old ones so that non-negativity is obtained on a slightly smaller domain. Given the fact that the non-negativity theorem being discussed here is local in nature, this is not a major disadvantage.

3.2. The result of Bichteler and a new approach

In Bichteler’s paper [4], it was proved that under certain hypotheses a local solution of the Boltzmann equation exists on a given curved spacetime, and that it is non-negative. A four-dimensional manifold M and a Lorentzian metric $g_{\alpha\beta}$ were assumed to be given, and the Boltzmann equation was written in the form

$$\mathcal{L}_X f = \iiint W(12 \rightarrow 34)\delta(1 + 2 - 3 - 4)(f(3)f(4) - f(1)f(2))d2 d3 d4, \quad (3.16)$$

where 1, 2, 3 and 4 stand for p^α , q^α , p'^α and q'^α respectively. The left-hand side of (3.16) means that the derivative of f along the integral curves of the vector field $X = (p^\alpha, -\Gamma_{\beta\gamma}^\alpha p^\beta p^\gamma)$ exists at almost all points of the domain of f . The transition rate W is given as

$$W(12 \rightarrow 34) = k\sigma(12 \rightarrow 34),$$

where k is some kinematical factor and σ can be regarded as the same quantity as the scattering kernel defined in (2.1). The volume forms $d2$, $d3$ and $d4$ are same as in our case:

$$d2 = \frac{|g|^{\frac{1}{2}}}{-q_0} dq$$

and $d3$ and $d4$ are similarly defined. The main theorem of [4] is the following.

Theorem 3.4 ([4]). *Let $f(0)$ be a measurable function on $\hat{\omega}_0$, and suppose that there exists a continuous timelike vector field $\beta^\alpha(x)$ on ω_0 such that*

$$f(x, p) \leq C e^{\beta_\alpha(x)p^\alpha} \quad \text{on } \hat{\omega}_0$$

and the scattering kernel satisfies the following property:

$$\iint \sigma(12 \rightarrow 34)\delta(1 + 2 - 3 - 4)d3d4 \leq \text{const.} \quad (3.17)$$

Then, there exists a solution f to the Boltzmann equation (3.16) on a domain $\hat{\Omega}$, which is again exponentially bounded. Moreover, if $f(0) \geq 0$, then also $f \geq 0$.

The condition (3.17) can be stated in our notation as

$$\int_{S^2} \sigma(\varrho, \theta)d\Omega \leq \text{const.} \quad (3.18)$$

The corresponding conditions on the scattering kernel for the EB case are given in Lemma 2.6 with the relation (2.1) between the scattering kernel and the collision cross-section. Although the conditions in Lemma 2.6 are much more complicated than (3.18), we can see that they do not imply (3.18). Hence Theorem 3.2 cannot be applied directly to prove Theorem 3.1.

If there exists a solution of the Boltzmann equation with non-negative initial data which becomes negative at some point in phase space then it is natural to follow the particle path through that point backwards in time. This curve must

meet the Cauchy surface and thus there is at least one point on it where f is non-negative. Hence there exists a point (t_*, x_*, p_*) on the curve where it is zero and immediately afterwards negative. At this point the derivative of f in the direction of the particle path is non-negative because the loss term vanishes. It might however be zero. This type of consideration plays a role in the proof of positivity in [4].

A small modification of this idea can be used to give a relatively simple proof of non-negativity for the EB system. We first introduce the following notation,

$$\mathcal{L} = \frac{\partial}{\partial t} + \frac{p^i}{p^0} \frac{\partial}{\partial x^i} - \Gamma_{\alpha\beta}^i \frac{p^\alpha p^\beta}{p^0} \frac{\partial}{\partial p^i}$$

and write the Boltzmann equation as $\mathcal{L}f = (p^0)^{-1}Q(f, f)$. We then modify the Boltzmann equation by adding a small quantity on the right-hand side.

$$\mathcal{L}f_\eta = \frac{1}{p^0}Q(f_\eta, f_\eta) + \eta e^{-|p|^2}, \tag{3.19}$$

where $\eta > 0$ is a small parameter. The above equation is understood to be defined on $\hat{\Omega}$ with a metric $g_{\alpha\beta}$ constructed as in Theorem 3.1. Since the quantity $\eta e^{-|p|^2}$ is sufficiently smooth and square integrable on $\hat{\Omega}$, existence of solutions is easily proved by the same argument of [3] using the following lemma.

Lemma 3.5 ([2]). *Let $g_{\alpha\beta}$ and $\hat{\Omega}$ be the metric and the domain given in Theorem 3.1, and consider the Cauchy problem for $\mathcal{L}f = g$ with $f(0)$ defined on $\hat{\omega}_0$. Then, for any $0 \leq s \leq t$, there exists a solution in $H_{\mu,N}(\hat{\Omega})$ satisfying the energy inequality,*

$$\|f(s)\|_{H_{\mu,N}(\hat{\omega}_s)}^2 \leq C \left(\|f(0)\|_{H_{\mu,N}(\hat{\omega}_0)}^2 + \int_0^t \|g(\tau)\|_{H_{\mu,N}(\hat{\omega}_\tau)}^2 d\tau \right),$$

where C is a constant depending only on $g_{\alpha\beta}$, Ω and μ .

As a result, we have two distribution functions f and f_η in a common domain $\hat{\Omega}$, and we can see that f_η is non-negative by reasoning as above. For in the case of the modified equation the derivative of f along the particle path is strictly positive at the point (t_*, x_*, p_*) , a contradiction. Non-negativity of f is now proved by showing continuous dependence on η for f_η .

Non-negativity of f . We subtract the modified equation from the original one and write $F = f - f_\eta$ to get

$$\mathcal{L}F = \frac{1}{p^0}Q(f, f) - \frac{1}{p^0}Q(f_\eta, f_\eta) - \eta e^{-|p|^2}.$$

By applying Lemma 3.5, we obtain the following integral inequality:

$$\|F(t)\|_{H_{\mu,N}(\hat{\omega}_t)}^2 \leq C \left(\eta + \int_0^t \left\| \frac{1}{p^0}Q(f, f)(\tau) - \frac{1}{p^0}Q(f_\eta, f_\eta)(\tau) \right\|_{H_{\mu,N}(\hat{\omega}_\tau)}^2 d\tau \right),$$

where we used $F(0) = f(0) - f_\eta(0) \equiv 0$. From the bilinearity of Q , the above difference of collision terms can be written as

$$Q(f, f) - Q(f_\eta, f_\eta) = Q(f, F) + Q(F, f_\eta).$$

We now apply Lemma 2.6 to obtain

$$\|F(t)\|_{H^{\mu, N}(\hat{\omega}_t)}^2 \leq C \left(\eta + \int_0^t \|F(\tau)\|_{H^{\mu, N}(\hat{\omega}_\tau)}^2 d\tau \right),$$

which implies that $\lim_{\eta \rightarrow 0} \|F(t)\|_{H^{\mu, N}(\hat{\omega}_t)} = 0$, and consequently $\lim_{\eta \rightarrow 0} \|F(t)\|_{H^{\mu, N}(\hat{\Omega})} = 0$. Since $\mu \geq 5$, we obtain

$$\lim_{\eta \rightarrow 0} \|f - f_\eta\|_{L^\infty(\hat{\Omega})} = 0$$

and this proves non-negativity of f .

The above argument actually proves a stronger statement. Consider a situation where the initial data are not everywhere non-negative. If (t, x, p) is a point of phase space such the distribution function is non-negative at the point where the particle path through (t, x, p) meets the Cauchy surface then we can argue as before to get a contradiction. Thus a statement is obtained about the non-negativity of f at certain points of phase space.

4. Further Considerations

An important concept in general relativity is that of general covariance. In the context of mathematical relativity this means that it should be possible to express the Einstein-matter equations in a form which is invariant under diffeomorphisms. This assertion applies in particular to the EB system. The results of the previous sections are concerned with solutions of the EB system expressed in a local coordinate system. In other words they are results about the reduced Einstein equations in a harmonic coordinate system. These can be used to establish properties of the EB system which are diffeomorphism invariant by standard procedures (cf. [16, Chap. 9]). From this point of view it is natural to prove a global positivity result for the EB system and for this it is enough to show that a solution of the Boltzmann equation on a globally hyperbolic spacetime with non-negative initial data is non-negative. A problem in formulating a theorem of this kind is to identify a suitable class of collision cross-sections.

The conditions on the collision cross-section used in [3] and in the previous sections are conditions on the function S and, as such, are coordinate dependent. For a global theorem it is necessary to formulate an invariant condition and the function S is not invariant since it depends explicitly on the spacetime coordinates. A better alternative would be to formulate a condition on σ . In the formulation of the reduced EB system using a coordinate basis the function σ still depends on the metric components. If instead the EB system is formulated in an orthonormal frame in the way explained in the introduction this dependence on the metric is eliminated.

This formulation also has an advantage for the consideration of the question of what collision cross-sections are physically reasonable. When the orthonormal frame approach is used the scattering kernel in general relativity is identical to that in special relativity. Thus the problem of identifying physically reasonable scattering kernels is reduced to the corresponding problem in special relativity which is better understood.

What has just been said provides a strong motivation for considering the reduced system in the orthonormal frame formulation in more detail. The method of proof of the theorems of [3] extends easily to this case. The inequalities on the derivatives $D_{x,p}^k S$ in Lemma 2.6 are replaced by the analogous estimates for $D_v^k S$ which are actually somewhat simpler. In fact there is one estimate which is necessary for those proofs which is not included in [3]. In the iteration used in that paper there is a new metric in each step of the iteration. In estimating the differences of iterates it is in particular necessary to estimate the change resulting from making a change of the metric in the collision term. This is not mentioned in [3]. The orthonormal frame formulation eliminates this problem since in that case the collision term does not depend on the metric components. The conditions on $D_v^k S$ which are analogous to those of Lemma 2.6 are coordinate independent and are thus appropriate for formulating a global theorem. Let us call these the orthonormal frame regularity conditions. From the point of view of comparing with physical conditions on the collision term it would be of interest to know what conditions on σ are required to imply the orthonormal frame regularity conditions. For instance, as a simple case an assumption on the support of ϱ can be made. Let us consider the orthonormal frame regularity conditions in more detail. The conditions of Lemma 2.6 are replaced as follows for the case $N = \infty$. Under the same conditions on μ, k and r , the following inequality should hold:

$$\iint |\partial_\varrho^i \hat{\sigma}(\varrho, \theta)|^2 \varrho^{2i-4|k'|} e^{-2q^0} (q^0)^{14|r|+2|k|} (p^0)^{4|r|-1} d\Omega dq \leq C,$$

where $i \leq |k'| \leq |k|$ and, motivated by the form of the expression $S = \lambda \varrho \sigma$, the quantity $\hat{\sigma} = \varrho \sigma$ has been introduced. Suppose now that σ is a smooth function and has support contained in the set defined by $m \leq \varrho \leq M$ for some positive numbers m and M . Then, the term $\varrho^{2i-4|k'|}$ is clearly bounded. Moreover, the condition $\varrho \leq M$ gives the following estimate. Since we are working in an orthonormal frame, it implies that

$$p^0 \leq \frac{1}{q^0} (q \cdot p + M') \leq \frac{1}{q^0} (|q||p| + M') \quad \text{for } M' = \frac{1}{2} M^2 + 1.$$

After some calculations and using the inequality $|p| \leq p^0$ we obtain $|p| \leq 2M'q^0$, and this implies in turn that $p^0 \leq Cq^0$ for some C . Hence the orthonormal frame regularity conditions hold.

Consider a solution f of the Boltzmann equation on a globally hyperbolic space-time and suppose that the scattering kernel satisfies the orthonormal frame regularity conditions. If the initial data are non-negative then f is non-negative. We

suppose that the statement is false and obtain a contradiction. If the statement is false there is a point (t_1, x_1, p_1) with $f(t_1, x_1, p_1) < 0$. Let γ be the particle path passing through (t_1, x_1, p_1) . If (t, x, p) is a point of γ sufficiently close to the initial hypersurface then local coordinates can be defined such that (t, x) is contained in their domain of definition and the conditions of the theorem of [3] are satisfied. Hence $f(t, x, p) \geq 0$ close to the initial hypersurface. Let (t_2, x_2, p_2) be a point on γ with $f(t, x, p) \geq 0$ for all points on γ with $t \leq t_2$ and $f(t, x, p) < 0$ for some points on γ with t arbitrarily close to t_2 . We can choose a local Cauchy surface through the point (t_2, x_2) and a local coordinate system on a neighborhood of (t_2, x_2) such that the conditions of the theorem of [3] are satisfied. It then follows by the result stated at the end of Sec. 3 that $f \geq 0$ at all points on γ with t slightly greater than t_2 , a contradiction.

It should be noted that as soon as results on well-posedness of the Cauchy problem for the EB system can be extended to wider classes of scattering kernels the positivity result can also be extended, provided the theorem includes a statement about continuous dependence of the solution on the scattering kernel in a suitable sense. For then it suffices to approximate a scattering kernel of the new class by a sequence of kernels of the class previously treated.

We hope that by clarifying a number of issues the results of this paper will contribute to the development of a mature theory of the local and global Cauchy problem for the EB system in the near future.

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