Loop quantum cosmology of the Bianchi I model: complete quantization

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Abstract. We complete the canonical quantization of the vacuum Bianchi I model within the improved dynamics scheme of loop quantum cosmology, characterizing the Hilbert structure of the physical states and providing a complete set of observables acting on them. In order to achieve this task, it has been essential to determine the structure of the separable superselection sectors that arise owing to the polymeric quantization, and to prove that the initial value problem obtained when regarding the Hamiltonian constraint as an evolution equation, interpreting the volume as the evolution parameter, is well-posed.

1. Introduction

Loop quantum cosmology (LQC) [1, 2, 3] adapts the techniques of loop quantum gravity [4] in the quantization of models with high degree of symmetry, such us homogeneous models. Remarkably, the quantization of (homogeneous and isotropic) Friedman-Lemaître-Robertson-Walker models within (the improved dynamics of) LQC succeeds in solving the singularity problem: the classical big bang turns out to be replaced by a quantum bounce happening at Planck scales, and no observable diverges in the quantum theory, as shown for the first time in the seminal work of reference [5].

In order to extent this quantization to (homogeneous but anisotropic) Bianchi models, we have focused our attention in the simplest case: the vacuum Bianchi I model. Our interest in this model also comes from the necessity of including inhomogeneities in LQC. Actually, with that aim we have analyzed from the LQC perspective the simplest inhomogeneous cosmologies: the linearly polarized Gowdy model with three-torus topology [6, 7]. This model can be regarded as a homogeneous Bianchi I background with three-torus topology filled with linearly polarized gravitational waves traveling in a single direction. The above quantization of the Gowdy model employs the polymeric quantization of the Bianchi I model when representing the homogeneous background and, therefore, first we need to have under control the Bianchi I model itself.

The quantization of the Bianchi I model in LQC is subject to several ambiguities. One of them concerns the representation of the curvature tensor of the connection in the quantum theory. Different definitions of this object lead to different schemes of quantization, being the most satisfactory one the *improved dynamics* put forward in Ref. [8], where although the Hamiltonian constraint was constructed, no further analysis of the physical solutions was carried out.

In this note we further study this quantization of the Bianchi I model. It summarizes some results already presented in [6, 7]. We will start by adopting a different factor ordering of that of [8] when symmetrizing the Hamiltonian constraint operator. With our factor ordering the operator is well-defined in the octant of $\mathbb{R}^3$ given by the positive eigenvalues of the operators representing the coefficients of the densitized triad. In this way our model displays sectors of superselection that are simpler than those of [8]. We will analyze the structure of the superselection sectors (aspect not studied in [8]) and see that they are separable.

The Hamiltonian constraint provides a difference equation in the volume. Then, it seems natural to interpret it as an internal time and the constraint as an evolution equation with respect to it. We will see that this interpretation is valid inasmuch as the corresponding initial value problem is well-possed: physical solutions are completely determined by a countable set of data given in the initial section of the volume (which displays a non-vanishing minimum value). The previous result allows us to identify the physical Hilbert space with the Hilbert space of initial data, whose inner product is determined by imposing reality conditions in a (over) complete set of physical observables.

2. Quantization of the model

We consider the vacuum Bianchi I model with three-torus spatial topology. Then, we use global coordinates $\{t, \theta, \sigma, \delta\}$, with $\theta, \sigma, \delta \in S^1$. In order to prepare the model for its loop quantization, we describe it in the Ashtekar-Barbero formalism [4]. Using a diagonal gauge, the nontrivial components of the densitized triad are $p_i/4\pi^2$, with $i = \theta, \sigma, \delta$, and $c_i/2\pi$ are those of the $su(2)$ connection. They satisfy $\{c_i, p_j\} = 8\pi G\gamma\delta_{ij}$, where γ is the Immirzi parameter and G is the Newton constant. The spacetime metric in these variables reads

$$ds^2 = -dt^2 + \frac{|p_\theta p_\sigma p_\delta|}{4\pi^2}\left(\frac{d\theta^2}{p_\theta^2} + \frac{d\sigma^2}{p_\sigma^2} + \frac{d\delta^2}{p_\delta^2}\right). \tag{1}$$

The phase space is constrained by the Hamiltonian constraint

$$C_{\rm BI} = -\frac{2}{\gamma^2}\frac{c^\theta p_\theta c^\sigma p_\sigma + c^\theta p_\theta c^\delta p_\delta + c^\sigma p_\sigma c^\delta p_\delta}{V} = 0. \tag{2}$$

In this expression, $V = \sqrt{|p_\theta p_\sigma p_\delta|}$ is the volume of the compact spatial sections.

In order to represent the phase space in the quantum theory we choose the kinematical Hilbert space of the Bianchi I model constructed in LQC (see e.g. [8]), that we call $\mathcal{H}_{\rm kin}$. We recall that, on $\mathcal{H}_{\rm kin}$, the operators $\hat{p}_i$ have a discrete spectrum equal to the real line. The corresponding eigenstates, $|p_\theta, p_\sigma, p_\delta\rangle$, form an orthonormal basis (in the discrete norm) of $\mathcal{H}_{\rm kin}$. Owing to this discreteness, there is no well-defined operator representing the connection, but rather its holonomies. They are computed along straight edges in the fiducial directions. The so-called improved dynamics prescription states that, when writing the curvature tensor in terms of holonomies, we have to evaluate them along edges with a certain minimum dynamical (state dependent) length $\bar{\mu}_i$. We use the specific improved dynamics prescription put forward in [8]: the elementary operators which represent the matrix elements of the holonomies, called $\hat{\mathcal{N}}_{\bar{\mu}_i}$, produce all a constant shift in volume V. In order to simplify the analysis, it is convenient to relabel the basis states in the form $|v, \lambda_\sigma, \lambda_\delta\rangle$, where $|v|$ is proportional to V such that the operators $\hat{\mathcal{N}}_{\pm\bar{\mu}_i}$ cause a shift on it equal to ± 1, and the parameters λ_i are all equally defined in terms of the corresponding parameters p_i, and verify that $v = 2\lambda_\theta\lambda_\sigma\lambda_\delta$.

Out of the basic operators $\hat{p}_i$, $\hat{\mathcal{N}}_{\bar{\mu}_i}$ we represent the Hamiltonian constraint as an operator $\widehat{\mathcal{C}}$. We choose a very suitable symmetric factor ordering such that $\widehat{\mathcal{C}}$ decouples the states of $\mathcal{H}_{\text{kin}}$ with support in $v = 0$, namely, the states with vanishing volume [6, 7]. Moreover, our operator $\widehat{\mathcal{C}}$ does not relate states with different orientation of any of the eigenvalues of the operators $\hat{p}_i$. Owing to this property we can then restrict the domain of definition of $\widehat{\mathcal{C}}$ to e.g. the space spanned by the states $|v > 0, \lambda_\sigma > 0, \lambda_\delta > 0\rangle$. We call the resulting Hilbert space $\mathcal{H}^+_{\text{kin}}$. Remarkably, for this restriction we do not need to impose any particular boundary condition or appeal to any parity symmetry. Note that both $\mathcal{H}_{\text{kin}}$ and $\mathcal{H}^+_{\text{kin}}$ are non-separable Hilbert spaces, feature not desirable for a physical theory. This problem is overcome by the own action of the Hamiltonian constraint operator. Indeed, $\widehat{\mathcal{C}}$ defined on $\mathcal{H}^+_{\text{kin}}$ turns out to leave invariant some separable Hilbert subspaces that provide superselection sectors. In concrete, they are spanned by the states $|v = \varepsilon + 4n, \lambda_\sigma = \lambda^\star_\sigma\omega_\varepsilon, \lambda_\delta = \lambda^\star_\delta\omega_\varepsilon\rangle$. Here $\varepsilon \in (0, 4]$ and $n \in \mathbb{N}$, and therefore v takes support in semilattices of constant step equal to 4 starting in a minimum non-vanishing value ε. In addition, $\lambda^\star_a$ $(a = \sigma, \delta)$ is some positive real number and ω_ε runs over the subset of $\mathbb{R}^+$ given by

$$\mathcal{W}_\varepsilon = \left\{ \left(\frac{\varepsilon - 2}{\varepsilon}\right)^z \prod_k \left(\frac{\varepsilon + 2m_k}{\varepsilon + 2n_k}\right)^{p_k} \right\},$$

where $m_k, n_k, p_k \in \mathbb{N}$, and $z \in \mathbb{Z}$ when $\varepsilon > 2$, while $z = 0$ otherwise. One can check that in deed this set is dense in $\mathbb{R}^+$ and countable [6]. Therefore, any of these sectors provide separable Hilbert spaces contained in $\mathcal{H}^+_{\text{kin}}$. We denote them as $\mathcal{H}_{\varepsilon,\lambda^\star_\sigma,\lambda^\star_\delta} = \mathcal{H}_\varepsilon \otimes \mathcal{H}_{\lambda^\star_\sigma} \otimes \mathcal{H}_{\lambda^\star_\delta}$. Note that the removal of the states with support in $v = 0$ means that there is no analog of the cosmological singularity (classically located in $p_i = 0$) in our quantum theory. We thus solve the singularity already at the level of superselection in a very simple way.

We then restrict the study to any of these superselection sectors, and expand a general solution in the form $(\psi| = \sum_{v\in\mathcal{L}^+_\varepsilon} \sum_{\omega_\varepsilon\in\mathcal{W}_\varepsilon} \sum_{\bar{\omega}_\varepsilon\in\mathcal{W}_\varepsilon} \psi(v, \omega_\varepsilon\lambda^\star_\sigma, \bar{\omega}_\varepsilon\lambda^\star_\delta)\langle v, \omega_\varepsilon\lambda^\star_\sigma, \bar{\omega}_\varepsilon\lambda^\star_\delta|$. One obtain that the constraint $(\psi|\widehat{\mathcal{C}}^\dagger = 0$ leads to a recurrence equation that relates the combination of states

$$\begin{aligned} \psi_+(v+4, \lambda_\sigma, \lambda_\delta) = {} & \psi\left(v+4, \lambda_\sigma, \frac{v+4}{v+2}\lambda_\delta\right) + \psi\left(v+4, \frac{v+4}{v+2}\lambda_\sigma, \lambda_\delta\right) + \psi\left(v+4, \frac{v+2}{v}\lambda_\sigma, \lambda_\delta\right) \\ & + \psi\left(v+4, \lambda_\sigma, \frac{v+2}{v}\lambda_\delta\right) + \psi\left(v+4, \frac{v+2}{v}\lambda_\sigma, \frac{v+4}{v+2}\lambda_\delta\right) + \psi\left(v+4, \frac{v+4}{v+2}\lambda_\sigma, \frac{v+2}{v}\lambda_\delta\right) \end{aligned} \tag{3}$$

with data in the previous sections v and $v - 4$. Therefore, if we regard v as an internal time the constraint can be interpreted as an evolution equation in it. It turns out that, when particularized to the initial time $v = \varepsilon$, the constraint simply gives the combination of states $\psi_+(\varepsilon + 4, \lambda_\sigma, \lambda_\delta)$ in terms of some initial data $\psi(\varepsilon, \lambda'_\sigma, \lambda'_\delta)$. From the combinations $\psi_+(\varepsilon + 4, \lambda_\sigma, \lambda_\delta)$, it is possible to determine any of the individual terms $\psi(\varepsilon + 4, \lambda_\sigma, \lambda_\delta)$, since the system of equations that relate the former ones with the latter ones is formally invertible. This issue is very non-trivial. Actually, both the separability of the support of λ_a and the fact that it is dense in $\mathbb{R}^+$ have been essential to show it (the details can be found in [7]). Therefore, the initial value problem is well-posed and it is indeed feasible to make the above interpretation of the constraint as an evolution equation in v.

In conclusion, the physical solutions of the Hamiltonian constraint are completely determined by the set of initial data $\{\psi(\varepsilon, \lambda_\sigma, \lambda_\delta) = \psi(\varepsilon, \omega_\varepsilon\lambda^\star_\sigma, \bar{\omega}_\varepsilon\lambda^\star_\delta), \, \omega_\varepsilon, \bar{\omega}_\varepsilon \in \mathcal{W}_\varepsilon\}$, and we can identify solutions with this set. We can also characterize the physical Hilbert space as the Hilbert space of the initial data. In order to endow the set of initial data with a Hilbert structure, one can take a complete set of observables forming a closed algebra, and impose that the quantum counterpart of their complex conjugation relations become adjointness relations between operators. This determines a unique (up to unitary equivalence) inner product.

Before doing that, it is most convenient changing the notation. Let us introduce the variables $x_a = \ln(\lambda_a) = \ln(\lambda_a^\star) + \rho_\varepsilon$. Note that ρ_ε takes values in a dense set of the real line, given by the logarithm of the points in the set $\mathcal{W}_\varepsilon$. We will denote that set by $\mathcal{Z}_\varepsilon$. Then, a set of observables acting on the initial data $\tilde{\psi}(x_\sigma, x_\delta) := \psi(\varepsilon, x_\sigma, x_\delta)$ is that formed by the operators $\widehat{e^{ix_a}}$ and $\widehat{U}_a^{\rho_a}$, with $\rho_a \in \mathcal{Z}_\varepsilon$ and $a = \sigma, \delta$, defined as

$$\widehat{e^{ix_\sigma}}\tilde{\psi}(x_\sigma, x_\delta) = e^{ix_\sigma}\tilde{\psi}(x_\sigma, x_\delta), \qquad \widehat{U}_\sigma^{\rho_\sigma}\tilde{\psi}(x_\sigma, x_\delta) = \tilde{\psi}(x_\sigma + \rho_\sigma, x_\delta), \tag{4}$$

and similarly for $\widehat{e^{ix_\delta}}$ and $\widehat{U}_\delta^{\rho_\delta}$. These operators provide an overcomplete set of observables and are unitary in $\mathcal{H}_{\lambda_\sigma^\star} \otimes \mathcal{H}_{\lambda_\delta^\star}$, according with their reality conditions. Therefore, we conclude that this Hilbert space is precisely the physical Hilbert space of the vacuum Bianchi I model.

3. Conclusions

Adopting the improved dynamics of reference [8], we have completed the quantization of the Bianchi I model in LQC providing the physical Hilbert space and a complete set of physical observables, task so far not achieved.

With that aim, first we have needed to determine the structure of the superselection sectors that arise in the quantum model owing to the polymeric representation of the geometry. These sectors display an involved structure concerning the anisotropies, as a consequence of the complicated action of the Hamiltonian constraint on those variables, inherent to the adopted improved dynamics. Actually, the values of the anisotropies run over a dense set of the positive real line that turns out to be countable. In contrast, the volume takes values in simple semilattices of constant step. As a result, the superselection sectors are separable.

The Hamiltonian constraint provides a difference equation in the volume, and can be interpreted as an evolution equation, being the volume the variable playing the role of the internal time. The separability of the support of the anisotropies and the fact that it is dense in $\mathbb{R}^+$ are essential to show that the associated initial value problem is well-posed, and therefore the above interpretation is valid. Since the initial data determines the solution, we can characterize the physical Hilbert space as the Hilbert space of the initial data. The physical inner product can be determined by imposing reality conditions on a complete set of observables. The result is that the physical Hilbert space coincides with the tensor product of the superselection sectors of the two anisotropy variables.

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4. References

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