PHYSICAL REVIEW D **85,** 124040 (2012)

Polycritical Gravities

Teake Nutma*

Max-Planck-Institut für Gravitationsphysik (Albert Einstein Institut), Am Mühlenberg 1, 14476 Golm, Germany

(Received 10 April 2012; published 19 June 2012)

We present higher-derivative gravities that propagate an arbitrary number of gravitons of different mass on (anti-)de Sitter backgrounds. These theories have multiple critical points, at which the masses degenerate and the graviton energies are non-negative. For six derivatives and higher there are critical points with positive energy.

DOI: 10.1103/PhysRevD.85.124040 PACS numbers: 04.50.−h

I. INTRODUCTION

Two-derivative Einstein gravity in four dimensions is nonrenormalizable. It can be made perturbatively renormalizable by adding four-derivative terms to the Lagrangian [1,2]. However, the addition of the curvature-squared terms spoils unitarity: around a Minkowski background they introduce a massive spin-0 and spin-2 mode. These massive modes have norm opposite of the massless spin-2 mode, and thus are ghosts. The spin-0 mode can be eliminated by tuning the coefficients of the curvature-squared terms, but the massive spin-2 modes cannot.

Recently, a consistent four-derivative theory of gravity in three dimensions, called "new massive gravity" was introduced in [3]. New massive gravity is ghost-free due to the fact that massless gravitons have no propagating degrees of freedom in three dimensions, which makes it possible to choose the overall sign of the action such that the massive gravitons have positive energy. This, however, is not possible in higher dimensions, as there both massive and massless gravitons propagate.

One way around this problem is to perturb around an (anti-)de Sitter [(A)dS] background, instead of a Minkowski background. The cosmological constant and the coefficients of the curvature-squared terms can then be tuned such that the massive modes becomes massless [4]. This is known as "critical gravity" [5]. As the massive modes disappear at the critical point, the theory is potentially unitary.

However, at the critical point the massive modes are replaced by so-called log modes [6–8]. As it turns out, these log modes are ghosts [9], and must the truncated to restore unitarity. As their falloff in the radial AdS coordinate is logarithmic (hence the name), this may be done by imposing certain boundary conditions.

The resulting theory is then unitary, but, unfortunately, also empty. Namely, at the critical point the energy of the massless graviton modes vanishes, together with the mass of the Schwarzschild black hole. It was recently argued from a conformal field theory perspective [10] that this is essentially due to the fact that critical gravity is of rank two (with the rank being half the number of maximum derivatives). Instead, gravity theories of odd rank should not suffer from this "zero-energy-problem."

The purpose of this paper is to investigate the criticality conditions for higher-rank theories of gravity. It is organized as follows. We first give a nonlinear Lagrangian for arbitrary rank r, that, on (A)dS backgrounds, propagates one massless and $r-1$ massive gravitons, but not the scalar ghost mode. Next, we show that the quadratic perturbation of this Lagrangian and its linear equations of motion can be concisely written in terms of the so-called *Schouten operator.* This reformulation enables us to calculate the global charges (such as black hole masses) and graviton energies for arbitrary rank. From the latter we deduce that the theory is critical, i.e., all energies are nonnegative, whenever sufficiently enough graviton masses are degenerate. In general there will be more than one critical point; hence the name *polycritical* gravities.

II. NON-LINEAR ACTION

For a gravity theory of rank r (thus containing at most $2r$ derivatives), we would like its linear equations of motion to be[1]

$$\prod_{n=0}^{r-1}(\bar{\Box} - 2\Lambda - m_n^2)h_{\mu\nu} = 0, \tag{1a}$$

$$\bar{\nabla}^{\mu}h_{\mu\nu} = 0, \tag{1b}$$

$$\bar{g}^{\mu\nu}h_{\mu\nu} = 0. \tag{1c}$$

This a straightforward generalization of the Fierz-Pauli equations of motion for a single massive graviton [11]. Here however we have r graviton modes with *a priori* different masses $m_0, m_1, \ldots, m_{r-1}$. Because these equations of motion should follow from some covariant nonlinear Lagrangian, one of the masses (say m_0) will always be zero. This is due to the diffeomorphism invariance of the nonlinear theory. We will use the index $n = 0, 1, \ldots, r-1$ to indicate all gravitons, and the index $i = 1, \ldots, r-1$ for only the massive gravitons.

*teake.nutma@aei.mpg.de

[1] See the Appendix for our conventions on linearization.

 © 2012 American Physical Society

In four dimensions and higher, a nonlinear Lagrangian whose linearized equations of motion are those given above, is

$$\mathcal{L} = \sqrt{-g}\Big[R - (d-2)(d-1)\Lambda + \frac{1}{4}C_{\mu\nu\rho\sigma}\Big(\sum_{i=1}^{r-1} a_i \Box^{i-1}\Big)C^{\mu\nu\rho\sigma}\Big]. \tag{2}$$

Here C denotes the Weyl tensor, and the coefficients $a_i = a_i(r, d, \Lambda, m_i)$ are functions of the rank, the dimension, the cosmological constant, and the graviton masses. For $r = 2$, this action was already written down in [12]. We will give explicit values for the coefficients a_i for $r = 3$ below.

Note that we use the canonical sign for the Einstein-Hilbert term in the above action. Flipping its sign is equivalent to changing the overall sign of the action, upon redefining Λ and a_i accordingly. Such a change of sign also changes the sign of the energy of the solutions (see Sec. VI), and, as noted in the Introduction, is particularly important in the $d = 3$, $r = 2$ case [3]. There it is customary to leave the sign of the Einstein-Hilbert term arbitary. Here, however, we have fixed the sign, keeping in mind that we can always flip the overall sign of the action in order to choose which modes have positive energy and which negative.

We have two main reasons for using only Weyl tensors in the higher-order terms. Both stem from the fact that the Weyl tensor vanishes identically on (A)dS spaces. First, this ensures the uniqueness of the (A)dS vacuum. Second, for perturbations around such a background, the higher-order terms do not contribute to the trace of the equations of motion. This comes about as follows.

The full nonlinear equations of motion that follow from (2) are

$$E_{\mu\nu} = -\frac{1}{\sqrt{-g}}\frac{\delta\mathcal{L}}{\delta g^{\mu\nu}} = G_{\mu\nu} + \sum_{i=1}^{r-1} a_i K^i_{\mu\nu} = 0. \tag{3}$$

Here $G_{\mu\nu}$ is the cosmological Einstein tensor (see the Appendix), and $K^i_{\mu\nu}$ are the contributions from the higher-order terms. Suppressing indices on the Weyl tensor, these contributions have the generic form

$$K^i_{\mu\nu} = 2\frac{\delta C}{\delta g^{\mu\nu}}\Box^i C + C\Big(\frac{\delta\Box^i}{\delta g^{\mu\nu}} - \frac{1}{2}g_{\mu\nu}\Big)C. \tag{4}$$

Thus $K^i_{\mu\nu}$ consists of a part that is linear in the Weyl tensor, and part that is quadratic. For an (A)dS space, both parts are zero. So, on these backgrounds, just the Einstein-Hilbert contribution of (3) survives. This uniquely fixes the background curvature to be Λ.

For linear perturbations around (A)dS solutions, the part of $K^i_{\mu\nu}$ that is quadratic in the Weyl tensor vanishes. The linearized higher-order contributions to $E^L_{\mu\nu}$ come then only from the first term on the right-hand side of (4), which evaluates to

$$(K^i_{\mu\nu})^L = \bar\nabla^\rho\bar\nabla^\sigma\bar\Box^i C^L_{\mu\rho\nu\sigma}. \tag{5}$$

The linear Weyl tensor C^L is, just like its nonlinear variant, traceless. Upon taking the trace of the linear equations of motion, it follows that the linear Ricci scalar vanishes on shell

$$\bar g^{\mu\nu}E^L_{\mu\nu} = \Big(1 - \frac{d}{2}\Big)R^L = 0. \tag{6}$$

As in Einstein gravity, this allows us to impose the transverse-traceless gauge [13], i.e., Eqs. (1b) and (1c), for the linear graviton fluctuations $h_{\mu\nu}$. Hence the scalar mode h, which would otherwise be a ghost, does not propagate.

In the remainder of this section we will show that the linear equations of motion take the form (1a), and give explicit values of the Lagrange parameters a_i for the rank $r = 3$. The linearized equations of motion can be written entirely in terms of $G^L_{\mu\nu}$ and R^L by using the identities

$$\bar\nabla^\rho\bar\nabla^\sigma\bar\Box^i C^L_{\mu\rho\nu\sigma} = [\bar\Box + 2(d-2)\Lambda]^i\bar\nabla^\rho\bar\nabla^\sigma C^L_{\mu\rho\nu\sigma}, \tag{7a}$$

$$\bar\nabla^\rho\bar\nabla^\sigma C^L_{\mu\rho\nu\sigma} = \frac{1}{2}\frac{d-3}{d-1}[\bar g_{\mu\nu}(\bar\Box - (d-1)\Lambda) - \bar\nabla_\mu\bar\nabla_\nu]R^L + \frac{d-3}{d-2}[\bar\Box - d\Lambda]G^L_{\mu\nu}. \tag{7b}$$

The former follows from commuting covariant derivatives, while the latter is a consequence of the Bianchi identities. Furthermore, in the transverse-traceless gauge the linear Einstein tensor takes on the form

$$G^L_{\mu\nu} = -\frac{1}{2}(\bar\Box - 2\Lambda)h_{\mu\nu}. \tag{8}$$

And, as $R^L = 0$ on shell, the linear equation of motion $E^L_{\mu\nu}$ is a polynomial in $\bar\Box$ that acts on $h_{\mu\nu}$. We may always choose the parameters a_i such that it factorizes into the form (1a). Indeed, for $r = 3$, the linear equation of motion becomes

$$E^L_{\mu\nu} = -\frac{1}{2\tau}(\bar\Box - 2\Lambda)\prod_{i=1}^{r-1}(\bar\Box - 2\Lambda - m_i^2)h_{\mu\nu} = 0, \tag{9}$$

where the parameter τ is given by

$$\tau = \prod_{i=1}^{r-1}[m_i^2 - (d-2)\Lambda], \tag{10}$$

and the squared masses by

$$m_1^2 = -\frac{d\Lambda}{2} - \frac{a_1 + \sqrt{b}}{2a_2}, \tag{11a}$$

$$m_2^2 = -\frac{d\Lambda}{2} - \frac{a_1 - \sqrt{b}}{2a_2}, \tag{11b}$$

$$b = [a_1 + (3d-4)\Lambda a_2]^2 - 4\frac{d-2}{d-3}a_2. \tag{11c}$$

Inverting the above equations for a_0 and a_1 gives finally

$$a_1 = -\frac{1}{\tau}\frac{d-2}{d-3}(m_1^2 + m_2^2 + d\Lambda), \tag{12a}$$

$$a_2 = +\frac{1}{\tau}\frac{d-2}{d-3}. \tag{12b}$$

The factors of $d-3$ in the denominator indicate that the nonlinear Lagrangian (2) is only valid for $d \geq 4$. This makes sense, as the Weyl tensor vanishes identically in three dimensions and lower. Similar explicit values for the parameters a_i of higher-rank theories can be computed along the same lines. While these explicit values are needed for the nonlinear action, they are not needed for its quadratic perturbation. As we will see in the next section, the latter can be written concisely in closed form using the mass parameters m_i instead of the parameters a_i. Furthermore, the quadratic Lagrangian will also be valid in three dimensions.

III. QUADRATIC ACTION

Before we set out to calculate the conserved charges and energies of our higher-rank theory, it is convenient to rewrite linear equations of motion a bit. We start by rearranging the quadratic perturbation of the nonlinear Lagrangian (2). It is given by

$$\mathcal{L}_2 = -\frac{1}{2}h^{\mu\nu}G^L_{\mu\nu} - \frac{1}{2}\bar\nabla^\sigma\bar\nabla^\nu h^{\mu\rho}\left(\sum_{i=1}^{r-1} a_i\bar\Box^{i-1}\right)C^L_{\mu\nu\rho\sigma}. \tag{13}$$

We have dropped a total derivative in the Einstein part, and expanded one of the linear Weyl tensors in terms of the graviton fluctuations $h_{\mu\nu}$. There are more contributions to this expansion than $\bar\nabla^\sigma\bar\nabla^\nu h^{\mu\rho}$, but they drop out because of the contraction with the other Weyl tensor.

Like the linear equations of motion, the quadratic Lagrangian can be written entirely in terms of $G^L_{\mu\nu}$ and R^L by using the identities (7). The resulting expression can be simplified further to

$$\mathcal{L}_2 = -\frac{1}{2\tau}G_L^{\mu\nu}\left(\prod_{i=1}^{r-1} 2\mathcal{S} + m_i^2\right) \circ h_{\mu\nu}. \tag{14}$$

Here τ is given as in (10), and we have introduced the Schouten operator $\mathcal{S}$. It is defined such that when it acts on the graviton fluctuations $h_{\mu\nu}$, it gives the linear cosmological Schouten tensor

$$\mathcal{S} \circ h_{\mu\nu} \equiv S^L_{\mu\nu}. \tag{15}$$

The cosmological Schouten tensor $S_{\mu\nu}$ is in turn defined such that for vanishing Λ it reduces to the normal Schouten tensor, and that it is zero on (A)dS backgrounds, i.e. $\bar S_{\mu\nu} = 0$. See also the Appendix . Surprisingly, the quadratic action (14) is also valid in three dimensions, whereas the nonlinear action (2) was not. For $d=3$, $r=2$, and $\Lambda = 0$ it coincides with the quadratic action given in [14].

Before deriving equations of motion from (14), we first list some useful properties of the Schouten operator and the Einstein operator $\mathcal{G}$. The latter is defined in a similar fashion as the Schouten operator

$$\mathcal{G} \circ h_{\mu\nu} \equiv G^L_{\mu\nu}. \tag{16}$$

For arbitrary symmetric tensors $A_{\mu\nu}$ and $B_{\mu\nu}$, we have

$$B^{\mu\nu}\mathcal{S}\circ A_{\mu\nu} = A^{\mu\nu}\left[\mathcal{S}\circ B_{\mu\nu} + \frac{1}{2}\frac{d-2}{d-1}(\bar\nabla_\mu\bar\nabla_\nu B - \bar g_{\mu\nu}\bar\nabla^\rho\bar\nabla^\sigma B_{\rho\sigma})\right], \tag{17a}$$

$$B^{\mu\nu}\mathcal{G}\circ A_{\mu\nu} = A^{\mu\nu}\mathcal{G}\circ B_{\mu\nu}, \tag{17b}$$

$$\bar\nabla^\mu\mathcal{G}\circ A_{\mu\nu} = 0, \tag{17c}$$

$$[\mathcal{G},\mathcal{S}]A_{\mu\nu} = \frac{1}{2}\frac{d-2}{d-1}\bar\nabla_\mu\bar\nabla_\nu\bar g^{\rho\sigma}\mathcal{G}\circ A_{\rho\sigma}. \tag{17d}$$

In the first two lines we have dropped a total derivative while integrating by parts. Note that for $B_{\mu\nu} = \mathcal{G}\circ A_{\mu\nu}$ the last term of (17a) vanishes, and the middle term is the same as the commutator (17d). Thanks to this subtle interplay between the Schouten and Einstein operators, the linear equation of motion reads

$$E^L_{\mu\nu} = -\frac{\delta\mathcal{L}_2}{\delta h^{\mu\nu}} = \frac{1}{\tau}\mathcal{G}\circ\left(\prod_{i=1}^{r-1} 2\mathcal{S} + m_i^2\right)\circ h_{\mu\nu}. \tag{18}$$

Upon taking the trace of these equations, we should recover (6), that is, $R^L = 0$. To see how this comes about, we need three additional properties of the Schouten and Einstein operators

$$\bar g^{\mu\nu}\mathcal{G}\circ A_{\mu\nu} = -(d-1)\bar g^{\mu\nu}\mathcal{S}\circ A_{\mu\nu}, \tag{19a}$$

$$\bar g^{\mu\nu}\mathcal{S}\circ A_{\mu\nu} = \frac{1}{2}\frac{d-2}{d-1}[\bar\nabla^\mu\bar\nabla^\nu A_{\mu\nu} - \bar\Box A - (d-1)\Lambda A], \tag{19b}$$

$$\bar\nabla^\mu\mathcal{S}\circ A_{\mu\nu} = \bar\nabla_\nu\bar g^{\rho\sigma}\mathcal{S}\circ A_{\rho\sigma}, \tag{19c}$$

from which it follows that

$$\bar g^{\mu\nu}\mathcal{G}\circ\mathcal{S}\circ A_{\mu\nu} = -\frac{d-2}{2}\Lambda\bar g^{\mu\nu}\mathcal{G}\circ A_{\mu\nu}. \tag{20}$$

A short calculation shows that we indeed recover (6)

$$\bar g^{\mu\nu}E^L_{\mu\nu} = \frac{1}{\tau}\bar g^{\mu\nu}\left(\prod_{i=1}^{r-1} m_i^2 - (d-2)\Lambda\right)\mathcal{G}\circ h_{\mu\nu} = \left(1-\frac{d}{2}\right)R^L = 0. \tag{21}$$

As the linear Ricci scalar vanishes on shell, we may go to the transverse-traceless gauge (1b) and (1c). In this gauge the Schouten and Einstein operators become equal [compare Eq. (8)],

$$\mathcal{S} \circ h_{\mu\nu} = \mathcal{G} \circ h_{\mu\nu} = -\frac{1}{2}(\bar{\Box} - 2\Lambda)h_{\mu\nu}. \tag{22}$$

The complete linear equations of motion (18) can then be written as

$$\frac{(-1)^r}{2\tau}\prod_{n=0}^{r-1}(\bar{\Box} - 2\Lambda - m_n^2)h_{\mu\nu} = 0, \tag{23a}$$

$$\bar{\nabla}^\mu h_{\mu\nu} = 0, \tag{23b}$$

$$\bar{g}^{\mu\nu} h_{\mu\nu} = 0, \tag{23c}$$

with $m_0 = 0$.

IV. CONFORMAL INVARIANCE

The overall factor $\frac{1}{\tau}$ in the quadratic Lagrangian (14) comes from demanding that the Ricci scalar in the nonlinear action (2) has the usual normalization. The advantage of this normalization is that we recover Einstein gravity upon decoupling the massive gravitons by sending their the masses to infinity. We will see later in Secs. V and VI that the conserved charges and graviton energies also reduce to their two-derivative "Einstein" values in this limit.

However, an obvious drawback of the overall factor $\frac{1}{\tau}$ is that it has poles at the mass values

$$m_i^2 = (d-2)\Lambda. \tag{24}$$

One easy way to get rid of the poles is to simply replace the overall factor $\frac{1}{\tau}$ by some other factor $\frac{1}{\tau'}$ that has the same mass dimension, but no explicit dependence on m_i and thus no poles. We can then freely let the masses take the values (24), with the drawback that we do not recover Einstein gravity upon decoupling the massive gravitons. Another possible drawback could be that for the mass values (24) the trace of the linear equations of motion (21) vanishes identically, and does not eliminate the scalar mode of the graviton.

Luckily, the latter does not happen. Instead, for the values (24) the linear theory develops a conformal invariance. To see how this happens, consider the linear conformal transformation

$$\delta_\omega h_{\mu\nu} = \bar{g}_{\mu\nu}\omega. \tag{25}$$

We would like to know the variation of the equations of motion (18) under this transformation. To this end we first compute the variation of a single Schouten operator,

$$\begin{aligned}\delta_\omega(\mathcal{S} \circ h_{\mu\nu}) &= \mathcal{S} \circ (\bar{g}_{\mu\nu}\omega)\\ &= -\frac{d-2}{2}(\Lambda \bar{g}_{\mu\nu} + \bar{\nabla}_\mu\bar{\nabla}_\nu)\omega. \end{aligned} \tag{26}$$

Next, we notice the identities

$$\mathcal{G} \circ (\bar{\nabla}_\mu\bar{\nabla}_\nu\omega) = \mathcal{S} \circ (\bar{\nabla}_\mu\bar{\nabla}_\nu\omega) = 0. \tag{27}$$

Thus for the repeated composition of the Schouten operator only the first term on the right-hand side of (26) is important. The variation of the equations of motion (18) then becomes

$$\delta_\omega E^L_{\mu\nu} = \frac{1}{\tau}\left(\prod_{i=1}^{r-1} m_i^2 - (d-2)\Lambda\right)\delta_\omega(\mathcal{G} \circ h_{\mu\nu}). \tag{28}$$

This is zero for the mass values (24) and the redefinition of τ mentioned above. This makes it possible to choose the conformal gauge $\omega = \frac{h}{d}$, such that the scalar mode vanishes everywhere.

The extra conformal gauge symmetry is somewhat reminiscent of the "partially massless" modes that occur in two-derivative Fierz-Pauli theory [15,16]. At the critical value of the Higuchi-bound Fierz-Pauli theory also develops an extra gauge symmetry [17], although not a conformal one. The extra gauge symmetry for the higher-derivative theories considered here can be thought of as a generalization of the two-derivative partially massless case.

V. CONSERVED CHARGES

We now derive the conserved charges of our theory. They can be calculated via the Abbott-Deser method [18], which is an extension of the Arnowitt-Deser-Misner energy [19,20] to backgrounds with constant curvature. In this method the linearized equations of motion $E^L_{\mu\nu}$ are treated as an effective energy-momentum tensor. This allows us to compute conserved charges Q^μ as follows:

$$Q^\mu(\bar{\xi}) = \int_\Sigma \mathrm{d}^{d-1}x\sqrt{-\bar{g}}E_L^{\mu\nu}\bar{\xi}_\nu. \tag{29}$$

Here $\bar{\xi}_\mu$ is a Killing vector of the background, and Σ is a spatial $(d-1)$ dimensional hypersurface. For instance, the global mass of a solution is then given by Q_0 for a timelike Killing vector. The trick for calculating the conserved charges is to show that the integrand can be written as a divergence of a two-form $F_{\mu\nu}$

$$E_L^{\mu\nu}\bar{\xi}_\nu = \bar{\nabla}_\nu F^{\mu\nu}. \tag{30}$$

The integral in (29) then reduces to a surface integral at spatial infinity,

$$Q^\mu(\bar{\xi}) = \int_{\partial\Sigma} \mathrm{d}S_\alpha F^{\mu\alpha}, \tag{31}$$

where $\partial\Sigma$ is the $(d-2)$ dimensional boundary of Σ. For Einstein-Hilbert gravity, whose linear equation of motion is simply $\mathcal{G} \circ h_{\mu\nu} = 0$, the two-form is

$$\begin{aligned}F^{\mathrm{EH}}_{\mu\nu} &= \bar{\xi}^\rho\bar{\nabla}_{[\mu}h_{\nu]\rho} + \bar{\xi}_{[\mu}\bar{\nabla}_{\nu]}h - \bar{\xi}_{[\mu}\bar{\nabla}^\rho h_{\nu]\rho}\\ &\quad + h^\rho{}_{[\mu}\bar{\nabla}_{\nu]}\bar{\xi}_\rho + \frac{1}{2}h\bar{\nabla}_\mu\bar{\xi}_\nu \equiv \mathcal{F}_{\bar{\xi}} \circ h_{\mu\nu}. \end{aligned} \tag{32}$$

Here we have introduced the two-form operator $\mathcal{F}_{\bar{\xi}}$. From the definition above, we have the following property. When it acts on a symmetric tensor, it gives a two-form whose divergence is the contraction of the Einstein operator with a Killing vector

$$\bar{\nabla}^{\nu}\mathcal{F}_{\bar{\xi}} \circ A_{\mu\nu} = \bar{\xi}^{\nu}\mathcal{G} \circ A_{\mu\nu}. \tag{33}$$

In our case the linear equation of motion is (18). Its general structure is the same as that of Einstein gravity, namely, a symmetric tensor hit by the Einstein operator. Hence the corresponding two-form simply reads

$$F_{\mu\nu} = \frac{1}{\tau}\mathcal{F}_{\bar{\xi}} \circ \left(\prod_{i=1}^{r-1} 2\mathcal{S} + m_i^2\right) \circ h_{\mu\nu}. \tag{34}$$

Like [21,22] we restrict to solutions that are asymptotically (A)dS. That is, at spatial infinity the vacuum Einstein equations are satisfied

$$G^L_{\mu\nu}|_{\partial\Sigma} = 0, \qquad R^L|_{\partial\Sigma} = 0, \qquad S^L_{\mu\nu}|_{\partial\Sigma} = 0. \tag{35}$$

The last equation follows from the fact that the linear Schouten tensor can be decomposed as $S^L_{\mu\nu} = G^L_{\mu\nu} + \frac{1}{2} \times \frac{d-2}{d-1}\bar{g}_{\mu\nu}R^L$. So the terms with Schouten operators in (34) are zero in the asymptotic region, and all that remains is the product of the squared masses m_i^2. Suppressing the dependency on the Killing vector, we obtain

$$Q^{\mu} = \frac{Q^{\mu}_{\rm EH}}{\tau}\prod_{i=1}^{r-1} m_i^2. \tag{36}$$

Thus the conserved charges are equal to those of two-derivative Einstein-Hilbert gravity, up to a renormalization factor. In the limit when all extra graviton modes become infinitely heavy and decouple, the renormalization factor goes to one by (10). Furthermore, the conserved charges vanish when one of the graviton masses is zero, which is what happens at the critical point in four-derivative critical gravity [5,12].

VI. GRAVITON ENERGIES

In this section we will derive the energies associated with the different graviton modes $h^{(n)}_{\mu\nu}$. These modes are annihilated by a single factor of the product in the complete equation of motion (23a),

$$(\bar{\Box} - 2\Lambda - m_n^2)h^{(n)}_{\mu\nu} = 0. \tag{37}$$

In [4,5,23] the graviton energies were computed by first deriving the Hamiltonian from an effective action. The Hamiltonian was then evaluated on shell for the different graviton modes in order to give the energy. The disadvantage of this approach is that one has to make an Arnowitt-Deser-Misner-like split of the indices and variables, and use the Ostrogradsky method to deal with the higher derivatives.

Here we will follow a different route, as outlined in [24], that circumvents these inconveniences. First we compute the energy-momentum tensor by varying the quadratic action (14) with respect to the background metric

$$T_{\mu\nu} = 2\frac{\delta\mathcal{L}_2}{\delta\bar{g}^{\mu\nu}}. \tag{38}$$

The energy is then obtained by integrating this energy-momentum tensor $T_{\mu\nu}$ over a Cauchy surface,

$$\mathcal{E} = \int_{\Sigma} \mathrm{d}^{d-1}x T_{\mu\nu}n^{\mu}\bar{\xi}^{\nu}. \tag{39}$$

Here n^{μ} is the unit normal to Σ and $\bar{\xi}^{\nu}$ is a timelike Killing vector.

For Einstein-Hilbert gravity, the on shell energy-momentum tensor is

$$T^{\rm EH}_{\mu\nu} = -h^{\rho\sigma}\frac{\delta G^L_{\rho\sigma}}{\delta\bar{g}^{\mu\nu}} = -h^{\rho\sigma}\frac{\delta\mathcal{G}}{\delta\bar{g}^{\mu\nu}} \circ h_{\rho\sigma}. \tag{40}$$

For deducing the energy-momentum tensor of our theory we need one last identity involving the Schouten and Einstein operators. First note from (19) that when the Schouten and Einstein operators act on a transverse and traceless tensor, the resulting tensor is also transverse and traceless. Furthermore, by Eq. (22), their action on transverse-traceless tensors gives the same result. This implies that for arbitrary transverse-traceless symmetric tensors $A_{\mu\nu}$ and $B_{\mu\nu}$, we have

$$B^{\mu\nu}\delta_{\bar{g}}(\mathcal{S} \circ A_{\mu\nu}) = B^{\mu\nu}(\delta_{\bar{g}}\mathcal{G}) \circ A_{\mu\nu} + (\delta_{\bar{g}}A_{\mu\nu})\mathcal{G} \circ B^{\mu\nu}. \tag{41}$$

Because we will evaluate the energy-momentum tensor on shell, $h_{\mu\nu}$ is transverse and traceless by (23). Thus we are allowed to use the above identities in deriving the energy-momentum tensor. Lastly, from Eqs. (22) with (37), we have on shell

$$\mathcal{S} \circ h^{(n)}_{\mu\nu} = \mathcal{G} \circ h^{(n)}_{\mu\nu} = -\frac{1}{2}m_n^2 h^{(n)}_{\mu\nu}. \tag{42}$$

Combining the above equations, the energy-momentum tensor becomes

$$T^{(n)}_{\mu\nu} = \frac{1}{\tau}\left[\prod_{i=1}^{r-1}(m_i^2 - m_n^2) - m_n^2\sum_{i=1}^{r-1}\prod_{\substack{j=1\\ j\neq i}}^{r-1}(m_j^2 - m_n^2)\right]T^{\rm EH}_{\mu\nu}. \tag{43}$$

The superscript (n) indicates that it is evaluated on shell for the mode $h^{(n)}_{\mu\nu}$. When $m_n = 0$, only the first product contributes to the energy-momentum tensor. For nonzero masses m_i, the first product vanishes and only one term in the sum is nonzero. This gives the following energies for the massless and massive gravitons:

$$\mathcal{E}^{(0)} = +\frac{\mathcal{E}_{\rm EH}}{\tau}\prod_{j=1}^{r-1} m_j^2 \tag{44a}$$

$$\mathcal{E}^{(i)} = -\frac{\mathcal{E}_{\rm EH}}{\tau}m_i^2\prod_{\substack{j=1\\ j\neq i}}^{r-1}(m_j^2 - m_i^2). \tag{44b}$$

The energy of the massless graviton $h^{(0)}_{\mu\nu}$ has the same overall factor as the conserved charge Q^{μ} (36). So it seems that the massive gravitons $h^{(i)}_{\mu\nu}$ do not contribute to the conserved charge. This is to be expected, as the massive gravitons fall off too fast towards spatial infinity to contribute to the surface integral (31) in the asymptotic region.

For four-derivative theories ($r = 2$), the above energies reduce to $\mathcal{E}^{(0)} = -\mathcal{E}^{(1)}$, which matches with the energies found in [4,5]. So, for rank two, the only way to obtain energies with the same sign is to set the mass m_1 to zero. Both energies are then zero, and the conserved charge also vanishes, rendering the theory trivial.

In the six-derivative case, $r = 3$, this zero-energy problem does not occur. The graviton energies (44) then, namely, read

$$\mathcal{E}^{(0)} = \frac{\mathcal{E}_{\rm EH}}{\tau} m_1^2 m_2^2, \tag{45a}$$

$$\mathcal{E}^{(1)} = \frac{\mathcal{E}_{\rm EH}}{\tau} m_1^2 (m_1^2 - m_2^2), \tag{45b}$$

$$\mathcal{E}^{(2)} = \frac{\mathcal{E}_{\rm EH}}{\tau} m_2^2 (m_2^2 - m_1^2). \tag{45c}$$

These energies are plotted in Fig. 1. There are two distinct points where they have same sign: either when $m_1^2 = m_2^2$ or when $m_1^2 = 0 \vee m_2^2 = 0$. In the last critical point the conserved is zero, whereas it can be positive in the first.

For yet higher-rank theories there are even more critical points. However, it will never be possible to have the same sign for all energies without degeneracies in the masses. If we, namely, arrange the masses by size,

$$m_1^2 < \ldots < m_i^2 < \ldots < m_{r-1}^2, \tag{46}$$

the sign of the energies (44) alternates

$$\operatorname{sgn}(\mathcal{E}^{(i)}) = -\operatorname{sgn}(\mathcal{E}^{(i+1)}). \tag{47}$$

Thus the situation with all masses different (46) leads to ghosts; to avoid this we need at least some degeneracy of the masses.

But whenever there is a mass m_n with multiplicity $\mu_n > 1$, a so-called log mode $h_{\mu\nu}^{(n,p)}$ appears [8]. These log modes are annihilated not by a single factor, but by multiple factors of the product in the equations of motion

$$(\bar{\Box} - 2\Lambda - m_n^2)^p h_{\mu\nu}^{(n,p)} = 0. \tag{48}$$

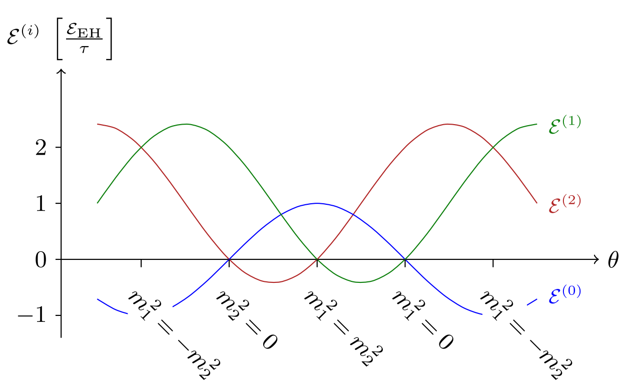

FIG. 1 (color online). Graviton energies for $r = 3$ in polar coordinates. Here $\theta = \tan^{-1}(\frac{m_2^2}{m_1^2})$ is the angle in the (m_1^2, m_2^2) plane. The masses are given by $m_1^2 = 2\cos\theta$ and $m_2^2 = 2\sin\theta$. There are three points where all energies are non-negative: $m_1^2 = m_2^2$, $m_1^2 = 0$, or $m_2^2 = 0$.

The label p can take the values $p = 2, \ldots, \mu_n$, as $p = 1$ simply gives the nonlogarithmic graviton mode $h_{\mu\nu}^{(n)}$. From the four-derivative case the log modes are expected to be ghosts [9], and, if possible, need to be truncated out in order to restore unitarity.

VII. CONCLUSIONS

In this paper we have studied gravities of arbitrary rank, meaning they propagate any number of gravitons on (A)dS backgrounds. Besides giving a quadratic and a nonlinear action, we have calculated the conserved charges and the graviton energies. From the energies we deduce that there will be ghosts unless the masses have critical values. At these critical points some of the gravitons have degenerate mass. But as mass degeneracies lead to logarithmic graviton modes, the untruncated theory will never be unitary. By truncating the log modes by imposing appropriate boundary conditions one could obtain a unitary subsector of the theory. We leave the exact form of both the higher-rank log modes and boundary conditions to future study.

When the rank is two, there is only one critical point, and all the energies vanish [4,5,12]. One can interpret the triviality of this theory as being to due to the proposed equivalence of Einstein gravity and conformal gravity [25,26]. We have shown that for higher-rank theories there are critical points where the conserved charges and graviton energies do not vanish. But in the fully degenerate case where all the graviton masses are zero, the theory will always be empty. Like the proposed equivalence of Einstein and conformal gravity, this "emptiness" of higher-rank theories could in principle be used to construct a chain of equivalence relations between gravity theories of different rank.

ACKNOWLEDGMENTS

We would like to thank Sjoerd de Haan, Axel Kleinschmidt, Ilarion Melnikov, Wout Merbis, and Stefan Theisen for useful discussions. As some of the calculations of this paper have been carried out using XACT [27], we thank Jose Martin-Garcia for making this rather excellent piece of software freely available. Lastly, we thank Amitabh Virmani and Andrea Campoleoni for discussions and feedback on early drafts.

APPENDIX A: CONVENTIONS

We use the "mostly plus" metric signature $(-, +, \ldots, +)$. The conventions for the Riemann tensor are the default of the XACT software package [27], which in turn follows Wald's conventions [13]

$$[\nabla_\mu, \nabla_\nu] T_\rho = R_{\mu\nu\rho}^{\sigma} T_\sigma, \qquad R_{\mu\nu} = R_{\mu\rho\nu}^{\rho}. \tag{A1}$$

Barred objects are background quantities (i.e., $\bar{g}$ denotes the background metric). AdS and dS backgrounds are chosen as follows:

$$\bar{R}_{\mu\nu\rho\sigma} = \Lambda(\bar{g}_{\mu\rho}\bar{g}_{\nu\sigma} - \bar{g}_{\mu\sigma}\bar{g}_{\nu\rho}), \quad \text{(A2a)}$$

$$\bar{R}_{\mu\nu} = (d-1)\Lambda\bar{g}_{\mu\nu}, \quad \text{(A2b)}$$

$$\bar{R} = d(d-1)\Lambda. \quad \text{(A2c)}$$

Perturbations around these backgrounds are defined as

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + g^L_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}. \quad \text{(A3)}$$

The superscript L indicates linear perturbations. Thus the linear perturbation of the metric is given by $h_{\mu\nu}$.

The cosmological Einstein tensor is the usual Einstein tensor plus a term proportional to the cosmological constant, such that it vanishes on the above backgrounds

$$G^{\Lambda}_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}[R - (d-2)(d-1)\Lambda], \quad \text{(A4a)}$$

$$\bar{G}^{\Lambda}_{\mu\nu} = 0. \quad \text{(A4b)}$$

The cosmological Schouten tensor is defined similarly

$$S^{\Lambda}_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\left[\frac{R}{d-1} + (d-2)\Lambda\right], \quad \text{(A5a)}$$

$$\bar{S}^{\Lambda}_{\mu\nu} = 0. \quad \text{(A5b)}$$

The Schouten tensor is usually given with an additional overall factor $\frac{1}{d-2}$. However, for our purposes the above definition is more convenient. In the main text the superscripts Λ are dropped from the cosmological Einstein and Schouten tensors. Thus by $G_{\mu\nu}$ and $S_{\mu\nu}$ we always mean their cosmological versions.

For completeness, we give the linear perturbations of the cosmological Einstein and Schouten tensors

$$G^L_{\mu\nu} = R^L_{\mu\nu} - (d-1)\Lambda h_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}R^L, \quad \text{(A6a)}$$

$$S^L_{\mu\nu} = R^L_{\mu\nu} - (d-1)\Lambda h_{\mu\nu} - \frac{1}{2(d-1)}\bar{g}_{\mu\nu}R^L, \quad \text{(A6b)}$$

with

$$R^L_{\mu\nu} = \bar{\nabla}_\rho\bar{\nabla}_{(\mu}h^{\rho}_{\nu)} - \frac{1}{2}\bar{\Box}h_{\mu\nu} - \frac{1}{2}\bar{\nabla}_\mu\bar{\nabla}_\nu h, \quad \text{(A7a)}$$

$$R^L = \bar{\nabla}_\rho\bar{\nabla}_\sigma h^{\rho\sigma} - \bar{\Box}h - (d-1)\Lambda h. \quad \text{(A7b)}$$

[1] K. S. Stelle, Phys. Rev. D **16**, 953 (1977).
[2] K. S. Stelle, Gen. Relativ. Gravit. **9**, 353 (1978).
[3] E. A. Bergshoeff, O. Hohm, and P. K. Townsend, Phys. Rev. Lett. **102**, 201301 (2009).
[4] Y. Liu and Y.-w. Sun, J. High Energy Phys. 04 (2009) 106.
[5] H. Lu and C. N. Pope, Phys. Rev. Lett. **106**, 181302 (2011).
[6] D. Grumiller and N. Johansson, J. High Energy Phys. 07 (2008) 134.
[7] Y. Liu and Y.-W. Sun, J. High Energy Phys. 05 (2009) 039.
[8] E. A. Bergshoeff, O. Hohm, J. Rosseel, and P. K. Townsend, Phys. Rev. D **83**, 104038 (2011).
[9] M. Porrati and M. M. Roberts, Phys. Rev. D **84**, 024013 (2011).
[10] E. A. Bergshoeff, S. de Haan, W. Merbis, M. Porrati, and J. Rosseel, J. High Energy Phys. 04 (2012), 134.
[11] K. Hinterbichler, Rev. Mod. Phys. **84**, 671 (2012).
[12] S. Deser, H. Liu, H. Lu, C. Pope, T. C. Sisman, and B. Tekin, Phys. Rev. D **83**, 061502 (2011).
[13] R. M. Wald, *General Relativity* (The University of Chicago Press, Chicago, 1984).
[14] E. Bergshoeff, O. Hohm, and P. Townsend, J. Phys. Conf. Ser. **229**, 012005 (2010).
[15] S. Deser and A. Waldron, Phys. Rev. Lett. **87**, 031601 (2001).
[16] S. Deser and A. Waldron, Nucl. Phys. **B607**, 577 (2001).
[17] S. Deser and R. I. Nepomechie, Ann. Phys. (N.Y.) **154**, 396 (1984).
[18] L. Abbott and S. Deser, Nucl. Phys. **B195**, 76 (1982).
[19] R. L. Arnowitt, S. Deser, and C. W. Misner, *Gravitation: An Introduction To Current Research*, edited by Louis Witten (Wiley, New York, 1962), Chap. 7, pp 227.
[20] T. Regge and C. Teitelboim, Ann. Phys. (N.Y.) **88**, 286 (1974).
[21] S. Deser and B. Tekin, Phys. Rev. Lett. **89**, 101101 (2002).
[22] S. Deser and B. Tekin, Phys. Rev. D **67**, 084009 (2003).
[23] W. Li, W. Song, and A. Strominger, J. High Energy Phys. 04 (2008) 082.
[24] V. Iyer and R. M. Wald, Phys. Rev. D **50**, 846 (1994).
[25] J. Maldacena, arXiv:1105.5632.
[26] H. Lu, Y. Pang, and C. Pope, Phys. Rev. D **84**, 064001 (2011).
[27] J. M. Martin-Garcia, XACT: Efficient tensor computer algebra for MATHEMATICA.