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Subtracted Geometry From Harrison Transformations

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Abstract

We consider the rotating non-extremal black hole of N=2 D=4 STU supergravity carrying three magnetic charges and one electric charge. We show that its subtracted geometry is obtained by applying a specific SO(4,4) Harrison transformation on the black hole. As previously noted, the resulting subtracted geometry is a solution of the N=2 S=T=U supergravity.

Contents

1 Introduction

Over the years there has been slow but steady progress in our understanding of relations between black holes and two dimensional conformal field theories. Several universal properties of black holes have been found to be related to universal properties of 2d CFTs. String theory has provided significant insights in this quest. Arguably, one of the most spectacular successes of string theory is the Sen-Strominger-Vafa counting of the microscopic configurations, and thereby providing a statistical mechanical explanation of the entropy of certain extremal and near-extremal black holes [1, 2]. Since then, many different types of black holes have been studied and the agreement between the Bekenstein-Hawking entropy and the statistical mechanical entropy has been shown to hold in a variety of cases.

These achievements, very impressive as they are, need to be contrasted with the challenge of microscopically understanding general non-extremal black holes. The methods advocated in [1, 2] cannot be directly applied to such general settings. More recently, considerable progress has been made in addressing general extremal black holes. These developments go under the name of the Kerr/CFT correspondence [3]; see [4] for a concise review and see [5] for a more comprehensive review[1]. Once again, these developments rely on certain specific structure of extremal black holes, and cannot be directly applied to non-extremal settings. In the case of the Kerr/CFT, existence of the decoupled near-horizon geometry is crucial. In settings far away from extremality one cannot decouple the near-horizon region from the asymptotic region. As a result, it remains unclear how the considerations of Kerr/CFT are useful for describing such general settings.

It comes as a surprise that even for black holes far away from extremality, certain tantalizing clues have been found for the presence of a conformal symmetry. It was observed in [6] that in certain low-energy near-horizon regimes the dynamics of a probe scalar field enjoys certain local hidden non-geometric SL(2,$\mathbb{R}$)$\times$SL(2,$\mathbb{R}$) symmetry. The precise meaning of this symmetry is a topic of future research, but the picture put forward in [6] shows remarkable coherence. These hidden symmetries only appear in a region close enough to the

[1]In these reviews further references on these and related developments can also be found.

horizon. It has been suggested [7, 8, 9] that one can consistently deform the geometry of an asymptotically flat black hole so that these hidden symmetries appear manifestly in the deformed geometries. These geometries are dubbed "subtracted geometries." The subtracted geometries are not asymptotically flat. They are supported by additional matter fields. In this work we explore these geometries and their relation to the original black holes.

The main aim of this paper is to establish that the subtracted geometries can be obtained from the original black hole by applying solution generating transformations. For concreteness we consider the case of four-charge rotating non-extremal four-dimensional asymptotically flat black holes of N=2 STU supergravity. Moreover we restrict ourselves to the black hole carrying three magnetic and one electric charge. This is just a choice; we expect our considerations to straightforwardly apply to other combinations of in total four electric and magnetic charges.

The motivation for looking at the 4d solution carrying three magnetic charges (and one electric charge) is manifold. Not only we can perform a study of its subtracted geometry, but also we can use it to perform various other studies; most notably in relation to a string theory realization of the Kerr/CFT correspondence and black rings. It was shown in [10] that the spinning magnetic one-brane of five-dimensional minimal supergravity admits a near-horizon limit that smoothly interpolates between a self-dual supersymmetric null orbifold of $\mathrm{AdS}_3 \times \mathrm{S}^2$ and the near-horizon limit of the extremal Kerr black hole times a circle. It is of interest to generalize this observation to a multicharge configuration. We present such a generalization in appendix D.

As for the construction of the rotating four-charge black hole carrying three magnetic and one electric charge, there are several ways in which one can approach this problem. The first, and perhaps also the most direct, approach that comes to mind is to use boosts and string dualities. One quickly realizes that to add three independent magnetic charges, the number of boosts and dualities steps required is in fact quite large (approximately 20). To perform all these steps coherently is a computational challenge[2].

There are other somewhat less computationally intensive possibilities. For example, a second possibility is to perform an electro-magnetic duality in four-dimensional N=2 STU supergravity and convert the two-electric two-magnetic rotating solution as presented in [12] to three-magnetic and one-electric one. Finally, a third possibility is to use the powerful machinery of three-dimensional hidden symmetries of the STU model to generate

[2] A construction along these lines of the spinning magnetic one-brane in five-dimensional $\mathrm{U}(1)^3$ supergravity with three independent M5 charges was attempted in [11]. However, the author did not completely succeed in achieving this goal. The expressions presented in [11] do not solve the supergravity equations.

this solution. It is the third path that was used to construct the solution carrying two electric and two magnetic charges [12]. In our opinion the second and the third routes are of almost equal computational complexity. Since the approach of three-dimensional hidden symmetries also allows us to relate to its subtracted geometry rather directly, we follow the third route in this paper.

For the ease of readability of the paper almost all technicalities related to the construction of the solution are presented in appendices. Appendix A presents the set-ups we work with in considerable detail. Here we also present an implementation of the SO(4,4) nonlinear sigma model. The group SO(4,4) is relevant because it is the group of hidden symmetries of the N=2 STU supergravity when the theory is dimensionally reduced on a Killing vector. The rest of the paper is organized as follows. We first construct the spinning M5-M5-M5 solution in section 2. We present it as a configuration in five-dimensional $\mathrm{U}(1)^3$ supergravity. Then we show how to add the fourth charge. In section 3 we obtain its subtracted geometry by applying a series of solution generating transformations. Three-dimensional sigma model fields for the M5-M5-M5 solution are presented in appendix B. Three dimensional fields for the subtracted geometry are presented in appendix C. We conclude in section 4.

2 Four-Charge Black Hole

Although four charge black holes of ungauged four dimensional supergravity theories are well studied in the literature [13, 12, 14, 8], to the best of our knowledge expressions for all fields when the black hole carries three independent magnetic charges have not been explicitly presented anywhere. We fill this gap in this section. For many purposes, e.g., in relation to black rings, or in relation to (0,4) MSW/D1-D5-KKM CFT [15], such a presentation is useful.

2.1 M5-M5-M5

We consider the M-theory frame and describe the configuration as a solution of five-dimensional $\mathrm{U}(1)^3$ supergravity. Upon reducing over the string direction we obtain a rotating 4d black hole carrying three independent magnetic charges. For various reasons we prefer to present the 5d lift of the 4d solution.

The Theory

We follow the conventions in which the $\mathrm{U}(1)^3$ supergravity Lagrangian takes the a manifestly triality-invariant form

$$\mathcal{L}_5 = R_5 \star_5 \mathbf{1} - \frac{1}{2} G_{IJ} \star_5 dh^I \wedge dh^J - \frac{1}{2} G_{IJ} \star_5 F^I_{[2]} \wedge F^J_{[2]} + \frac{1}{6} C_{IJK} F^I_{[2]} \wedge F^J_{[2]} \wedge A^K_{[1]}. \quad (2.1)$$

The symbol C_{IJK} is pairwise symmetric in its indices with $C_{123} = 1$ and is zero otherwise. The metric G_{IJ} on the scalar moduli space is diagonal with entries $G_{II} = (h^I)^{-2}$, where these scalars satisfy the constraint $h^1 h^2 h^3 = 1$. This constraint must be solved before computing variations of the action to obtain EOMs for various fields.

We construct the M5-M5-M5 solution using the familiar coset model solution generating techniques. We reduce the theory (2.1) on commuting Killing vectors to three dimensions. We do this reduction first over a spacelike Killing vector and then over a timelike Killing vector. The theory reduces to 3d gravity coupled to SO(4,4)/(SO(2,2)×SO(2,2)) non-linear sigma model. Acting with an appropriate group elements of SO(4,4) on the Kerr string we get the non-extremal spinning magnetic one-brane of $\mathrm{U}(1)^3$ supergravity. Details on the set-up and the explicit form of the group element can be found in appendix A. For five-dimensional minimal supergravity such constructions have been extensively discussed in our previous work [16, 17, 18, 19, 20, 10].

The Solution

Let $s_I = \sinh \alpha_I$ and $c_I = \cosh \alpha_I$ with $I = 1, 2, 3$, then the spinning magnetic one-brane with three-independent M5-charges is given as

$$ds_5^2 = f^2 (dz + {A_4}^0)^2 + f^{-1} (-e^{2U} (dt + \omega_3)^2 + e^{-2U} ds_3^2(\mathcal{B})), \quad (2.2)$$

where

$$ds_3^2(\mathcal{B}) = \frac{\Delta_2}{\Delta} dr^2 + \Delta_2 d\theta^2 + \Delta \sin^2\theta d\phi^2, \quad (2.3)$$

is the three-dimensional base metric obtained by reducing the Kerr string on ∂_z first and then over ∂_t, and

$$\begin{aligned} \Delta &= r^2 - 2mr + a^2, \qquad \Delta_2 = \Delta - a^2 \sin^2\theta & (2.4)\\ f^2 &= 4\xi(\Omega_1\Omega_2\Omega_3)^{-2/3}, \qquad e^{4U} = \frac{{\Delta_2}^2}{\xi} & (2.5)\\ \omega_3 &= c_1 c_2 c_3 \frac{2amr\sin^2\theta}{\Delta_2} d\phi, & (2.6) \end{aligned}$$

are the metric functions appearing in the line element. The rest of the metric functions take the form

$$\begin{aligned} \xi &= (r^2 + a^2\cos^2\theta)^2 + 2mr(r^2 + a^2\cos^2\theta)\left(\sum_{I=1}^{3} s_I^2\right) \\ &\quad +4m^2r^2\left(s_1^2s_2^2 + s_2^2s_3^2 + s_1^2s_3^2\right) + 4m^2(2mr - a^2\cos^2\theta)\left(\prod_{I=1}^{3} s_I^2\right), \end{aligned} \tag{2.7}$$

$$\Omega_1 = 2(a^2\cos^2\theta + (r + 2ms_2^2)(r + 2ms_3^2)), \tag{2.8}$$

and cyclic permutations. Furthermore we have

$$A_4{}^0 = \zeta^0(dt + \omega_3) + 2s_1s_2s_3\frac{am(r-2m)}{\Delta_2}\sin^2\theta d\phi, \tag{2.9}$$

with

$$\zeta^0 = 4c_1c_2c_3s_1s_2s_3\frac{a^2m^2\cos^2\theta}{\xi}. \tag{2.10}$$

The Maxwell potentials A^I's of the five-dimensional theory take the form

$$A^I = \chi^I(dz + A_4^0) + \zeta^I(dt + \omega_3) + 2ms_Ic_I\frac{\Delta}{\Delta_2}\cos\theta d\phi \tag{2.11}$$

with

$$\chi^1 = 4c_1s_2s_3\frac{am\cos\theta}{\Omega_1}, \tag{2.12}$$

$$\zeta^1 = -2s_1c_2c_3(r^2 + a^2\cos^2\theta + 2rms_1^2))\frac{am\cos\theta}{\xi}, \tag{2.13}$$

and obvious cyclic permutations. Finally, the three scalars in the $\mathrm{U}(1)^3$ theory take the form

$$h^I = (\Omega_1\Omega_2\Omega_3)^{1/3}\Omega_I^{-1}. \tag{2.14}$$

The solution is sufficiently complicated, and it is non-trivial to check that all supergravity equations are solved. We have checked that they are solved.

Setting any two of the three charges to zero, while keeping the angular momentum non-zero, the resulting solution can be compared to reference [21]. In this special case the solution also admits a lift to vacuum gravity in six dimensions. By setting the three charges equal the solution can be compared with [16]. Certain physical properties of the solution and its near horizon geometry in the extremal limit are studied in appendix D.

2.2 Adding the Fourth Charge

By boosting the string configuration (2.2) in (t, z), and then dimensional reducing over the z direction we obtain a four-charge four-dimensional black hole. The 4d black hole carries three-magnetic charges and one-electric charge. From the hidden symmetries point of view this procedure is equivalent to performing

$$\mathcal{M}_{3-\text{charge}} \to \mathcal{M}_{4-\text{charge}} = g_4^\sharp \cdot \mathcal{M}_{3-\text{charge}} \cdot g_4, \tag{2.15}$$

with

$$g_4 = \exp\left[-\alpha_0(E_{q_0} + F_{q_0})\right]. \tag{2.16}$$

Here $\mathcal{M}_{3-\text{charge}}$ denotes the SO(4,4) coset matrix for the above three-charge configuration[3]. The explicit expressions for the resulting fields are fairly complicated. For the case of two-electric and two-magnetic charges these expression are presented in full detail in [12]. Fortunately, we will not need the explicit expressions in what follows.

3 Subtracted Geometry From Harrison Transformations

To obtain the subtracted geometry of the above described four-charge black hole we act on it with a series of solution generating transformations. These transformations perform the required Harrison boosts that give the subtracted geometry. The precise sequence of transformations is somewhat involved. We perform them in a certain specific order explained below to maintain the complexity of intermediate expressions under control.

This investigation was systematically initiated in [9]. There it was suggested that the subtracted geometry of the four-charge black hole can be obtained by certain Harrison boosts. The subtracted geometry of the Schwarzschild and Kerr solutions were obtained in Einstein-Maxwell-Dilaton theories by applying certain infinite Harrison boosts. The key technical observation we take from that work is their equation (33), i.e., that the Harrison boosts used are of the *lower triangular* form. From the point of view of the SO(4,4) Lie algebra this suggests that the specific Harrison transformation that leads to the subtracted geometry of the four-charge black hole belongs to certain 'lowering' generators. This is indeed the case, as we explain next.

3.1 Charging, Harrison Boosts, and Scaling

The most important transformation on the four-charge black hole to obtain its subtracted geometry is of the form

$$\mathcal{M}_{4-\text{charge}} \to \mathcal{M}' = g_H^\sharp \cdot \mathcal{M}_{4-\text{charge}} \cdot g_H, \tag{3.1}$$

[3]The notation $g^\sharp$ denotes a generalized transposition. The transposition is defined on the generators of the $\mathfrak{so}(4,4)$ Lie algebra by $\sharp(x) = -\tau(x) \forall x \in \mathfrak{so}(4,4)$, where τ is the involution of the Lie algebra that defines the coset. More details can be found in appendix A.

with the Harrison transformation g_H

$$g_H = \exp[(F_{p^1} + F_{p^2} + F_{p^3})]. \tag{3.2}$$

Note that despite the fact that the four-charge black hole carries three independent M5 charges, the Harrison boosts in (3.2) are by the same 'amount' in p^1, p^2 and p^3 'directions.' In all these three directions the boosts are infinite, in the sense that the lowering generators F_{p^1}, F_{p^2} and F_{p^3} are exponentiated with unit coefficients, in line with [9]. Furthermore, note that we *do not* apply a Harrison boost in the q_0 'direction.' This is reminiscent of the near-extreme multi-charge black holes in the so-called dilute gas approximation [22, 23].

However, it so happens that performing the transformation (3.2) on the four-charge black hole resulting from (2.16) is quite intricate to implement. To bypass this purely technical complexity we make the following crucial observation: the generator that adds the fourth charge, namely, $(E_{q_0} + F_{q_0})$ commutes with all three generators of the Harisson boosts we want to perform F_{p^1}, F_{p^2} and F_{p^3}. As a result the transformation

$$\begin{aligned} \mathcal{M}' &= g_H^\sharp \cdot \mathcal{M}_{4-\text{charge}} \cdot g_H & (3.3) \\ &= g_H^\sharp \cdot g_4^\sharp \cdot \mathcal{M}_{3-\text{charge}} \cdot g_4 \cdot g_H & (3.4) \end{aligned}$$

is the same as doing

$$\mathcal{M}' = g_4^\sharp \cdot g_H^\sharp \cdot \mathcal{M}_{3-\text{charge}} \cdot g_H \cdot g_4, \tag{3.5}$$

where we have commuted g_4 past g_H. Physically there is absolutely no difference between (3.4) and (3.5), but computationally performing (3.5) is significantly simpler (at least in the way we have organized our computer implementation of the SO(4,4) coset model).

This is not the end of the story. One also needs to perform a further scaling transformation to get the subtracted geometry in precisely the form given in [9]. This last transformation is as follows

$$\mathcal{M}_{\text{subtracted}} = g_S^\sharp \cdot \mathcal{M}' \cdot g_S, \qquad g_S = \exp[-cH_0], \tag{3.6}$$

where c is given below. Having done all these solution generating transformations we need to change variables as suggested in [9] and choose the parameter c in (3.6) in a specific way. The choice

$$\begin{aligned} \alpha_1 = \alpha_2 = \alpha_3 &= -\frac{1}{2}\ln\left(\Pi_c^2 - \Pi_s^2\right), & (3.7) \\ \alpha_0 &= \sinh^{-1}\left(\frac{\Pi_s}{\sqrt{\Pi_c^2 - \Pi_s^2}}\right) & (3.8) \\ c &= -\ln\left(\Pi_c^2 - \Pi_s^2\right), & (3.9) \end{aligned}$$

leads to the subtracted geometry of the four-charge black hole[4] as presented in [9]. Perhaps a more general subtracted geometry is possible with $\alpha_1 \neq \alpha_2 \neq \alpha_3$. This issue needs further investigation.

We summarize. To obtain subtracted geometry of the four-charge black hole as presented in [9] we perform the following transformation on the 3-charge black hole

$$\mathcal{M}_{\text{subtracted}} = g_S^\sharp \cdot g_4^\sharp \cdot g_H^\sharp \cdot \mathcal{M}_{3-\text{charge}} \cdot g_H \cdot g_4 \cdot g_S. \tag{3.10}$$

For convenience and completeness all the resulting three-dimensional fields are listed in appendix C. In the the next section we present the final geometry in the four-dimensional language and compare it with the analysis of Cvetic and Gibbons.

3.2 Resulting Geometry

The resulting geometry in the four-dimensional language is most conveniently expressed as

$$ds_4^2 = -e^{2U}(dt + \omega_3)^2 + e^{-2U} ds_3^2(\mathcal{B}), \tag{3.11}$$

where

$$ds_3^2(\mathcal{B}) = \frac{\Delta_2}{\Delta} dr^2 + \Delta_2 d\theta^2 + \Delta \sin^2\theta d\phi^2, \tag{3.12}$$

is the three-dimensional base metric obtained by reducing the Kerr black hole over ∂_t, and

$$\Delta = r^2 - 2mr + a^2, \qquad \Delta_2 = \Delta - a^2 \sin^2\theta. \tag{3.13}$$

Rewriting the four-dimensional metric in the form as in [8, 9] we get

$$ds_4^2 = -\left(\frac{1}{e^{-2U}\Delta_2}\right)\Delta_2(dt+\omega_3)^2 + e^{-2U}\Delta_2\left(\frac{dr^2}{\Delta} + d\theta^2 + \frac{\Delta}{\Delta_2}\sin^2\theta d\phi^2\right). \tag{3.14}$$

The square of the factor $e^{-2U}\Delta_2$ is called the subtracted conformal factor in [8, 9]. From appendix C we read the value of $e^{-2U}\Delta_2$ to be

$$e^{-2U}\Delta_2 = 2m\sqrt{\tilde{\Delta}_s}, \tag{3.15}$$

with

$$\tilde{\Delta}_s = 2mr(\Pi_c^2 - \Pi_s^2) + 4m^2\Pi_s^2 - a^2x^2(\Pi_c - \Pi_s)^2. \tag{3.16}$$

Our $4m^2\tilde{\Delta}_s$ precisely corresponds to the subtracted conformal factor used in [8, 9]. For the four-dimensional axions and dilaton fields we find

$$\chi^1 = \chi^2 = \chi^3 = \frac{ax(\Pi_c - \Pi_s)}{m}, \tag{3.17}$$

[4]The explicit product expressions [8, 9] $\Pi_c = \prod_{I=0}^4 \cosh\alpha_I, \Pi_s = \prod_{I=0}^4 \sinh\alpha_I$, are not needed in our computations, because the final geometry is parameterized solely in terms of Π_c and Π_s. See also footnote 3 of [9].

and

$$y_1 = y_2 = y_3 = \frac{\sqrt{\tilde{\Delta}_s}}{2m}, \tag{3.18}$$

which again precisely matches with the expressions reported in [9], once we make a translation of conventions. Finally, the four dimensional vector fields take the form

$$\check{A}^0_{[1]} = \frac{4am^2\sin^2\theta(\Pi_c-\Pi_s)}{\tilde{\Delta}_s}d\phi + \frac{a^2\cos^2\theta(\Pi_c-\Pi_s)^2+4m^2\Pi_c\Pi_s}{(\Pi_c^2-\Pi_s^2)\tilde{\Delta}_s}, \tag{3.19}$$

and

$$\check{A}^1_{[1]} = \check{A}^2_{[1]} = \check{A}^3_{[1]}, \tag{3.20}$$

$$\begin{aligned} \check{A}^1_{[1]} &= 2m\cos\theta\frac{2m(2m\Pi_s^2+r(\Pi_c^2-\Pi_s^2))-a^2(\Pi_c-\Pi_s)^2}{\tilde{\Delta}_s}d\phi \\ &\quad -\frac{a\cos\theta(2m\Pi_s+r(\Pi_c-\Pi_s))}{\tilde{\Delta}_s}dt. \end{aligned} \tag{3.21}$$

As far as the expressions for the vector fields can be compared with the corresponding expressions in [9], they perfectly match. Since our vector field $\check{A}^1_{[1]}$ is magnetically sourced, whereas in [9] the corresponding vector is electrically sourced a direct comparison is not possible. We have explicitly checked that our subtracted solution solves all supergravity equations. Furthermore, since the dilatons are all obtained to be equal and so are the axions and the three vectors $\check{A}^1_{[1]} = \check{A}^2_{[1]} = \check{A}^3_{[1]}$, the resulting solution is in fact a solution of the N=2 S=T=U supergravity. This fact has been previously noted as well [8, 9].

4 Conclusions

The key result of this paper is to show that the multicharge subtracted geometry can be obtained via a series of solution generating transformations on the original black hole field configuration. There are number of ways in which our study can be extended. In this work we have concentrated on a four-charge four-dimensional black hole carrying three magnetic charges and one electric charge. It is fairly clear from our work how to implement the same procedure for the black hole carrying two electric and two magnetic charges. It can be a useful exercise to fill in all details. In this regard understanding the precise meaning of equations (3.7)–(3.9), and how they can be relaxed is an important future direction. In the same line of thought, it is interesting to explore a similar series of transformations for the non-extremal rotating three-charge five-dimensional asymptotically flat black hole.

As explained in the previous work of Cvetic and Larsen [7, 8] and Cvetic and Gibbons [9], the entropy and thermodynamic properties of the black hole are preserved by the transformations leading to the subtracted geometries. It is hoped that the dual CFT description of the black hole is also somehow preserved. With these motivations it is of interest to further study these geometries and in particular to explore the existence of asymptotic Virasoro algebras in the subtracted geometries. It will then be of interest to know how the asymptotic Virasoro symmetries get transformed under the inverse solution generating transformations. Such a line of investigation can teach us some general and important lessons about non-extremal rotating black holes in string theory and their relation to two-dimensional conformal field theories. We hope to report on some of these issues in our future work.

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A The Set-Up

In this section we present the set-ups we work with. We also present certain details on the implementation of the SO(4,4) coset model.

A.1 A Chain of Dimensional Reductions

Various relations through dimensional reduction, truncations, and oxidations are presented. All results of this section are already well known in the literature. For this reason we shall be brief. The main purpose of this section to set the notation and conventions for the main text of the paper.

Truncation of IIB Theory on T^4

A well known consistent truncation of the IIB theory on a four-torus is as follows

$$ds^2_{10,\,\mathrm{string}} = ds^2_6 + e^{\frac{\Phi}{\sqrt{2}}} ds^2_4, \qquad \Phi_{10} = \frac{\Phi}{\sqrt{2}}, \qquad C^{\mathrm{RR}}_{[2]} = C_{[2]}, \tag{A.1}$$

where ds^2_4 denotes the metric on the four-torus and $C^{\mathrm{RR}}_{[2]}$ is the Ramond-Ramond two-form of the IIB theory. The rest of the IIB fields are set to zero. The two-form $C_{[2]}$ is the descendant from the IIB Ramond-Ramond $C^{\rm RR}_{[2]}$ to six-dimensions. The resulting six-dimensional theory contains a graviton, an antisymmetric tensor and a dilaton. The bosonic part of the Lagrangian is [24]

$$\mathcal{L}_{6B} = R_6 \star_6 1 - \frac{1}{2} \star_6 d\Phi \wedge d\Phi - \frac{1}{2} e^{\sqrt{2}\Phi} \star_6 F_{[3]} \wedge F_{[3]}, \tag{A.2}$$

with the three-form field strength $F_{[3]} = dC_{[2]}$. Upon further dimensional reduction on a two-torus the six-dimensional theory (A.2) reduces to the N=2 STU model in four-dimensions. We present certain details of this construction in the following.

Five-dimensional $\mathrm{U}(1)^3$ supergravity

M-theory on a six-torus admits a truncation to five-dimensional $\mathrm{U}(1)^3$ supergravity. For relevant details see e.g. [25]. It can also be obtained by circle reduction of the Lagrangian (A.2). We follow this route here. Using the standard Kaluza-Klein ansatz for the six-dimensional fields [26]

$$\begin{aligned} ds_6^2 &= e^{-\sqrt{\frac{3}{2}}\Psi}(dz_6 + A^1_{[1]})^2 + e^{\frac{1}{\sqrt{6}}\Psi} ds_5^2 && \text{(A.3)} \\ F_{[3]} &= F^{(5\mathrm{d})}_{[3]} + dA^2_{[1]} \wedge (dz + A^1_{[1]}) && \text{(A.4)} \end{aligned}$$

with

$$F^{(5\mathrm{d})}_{[3]} = dC^{(5\mathrm{d})}_{[2]} - dA^2_{[1]} \wedge A^1_{[1]}, \tag{A.5}$$

we obtain the following five-dimensional Lagrangian

$$\begin{aligned} \mathcal{L}_5 = \; & R_5 \star_5 1 - \frac{1}{2} \star_5 d\Phi \wedge d\Phi - \frac{1}{2} \star_5 d\Psi \wedge d\Psi - \frac{1}{2} e^{-2\sqrt{\frac{2}{3}}\Psi} \star_5 F^1_{[2]} \wedge F^1_{[2]} \\ & - \frac{1}{2} e^{-\sqrt{\frac{2}{3}}\Psi + \sqrt{2}\Phi} \star_5 F^{(5\mathrm{d})}_{[3]} \wedge F^{(5\mathrm{d})}_{[3]} - \frac{1}{2} e^{\sqrt{\frac{2}{3}}\Psi + \sqrt{2}\Phi} \star_5 F^2_{[2]} \wedge F^2_{[2]}, \end{aligned} \tag{A.6}$$

where $F^I_{[2]} = dA^I_{[1]}$ and $I = 1, 2$. Now, in five-dimensions the two-form $C^{(5\mathrm{d})}_{[2]}$ is dual to a one-form, which we denote as $A^3_{[1]}$. After this dualization we end up with three one-forms in five-dimensions. We use the notation $A^I_{[1]}$, where now the index I runs as $I = 1, 2, 3$. We see the triality structure of the $\mathrm{U}(1)^3$ supergravity emerging. The Chern-Simons term of the $\mathrm{U}(1)^3$ supergravity is also obtained through this dualization.

To see this, recall that in the process of dualisation, Bianchi identities exchange role with the equations of motion. The Bianchi identity for $F^{(5\mathrm{d})}_{[3]}$ is

$$dF^{(5\mathrm{d})}_{[3]} + F^2_{[2]} \wedge F^1_{[2]} = 0. \tag{A.7}$$

The easiest way to do the dualization is to introduce $A^3_{[1]}$ as a Lagrange multiplier for the Bianchi identity (A.7). Adding the appropriate Lagrange multiplier term to (A.6) we get

$$\mathcal{L}'_5 = \mathcal{L}_5 + A^3_{[1]} \wedge (dF^{(5\mathrm{d})}_{[3]} + F^2_{[2]} \wedge F^1_{[2]}). \tag{A.8}$$

As the next step, we treat the field strength $F^{(5\mathrm{d})}_{[3]}$ as a fundamental fields. Varying $\mathcal{L}'_5$ with respect to $F^{(5\mathrm{d})}_{[3]}$ we find

$$F^3_{[2]} - e^{-\sqrt{\frac{2}{3}}\Psi+\sqrt{2}\Phi} \star_5 F^{(5\mathrm{d})}_{[3]} = 0. \tag{A.9}$$

Substituting this back into the Lagrangian (A.8), we get

$$\begin{aligned}\mathcal{L}'_5 \;\; = \;\; & R_5 \star_5 1 - \frac{1}{2}\star_5 d\Phi \wedge d\Phi - \frac{1}{2}\star_5 d\Psi \wedge d\Psi \\ & -\frac{1}{2}e^{-2\sqrt{\frac{2}{3}}\Psi}\star_5 F^1_{[2]} \wedge F^1_{[2]} - \frac{1}{2}e^{\sqrt{\frac{2}{3}}\Psi+\sqrt{2}\Phi}\star_5 F^2_{[2]}\wedge F^2_{[2]} \\ & -\frac{1}{2}e^{\sqrt{\frac{2}{3}}\Psi-\sqrt{2}\Phi}\star_5 F^3_{[2]}\wedge F^3_{[2]} + A^3_{[1]}\wedge F^2_{[2]}\wedge F^1_{[2]}.\end{aligned} \tag{A.10}$$

Lagrangian (A.10) is equivalent to five-dimensional $\mathrm{U}(1)^3$ supergravity with the parameterization of the real special manifold as

$$h^1 = e^{\sqrt{\frac{2}{3}}\Psi}, \quad h^2 = e^{-\sqrt{\frac{1}{6}}\Psi-\sqrt{\frac{1}{2}}\Phi}, \quad h^3 = e^{-\sqrt{\frac{1}{6}}\Psi+\sqrt{\frac{1}{2}}\Phi}. \tag{A.11}$$

Clearly $h^1h^2h^3 = 1$. A manifestly triality-invariant form now be written as (we drop the prime on $\mathcal{L}'_5$ from now on)

$$\mathcal{L}_5 = R_5 \star_5 \mathbf{1} - \frac{1}{2}G_{IJ}\star_5 dh^I \wedge dh^J - \frac{1}{2}G_{IJ}\star_5 F^I_{[2]}\wedge F^J_{[2]} + \frac{1}{6}C_{IJK}F^I_{[2]}\wedge F^J_{[2]}\wedge A^K_{[1]}. \tag{A.12}$$

The symbol C_{IJK} is pairwise symmetric in its indices with $C_{123} = 1$ and is zero otherwise. The metric G_{IJ} on the scalar moduli space is diagonal with entries $G_{II} = (h^I)^{-2}$.

For completeness, let us also write the six-dimensional field strength $F_{[3]}$ in terms of the five-dimensional fields introduced above. We obtain

$$F_{[3]} = -(h^3)^{-2}\star_5 dA^3_{[1]} + dA^2_{[1]}\wedge(dz_6 + A^1_{[1]}). \tag{A.13}$$

Together with (A.3), equation (A.13) allows us to uplift any solution of five-dimensional $\mathrm{U}(1)^3$ supergravity to the IIB theory. Examining the RR 3-form (A.13) reveals that the electric charge that couples to the two-form $F^3_{[2]}$ arises from D5-branes wrapped on T^5: $(z_6, z_7, z_8, z_9, z_{10})$. Similarly, the electric charge that couples to the two-form $F^2_{[2]}$ arises from D1-branes wrapped along the z_6-circle. The appearance of $A^1_{[1]}$ in the metric reveals that electric charge that couples to $F^1_{[2]}$ arises from momentum (P) around the z_6-circle. The interpretation of magnetic couplings is readily obtained. The M-theory interpretation of these couplings is reviewed at several places. See e.g. [25].

Four-dimensional STU Model

Further dimensional reduction of the five-dimensional $\mathrm{U}(1)^3$ supergravity to four dimensions gives the so-called STU model. The STU model is a particular N=2 supergravity in four dimensions coupled to three vector multiplets.

To fix our notation we quickly review here the N=2 supergravity action. Four-dimensional N=2 supergravity coupled to n_v vector-multiplets is governed by a prepotential function F depending on (n_v+1) complex scalars X^Λ ($\Lambda = 0, 1, \ldots, n_v$). The bosonic degrees of freedom are the metric $g_{\mu\nu}$, the complex scalars X^Λ and a set of (n_v+1) one-forms $\check{A}^\Lambda_{[1]}$. The bosonic part of the action is given as [27]

$$\mathcal{L}_4 = R \star_4 \mathbf{1} - 2 g_{I\bar{J}} \star_4 dX^I \wedge d\bar{X}^{\bar{J}} + \frac{1}{2} \check{F}^\Lambda_{[2]} \wedge \check{G}_{\Lambda\,[2]}, \tag{A.14}$$

where $\check{F}^\Lambda_{[2]} = d\check{A}^\Lambda_{[1]}$. The ranges of the indices are $I, J = 1, \ldots, n_v$, and $g_{I\bar{J}} = \partial_I \partial_{\bar{J}} K$ is the Kähler metric with the Kähler potential $K = -\log\left[-i(\bar{X}^\Lambda F_\Lambda - \bar{F}_\Lambda X^\Lambda)\right]$. The two-forms $\check{G}_{\Lambda\,[2]}$ are defined as

$$\check{G}_{\Lambda\,[2]} = (\mathrm{Re}N)_{\Lambda\Sigma} \check{F}^\Sigma_{[2]} + (\mathrm{Im}N)_{\Lambda\Sigma} \star_4 \check{F}^\Sigma_{[2]} \ , \tag{A.15}$$

where the complex symmetric matrix $N_{\Lambda\Sigma}$ is constructed from the prepotential $F(X)$ as

$$N_{\Lambda\Sigma} = \bar{F}_{\Lambda\Sigma} + 2i \frac{(\mathrm{Im}F \cdot X)_\Lambda (\mathrm{Im}F \cdot X)_\Sigma}{X \cdot \mathrm{Im}F \cdot X} \ , \tag{A.16}$$

with $F_\Lambda = \partial_\Lambda F$ and $F_{\Lambda\Sigma} = \partial_\Lambda \partial_\Sigma F$. For the system we are interested in $n_v = 3$ and the prepotential is

$$F(X) = -\frac{X^1 X^2 X^3}{X^0}. \tag{A.17}$$

Let us now make contact of this Lagrangian with the circle reduction of the five-dimensional U(1)^3 supergravity. We parametrize our five-dimensional space-time as

$$ds_5^2 = f^2 (dz + \check{A}^0_{[1]})^2 + f^{-1} ds_4^2, \tag{A.18}$$

and the vectors as

$$A^I_{[1]} = \chi^I (dz + \check{A}^0_{[1]}) + \check{A}^I_{[1]}. \tag{A.19}$$

Together the graviphoton $\check{A}^0_{[1]}$ and the vectors $\check{A}^I_{[1]}$ form a symplectic vector $\check{A}^\Lambda_{[1]}$ with $\Lambda = 0, 1, 2, 3$ in four dimensions.

Upon circle reduction of the above 5d theory we obtain (with $\check{F}^\Lambda_{[2]} = d\check{A}^\Lambda_{[1]}$)

$$\begin{aligned}
\mathcal{L}_4 \ = \ & R \star_4 \mathbf{1} - \frac{1}{2} G_{IJ} \star_4 dh^I \wedge dh^J - \frac{3}{2f^2} \star_4 df \wedge df - \frac{f^3}{2} \star_4 \check{F}^0_{[2]} \wedge \check{F}^0_{[2]} \\
& - \frac{1}{2f^2} G_{IJ} \star_4 d\chi^I \wedge d\chi^J - \frac{f}{2} G_{IJ} \star_4 (\check{F}^I_{[2]} + \chi^I \check{F}^0_{[2]}) \wedge (\check{F}^J_{[2]} + \chi^J \check{F}^0_{[2]}) \\
& + \frac{1}{2} C_{IJK} \chi^I \check{F}^J_{[2]} \wedge \check{F}^K_{[2]} + \frac{1}{2} C_{IJK} \chi^I \chi^J \check{F}^0_{[2]} \wedge \check{F}^K_{[2]} + \frac{1}{6} C_{IJK} \chi^I \chi^J \chi^K \check{F}^0_{[2]} \wedge \check{F}^0_{[2]} \, .
\end{aligned} \tag{A.20}$$

The scalars χ^I and h^I combine to form the complex scalars $z^I = X^I / X^0$ in the STU theory according to $z^I = -\chi^I + i f h^I$. Using the gauge fixing condition $X^0 = 1$ and the replacement $X^I \to z^I$ the action (A.14) for the prepotential (A.17) can be shown to be exactly equivalent to the action (A.20). In order to perform the above computation we found appendix A of reference [28] useful.

A.2 SO(4,4)/(SO(2,2) × SO(2,2)) Coset Model in 3d

In this section we discuss how to obtain the SO(4,4)/(SO(2,2) × SO(2,2)) coset model in three-dimensions by performing further dimensional reduction over time direction of the STU action (A.20). We parametrize our four-dimensional space-time as

$$ds_4^2 = -e^{2U}(dt+\omega_3)^2 + e^{-2U}ds_3^2, \tag{A.21}$$

and the four-dimensional vectors as

$$\check{A}^{\Lambda}_{[1]} = \zeta^{\Lambda}(dt+\omega_3) + A_3^{\Lambda}, \tag{A.22}$$

where ω_3 and A_3^{Λ} are one-forms in three-dimensions.

Following [29, 30] we dualize the three dimensional vectors as

$$-d\tilde{\zeta}_{\Lambda} = e^{2U}(\mathrm{Im}N)_{\Lambda\Sigma} \star_3 (d{A_3}^{\Sigma} + \zeta^{\Sigma}d\omega_3) + (\mathrm{Re}N)_{\Lambda\Sigma}d\zeta^{\Sigma} \tag{A.23}$$

where $\tilde{\zeta}_{\Lambda}$ are pseudo-scalars. Similarly we define the pseudo-scalar σ dual to ω_3 as

$$-d\sigma = 2e^{4U}\star_3 d\omega_3 - \zeta^{\Lambda}d\tilde{\zeta}_{\Lambda} + \tilde{\zeta}^{\Lambda}d\zeta_{\Lambda}. \tag{A.24}$$

The full set of three-dimensional scalar fields are now $\varphi^a = \{U, z^I, \bar{z}^I, \zeta^{\Lambda}, \tilde{\zeta}_{\Lambda}, \sigma\}$. The resulting three-dimensional Lagrangian takes the form

$$\mathcal{L}_3 = R\star_3 \mathbf{1} - \frac{1}{2}G_{ab}\partial\varphi^a\partial\varphi^b, \tag{A.25}$$

where the target space Lorentzian manifold parametrized by scalars φ^a is of signature $(8,8)$. It is an analytic continuation of the c-map of Ferrara and Sabharwal [31]. The metric in our conventions is[5]

$$\begin{aligned} G_{ab}d\varphi^a d\varphi^b \;=\;& 4dU^2 + 4g_{I\bar{J}}dz^I dz^{\bar{J}} + \frac{1}{4}e^{-4U}\left(d\sigma + \tilde{\zeta}_{\Lambda}d\zeta^{\Lambda} - \zeta^{\Lambda}d\tilde{\zeta}_{\Lambda}\right)^2 \\ &+e^{-2U}\left[(\mathrm{Im}N)_{\Lambda\Sigma}d\zeta^{\Lambda}d\zeta^{\Sigma} + ((\mathrm{Im}N)^{-1})^{\Lambda\Sigma}\left(d\tilde{\zeta}_{\Lambda} + (\mathrm{Re}N)_{\Lambda\Xi}d\zeta^{\Xi}\right)\left(d\tilde{\zeta}_{\Sigma} + (\mathrm{Re}N)_{\Sigma\Xi}d\zeta^{\Xi}\right)\right]. \end{aligned} \tag{A.26}$$

This symmetric space can be parametrized in the Iwasawa gauge by the coset element [30]

$$\mathcal{V} = e^{-U\,H_0}\cdot\left(\prod_{I=1,2,3} e^{-\frac{1}{2}(\log y^I)H_I}\cdot e^{-x^I E_I}\right)\cdot e^{-\zeta^{\Lambda}E_{q_{\Lambda}}-\tilde{\zeta}_{\Lambda}E_{p^{\Lambda}}}\cdot e^{-\frac{1}{2}\sigma E_0}, \tag{A.27}$$

where we use the notation $z^I = x^I + iy^I$ (so, $y^I = fh^I$, $x^I = -\chi^I$). The Iwasawa parameterization only covers an open subset of the full manifold. This is because the target space

[5] Our conventions are identical to that of [30]. There is a minor typo of a factor of 1/2 in equation (4.4) of [30].

is not precisely the c-map but an analytic continuation of it. The metric (A.26) is obtained from the Maurer-Cartan one-form $\theta = d\mathcal{V}\cdot\mathcal{V}^{-1}$,

$$G_{ab}d\varphi^a d\varphi^b = \mathrm{Tr}(P_*\, P_*) \ , \quad P_* = \frac{1}{2}(\theta + \eta'\,\theta^T\eta'^{-1}) \ , \quad \eta' = \mathrm{diag}(-1,-1,1,1,-1,-1,1,1), \tag{A.28}$$

where η' is the quadratic form preserved by SO(2,2)×SO(2,2). The matrix $\mathcal{M}$ is defined as $\mathcal{M} = (\mathcal{V}^\sharp)\mathcal{V}$, with $\theta^\sharp = \eta'\theta^T\eta'^{-1}$ for all $\theta \in \mathfrak{so}(4,4)$. For convenience we explicitly list the matrix representation of SO(4,4) in appendix A.3.

A.3 Matrix representation of $\mathfrak{so}$(4,4) Lie algebra

An explicit realization of the generators of $\mathfrak{so}(4,4)$ is as follows. Calling E_{ij} the 8×8 matrix with 1 in the i-th row and j-th column and 0 elsewhere, the $\mathfrak{so}(4,4)$ generators in the fundamental representation is given by

$$\begin{aligned} H_0 &= E_{33}+E_{44}-E_{77}-E_{88} \qquad & H_1 &= E_{33}-E_{44}-E_{77}+E_{88} \\ H_2 &= E_{11}+E_{22}-E_{55}-E_{66} \qquad & H_3 &= E_{11}-E_{22}-E_{55}+E_{66} \end{aligned} \tag{A.29}$$

$$\begin{aligned} E_0 &= E_{47}-E_{38} \qquad & E_1 &= E_{87}-E_{34} \\ E_2 &= E_{25}-E_{16} \qquad & E_3 &= E_{65}-E_{12} \end{aligned} \tag{A.30}$$

$$\begin{aligned} F_0 &= E_{74}-E_{83} \qquad & F_1 &= E_{78}-E_{43} \\ F_2 &= E_{52}-E_{61} \qquad & F_3 &= E_{56}-E_{21} \end{aligned} \tag{A.31}$$

$$\begin{aligned} E_{q_0} &= E_{41}-E_{58} \qquad & E_{q_1} &= E_{57}-E_{31} \\ E_{q_2} &= E_{46}-E_{28} \qquad & E_{q_3} &= E_{42}-E_{68} \end{aligned} \tag{A.32}$$

$$\begin{aligned} F_{q_0} &= E_{14}-E_{85} \qquad & F_{q_1} &= E_{75}-E_{13} \\ F_{q_2} &= E_{64}-E_{82} \qquad & F_{q_3} &= E_{24}-E_{86} \end{aligned} \tag{A.33}$$

$$\begin{aligned} E_{p^0} &= E_{17}-E_{35} \qquad & E_{p^1} &= E_{18}-E_{45} \\ E_{p^2} &= E_{67}-E_{32} \qquad & E_{p^3} &= E_{27}-E_{36} \end{aligned} \tag{A.34}$$

$$\begin{aligned} F_{p^0} &= E_{71}-E_{53} \qquad & F_{p^1} &= E_{81}-E_{54} \\ F_{p^2} &= E_{76}-E_{23} \qquad & F_{p^3} &= E_{72}-E_{63}. \end{aligned} \tag{A.35}$$

This basis of representation is identical to the one given in [30]. For more details we refer the reader to this reference. Other implementations of the SO(4,4) coset model can be found in [12, 32, 33].

A.4 Group element for the M5-M5-M5 black hole

On the Kerr matrix $\mathcal{M}_{\text{Kerr}}$ we act with the group element

$$g = \exp\left[\alpha_1(E_{p^1} + F_{p^1})\right] \cdot \exp\left[\alpha_2(E_{p^2} + F_{p^2})\right] \cdot \exp\left[\alpha_3(E_{p^3} + F_{p^3})\right], \tag{A.36}$$

as

$$\mathcal{M}_{\text{Kerr}} \to \mathcal{M}_{3-\text{charge}} = g^\sharp \cdot \mathcal{M}_{\text{Kerr}} \cdot g. \tag{A.37}$$

Reading off the new scalars from the new matrix $\mathcal{M}_{3-\text{charge}}$ and performing the inverse dualization through (A.23)–(A.24) we obtain the spinning magentic one-brane of five-dimensional $\text{U}(1)^3$ supergravity as presented in section 2.

B Three Dimensional Fields: 4d Asymptotically Flat

For convenience and completeness we list all the resulting three-dimensional fields obtained after the action of the group element (A.36) on the coset matrix $\mathcal{M}_{\text{Kerr}}$:

$$x_1 = -4c_1s_2s_3\frac{am\cos\theta}{\Omega_1}, \tag{B.1}$$

$$x_2 = -4c_2s_3s_1\frac{am\cos\theta}{\Omega_2}, \tag{B.2}$$

$$x_3 = -4c_3s_1s_2\frac{am\cos\theta}{\Omega_3}, \tag{B.3}$$

$$y_1 = \frac{2}{\Omega_1}\sqrt{\xi}, \qquad y_2 = \frac{2}{\Omega_2}\sqrt{\xi}, \qquad y_3 = \frac{2}{\Omega_3}\sqrt{\xi}, \tag{B.4}$$

$$\zeta^0 = 4c_1c_2c_3s_1s_2s_3\frac{a^2m^2\cos^2\theta}{\xi}, \tag{B.5}$$

$$\zeta^1 = -2s_1c_2c_3(r^2 + a^2\cos^2\theta + 2mrs_1^2)\frac{am\cos\theta}{\xi}, \tag{B.6}$$

$$\zeta^2 = -2s_2c_3c_1(r^2 + a^2\cos^2\theta + 2mrs_2^2)\frac{am\cos\theta}{\xi}, \tag{B.7}$$

$$\zeta^3 = -2s_3c_1c_2(r^2 + a^2\cos^2\theta + 2mrs_3^2)\frac{am\cos\theta}{\xi}, \tag{B.8}$$

$$\tilde{\zeta}_0 = 2ma\cos\theta s_1s_2s_3\frac{\Delta_2}{\xi}, \tag{B.9}$$

$$\tilde{\zeta}_1 = \frac{mc_1s_1}{\xi}\left(4ma^2\cos^2\theta s_2^2s_3^2 - r\Omega_1\right), \tag{B.10}$$

$$\tilde{\zeta}_2 = \frac{mc_2s_2}{\xi}\left(4ma^2\cos^2\theta s_1^2s_3^2 - r\Omega_2\right), \tag{B.11}$$

$$\tilde{\zeta}_3 = \frac{mc_3s_3}{\xi}\left(4ma^2\cos^2\theta s_1^2s_2^2 - r\Omega_3\right), \tag{B.12}$$

and finally

$$e^{2U} = \frac{\Delta_2}{\sqrt{\xi}}, \tag{B.13}$$

$$\sigma = -4ma\cos\theta c_1c_2c_3\frac{r^2 + a^2\cos^2\theta + mr(s_1^2 + s_2^2 + s_3^2)}{\xi}, \tag{B.14}$$

where Ω_1,Ω_2, and Ω_3 are defined in (2.8) and Δ_2 and ξ are defined respectively in (3.13) and (2.8). Finally, $c_1 = \cosh\alpha_1$, $c_2 = \cosh\alpha_2$, $c_3 = \cosh\alpha_3$ and $s_1 = \sinh\alpha_1$, $c_2 = \sinh\alpha_2$, $c_3 = \sinh\alpha_3$.

C Three Dimensional Fields: 4d Subtracted Geometry

Here for completeness we list all the resulting three-dimensional fields obtained after the action of the group element (3.10) with the choices (3.7)–(3.9) on the coset matrix $\mathcal{M}_{3-\text{charge}}$ (where $x = \cos\theta$):

$$x_1 = x_2 = x_3 = -\frac{ax(\Pi_c - \Pi_s)}{m}. \tag{C.1}$$

Defining

$$\tilde{\Delta}_s = 2mr(\Pi_c^2 - \Pi_s^2) + 4m^2\Pi_s^2 - a^2x^2(\Pi_c - \Pi_s)^2, \tag{C.2}$$

we have

$$y_1 = y_2 = y_3 = \frac{\sqrt{\tilde{\Delta}_s}}{2m}, \tag{C.3}$$

$$\zeta^0 = \frac{4m^2\Pi_c\Pi_s + a^2x^2(\Pi_c - \Pi_s)^2}{\tilde{\Delta}_s(\Pi_c^2 - \Pi_s^2)}, \tag{C.4}$$

$$\zeta^1 = \zeta^2 = \zeta^3 = -\frac{ax(2m\Pi_s + r(\Pi_c - \Pi_s))}{\tilde{\Delta}_s}, \tag{C.5}$$

$$\tilde{\zeta}_0 = \frac{ax}{2m\tilde{\Delta}_s}\left((\Pi_c - \Pi_s)^2(\Pi_c + \Pi_s)(r^2 + a^2x^2) - 2mr(\Pi_c^3 - 2\Pi_c\Pi_s^2 + \Pi_s^3) - 4m^2\Pi_c\Pi_s^2\right), \tag{C.6}$$

$$\tilde{\zeta}^1 = \tilde{\zeta}^2 = \tilde{\zeta}^3, \tag{C.7}$$

$$\begin{aligned}\tilde{\zeta}^1 = {} & \frac{1}{2\tilde{\Delta}_s(\Pi_c^2 - \Pi_s^2)}\Big[-2mr(\Pi_c^2 - \Pi_s^2)(1 + \Pi_c^2 - 3\Pi_s^2) + 4m^2\Pi_s^2(\Pi_s^2 - \Pi_c^2 - 1) \\ & +(\Pi_c - \Pi_s)^2(2r^2(\Pi_c + \Pi_s)^2 + a^2x^2(1 + (\Pi_c + \Pi_s)^2))\Big],\end{aligned} \tag{C.8}$$

and finally

$$\begin{aligned}\sigma = {} & \frac{ax}{2m(\Pi_c^2 - \Pi_s^2)\tilde{\Delta}_s}\Big[(r^2 + a^2x^2)(\Pi_c - \Pi_s)^2(\Pi_c + \Pi_s) - mr(\Pi_c - \Pi_s)(3 + 3\Pi_c^2 - \Pi_s^2) \\ & -2m^2\Pi_s(3 + \Pi_c^2 + \Pi_s^2)\Big],\end{aligned} \tag{C.9}$$

$$e^{2U} = \frac{r^2 + a^2x^2 - 2mr}{2m\sqrt{\tilde{\Delta}_s}}. \tag{C.10}$$

D Magnetic One Brane of U(1)^3 Theory

We provide an analysis of physical properties and near horizon limit of the rotating magnetic string (2.2).

D.1 Physical Properties

From the g^{rr} component of the metric it is seen that the solution has a regular outer horizon at $r = r_+ := m + \sqrt{m^2 - a^2}$ and an inner horizon at $r = r_- := m - \sqrt{m^2 - a^2}$. The extremal limit is when the two horizons coincide, i.e., $m = a$. The ADM stress tensor takes the form

$$T_{tt} = \frac{m}{2G}\left(2 + s_1^2 + s_2^2 + s_3^2\right), \quad T_{zz} = -\frac{m}{2G}\left(1 + s_1^2 + s_2^2 + s_3^2\right), \quad T_{tz} = 0, \tag{D.1}$$

where T_{tt} and T_{tz} are respectively the energy and linear momentum density along the string. T_{zz} is the pressure density; the ADM tension is $\mathcal{T} = -T_{zz}$. Physical properties of the solution such as mass, inner and outer horizon areas, angular momentum, and angular velocities can be straightforwardly calculated. For the asymptotic quantities one finds

$$M = 2\pi R T_{tt} = \frac{\pi m R}{G}\left(2 + s_1^2 + s_2^2 + s_3^2\right), \tag{D.2}$$

$$P_z = 2\pi R T_{tz} = 0, \quad J_\phi = \frac{2\pi R m a}{G} c_1 c_2 c_3, \tag{D.3}$$

and for the quantities at the outer ($r = r_+$) and inner ($r = r_-$) horizon one finds

$$\Omega_\phi^\pm = \frac{a}{2 c_1 c_2 c_3 m r_\pm}, \quad v_z^\pm = -\frac{a^2 s_1 s_2 s_3}{r_\pm^2 c_1 c_2 c_3}, \quad A_H^\pm = 8\pi^2 R\,(r_\pm^2 + a^2) c_1 c_2 c_3. \tag{D.4}$$

Temperatures of the inner and the outer horizons can be calculated from surface gravities,

$$T_H^\pm = \frac{r_\pm - r_\mp}{4\pi (r_\pm^2 + a^2) c_1 c_2 c_3}\ . \tag{D.5}$$

Magentic charges are defined as $Q_M^I = \frac{1}{4\pi G}\int_{S^2_\infty} F^I = -2mG^{-1} s_I c_I$. The magnetic potentials dual to these charges can be guessed, say using the Smarr relation[6]

$$M = \frac{3}{2}\left(\frac{1}{4G} T_H^+ A_H^+ + \Omega_\phi^+ J_\phi\right) + \frac{1}{2}\mathcal{T}(2\pi R) + \frac{1}{2}\sum_{I=1}^{3} \Phi^I Q_M^I. \tag{D.6}$$

This guess is then confirmed by explicitly verifying the first law

$$dM = \frac{1}{4G} T_H^+ dA_H^+ + \Omega_\phi^+ J_\phi + \sum_{I=1}^{3} \Phi^I dQ_M^I + 2\pi \mathcal{T} dR. \tag{D.7}$$

[6]A first principle calculation of the magnetic potentials requires appropriately generalizing the formalism of [34] (see also [16]) to the U(1)^3 theory. Such a generalization is beyond the aspirations of the present study.

We find $\Phi_I = -\frac{\pi R s_I}{2c_I}$. Moreover, the product $A_H^+ A_H^- = 4(8\pi^2 R)^2 m^2 a^2 c_1^2 c_2^2 c_3^2 = (8\pi G J_\phi)^2$ takes the expected form [35].

Of particular interest is the fact that for the un-boosted solution the linear velocities $v_z^\pm$ (D.4) are non-zero, while the ADM momentum P_z is zero. Since $v_z^\pm$ vanish if either $a = 0$ or any of the $\alpha_I = 0$, this is a cumulative effect of rotation and *all three* magnetic charges.

D.2 Near Horizon Limit

The near-horizon limit of the solution in section 2.1 is obtained as follows. First, we write the extremal rotating solution ($m = a$) in comoving coordinates and second we zoom in close to the horizon. More precisely, we perform

$$r \to a + \mu r, \quad t \to \frac{t}{\mu}, \quad \phi \to \phi + \Omega_\phi \frac{t}{\mu}, \quad z \to z + v_z \frac{t}{\mu} \tag{D.8}$$

with

$$\Omega_\phi = \frac{1}{2ac_1c_2c_3}, \quad v_z = -\frac{s_1s_2s_3}{c_1c_2c_3} \tag{D.9}$$

and send $\mu \to 0$. In this limit asymptotically flat region is dispensed with. The resulting configuration is a solution of the U(1)^3 supergravity. The geometry has enhanced isometry SL(2, $\mathbb{R}$)×U(1)×U(1), as is familiar from general near-horizon limits [36, 37]. The solution reads as

$$\begin{aligned} ds^2_{\text{nh}} &= \Gamma(x)\left[-(k_\phi)^2 r^2 dt^2 + \frac{dr^2}{r^2} + \frac{dx^2}{1-x^2}\right] + \gamma_{\phi\phi}(x)e_\phi^2 + 2\gamma_{\phi z}(x)e_\phi\, e_z + \gamma_{zz}(\theta)e_z^2 \\ A^I &= f_\phi^I(x)e_\phi + f_z^I(x)e_z, \qquad h^I = h^I(x) \end{aligned} \tag{D.10}$$

where $e_\phi = d\phi + k_\phi r dt$, $e_z = dz + k_z r dt$. All functions are expressed most easily expressed as ($x = \cos\theta$)

$$\begin{aligned} k_\phi &= \frac{1}{2a^2c_1c_2c_3}, \quad k_z = -\frac{2s_1s_2s_3}{ac_1c_2c_3}, \quad \Gamma(x) = \frac{1}{2}\left(\Omega_1\Omega_2\Omega_3\right)^{1/3} \\ \gamma_{zz} &= \frac{4\xi}{(\Omega_1\Omega_2\Omega_3)^{2/3}}, \quad \gamma_{z\phi} = 8as_1s_2s_3\frac{\xi - 4a^4x^2c_1^2c_2^2c_3^2}{(\Omega_1\Omega_2\Omega_3)^{2/3}} \\ \gamma_{\phi\phi} &= \frac{16a^2s_1^2s_2^2s_3^2(\xi - 4a^4x^2c_1^2c_2^2c_3^2)^2 + 2a^4c_1^2c_2^2c_3^2(1-x^2)\Omega_1\Omega_2\Omega_3}{\xi(\Omega_1\Omega_2\Omega_3)^{2/3}} \\ f_\phi^1 &= 4xs_1c_1a^3\frac{1+(1+2s_2^2)(1+2s_3^2)}{\Omega_1}, \quad f_z^1 = 4xa^2\frac{c_1s_2s_3}{\Omega_1}, \\ h^I &= (\Omega_1\Omega_2\Omega_3)^{1/3}\Omega_I^{-1} \end{aligned} \tag{D.11}$$

The rest of the functions f_ϕ^2, f_ϕ^3 and f_z^2, f_z^3 are obtained by obvious cyclic permutations. In all expressions in (D.11) the functions Ω_I and ξ are computed at $r = a$. An alternative presentation of these function can also be given as in [10]. Now let us look at various interesting limiting cases:

1. Upon setting all three M5 charges equal one recovers exactly the expressions previously obtained in (11) of [10].

2. When M5 charges are set to zero the solution reduces to the NHEK geometry [38] times a circle, as expected.

3. The non-trivial observation of [10] is that in the limit of no rotation, while keep the number of M5 branes n_I fixed, the solution reduces to a null orbifold of $\mathrm{AdS}_3 \times \mathrm{S}^2$,

$$\begin{aligned} ds^2 &= l^2 \left(\frac{dr^2}{4r^2} - 2r dt dz \right) + l_{S_2}^2 d\Omega_2, \\ A^I &= -\frac{n_I}{2} x d\phi, \quad z \sim z + 2\pi, \qquad \Phi^I = \frac{n_I}{l} \end{aligned} \tag{D.12}$$

where the two sphere has radius $l_{S_2} = \frac{1}{2}\ell_p (n_1 n_2 n_3)^{1/3}$, the AdS_3 radius is $l = \ell_p (n_1 n_2 n_3)^{1/3}$, with $\ell_p = (4G/\pi)^{1/3}$. This solution has zero entropy and zero angular momentum.

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