

**Group field theory and simplicial gravity path integrals: A model for Holst-Plebanski gravity**Aristide Baratin<sup>1</sup> and Daniele Oriti<sup>2</sup><sup>1</sup>*Centre de Physique Théorique, CNRS UMR 7644, Ecole Polytechnique, F-9112 Palaiseau Cedex, France*<sup>2</sup>*Max Planck Institute for Gravitational Physics, Albert Einstein Institute Am Mühlenberg 1, 14467 Golm, Germany*

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In a recent work, a dual formulation of group field theories as noncommutative quantum field theories has been proposed, providing an exact duality between spin foam models and noncommutative simplicial path integrals for constrained BF theories. In light of this new framework, we define a model for four-dimensional gravity which includes the Immirzi parameter  $\gamma$ . It reproduces the Barrett-Crane amplitudes when  $\gamma = \infty$ , but differs from existing models otherwise; in particular, it does not require any rationality condition for  $\gamma$ . We formulate the amplitudes both as BF simplicial path integrals with explicit noncommutative  $B$  variables, and in spin foam form in terms of Wigner  $15j$  symbols. Finally, we briefly discuss the correlation between neighboring simplices, often argued to be a problematic feature, for example, in the Barrett-Crane model.

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**I. INTRODUCTION**

Group field theories [1] are quantum field theories showing up as a higher dimensional generalization of matrix models in background independent approaches to quantum gravity [2]. The perturbative Feynman expansion generates stranded graphs dual to simplicial complexes of all topologies, weighted by spin foam amplitudes [3]. Conversely, it can be shown [4] that any spin foam model admits a group field theory (GFT) formulation, which removes its dependence on the triangulation.

Most spin foam and GFT models for quantum gravity are based on modifications of the Ooguri model for four-dimensional BF theory.<sup>1</sup> This approach is motivated by the fact that classical four-dimensional gravity can be expressed as a constrained BF theory (Plebanski formulation) [6,7]

$$S(\omega, B, \lambda) = \int_{\mathcal{M}} \text{Tr} B \wedge F(\omega) + \lambda \mathcal{C}(B), \quad (1)$$

for  $\mathfrak{so}(4)$  valued<sup>2</sup> 1-form connection  $\omega$  and 2-form field  $B$ , where  $\mathcal{C}(B)$  are polynomial (so-called simplicity) constraints and  $\lambda$  is some Lagrange multiplier. The variation with respect to the Lagrange multiplier constrains  $B$  to be a function of a tetrad 1-form field  $B = *(e \wedge e)$ , turning BF to the Palatini action for gravity in the first-order formalism. The Immirzi parameter  $\gamma$ , which plays a crucial role in loop gravity, can be introduced by replacing  $B \rightarrow B + \frac{1}{\gamma} * B$  in the BF term of the action. Solving the constraints reproduces the Holst action [8], classically equivalent to Palatini gravity and the starting point for the quantization leading to loop quantum gravity.

The spin foam quantization stems from a discretization of the classical theory, by choosing a triangulation  $\Delta$  on

$\mathcal{M}$ . While the most direct route to quantization would be to include a discrete analogue of the constraints  $\mathcal{C}(B)$  into the definition of the measure of the discretized path integral [9,10]

$$I_{\Delta} = \int \mathcal{D}[\omega_{\Delta}, B_{\Delta}] \delta(\mathcal{C}(B_{\Delta})) e^{i \text{Tr} B_{\Delta} F_{\Delta}}, \quad (2)$$

the standard spin foam strategy consists of quantizing first the topological BF part of the discretized theory: the discretization and quantization of BF theories in any dimension are in fact well-understood [10,11]. The task is then to implement a quantum version of the constraints in order to recover the gravity degrees of freedom. This has been shown to be a quite subtle task, partly because of the very simplicial setting in which the construction takes place: in fact, no standard canonical quantization procedure exists in such discrete setting, and of course things only become more difficult when the classical system to be quantized is a background independent simplicial gravity theory.<sup>3</sup> Proposals for the implementation of the constraints motivated by the geometric quantization of simplicial structures [13,14] first led to the famous Barrett-Crane model [15], and more recently to the Engle-Pereira-Rovelli-Livine (EPRL) model [16], which includes the Immirzi parameter and reduces to Barrett-Crane when  $\gamma = \infty$ .

One of the main difficulties that this strategy encounters stems from the noncommutative nature of the geometrical variables in the BF Ooguri model [13]—beginning with the quantum  $B$  variables themselves represented as generators of the gauge group—which obscures the geometrical interpretation of the constraints. As an attempt to remedy this problem, Livine and Speziale suggested [17] to rewrite the

<sup>1</sup>With some exceptions, see for example [5].<sup>2</sup>We will restrict to the Riemannian signature in this paper.<sup>3</sup>See however [12] for a recent proposal of a general canonical formalism for simplicial gravity.

BF amplitudes in a basis of Perelomov group coherent states and to interpret the coherent state labels as classical bivectors (though with quantized norm) on which to impose the constraints. The realization of this idea led to the Freidel-Krasnov (FK) or  $FK_\gamma$  model [18] which, remarkably, coincides with EPRL when  $\gamma < 1$ . Using the coherent state representation, it can be shown that both models exhibit a path integral-like formulation [19]. Though it provides a powerful tool to relate the models to Regge gravity in the semiclassical limit [20], this formulation, which involves a quite specific and nonstandard action, is however far remote from the original path integral (2).

The goal of the series of papers [21,22] and the present work, in the spirit of earlier works [23] by Bonzom and Livine, is instead to dig deeper into the relation between spin foam models and simplicial path integrals of the type (2). Our main result is to show that generic spin foam models based on quantum BF theory have a dual formulation as a version of (2) in which functionals of the discrete bivectors  $B_\Delta$  are endowed with a noncommutative structure (star product) deforming the usual pointwise product. Such a formulation of the path integral,<sup>4</sup> in contrast to standard constructions with commutative variables [25], captures the key aspect of noncommutativity of bivectors in spin foam models, covariant counterpart of the noncommutativity of the flux variables in loop gravity [26,27]. It is important to note that this noncommutativity is not an anomaly of quantization, as it can be traced back to the classical theory [26,28–30].

This result is important for the study of spin foam models, in many respects. It gives a direct correspondence between purely algebraic amplitudes describing the quantum geometry on one hand, and a measure on the variables of the classical theory on the other. It allows for a direct comparison between the spin foam quantization and a proper path integral quantization of Holst-Plebanski gravity [31,32]. It also opens the way for a precise study of the consequences of noncommutativity of the geometry inherent to these models.

The duality between spin foam models and the path integrals (2) is realized at the level of the generating group field theories. We will thus work in this very general setting, though the construction could also be carried out directly at the level of the amplitudes. The mechanism is the following. As we will review in Sec. II, in addition to its usual formulations in terms of gauge-invariant group fields  $\varphi(g_1, \dots, g_4)$  or of its Peter-Weyl tensor components, the Ooguri GFT model for  $SO(4)$  BF theory has a dual formulation in terms of fields on four copies of the Lie algebra  $\mathfrak{so}(4)$ , obtained by a Fourier transform, where  $d$  denotes a differential,

$$\hat{\varphi}(x_1, \dots, x_4) := \int [dg_i]^4 \varphi(g_1, \dots, g_4) e^{i \text{Tr} x_1 g_1} \dots e^{i \text{Tr} x_4 g_4}, \quad (3)$$

which endows the space of fields with a noncommutative star product (dual to group convolution). It can be shown that the gauge invariance of the field translates into a closure condition  $x_1 + \dots + x_4 = 0$  for the  $\mathfrak{so}(4)$  variables, which have a direct interpretation as the discrete  $B$  variables labeling the faces of a tetrahedron. In fact, in this representation, the GFT Feynman amplitudes are simplicial BF path integrals [21]. In such a representation, where all the geometrical variables are explicit, constrained models for gravity take a very suggestive form in terms of constrained fields  $(\mathcal{S} \star \hat{\varphi})(x_j)$  for some functions  $\mathcal{S}(x_j)$  constraining the bivectors. By construction, the Feynman amplitudes are simplicial path integrals for constrained BF theories.

In principle, every GFT and spin foam model for gravity based on BF theory, thus including EPRL-FK, can be formulated this way, with more or less natural forms for the constraint functions  $\mathcal{S}$ . In this paper, however, we rather follow a constructive approach: in Sec. III, we *define*  $\mathcal{S}$  in the most natural way in this framework, namely, in terms of (noncommutative) Dirac distributions  $\mathcal{S}(x_j) = \delta(\mathcal{C}(x_j))$ , where  $\mathcal{C}(x_j)$  are the discrete simplicity constraints. The use of Dirac distributions effectively amounts to constraining the measure on the bivector variables. As we will see, the fact that the constraints are imposed on the group field, hence in all tetrahedra, will automatically lead to additional constraints on the connection in the path integral form of the amplitudes [32].

We will work with the linear form of the discrete simplicity constraints [16,18,33,34], with Immirzi parameter, and a minimal extension of the group field formalism to include the normals to tetrahedra as an additional variable of the field: this allows us to implement the constraints in a gauge covariant way. In this formalism, polynomial boundary observables are labeled by so-called projected spin networks [35]. We thus obtain a constrained GFT formulated as a theory of dynamical (noncommutative) geometric tetrahedra, which interact in the simplest possible way, as dictated by the star product. Its Feynman amplitudes define a spin foam model for gravity with Immirzi parameter  $\gamma$ , which gives a variant of the Barrett-Crane model when  $\gamma = \infty$  but differs from the existing models for generic values of  $\gamma$ . In particular it does not require any rationality condition for  $\gamma$ . This model is formulated both as a path integral (2) and in terms of Wigner  $15j$  symbols, in Eqs. (50) and (51) below: this is the main result of the paper.

The framework will also allow us to take a new (covariant) look at the peculiar features of the path integral amplitudes induced by noncommutativity. In particular we briefly discuss in Sec. IV how the so-called “ultra-locality problem” anticipated for the Barrett-Crane model

<sup>4</sup>This formulation is adapted to the quantization of classical systems with a “curved” phase space. See [24] for the simple example of a quantum system on the group  $SO(3)$ .

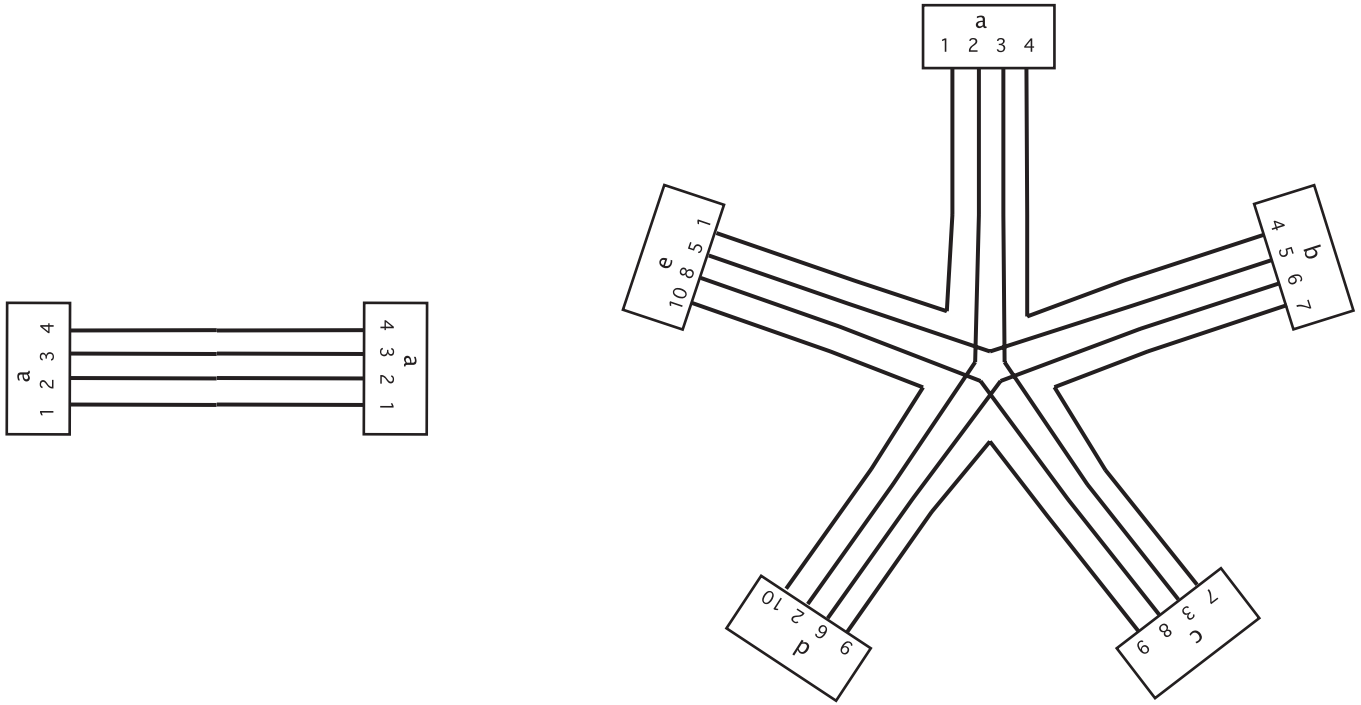


FIG. 1. GFT propagator and vertex.

manifests itself in our framework and argue that it may disappear in a suitable semiclassical limit involving a commutative limit.

In Sec. V we conclude and sketch some directions for future work.

**II. GFT MODELS FOR BF THEORY**

In this section, we start by recalling the standard Ooguri GFT for BF theory and its noncommutative bivector representation. We then present an extension of the GFT formalism, where the usual field variables, associated with the four triangles of a tetrahedron, are supplemented by an  $S^3$  vector playing the role of the normal to the tetrahedron. As we will see, it will allow us to implement the linear simplicity constraints (26) in a covariant way.

Our notations and conventions are as follows. We identify functions on  $SO(4)$  with functions on  $SU(2)^- \times SU(2)^+ / \mathbb{Z}_2$  and denote by  $g = (g^-, g^+)$  the  $SU(2)$  decomposition of the field variables. We also use the decomposition of  $\mathfrak{so}(4)$  in anti-self-dual and self-dual sectors  $\mathfrak{so}(4) = \mathfrak{su}(2)^+ \oplus \mathfrak{su}(2)^-$  and denote by  $x = (x^-, x^+)$  the corresponding decomposition of its elements. From Sec. II B on, based on the  $SO(3)$  Fourier transform<sup>5</sup> [38],

<sup>5</sup>An extension of the group transform to the whole  $SU(2)$  has been developed in [36]. We do not use it in this work, because on the one hand we do not expect the results to be very much modified by such an extension, and on the other hand the general case would entail a more involved notation. Note also that a different  $SU(2)$  transform has been proposed and studied in [37].

we further assume an invariance of group functions under  $g \rightarrow -g$ , so that they are effectively functions on  $SO(3) \times SO(3)$ .

**A. Connection and spin formulations**

In the standard connection formulation, the Ooguri GFT model [11] for BF theory is described in terms of a field  $\varphi(g_1, \dots, g_4)$  on four copies of the gauge group, satisfying the gauge invariance condition,

$$\forall h \in SO(4), \quad \varphi(g_1, \dots, g_4) = \varphi(hg_1, \dots, hg_4). \quad (4)$$

The dynamics is governed by the action

$$S = \frac{1}{2} \int [dg_i]^4 \varphi_{1234}^2 + \frac{\lambda}{5!} \times \int [dg_i]^{10} \varphi_{1234} \varphi_{4567} \varphi_{7389} \varphi_{96210} \varphi_{10851}, \quad (5)$$

where  $\varphi_{1234}$  is a shorthand notation for  $\varphi(g_1, \dots, g_4)$ ,  $dg$  is the normalized Haar measure, and  $\lambda$  is a coupling constant. The perturbative expansion in  $\lambda$  generates four-stranded graphs dual to four-dimensional simplicial complexes (see Fig. 1): if one associates the field variables to the four triangles of a tetrahedron, the quintic interaction sticks five tetrahedra together on a common triangle to form a 4-simplex; the kinetic term dictates the gluing rules for 4-simplices along tetrahedra.

By using the harmonic analysis on  $SO(4)$ , the gauge-invariant field is expanded into four  $SO(4)$  irreducible representations, labeled by pairs of  $SU(2)$  spins  $J = (j^-, j^+)$ , and 4-valent intertwiners  $\iota = (\iota^-, \iota^+)$  labeled by

a pair of intermediate SU(2) spins. The interaction vertex is expressed in terms of the SU(2) Wigner 15j symbols [11]. In this formulation, the Feynman amplitudes takes the form of a state sum model on the simplicial complex dual  $\Delta$  to the graph

$$I_\Delta = \sum_{\{J_i, \iota_\tau\}} \prod_{\tau} d_{j_i^-} d_{j_i^+} \prod_{\sigma} \{15j\}_{\sigma}^- \{15j\}_{\sigma}^+. \quad (6)$$

The sum is over the SO(4) representations  $J_i$  and intertwiners  $\iota_\tau$  labeling the triangles and the tetrahedra;  $d_j = 2j + 1$  is the dimension of the SU(2) representation  $j$ . The symbol for the vertex amplitude (associated with each 4-simplex), is the product of two SU(2) Wigner 15j symbols.

### B. Noncommutative Fourier transform and bivector formulation

The simplicial geometry encoded in the model (5) is best understood in a dual formulation, coined ‘‘metric representation’’ in [21], obtained by a group Fourier transform of the field. The relevant Fourier transform here is the obvious extension of the noncommutative SO(3) Fourier transform [36,38,39] to the group  $[\text{SO}(3) \times \text{SO}(3)]^4$ ,

$$\hat{\phi}(x_1, \dots, x_4) := \int [dg_i]^4 \varphi(g_1, \dots, g_4) e^{i \text{Tr} x_1 g_1} \dots e^{i \text{Tr} x_4 g_4}. \quad (7)$$

The variables  $x_i$  belong to the Lie algebra  $\mathfrak{so}(4) = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ . The kernel of the Fourier transform is a product of ‘‘plane waves’’  $E_g(x) = e^{i \text{Tr} x g}$ , where the trace  $\text{Tr}$  is defined in terms of the usual trace of  $2 \times 2$  matrices<sup>6</sup> as  $\text{Tr} x g = \sum_{\pm} \epsilon_{g^{\pm}} \text{tr}[x^{\pm} g^{\pm}]$  with  $\epsilon_{g^{\pm}} = \text{sgn}(\text{tr} g^{\pm})$ . Thus  $E_g(x)$  is itself a product of two SO(3) plane waves  $e_{g^{\pm}}(x^{\pm}) := e^{i \epsilon_{g^{\pm}} \text{tr} x^{\pm} g^{\pm}}$ . The plane waves satisfy the properties

$$\int d^6 x E_g(x) = \delta(g), \quad E_{g^{-1}}(x) = E_g(-x).$$

Here  $d^6 x$  is the Lebesgue measure on  $\mathfrak{so}(4) \sim \mathbb{R}^6$  and  $\delta(g) := \delta_{\text{SO}(3)}(g^+) \delta_{\text{SO}(3)}(g^-)$  acts as the delta distribution on group fields. We deduce from these the following expression of the GFT action (5) in terms of the dual field  $\hat{\phi}$ :

$$S = \frac{1}{2} \int [d^6 x_i]^4 \hat{\phi}_{1234} \star \hat{\phi}_{-1-2-3-4} + \frac{\lambda}{5!} \int [d^6 x_i]^{10} \hat{\phi}_{1234} \star \hat{\phi}_{-4567} \star \hat{\phi}_{-7-389} \star \hat{\phi}_{-9-6-210} \star \hat{\phi}_{-10-8-5-1}. \quad (8)$$

Notations are as follows.  $\hat{\phi}_{\pm 1 \pm 2 \pm 3 \pm 4}$  is a shorthand notation for  $\hat{\phi}(\pm x_1, \dots, \pm x_4)$ . The star product is defined on

<sup>6</sup>Let  $\tau_j$  be  $i$  times the Pauli matrices, then  $\text{tr} \tau_i \tau_j = -\delta_{ij}$ . Given an SU(2) element  $u = e^{\theta n / \tau_j}$  parametrized by the angle  $\theta \in [0, \pi]$  and the unit  $\mathbb{R}^3$  vector  $\vec{n}$  and  $a = a^i \tau_j$  in the algebra  $\mathfrak{su}(2)$ , we thus have  $\text{tr}[au] = -\sin \theta \vec{n} \cdot \vec{a}$ . Also  $\epsilon_u := \text{sgn}(\text{tr} u) = \text{sgn}(\cos \theta)$ .

SO(3) plane waves as  $e_g \star e_{g'}(x) = e_{gg'}$ , extended to  $E_g \star E_{g'}(x) = E_{gg'}(x)$  and by linearity to the algebra functions. In the expression above, it is understood that the star product pairs repeated indices: for example, the first product of the interaction term is a product of functions of the variable  $x_4$ ,  $\hat{\phi}_{1234} \star_{x_4} \hat{\phi}_{-4567}$ . To recover (5) from (8), one expands the dual fields in group modes; the integration over the variables  $x_i$  produces delta functions  $\delta(g_i^{-1} g_i') = \int dx_i E_{g_i^{-1} g_i'}(x_i)$  which identify the group elements associated with the same index.

Gauge invariance (4) translates into the invariance of the dual field under  $\star$ -multiplication by a product of four plane waves  $E_h(x_1) \dots E_h(x_4) = E_h(x_1 + \dots + x_4)$  labeled by the same  $h$ ,

$$\forall h \in \text{SO}(4), \quad \hat{\phi} = E_h \dots E_h \star \hat{\phi}. \quad (9)$$

Integrating over  $h$  on both sides of this equality gives

$$\hat{\phi} = \delta(x_1 + \dots + x_4) \star \hat{\phi}, \quad \delta(x) := \int dh E_h(x), \quad (10)$$

where  $\delta$  plays the role of noncommutative delta function on algebra functions  $\delta \star \phi(x) = \phi(0) \delta(x)$ . In other words, gauge invariance corresponds to a constraint on the dual fields imposing the closure  $x_1 + \dots + x_4 = 0$  of its variables.<sup>7</sup> It is interesting to note that, in the noncommutative setting, the closure constraint is implemented by a projector, since  $\delta \star \delta = \delta$ . Geometrically,  $\hat{\phi}$  represents a tetrahedron whose four faces are labeled by a bivector  $x_i^J$ . The gluing rules for tetrahedra dictated by the action (8) corresponds to the identification of the face bivectors, modulo a sign encoding a flip of the face orientation.

More precisely, propagator and vertex in this representation are given by

$$P(x, x') = \prod_{i=1}^4 \delta_{-x_i}(x'_i), \quad (11)$$

$$V(x, x') = \int [dh_\ell]^{10} \prod_{i=1}^{10} (\delta_{-x_i^\ell} \star E_{h_\ell h_{\ell'}^{-1}})(x_i^{\ell'}),$$

where  $i$  labels the oriented strands (triangles) and  $\ell$  the half lines (tetrahedra) of the graphs in Fig. 1 and  $\delta_x(y) := \delta(x - y)$ , with  $\delta$  defined as in (10). In terms of simplicial geometry, the vertex function encodes the identification, for each of the ten triangles  $i$  of a 4-simplex, of the two bivectors  $x_i^\ell, x_i^{\ell'}$  associated with it, corresponding to the two tetrahedra  $\ell, \ell'$  sharing the triangle, up to parallel transport  $h_\ell h_{\ell'}^{-1}$  from one tetrahedron to another [21]. The sign difference reflects the fact, in an oriented 4-simplex, a triangle inherits opposite orientations from the two tetrahedra sharing it. The integration over  $h_\ell$  implements the gauge invariance (9). We have chosen to

<sup>7</sup>In terms of the canonical theory, this is indeed just the standard Gauss law corresponding to SO(4) gauge invariance in flux variables [27].

gauge-average the vertex here, but since gauge-averaging is a projection  $\hat{C}^2 = \hat{C}$ , one could instead gauge-average the propagator or both vertex and propagator without affecting the amplitudes.

The Feynman amplitudes are obtained by taking the star product of propagator and vertex functions, following the strands of the graph in Fig. 1 [21]. The structure of the star product gives a clear geometrical meaning to the algebraic expressions. In particular, commutation with a plane wave signifies a change of frame:  $E_h \star \varphi = \varphi^h \star E_h$  with  $\varphi^h(x) = \varphi(h^{-1}xh)$ .

For a given closed graph dual to a simplicial complex, this results in integrals over group elements  $h_{\tau\sigma}$  labeled by adjacent pairs {tetrahedron, 4-simplex}, interpreted as parallel transport from the center of the 4-simplex  $\sigma$  to the center of its boundary tetrahedron  $\tau$ , and over  $\mathfrak{so}(4)$  variables  $x_t^\tau, x_t^\sigma$ , interpreted as the same bivector of  $t$  seen in different frames associated with the tetrahedra and 4-simplices sharing  $t$ . In what follows we set  $h_{\sigma\tau} := h_{\tau\sigma}^{-1}$  and denote by  $h_{\tau\tau'} = h_{\tau\sigma} h_{\sigma\tau'}$  the holonomy between two neighboring tetrahedra through an adjacent 4-simplex. The integrand factorizes into contributions of each loop of strands, dual to a triangle  $t$ , taking the form of a star product of delta functions identifying all the variables  $x_t^\sigma, x_t^\tau$  associated with the triangle  $t$ .

After integration over all variables but one per triangle  $x_t := x_t^{\tau_0}$  associated with a ‘‘reference’’ tetrahedron  $\tau_0(t)$ , the amplitude reads

$$I_{\text{BF}} = \int \prod_{\langle\tau\sigma\rangle} dh_{\tau\sigma} \prod_t d^6 x_t e^{i \sum_t \text{Tr}_x H_t}, \quad (12)$$

where  $H_t = h_{\tau_0\tau_1} \cdots h_{\tau_{N_t}\tau_0}$  is the holonomy along the loop of  $N_t + 1$  tetrahedra sharing  $t$  calculated for a choice of orientation and reference tetrahedron.<sup>8</sup> The integrand is the exponential of the discrete BF action, resulting from the star product of  $N_t$  plane waves for each  $t$ ,

$$E_{h_{\tau_0\tau_1}} \star \cdots \star E_{h_{\tau_{N_t}\tau_0}}(x_t) = e^{i \text{Tr}_x H_t}. \quad (13)$$

The GFT amplitudes in the bivector representation thus take the form of simplicial path integrals for BF theory, where field variables  $x \in \mathfrak{so}(4)$  and group elements  $h \in \text{SO}(4)$  arising from gauge invariance play the respective roles of discrete  $B$  field and discrete connection.

The bivector formulation of a GFT suggests clear routes for defining geometrical models, by means of constraint operators implementing the simplicity constraint on the field variables. The linear constraints (26), however, involve an another geometrical variable: the normal to the tetrahedron. In the next section, we review an extension of the usual GFT formalism introduced in [22], which includes the normals as an additional field variable. Although the extended GFT generates the same BF

amplitudes for closed graphs, it will allow us to impose the constraints covariantly, that is on fields that are gauge-invariant under a simultaneous  $\text{SO}(4)$  rotation of both bivectors and normal vector.

### C. Introducing normals: Extended GFT formalism

In the extended GFT formalism [22,40], the basic group field  $\varphi_k(g_1, \cdots g_4)$  is supplemented with a fifth variable  $k \in \text{SU}(2) \sim \mathcal{S}^3$ , viewed as a unit vector in  $\mathbb{R}^4$ . In geometrical models,  $k$  will be interpreted as the normal to a tetrahedron. Gauge invariance (4) is replaced by a gauge covariance with respect to the normal  $k$ ,

$$\forall h, \quad \varphi_k(g_1, \cdots g_4) = \varphi_{h \triangleright k}(hg_1, \cdots hg_4), \quad (14)$$

where  $h \triangleright k := h^+ k (h^-)^{-1}$  is the normal rotated by  $h$ . Clearly, the field obtained by integrating over the normals obeys the gauge invariance (4).

The dynamics is governed by the action

$$S[\varphi] = \frac{1}{2} \int [\text{dg}_i]^4 dk \varphi_{k1234}^2 + \frac{\lambda}{5!} \int [\text{dg}_i]^{10} \times [\text{dk}_i]^5 \varphi_{k_11234} \varphi_{k_24567} \varphi_{k_37389} \varphi_{k_496210} \varphi_{k_510851}, \quad (15)$$

where  $\varphi_{k1234}$  is a shorthand notation for  $\varphi_k(g_1, \cdots g_4)$ ,  $\text{dg}$  and  $\text{dk}$  are the Haar measures on  $\text{SO}(4)$  and  $\text{SU}(2)$ . Hence, whereas the interaction polynomial does not couple the normals, the kinetic term, which encodes the gluing rule of 4-simplices along a tetrahedron, identifies both group elements and normals. It is already clear from the structure of this action that the amplitudes of closed diagrams will not depend on the normals; the extended formulation only modifies the structure of boundary states.

Note that gauge covariance (14) induces an invariance under the stabilizer group  $\text{SO}(3)_k = \{h \in \text{SO}(4), h^+ k (h^-)^{-1} = k\}$  of the normal  $k$ , affecting only the four group arguments of the field. Upon Peter-Weyl decomposition, gauge-invariant fields are expanded into four irreducible  $\text{SO}(4)$  representations [given by pairs of  $\text{SU}(2)$  spins  $J_i = (j_i^-, j_i^+)$ ,  $i = 1 \cdots 4$ ], each of which can be further decomposed into  $\text{SO}(3)_k$  representations. A set of basis functions is given by

$$\Psi_{m_i^-, m_i^+}^{(J_i, k_i, j)}(g_i; k) = \left( \prod_{i=1}^4 D_{n_i^-, m_i^-}^{j_i^-}(g_i^-) D_{n_i^+, m_i^+}^{j_i^+}(g_i^+) \tilde{C}_{n_i^- n_i^+ p_i}^{j_i^- j_i^+ k_i}(k) \right) (\iota_j)_{p_i}^{k_i}, \quad (16)$$

where repeated lower indices are summed over.  $D^{j^\pm}(g^\pm)$  are the  $\text{SU}(2)$  Wigner matrices,  $(\iota_j)^{k_i}$  form a basis of 4-valent  $\text{SO}(3)$  intertwiners, labeled by an intermediate spin  $j$ . The  $k$ -dependent coefficients, defined in terms of the  $\text{SO}(3)$  Clebsch-Gordan coefficients  $C_{m\dot{n}p}^{j^- j^+ k}$  as

$$\tilde{C}_{m_i^- m_i^+ p_i}^{j_i^- j_i^+ k_i}(k) = \sum_m C_{m m_i^+ p_i}^{j_i^- j_i^+ k_i} D_{m m_i^-}^{j_i^-}(k), \quad (17)$$

<sup>8</sup>The amplitude does not depend on these choices.

define a tensor that intertwines the action of  $\text{SO}(3)_k$  in the representation  $j_i^- \otimes j_i^+$  and the action of  $\text{SO}(3)$  in the representation  $k_i$ . Namely, given  $\mathbf{u}_k = (k^{-1}uk, u) \in \text{SO}(3)_k$ , we have

$$\tilde{C}_{m_i^- m_i^+ p_i}^{j_i^- j_i^+ k_i}(k) D_{m_i^- n_i^-}^{j_i^-}(\mathbf{u}_k^-) D_{m_i^+ n_i^+}^{j_i^+}(\mathbf{u}_k^+) = \tilde{C}_{n_i^- n_i^+ q_i}^{j_i^- j_i^+ k_i}(k) D_{q_i p_i}^{k_i}(u). \quad (18)$$

Equation (16) corresponds to the vertex structure of the so-called projected spin networks of the covariant approach to loop quantum gravity [35], which thus define a basis for polynomial gauge-invariant observables (and thus boundary states) in the extended GFT formalism.

Just as in the standard formulation, the bivector representation of the GFT is obtained by Fourier transform of the field

$$\hat{\varphi}_k(x_1, \dots, x_4) := \int [dg]^4 \varphi_k(g_1, \dots, g_4) e^{i\text{Tr}x_1 g_1} \dots e^{i\text{Tr}x_4 g_4}. \quad (19)$$

Gauge invariance translates into

$$\forall h, \quad \hat{\varphi}_k = E_h \cdots E_h \star \hat{\varphi}_{h^{-1} \triangleright k}, \quad (20)$$

where  $h^{-1} \triangleright k = (h^+)^{-1} k h^-$ , thus implemented by the gauge invariance projector acting on extended fields as

$$(\hat{C} \triangleright \hat{\varphi})_k = \int dh E_h \cdots E_h \star \hat{\varphi}_{h^{-1} \triangleright k}. \quad (21)$$

Note that upon integration over the normal, gauge invariance gives the closure of the four bivector variables: in fact if  $\hat{\psi} = \int dk \hat{\varphi}_k$ , then  $\hat{\psi} = \delta(x_1 + \dots + x_4) \star \hat{\psi}$ . The action is the obvious extension of (8). It is interesting to write the interaction in terms of  $\hat{\psi}$ , to emphasize the fact it implements the closure constraints,

$$\begin{aligned} S &= \frac{1}{2} \int [d^6 x_i]^4 [dk] \hat{\varphi}_{k_{1234}} \star \hat{\varphi}_{k^{-1-2-3-4}} \\ &+ \frac{\lambda}{5!} \int [d^6 x_i]^{10} \hat{\psi}_{1234} \star \hat{\psi}_{-4567} \star \hat{\psi}_{-7-389} \\ &\star \hat{\psi}_{-9-6-210} \star \hat{\psi}_{-10-8-5-1}. \end{aligned} \quad (22)$$

The propagator of the extended GFT is supplemented with an additional strand which identifies the normals  $k$  up to parallel transport arising from gauge invariance. Just as in the nonextended case, gauge invariance can be implemented in the vertex only, or in the propagator, or in both vertex and propagator. Choosing the first case, the propagator then reads, in the bivector formulation,

$$P(x, k; x', k') = \prod_{i=1}^4 \delta_{-x_i}(x'_i) \delta(k' k^{-1}). \quad (23)$$

The additional contribution reduces the number of  $\text{SU}(2)$  variables to one per link, hence, in terms of the dual simplicial complex, to one  $k_\tau$  per tetrahedron. The integrals over the normals on the internal links (bulk tetrahedra) drop from the amplitudes; just like in the standard

formulation, the extended GFT generates simplicial BF path integrals as Feynman amplitudes. However, due to the extension of the GFT field to include the normal vectors to tetrahedra, the boundary states appearing in the amplitudes for GFT  $n$ -point functions are different from those of the standard Ooguri model, recovered only after averaging out the normal variables independently at each tetrahedron in the boundary.

### III. GEOMETRICAL MODELS

We have seen in the previous section that, in the bivector representation of the GFT for BF theory, the field  $\hat{\psi} = \int dk \hat{\varphi}_k$  represents a tetrahedron characterized by four bivectors  $x_j$ ,  $j = 1 \dots 4$  playing the role of a discrete  $B$  field; gauge invariance implements the closure condition  $\sum_j x_j = 0$ . We now propose a natural modification of the GFT (15) in terms of a constraint operator acting on the field by implementing the simplicity constraint of its bivector variables, for any positive value of the Immirzi parameter  $\gamma$ , allowing one to reconstruct a discrete tetrad for the tetrahedron. By construction, the Feynman expansion will generate simplicial path integrals for a constrained BF theory of the Holst-Plebanski type.

We start by recalling the discrete form of the (linear) simplicity constraints for a classical bivector geometry.

#### A. Discrete simplicity constraints

In the absence of Immirzi parameter, the simplicity constraints state that the Hodge duals  $*x_j^{IJ}$  are the area bivectors of a geometric (metric) tetrahedron: this is the discrete equivalent of  $B = *e \wedge e$ . Following [18,32], these constraints are implemented by requiring that the four  $*x_j^{IJ}$  lie in the same hypersurface normal to a given unit vector  $k^I$  in  $\mathbb{R}^4$ , namely  $*x_j^{IJ} k_J = 0$  for all  $j = 1 \dots 4$ . Using the self-dual/anti-self-dual decomposition of the algebra  $\mathfrak{so}(4)$ , this can be expressed as

$$\begin{aligned} \forall j \in \{1 \dots 4\}, \quad \exists k \in \text{SU}(2), \\ kx_j^- k^{-1} + x_j^+ = 0. \end{aligned} \quad (24)$$

The variable  $k \in \text{SU}(2) \sim \mathcal{S}^3$  is then the  $\text{SU}(2)$  representation of the unit vector  $k^I$  normal to the tetrahedron.<sup>9</sup> If ten bivectors labeling the faces of a 4-simplex satisfy simplicity and the closure constraint for each tetrahedron, then they define a geometric 4-simplex (for nondegenerate configurations). Furthermore, if the (constrained) bivectors associated with a given tetrahedron are also correctly identified across the two 4-simplices sharing it, then the reconstruction of a discrete tetrad can be carried out for the whole simplicial complex, again modulo degenerate configurations.

<sup>9</sup>Let  $\bar{k} := (\bar{k}^-, \bar{k}^+)$  be the  $\text{SO}(4)$  rotation mapping the vector  $N^I = (1, 0, 0, 0)$  to  $k^I$ , then  $k = \bar{k}^+ \bar{k}^{-1}$ .

The inclusion of the Immirzi parameter can be performed easily also at the discrete level [18]. We have mentioned that the Immirzi parameter is introduced in the continuum action (1) by a change of variables  $B \rightarrow \bar{B} = B + \frac{1}{\gamma} * B$  in the BF term. The action in the new variables looks again like a constrained BF theory, but where the constraints  $\mathcal{C}(B(\bar{B}))$  are now imposed on the following linear combination field  $\bar{B}$ :

$$B(\bar{B}) = \frac{\gamma}{1 - \gamma^2} (*\bar{B} - \gamma\bar{B}). \quad (25)$$

In the discrete setting, the simplicity condition with Immirzi parameter is thus obtained from (24) by replacing  $x_j$  by the linear combinations  $*x_j^{IJ} - \gamma x_j^{IJ}$ . In terms of the self-dual/anti-self-dual decomposition of the bivectors, it reads

$$\begin{aligned} \forall j \in \{1 \dots 4\}, \quad \exists k \in \text{SU}(2), \\ kx_j^- k^{-1} + \beta x_j^+ = 0, \end{aligned} \quad (26)$$

where the parameter  $\beta$  is related to the Immirzi parameter as

$$\beta = \frac{\gamma - 1}{\gamma + 1}. \quad (27)$$

Note that the relation (26) is invariant under simultaneous sign flip  $\gamma \rightarrow -\gamma$  of the Immirzi parameter and exchange  $x_j^+ \leftrightarrow x_j^-$  of the self-dual and anti-self-dual part of the bivectors. It is also well-defined for  $\gamma = 1$ , although then the change of variables (25) is singular and the geometrical interpretation is lost. In the following, we restrict to  $\gamma \in [0, \infty]$ , so that the parameter  $\beta$  takes its value in  $[-1, 1]$ .

### B. Constraint operator and noncommutative tetrahedra

Back to GFT, we now need to encode the simplicity condition (26) of the bivector variables  $x_j$  as a constraint on the field  $\hat{\varphi}_k$ , the idea being of course to identify the normal to the tetrahedron to the  $\text{SU}(2)$  variable  $k$  of the field. The natural way to do so while taking into account the non-commutativity of the fields is to use noncommutative delta functions, defined by their plane wave expansion (10). These delta functions act as distributions for the star product,  $\delta \star \phi(a) = \phi(0)\delta(a)$ , so using these to constrain the field will effectively amount to constraining the measure on the bivectors.

We thus introduce the following function of  $x = (x^-, x^+) \in \mathfrak{su}(4)$ :

$$S_k^\beta(x) := \delta_{-kx^- k^{-1}}(\beta x^+) = \int_{\text{SU}(2)} du e^{i \text{tr}[k^{-1} u k x^-]} e^{i \beta \text{tr}[u x^+]}, \quad (28)$$

where  $\delta_{-a}(b) := \delta(a + b)$  and  $\delta$  is the  $\mathfrak{su}(2)$  noncommutative delta function. Our geometrical GFT models will be

defined by constraining the field  $\hat{\varphi}_k(x_j)$  in the action (8), by means of an operator  $\hat{S}^\beta$  acting on it by  $\star$ -multiplication by the product  $S_k^\beta(x_1) \dots S_k^\beta(x_4)$  of four simplicity functions,

$$(\hat{S}^\beta \triangleright \hat{\varphi})_k(x_1, \dots, x_4) = \prod_{j=1}^4 S_k^\beta(x_j) \star \hat{\varphi}_k(x_1, \dots, x_4). \quad (29)$$

We give below the explicit expression of the star product (29) in terms of group and tensor Fourier components. But first, let us show that the action of this operator is well-defined. We will also see that it commutes with the  $\text{SO}(4)$  gauge transformations (20).

To be able to take the star product of the  $S_k^\beta$  with the field, we need the function (28) to be in the image of the group Fourier transform. To see why this is indeed the case, notice that, because  $|\beta| \leq 1$ , there exists  $u_\beta \in \text{SU}(2)$  such that  $\beta \text{tr}[au] = \text{tr}[au_\beta]$  for all  $a \in \mathfrak{su}(2)$ . Indeed, if  $u = e^{\theta n^i \tau_j}$  is parametrized by the angle  $\theta \in [0, \pi]$  and the unit  $\mathbb{R}^3$  vector  $\vec{n}$ , where the  $\tau_j$  are  $i$  times the Pauli matrices, we define  $u_\beta = e^{\theta_\beta n^i \tau_j}$ , where the parameters  $\theta_\beta$  and  $\vec{n}_\beta$  are

$$\sin \theta_\beta = |\beta| \sin \theta, \quad \text{sgn}(\cos \theta_\beta) = \text{sgn}(\cos \theta); \quad (30)$$

$$\vec{n}_\beta = \text{sgn}(\beta) \vec{n}.$$

The simplicity function (28) can thus be written as a superposition of plane waves  $E_g(x) = e^{i \text{Tr} g x}$ ,

$$\delta_{-kx^- k^{-1}}(\beta x^+) = \int_{\text{SU}(2)} du E_{u_\beta^k}(x), \quad (31)$$

where we introduced  $u_\beta^k = (k^{-1} u k, u_\beta) \in \text{SU}(2) \times \text{SU}(2)$ . Therefore it belongs to the image of the Fourier transform, and its star product with the field is well-defined.

The operator  $\hat{S}$  is not a projector for generic values of  $\beta$ , unless  $\beta = 0, 1$  (which corresponds to  $\gamma = 1, \infty$ ). Indeed, because of the nonlinearity of scaling by  $\beta$  in the definition (30), we have that  $(uv)_\beta \neq u_\beta v_\beta$ , and thus  $S_k^\beta \star S_k^\beta \neq S_k^\beta$ . Remarkably, however, the action of  $\hat{S}$  is well-defined on gauge-invariant fields, as it commutes with the gauge transformations (20),

$$\hat{S}^\beta \triangleright [E_h \cdots E_h \star \hat{\varphi}_{h^{-1}}] = E_h \cdots E_h \star (\hat{S}^\beta \triangleright \hat{\varphi})_{h^{-1} \triangleright k}, \quad (32)$$

thanks to the commutation relations between plane waves and simplicity functions,

$$E_h \star S_k^\beta = S_{h \triangleright k}^\beta \star E_h. \quad (33)$$

Geometrically, these relations express the fact that rotating a bivector which is simple with respect to a normal  $k$  gives a bivector which is simple with respect to the rotated normal  $h \triangleright k := h^+ k (h^-)^{-1}$ . This is the advantage of the extended GFT formalism, where the normals are explicit variables of the field: the linear simplicity constraints on the bivectors can be imposed a covariant way. This is not

the case in the standard formulation of the Barrett-Crane model, nor on the EPRL-FK model, where simplicity and gauge invariance are implemented by means of two non-commuting projectors.

Let us now examine the dual action of  $\hat{S}$  on the original group fields  $\varphi_k(g_j)$ . By using the plane wave expansion (31) of the simplicity functions and the definition of the star product, we obtain

$$\begin{aligned} (\hat{S}^\beta \triangleright \varphi)_k(g_1, \dots, g_4) \\ = \int_{\text{SU}(2)^4} [du_j]^4 \varphi_k(\mathbf{u}_{1\beta}^k g_1, \dots, \mathbf{u}_{4\beta}^k g_4), \end{aligned} \quad (34)$$

where  $\mathbf{u}_{j\beta}^k = (k^{-1}u_j k, u_{j\beta}) \in \text{SU}(2) \times \text{SU}(2)$  and  $u_{j\beta}$  is defined as in (30). For the particular value  $\beta = 1$  reached in the limit  $\gamma \rightarrow \infty$ ,  $\hat{S}^1$  reduces to the projector onto fields on four copies of the homogeneous space  $\text{SO}(4)/\text{SO}(3)_k$ . Using the invariance (14), one can gauge fix the normal to the value  $k = 1$  (time gauge). On such gauge-fixed fields, the simplicity operator coincides with the projector defining the standard GFT formulation of the Barrett-Crane model [15,22,41]. The difference here, however, is that the gauge-fixed extended fields are obviously not gauge-invariant under the full  $\text{SO}(4)$ —but only under the diagonal  $\text{SU}(2)$  subgroup.

Upon Peter-Weyl decomposition of the constrained field  $(\hat{S}^\beta \triangleright \varphi)_k$ , a set of basis functions is given by the action of the  $\hat{S}$  on the functions (16),

$$\begin{aligned} \hat{S}^\beta \triangleright \Psi_{m_i^-, m_i^+}^{(J_i, k_i, j)}(g_i; k) \\ = \left( \prod_{i=1}^4 D_{n_i^-, m_i^-}^{j_i^-} (g_i^-) D_{n_i^+, m_i^+}^{j_i^+} (g_i^+) F_{n_i^-, n_i^+}^{j_i^- j_i^+ k_i} (k) \right) (\iota_j)_{p_i}^{k_i}, \end{aligned} \quad (35)$$

where repeated lower indices are summed over. This expression is obtained from (16) by replacing the  $k$ -dependent coefficients  $\tilde{C}_{m_i^+ m_i^-}^{j_i^- j_i^+ k_i} (k) = C_{m_i^+ m_i^-}^{j_i^- j_i^+ k_i} D_{m_i^+ m_i^-}^{j_i^-} (k)$  by new ones given by

$$\begin{aligned} F_{n_i^-, n_i^+}^{j_i^- j_i^+ k_i} (k) = \int_{\text{SU}(2)} du D_{m_i^-, n_i^-}^{j_i^-} (k^{-1}uk) \\ \times D_{m_i^+, n_i^+}^{j_i^+} (u_\beta) \tilde{C}_{m_i^+ m_i^-}^{j_i^- j_i^+ k_i} (k), \end{aligned} \quad (36)$$

with  $u_\beta$  given as in (30). Just as in (16), these coefficients intertwine the action of stabilizer subgroup  $\text{SO}(3)_k$  in the representation  $j_i^- \otimes j_i^+$  and the action of  $\text{SO}(3)$  in the representation  $k_i$ . Namely, given  $\mathbf{u}_k = (k^{-1}uk, u) \in \text{SO}(3)_k$ , we have

$$F_{m_i^-, m_i^+}^{j_i^- j_i^+ k_i} (k) D_{m_i^-, n_i^-}^{j_i^-} (\mathbf{u}_k^-) D_{m_i^+, n_i^+}^{j_i^+} (\mathbf{u}_k^+) = F_{n_i^-, n_i^+}^{j_i^- j_i^+ k_i} (k) D_{q_i}^{k_i} (u). \quad (37)$$

Here, they also contain all the information about the simplicity constraints and the specific form of the operator that implements them. In particular the integral of the two Wigner matrices encodes a relation between the spins

$(j_i^-, j_i^+)$ , which depends on the Immirzi parameter; for example  $j_i^- = j_i^+$  when  $\beta \in \{-1, 1\}$ , namely, when  $\gamma \in \{0, \infty\}$ . However for generic values of  $\beta$ , it does not enforce the spin relations  $j_i^- = |\beta| j_i^+$  characteristic of the EPRL-FK models, an analogue of which we expect to recover only in the asymptotic regime. In particular, they do not impose any rationality condition on the Immirzi parameter  $\gamma$ . A detailed study of the properties of these coefficients, in particular, their asymptotic behavior for large spins, is left for future work.

To sum up this section, we have defined an operator  $\hat{S}^\beta$  acting on the field by imposing the linear simplicity condition (26), for any positive value of the Immirzi parameter, on its four bivector variables. Note that, because of gauge invariance, the closure constraint holds after integration over the normal: let  $\hat{\psi} := \int dk \hat{S}^\beta \triangleright \hat{\varphi}_k$ , then  $\hat{\psi} = \delta(x_1 + \dots + x_4) \star \hat{\psi}$ , where  $\delta$  is the noncommutative delta function  $\delta \star \phi(x) = \phi(0)\delta(x)$  defined in (10).

In relation to a canonical theory, one can link the definition of the constrained GFT field to a quantization of a tetrahedron characterized by its four constrained bivectors: in particular, one can check (using the Fourier duality with group fields) that the generators  $J_j^{IJ}$  of the gauge group act by  $\star$ -multiplication by the coordinate functions  $\hat{x}_j^{IJ}(x_j) = x_j^{IJ}$ . The quantization procedure consists of first quantizing classical configurations  $\{x_j\} \in \mathfrak{so}(4)$ ,  $k \in \text{SU}(2) \sim \mathcal{S}^3$  of bivectors and normal and then imposing geometricity (simplicity) constraints at the quantum level. The Hilbert space is the tensor product of  $L^2(\text{SU}(2))$  with

$$\bigotimes_{i=1}^4 L_\star^2(\mathbb{R}^6),$$

where the  $L_\star^2$  spaces, which also appear as state spaces in the flux representation of loop quantum gravity [27], are spaces of functions on  $\mathfrak{so}(4) \sim \mathbb{R}^6$  endowed with the scalar product  $\int d^6x (\tilde{f} \star g)(x)$ , where  $\tilde{f}(x) = f(-x)$ . The algebra structure encoded in the star product, which deforms the usual pointwise product, stems directly by Fourier transform from the algebra of group functions. It makes explicit the noncommutativity of the geometry inherent to spin foams and group field theories. This procedure is manifestly dual to geometric quantization: the advantage here is that the classical variables characterizing the geometry remain explicit, as arguments of the fields. Geometricity conditions are then implemented by using two commuting operators: the simplicity operator  $\hat{S}^\beta$  and the gauge projector  $\hat{C}$  defined by (21), leading to the unambiguous definition of a ‘‘geometricity operator’’  $\hat{G} = \hat{S}^\beta \hat{C} = \hat{C} \hat{S}^\beta$ . All constraints are imposed by means of noncommutative delta functions, acting as Dirac distributions for the star product, so that it effectively amounts to constraining the measures  $d^6x_j$  on the classical field variables.

The algebra structure and scalar product allow one to build up more involved polyhedra obtained by gluing



tetrahedra along a common triangle. Thus, gluing five geometric tetrahedra  $\hat{\Psi} := \int dk \hat{G} \triangleright \hat{\varphi}_k$  along common triangles as

$$\hat{\Psi}_{1234} \star \hat{\Psi}_{-4567} \star \hat{\Psi}_{-7-389} \star \hat{\Psi}_{-9-6-210} \star \hat{\Psi}_{-10-8-5-1}, \quad (38)$$

where  $\hat{\Psi}_{k_{\pm 1 \pm 2 \pm 3 \pm 4}}$  is a shorthand notation for  $\hat{\Psi}_k(\pm x_1, \dots, \pm x_4)$  and the star product pairs repeated lower indices give a straightforward ansatz for a quantum 4-simplex. This is precisely the interaction polynomial of our geometrical GFT, which we define now.

### C. GFT for gravity with no Immirzi parameter

We first focus on the particular case  $\beta = 1$ . It corresponds to  $\gamma \rightarrow \infty$ , namely, the case of gravity with no Immirzi parameter. In this case, the operator  $\hat{S}^1$  reduces to the projector onto the homogeneous space  $\text{SO}(4)/\text{SO}(3)_k$ . The corresponding geometrical GFT has been introduced and studied in [22]; we recall the basics here, before treating the general case.

The model is defined by constraining the field  $\hat{\varphi}_k$  in the action of the extended Ooguri model. Since  $\hat{S}^1$  is a projector, it can be inserted in the propagator, in the vertex, or in both, without affecting the amplitudes. In particular, these choices all lead to a unique form of edge amplitudes in the spin foam model. Here we set  $\hat{\Psi} := \int dk \hat{S}^1 \triangleright \hat{\varphi}_k$  and define the action

$$\begin{aligned} S = & \frac{1}{2} \int [d^6 x_i]^4 dk \hat{\varphi}_{k_{1234}} \star \hat{\varphi}_{k_{-1-2-3-4}} + \frac{\lambda}{5!} \\ & \times \int [d^6 x_i]^{10} \hat{\Psi}_{1234} \star \hat{\Psi}_{-4567} \star \hat{\Psi}_{-7-389} \\ & \star \hat{\Psi}_{-9-6-210} \star \hat{\Psi}_{-10-8-5-1}, \end{aligned} \quad (39)$$

where the star product pairs repeated indices.

We now calculate the Feynman amplitudes of this theory. We will obtain two dual representations of these, in terms of simplicial path integrals on one hand, and spin foam models on the other.

#### 1. Simplicial path integral representation of the amplitudes

The Feynman amplitudes of this theory are calculated with the same propagator as in (23) and the vertex

$$V(x_i^\ell; k_\ell) = \int [dh_\ell]^5 \prod_{i=1}^{10} (\delta_{-x_i^\ell} \star S_{k_\ell} \star E_{h_\ell h_{\ell'}^{-1}} \star S_{k_{\ell'}})(x_i^{\ell'}), \quad (40)$$

where  $i$  labels the oriented strands (triangles) and  $\ell$  the half lines (tetrahedra) of the vertex graph, and  $S_k = \delta_{-kx^- k^{-1}}(x^+)$  is the simplicity function for  $\beta = 1$ . The calculation is analogous to the unconstrained case. For a given closed graph dual to a simplicial complex  $\Delta$ , this

results in integrals over holonomies  $h_{\tau\sigma} \in \text{SO}(4)$ , tetrahedra normals  $k_\tau \in \text{SU}(2)$ , and bivectors  $x_t^\tau, x_t^\sigma$  on  $t$  seen in different frames associated with the tetrahedra and 4-simplices sharing  $t$ . After integration over all variables but one per triangle  $x_t := x_t^{\tau_0}$  associated with a reference tetrahedron  $\tau_0(t)$ , the amplitude reads [22]

$$I_\Delta = \int [dh_{\tau\sigma}] [dk_\tau] [d^6 x_t] \left[ \prod_t \star_{j=0}^{N_t} S_{h_{0j} \triangleright k_j}(x_t) \right] \star e^{i \sum_t \text{Tr} x_t H_t}. \quad (41)$$

The notations are that of Sec. II B:  $H_t = h_{\tau_0 \tau_1} \cdots h_{\tau_{N_t} \tau_0}$  is the holonomy along the (oriented) loop of  $N_t + 1$  tetrahedra sharing  $t$ , labeled by the integer  $j$ ;  $h_{0j} = h_{\tau_0 \tau_1} \cdots h_{\tau_{j-1} \tau_j}$  is the holonomy from the reference tetrahedron to the  $j$ -th tetrahedron sharing  $t$ .  $k_j := k_{\tau_j}$  is the normal of the  $j$ -th tetrahedron around  $t$ . The function  $S_{h_{0j} \triangleright k_j}(x_t)$  imposes on  $x_t$  the linear simplicity condition with respect to the rotated normal  $h_{0j} \triangleright k_j := h_{0j}^+ k_j (h_{0j}^-)^{-1}$ , namely, the pull-back of  $k_j$  in the frame of the reference tetrahedron  $\tau_0(t)$ . It amounts to imposing the linear simplicity of the pushed forward bivector  $h_{0j}^{-1} x_t h_{0j}$  with respect to  $k_j$ . The integrand results from taking, for each  $t$ , the alternate star product of  $N_t$  plane waves and  $N_t$  simplicity functions,

$$\begin{aligned} & S_{k_0} \star E_{h_{\tau_0 \tau_1}} \cdots \star S_{k_{N_t}} \star E_{h_{\tau_{N_t} \tau_0}}(x_t) \\ & = [\star_{j=0}^{N_t} S_{h_{0j} \triangleright k_j}] \star e^{i \text{Tr} x_t H_t}, \end{aligned} \quad (42)$$

where we used the commutation relation  $S_k \star E_h = S_{h \triangleright k} \star E_h$  to regroup all the plane waves on the right-hand side of the expression.

The Feynman amplitudes of the GFT (8) thus take the form of (noncommutative) simplicial path integrals for a constraint BF theory of Plebanski type. The constraints are noncommutative delta functions modifying the measures  $d^6 x_t$  on the bivectors, imposing the simplicity of each  $x_t$  with respect to the normals of all the tetrahedra sharing  $t$ .

Note that by construction, the integrand of (41) is invariant under  $\text{SO}(4)$  rotations  $\{g_\tau, g_\sigma\}$  of all local frames,

$$\begin{aligned} h_{\tau\sigma} & \mapsto g_\tau h_{\tau\sigma} g_\sigma, & k_\tau & \mapsto g_\tau^+ k_\tau (g_\tau^-)^{-1} \\ x_t & \mapsto g_{\tau(t)}^{-1} x_t g_{\tau(t)}, \end{aligned} \quad (43)$$

where  $\tau(t)$  is the reference tetrahedron of the triangle  $t$ . This includes the gauge invariance of the discrete BF action. The choice  $g_\tau = (g_\tau^-, g_\tau^+) := (k_\tau^{-1}, 1)$  leads to the ‘‘time gauge’’  $k_\tau = 1$ , which shows that the integral over the normals  $k_\tau$  drops out of the amplitude. Of course in the case of open graphs, dual to simplicial complexes with boundary, the amplitude still has an explicit dependence on the normals of the boundary tetrahedra.

It is interesting to distinguish two types of constraints on the bivector  $x_t$  of a given triangle. The  $j = 0$  contribution

$S_{k_0}(x_t) = \delta_{-k_0 x_t^- k_0^{-1}}(x_t^+)$  in (41) imposes the linear simplicity of each bivector  $x_t$  with respect to the normal to the reference tetrahedron  $\tau_0(t)$ . The remaining part, for a given set of bivectors, can be viewed as constraints on the holonomies modifying the measures  $dh_{\tau\sigma}$  on the discrete connections. The effective measure

$$\mathcal{D}^{x_t, k_\tau}[h_{\tau\sigma}] = [dh_{\tau\sigma}] \prod_t \star_{j=1}^{N_t} \delta_{-h_{0j} x_t^- (h_{0j})^{-1}}(x_t^+) \quad (44)$$

transforms covariantly under the gauge transformations (43). Using the cyclic invariance of the star product under integration, the amplitude (41) can be written in terms of this measure as

$$I_\Delta = \int [d^6 x_t] [dk_\tau] \mathcal{D}^{x_t, k_\tau}[h_{\tau\sigma}] \star \prod_t [e^{i \text{Tr} x_t H_t} \star \delta_{-k_{\tau_0(t)} x_t^- k_{\tau_0(t)}^{-1}}(x_t^+)], \quad (45)$$

where the star product pairs the variables  $x_t$ . Note that the appearance of a gauge covariant measure on the discrete connection is the result of using extended GFT fields, and thus of requiring a covariant imposition of the simplicity constraints. The need of a generalization of the closure constraint to achieve this has been noted on several occasions in the literature [32,34] (see also [31]).

## 2. Spin foam representation of the amplitudes

We have derived the GFT amplitudes starting from the bivector formulation of the group field theory, where the simplicial geometry is implicit. By construction, they take the form of simplicial path integrals. It should be clear, however, that the dual connection and spin formulations of the same GFT will generate dual expressions of the same amplitudes in terms of a lattice gauge theory and spin foam amplitudes.

The spin foam representation of the amplitudes (41) can be also be computed directly by Plancherel decomposition of the group functions into irreducible representations and integration over group and Lie algebra elements. As proved in [22], it gives the Barrett-Crane amplitudes

$$I_\Delta = I_{\text{BC}} := \sum_{\{j_i\}} \prod_t d_{j_t}^2 \prod_\tau \frac{1}{\prod_{i \in \partial\tau} d_{j_i}} \prod_\sigma \{10j\}_\sigma. \quad (46)$$

The sum is over  $\text{SU}(2)$  spins  $j_i$  labeled by triangles,  $d_{j_i} := 2j_i + 1$ . The products are over all triangles  $t$ , tetrahedra  $\tau$ , and 4-simplices  $\sigma$ . The 4-simplex weight  $\{10j\}_\sigma$  is the Barrett-Crane  $10j$  symbol [15]. This derivation singles out a specific edge amplitude (tetrahedral weight), which differs from the ones that appear in the literature with the exception of [23]. We emphasize again that, in our GFT construction, although the amplitudes of closed graphs reproduce the Barrett-Crane amplitudes, the requirement of a covariant imposition of the simplicity constraints imposed the use of extended boundary states which include

tetrahedra normals, hence labeled by projected spin networks.

This geometrical construction sheds an interesting new light on the Barrett-Crane model. In particular, the formula (41) gives a new simplicial path integral formulation of its amplitudes, making the simplicial geometry of the model manifest. We refer to [22] for a detailed analysis and discussion.

## D. Including the Immirzi parameter: GFT for Holst-Plebanski gravity

We now turn to the inclusion of the Immirzi parameter in the model. For generic values of  $\beta = \frac{\gamma-1}{\gamma+1}$ , the operator  $\hat{S}^\beta$  is no longer a projector. Depending on whether it is inserted in the propagator, in the vertex, or in both the vertex and propagator, in a single or in multiple copies, will lead to *a priori* different spin foam amplitudes. These will however have the same vertex amplitude (4-simplex weight) and differ only in the edge and face amplitudes (weights associated with tetrahedra and triangles). Here, just as in the previous section, we constrain the field in the interaction of the extended Ooguri model. Setting  $\hat{\Psi}^\beta := \int dk \hat{S}^\beta \triangleright \hat{\phi}_k$ , we thus consider

$$S = \frac{1}{2} \int [d^6 x_i] d^4 k \hat{\phi}_{k1234} \star \hat{\phi}_{k-1-2-3-4} + \frac{\lambda}{5!} \int [d^6 x_i]^{10} \hat{\Psi}_{1234} \star \hat{\Psi}_{-4567} \star \hat{\Psi}_{-7-389} \star \hat{\Psi}_{-9-6-210} \star \hat{\Psi}_{-10-8-5-1}, \quad (47)$$

where the star product pairs repeated indices.

### 1. Simplicial path integral representation of the amplitudes

The derivation of the Feynman amplitudes is analogous to the case  $\beta = 1$ . These amplitudes are calculated with the same propagator as in (23) and the vertex

$$V^\beta(x_\ell^i; k_\ell) = \int [dh_\ell] d^5 \prod_{i=1}^{10} (\delta_{-x_i^\ell} \star S_{k_\ell}^\beta \star E_{h_\ell h_\ell^{-1}} \star S_{k_\ell'}^\beta)(x_i^\ell), \quad (48)$$

where  $i$  labels the oriented strands (triangles) and  $\ell$  the half lines (tetrahedra) of the vertex graph, and  $S_k^\beta = \delta_{-kx^- k^{-1}}(\beta x^+)$  is the simplicity function. Using the same notations as in (41), we obtain, for the amplitude of a closed graph,

$$I_\Delta^\beta = \int [dh_{\tau\sigma}] [dk_\tau] [d^6 x_t] \left[ \prod_t \star_{j=0}^{N_t} S_{h_{0j} \triangleright k_j}^{\beta \star 2} \right] \star e^{i \sum_t \text{Tr} x_t H_t}, \quad (49)$$

where  $S^{\beta \star 2}$  denotes the squared function  $S^\beta \star S^\beta$ . For each triangle  $t$ , the constraints impose, by means of

noncommutative delta functions, the linear simplicity condition of  $x_t$  with respect to the normals of all the tetrahedra  $\{\tau_j\}_{j=0\dots N_t}$  sharing  $t$ . The square stems from the fact both 4-simplices sharing the tetrahedron  $\tau_j$  contribute to a factor  $S_{h_{0j} > k_j}^\beta(x_t)$ . The Feynman amplitudes of this theory thus take the form of simplicial path integrals for a constrained BF theory of Holst-Plebanski type with Immirzi parameter  $\gamma$ , with linear simplicity constraints [34].

Just as for the case with no Immirzi parameter, the integrands of (49) are invariant under the gauge transformations (4); in the case of closed graphs, the integration over the normals  $k_\tau \in \text{SU}(2)$  drops from the amplitude. Making explicit the form of the simplicity functions, we thus obtain

$$I_\Delta^\beta = \int [dh_{\tau\sigma}] [d^6x_t] \left[ \prod_t \star_{j=0}^{N_t} \delta_{-\bar{h}_{0j} x_t^- \bar{h}_{0j}^-}^{\star 2}(\beta x_t^+) \right] \star e^{i \sum_t \text{Tr} x_t H_t}, \quad (50)$$

where we wrote  $f^{\star 2}$  for the squared function  $f \star f$ , and  $\bar{h}_{0j} = h_{0j}^+(h_{0j}^-)^{-1}$ . Finally, by splitting the constraints into a part (the  $j = 0$  contributions) that is independent of the holonomies and a part playing the role of constraints on the holonomies, the amplitude  $I_\Delta^\beta$  can be put under a form analogous to (46), in terms of a covariant measure on the space of discrete connections.

## 2. Spin foam representation of the amplitudes

The spin foam representation of the amplitudes can be obtained either directly from (49) by inverse Fourier transform and Peter-Weyl decomposition of the group functions, or from the Feynman rules of the spin representation of the generating GFT. In this section we give the explicit form of the resulting spin foam amplitudes in terms of  $15j$  symbols and so-called fusion coefficients, which will allow a direct comparison with the existing models [16,18,19]. Their derivation from the GFT is straightforward: we only sketch it here.

In the spin foam representation, the amplitudes read

$$I_\Delta^\beta = \sum_{j_i^-, j_i^+, \lambda_{i\tau}, i_\tau} \prod_t d_{j_i^-} d_{j_i^+} \prod_{(t\tau)} d_{\lambda_{i\tau}} \prod_\sigma A_\sigma^\beta(j_i^\pm, \lambda_t, i_\tau; k_\tau), \quad (51)$$

where the 4-simplex weight (vertex amplitude) is given by

$$A_\sigma^\beta(j_i^\pm, \lambda_{i\tau}, i_\tau; k_\tau) = \sum_{i_\tau\sigma, i_\tau\sigma} \{15j\}_\sigma^- \{15j\}_\sigma^+ \prod_{\tau \subset \sigma} d_{i_\tau\sigma} d_{i_\tau\sigma}^+ f_{i_\tau\sigma, i_\tau\sigma}^{i_\tau} (j_i^\pm, \lambda_{i\tau}; k_\tau). \quad (52)$$

The notations are as follows.  $t$ ,  $\tau$ , and  $\sigma$  denote the triangles, tetrahedra, and 4-simplices of the simplicial complex  $\Delta$ . The sums are over  $\text{SO}(3)$  representation  $j$ ,  $\lambda$  and 4-valent  $\text{SO}(3)$  intertwiners  $i$ , all labeled by an integer spin, and  $d_j = 2j + 1$ . This gives a pair of spins  $(j_i^-, j_i^+)$

for each triangle, a spin  $\lambda_{i\tau}$  for each couple  $(t\tau)$  with  $t \subset \tau$ , a spin  $i_\tau\sigma$  for each tetrahedron, and a pair of spins  $(i_\tau^-, i_\tau^+)$  for each couple  $(\tau\sigma)$  with  $\tau \subset \sigma$ . We set  $d_j = 2j + 1$ . The variables  $k_\tau \in \text{SU}(2)$  are the normals to the tetrahedra: as we have seen, the dependence upon the normals for the bulk tetrahedra (internal links of the GFT graph) drop, hence we have not made the integrals over these explicit. The above amplitude may as well be evaluated in the time gauge  $k_\tau = 1$  for all bulk tetrahedra, though it then makes less transparent the nature of the boundary states, here labeled by projected spin networks.

The amplitude  $A_\sigma^\beta$  is defined in terms of  $\text{SU}(2)$  Wigner symbols  $\{15j\}_\sigma^\pm(j_i^\pm, i_\tau^\pm)$  and so-called fusion coefficients [16]  $f_{i_\tau^-, i_\tau^+}^{i_\tau}$ . These coefficients define a map from the space of  $\text{SO}(3)$  intertwiners between the representations  $\lambda_{i_1\tau}, \dots, \lambda_{i_4\tau}$  and the space of  $\text{SO}(4)$  intertwiners between the representations  $(j_1^-, j_1^+), \dots, (j_4^-, j_4^+)$ ,

$$f|i_\tau\rangle = \sum_{i_\tau\sigma, i_\tau\sigma} f_{i_\tau\sigma, i_\tau\sigma}^{i_\tau} |i_\tau\sigma^- \otimes i_\tau\sigma^+\rangle. \quad (53)$$

While the form (51) is quite general for a spin foam model defined as a constrained BF theory, the specificity of a model lies in the exact form of the fusion coefficients, which encode the way simplicity constraints are imposed. For the new model presented here, they are given by

$$\begin{aligned} f_{i_\tau\sigma, i_\tau\sigma}^{i_\tau}(j_i^\pm, \lambda_{i\tau}; k_\tau) &= \langle i_\tau\sigma^- \otimes i_\tau\sigma^+ | \otimes_i F^{j_i^- j_i^+ \lambda_{i\tau}}(k_\tau) | i_\tau \rangle \\ &= (i_\tau\sigma^-)_{m_i^-} (i_\tau\sigma^+)_{m_i^+} \left( \prod_i F_{m_i^- m_i^+ p_i}^{j_i^- j_i^+ \lambda_{i\tau}}(k_\tau) \right) (i_\tau)_{p_i}, \end{aligned} \quad (54)$$

where repeated lower indices are summed over. The tensor  $F^{j_i^- j_i^+ \lambda_{i\tau}}(k_\tau)$  are the ones defined in (36). They provide an embedding of  $\text{SO}(3)$  structures into  $\text{SO}(4)$  ones. In particular, because of the intertwining property (37), they realize  $\text{SO}(3)$  as the stabilizer subgroup  $\text{SO}(3)_k \subset \text{SO}(4)$  of the normal  $k_\tau$  to the tetrahedron. They also depend on the Immirzi parameter and encode the simplicity constraints.

This form (51) of the amplitudes follows from the GFT Feynman rules in the spin representation. In this representation, the bivectors  $x_t$  are replaced by pairs of spins  $J_t := (j_t^-, j_t^+)$  and magnetic numbers labeling the strands of the graphs. A way to read these rules is then the following. For a given graph  $\mathcal{G}$  labeled by spins  $J_t$  and tetrahedron normals  $k_\tau$ , they attach to each four-stranded line (tetrahedron) a propagator  $P^{\beta, J_i}(k_\tau) \in \text{end}(\bigotimes_{i=1}^4 J_{t_i})$  defined as an endomorphism of the tensor product of the representations labeling its strands (triangles). The amplitudes are obtained by taking the trace (i.e., index contractions) of all propagators, following the combinatorics of the graph and by summing over all spins (and normals) as

$$I_\Delta^\beta = \sum_{\{J_t\}} \prod_t d_{j_t^-} d_{j_t^+} \text{Tr}_{\mathcal{G}} \left[ \bigotimes_\tau P^{\beta, J_i}(k_\tau) \right], \quad (55)$$

where  $d_j = 2j + 1$ . The propagator decomposes as

$$P^{\beta, J_i}(k_\tau) = P_{\text{SO}(4)}^{J_i} \tilde{P}^{\beta, J_i}(k) P_{\text{SO}(4)}^{J_i}, \quad (56)$$

where  $P_{\text{SO}(4)}^{J_i}$  is the projector onto SO(4)-invariant tensors and  $\tilde{P}^{\beta, J_i}(k)$  is defined in terms of the tensors (36) as

$$\begin{aligned} & \tilde{P}_{m_i^-, m_i^+; n_i^-, n_i^+}^{\beta, J_i}(k) \\ &= \prod_{i=1}^4 \left( \sum_{\lambda_i} d_{\lambda_i} \bar{F}_{m_i^-, m_i^+; p_i}^{J_i \lambda_i}(k) F_{n_i^-, n_i^+; q_i}^{J_i \lambda_i}(k) \right) (P_{\text{SO}(3)}^{\lambda_i})_{p_i q_i}, \quad (57) \end{aligned}$$

where repeated lower indices are summed over.  $P_{\text{SO}(3)}^{k_i}$  is the projector onto SO(3) invariant tensors in  $\mathfrak{O}_i \lambda_i$ . Note that its insertion is actually redundant in the definition of the propagator. Indeed, because of the property (37), an SO(3) rotation in  $\mathfrak{O}_i \lambda_i$  is intertwined by  $F$  with a  $\text{SO}(3)_k$  rotation in  $\mathfrak{O}_i J_i$ , which can be reabsorbed into  $P_{\text{SO}(4)}^{J_i}$ . This form however allows us to split the trace in (55) into a product of 4-simplex weights as in (51). This is done by expanding the projectors  $P_{\text{SO}(4)}^{J_i}$  and  $P_{\text{SO}(3)}^{k_i}$  into 4-valent intertwiners ( $t^-, t^+$ ) and  $t$ .

To close this section, we emphasize again that, as in the extended BF case, the boundary states of the model are by construction (constrained) projected spin networks [35]. Even in the presence of the Immirzi parameter, we see therefore that the boundary states of the amplitudes (or the GFT polynomial observables) are different from the states of standard loop quantum gravity (LQG). The projected spin network structure is actually present also in the boundary states of all the new models [16,18], even if in a less explicit way, and even if their apparent coincidence with LQG states in representation space (because of the specific form of the simplicity constraints imposed there) is more emphasized.

### E. Limiting cases

The model presented in the previous sections corresponds to a candidate quantization of a simplicial version of the Plebanski-Holst formulation of four-dimensional gravity, for generic values of the Immirzi parameter  $\gamma$ . We have already seen that the case  $\beta = 1$ , which corresponds to  $\gamma = \infty$ , gives a variant of the Barrett-Crane model with a specific edge amplitude, where the boundary states are extended to include tetrahedra normals, hence are labeled by projected spin networks. Thus, the formula (41) not only gives a new simplicial path integral formulation of the Barrett-Crane model, but it also provides a natural deformation of that model which includes the Immirzi parameter. We now discuss briefly other limiting cases of this model: the self-dual case  $\gamma = 1$  and the topological case  $\gamma = 0$ .

For  $\gamma = 1$ , as mentioned the change of variables (25) becomes singular—so the contact with the classical Holst

theory is lost. Despite the lack of a clear geometric interpretation, the constraint operator  $S^\beta$  and the resulting model are well-defined for  $\beta = 0$ ; it acts on the field by  $\hat{\varphi}_k$  by projection of its bivector variables onto the self-dual part of  $\mathfrak{so}(4)$ . The constrained model reduces to the Ooguri model for topological SU(2) BF theory.

The case  $\gamma = 0$  corresponds to the so-called ‘‘topological sector’’ of Holst gravity. This denomination comes from the fact that the term of the classical Holst action that seemingly dominates in this limit is the one that vanishes on shell, due to the requirement of torsion freeness of the connection. As a consequence, one would expect that the resulting spin foam model/path integral would define a trivial dynamics for any boundary state. It is not totally obvious, however, that the above reasoning goes through in the quantum theory as well. It could also be argued [42] that the resulting quantum theory would rather correspond to a quantization of second order, metric gravity with no torsion. The rough argument is that in a path integral for the Holst action, the limit  $\gamma \rightarrow 0$  would force, analogously to a semiclassical limit, the same path integral to be dominated by solutions of the equations of motion coming from the topological term only, that is exactly the torsion freeness condition. While these arguments are obviously not conclusive, they suggest not to dismiss the resulting model as uninteresting. In our context, this corresponds to  $\beta = -1$ . As discussed also in [22], the constraint operator for  $\beta = -1$ : (a) projects onto simple SO(4) representation  $J = (j, j)$ ; and (b) does not impose any restriction on the expansion of  $(j, j)$  into SU(2) irreducible representations  $k = 0, \dots, 2j$ , and (c) acts on each component  $(J, k)$  by multiplication by the phase  $(-1)^{2j+k}$ . In computing the amplitudes, the phase factors cancel each other. The resulting model, distinct from the EPR amplitudes [16], is obtained from the SO(4) Ooguri model (6) by restricting the representations to simple ones  $J_i = (j_i, j_i)$ . It would be interesting to study what the geometric interpretation of such amplitudes may be.

The general GFT model for arbitrary  $\beta$  thus encompasses and generalizes several distinct models, and interpolates between them. It is tempting to speculate (see also [42,43]) that the model possesses a nontrivial renormalization group flow in parameter space  $(G_N, \beta)$ , where  $G_N$  is Newton’s constant, which is hidden in our formulation as we use dimensionless quantities throughout. The natural candidates for fixed points would then be these special values for the  $\beta$  parameter:  $(-1, 0, 1)$ , namely  $\gamma = 0, 1$ , or  $\infty$ . In particular, while the case  $\beta = -1$  is distinguished only for being in some sense extremal, and for its peculiar classical analogue, the other two values can be seen as special already at the level of the very definition of the corresponding model. In fact, as we have seen, in these two cases, and only then, the simplicity operator defines a projector, and the quantum amplitudes are insensitive to

the specific choice of insertion of this operator in the GFT action.

#### IV. SIMPLEX CORRELATIONS AND ULTRALOCALITY

The main advantage of the framework developed in this paper is that the variables encoding the (fuzzy) bivector simplicial geometry of GFT and spin foam models are explicit. In particular, with respect to other formalisms, this framework gives a more direct access to the way simplices are correlated in the model, namely, how the model relates the geometrical data of common subsimplices in the gluing of neighboring simplices.

It has in fact often been argued that, for example, the Barrett-Crane model suffers from a default of correlations between neighboring simplices. This ultralocality feature has been one of the reasons to discard this model in favor of the new models. Since the Barrett-Crane amplitudes show up in our model for the value  $\gamma = \infty$  of the Immirzi parameter, this feature can be made explicit and clarified in our framework. An indepth study of the ultralocality issue is beyond the scope of the present paper. In this section we however discuss how it manifests itself in our framework.

The interaction and kinetic polynomials of our GFT model are written as a simple star product of copies of the constrained field in which the bivector variables associated with the common triangles are *strictly* identified (modulo an orientation flip). At the level of the Feynman amplitudes however, after expansion of the gauge invariance operator, the Feynman rules dictate the relation between bivectors  $\{x_i^\tau, x_i^\sigma\}$  expressed in different frames related by holonomies  $h_{\tau\sigma}$ . For example in the BF model, two bivectors  $x_i^\tau, x_i^{\tau'}$  on the same triangle  $t$  but seen from different tetrahedra are related by a noncommutative delta function,

$$(\delta_{-x_i^\tau \star E_{h_{\tau\tau'}}})(x_i^{\tau'}), \quad (58)$$

where  $h_{\tau\tau'} = h_{\tau\sigma}h_{\sigma\tau'}$  parallel transports the frame of one tetrahedron to the frame of the other. The structure of the star product gives a clear geometrical meaning to the algebraic expressions. In particular, the algebraic operation corresponding to the parallel transport of bivectors is the commutation with plane waves:  $E_h \star f = f^h \star E_h$ , where  $f^h(x) = f(h^{-1}xh)$ .

In the amplitudes of the constrained theory, the plane waves are supplemented with simplicity functions imposing the linear simplicity condition of the bivectors in each frame,

$$E_{h_{\tau\tau'}} \rightarrow S_{k_\tau}^\beta(x_t) \star E_{h_{\tau\tau'}} \star S_{k_{\tau'}}^\beta(x_t), \quad (59)$$

so that the commutation with the plane wave encodes the parallel transport of *simple* bivectors. In the case  $\beta = 1$  (i.e.,  $\gamma = \infty$ ) corresponding to the Barrett-Crane amplitudes, the simplicity functions  $S_k := S_k^1$  satisfy

$S_k \star S_k = S_k$ . The definition (28) of these functions and the structure of the star product then lead to the identity

$$S_{k_\tau} \star E_{h_{\tau\tau'}} \star S_{k_{\tau'}} = S_{k_\tau} \star E_{\mathbf{u}_i^\tau h_{\tau\tau'} \mathbf{u}_i^{\tau'}} \star S_{k_{\tau'}} \quad (60)$$

for all  $\mathbf{u}_i^\tau = (k_\tau^{-1} u_i^\tau k_\tau, u_i^\tau)$  and  $\mathbf{u}_i^{\tau'} = (k_{\tau'}^{-1} u_i^{\tau'} k_{\tau'}, u_i^{\tau'})$  in the stabilizer subgroups  $\text{SO}(3)_{k_\tau}$  and  $\text{SO}(3)_{k_{\tau'}}$  of the normals. This is because  $\mathbf{u}_i^\tau, \mathbf{u}_i^{\tau'}$  can be reabsorbed into the group elements labeling the plane wave expansion (28) of  $S_{k_\tau}$  and  $S_{k_{\tau'}}$ . This identity can be understood as a relaxation of the parallel transport condition, or a weakening of the bivector correlations: upon parallel transport  $h_{\tau\tau'}$ , simple bivectors are identified only up to spatial rotations. This is the manifestation of ultralocality in this geometrical setting, here due to the interplay between simplicity and parallel transport conditions induced by the noncommutativity of the star product.

Note that the argument does not extend to general values of  $\beta$  in an obvious way, for the same reason that makes the constraint operator fail to be a projector  $S_k \star S_k \neq S_k$ . However it remains that upon commutation with (59), a Lie algebra function gets conjugated not only by the holonomy  $h_{\tau\tau'}$  but also by the Lagrangian multipliers of the simplicity functions.

Arguing whether or not this feature is a serious problem from the point of view of quantum geometry is not our point here. Our point is to emphasize that it appears in our framework as an unavoidable feature following a clear geometrical construction of a dynamical theory for noncommutative tetrahedra. In fact, as has been anticipated in the literature, the above argument shows that ultralocality is manifestly inherent to the noncommutativity of the bivector geometry, here entirely encoded into the star product. From the point of view of the canonical theory, where the tetrahedron states live in a tensor product of noncommutative spaces  $L_\star^2(\mathbb{R}^6)$ , it is tied to the choice of quantization map. The presence of the star product thus encodes also quantum corrections, of which the above effect is a manifestation.

This raises the question whether ultralocality survives in a semiclassical regime involving a commutative limit. The star product structure being dual to group composition, this limit corresponds to a linearization of the group. It can be formally defined by introducing a parameter  $\epsilon$  in the coordinates of the group manifold and to parametrize  $\text{SU}(2)^\pm$  group elements  $u = e^{i\theta\vec{n}\cdot\tau}$  for e.g. by  $\mathbb{R}^3$  vectors  $\vec{p}_u = \frac{1}{\epsilon} \sin\theta\vec{n}$ . The Fourier transform can be parametrized accordingly [38]; in the regime of small  $\epsilon$ , the  $\text{SO}(3)$  [respectively,  $\text{SO}(4)$ ] star product reduces to the usual pointwise product on functions of  $\mathbb{R}^3$  (respectively,  $\mathbb{R}^6$ ). The commutative regime should correspond by duality to the large spin limit of spin foam models. However since it amounts to linearizing the holonomies, it could also be viewed as an analogue of the continuum

limit in lattice gauge theories. Whether or not a proper development of the model around a commutative limit, at the level of the GFT [44] or its amplitudes, can be properly defined, and shown to tame the ultralocality feature, remains to be seen.

## V. CONCLUSIONS

In this paper, we exploited a dual formulation of group field theories in terms of noncommutative bivector variables, which provide a duality between spin foam models and simplicial path integrals for constrained BF theories, to derive a new model for four-dimensional gravity with Immirzi parameter. All geometrical variables remain explicit in this construction, which consists of inserting a constraint operator in a GFT for four-dimensional  $SO(4)$  BF theory implementing the discrete simplicity constraints turning quantum simplicial BF theory into quantum simplicial gravity.

Thanks to the framework chosen, we can keep the geometric content of all variables and of imposed constraints manifest at all stages of the construction. The resulting amplitudes for each simplicial complex, generated in the Feynman expansion of the GFT, give a quantum simplicial version of the Holst-Plebanski formulation of gravity. We formulated these amplitudes both as BF simplicial path integrals with explicit noncommutative  $B$  variables and in terms of Wigner  $15j$  symbols and fusion coefficients. The new model differs from existing ones in the literature; it imposes a different restriction on representation labels for quantum states. In particular, it does not lead to any rationality condition for the Immirzi parameter.

In light of this geometrical framework, we suggested a possible new perspective on the issue of the quantum correlations between neighboring simplices, often argued to be a problematic feature, for example, in the Barrett-Crane model. In our formalism, in fact, the relaxation of parallel transport condition is an unavoidable consequence of the very noncommutative nature of bivector variables and of the simplicity constraints, and is tied to their quantization. Moreover, our framework is best suited for studying the geometric interpretation and consequences of

this relaxation, as well as the semiclassical limit of the amplitudes.

While this problematic issue certainly needs to be investigated further, we believe that the construction we performed suggests to consider the resulting model seriously as a candidate model for quantum gravity. In fact, its features appear all natural from the point of view of quantum geometry. Of course, this new model should now be tested in all its aspects to support further or refute its validity. Note that its explicit formulation as a path integral for constrained BF theory, in contrast with the other existing models, should facilitate the study of its relation with a path integral quantization of continuum Holst-Plebanski gravity [31,32].

We also believe that the noncommutative formalism on which the construction is based, and that has proven useful already in different contexts and for different purposes, should itself be studied in depth, to unravel even more aspects of (simplicial) quantum geometry. We have in mind the issue of symmetries, in particular, the simplicial analogue of diffeomorphism symmetry. This has been studied in the BF context in [45] and the analysis should now be extended to the four-dimensional gravity model proposed here. Indeed, the bivector representation is the one most suited for defining such symmetries in a geometrically clear way.

Finally, the crucial question is whether this formalism leads to an effective continuum dynamics of geometry, hopefully governed by some form of general relativistic action, in the continuum limit. For this one has to study either the coarse graining of the lattice path integral appearing [46] in our amplitudes, or the renormalization flow and critical behavior of our GFT model [43]. Also in this respect, our result offers a new, promising concrete model to analyze.

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