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Diffusion in quantum gravity

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The change of the effective dimension of spacetime with the probed scale is a universal phenomenon shared by independent models of quantum gravity. Using tools of probability theory and multifractal geometry, we show how dimensional flow is controlled by a multiscale fractional diffusion equation, and physically interpreted as a composite stochastic process. The simplest example is a fractional telegraph process, describing quantum spacetimes with a spectral dimension equal to 2 in the ultraviolet and monotonically rising to 4 towards the infrared.

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The spectral properties of effective quantum spacetimes show that the ultraviolet (UV) finiteness of independent theories of quantum gravity is universally associated with a lower spectral dimension of spacetime (typically, $d_{\rm S}\sim 2$) at small scales, while $d_{\rm S}\sim 4$ is recovered in the infrared (IR). Instances are causal dynamical triangulations [1], asymptotic safety [2, 3], spin foams [4, 5], noncommutative geometry [6], Hořava-Lifshitz gravity [7], and other approaches or models [8].

The change of dimension with the probed scale is known as dimensional reduction or dimensional flow [9]. Understanding its physical meaning is an eventually important piece of the puzzle of quantum gravity, since the multiscale behavior is deeply related to the renormalization properties of these theories. Differential geometry and ordinary calculus, as employed in general relativity and field theory, are inadequate to study this and other properties of quantum spacetimes, and stochastic processes and multifractal geometry can offer powerful tools of analysis and novel insight. While there is the tendency to label all multiscale spaces as "fractal," the accumulated knowledge from these branches of physics and mathematics permit to make sharper statements about the geometric and physical properties of quantum-gravity models. This philosophy inspired the revisiting of a recent problem, the construction of quantum field theories in fractal spacetimes, under a fresh perspective focused on an effective continuum geometry [10], in particular via the formalism of multifractional spacetimes [11].

Here we reexamine the spectral dimension starting from its foundation, the diffusion equation. A critical appraisal of the latter in multifractional theory will allow us to classify quantum geometries in terms of stochastic processes on one hand, and to get a precise back-up to the notion of "fractal spacetime" on the other hand. For the process to be meaningful, the solution P of a given diffusion equation must be nonnegative at all points,

$$P\geq 0\,. \tag{1}$$

If P is normalized to 1, it is interpreted as the probability to find the diffusing particle (if the probe is pointwise) at a given point. This probability distribution describes a stochastic process, i.e., a sequence or collection of random variables. We shall use Eq. (1) as one of the guiding principles to identify the random process associated with a given behavior of quantum geometry. Here we do not pay attention to the techniques employed for solving the diffusion equations; an expanded discussion is in [12].

Classical spacetimes. In a smooth classical spacetime with D topological dimensions, the diffusion equation is

$$\left(\partial_\sigma-\nabla_x^2\right)P(x,x',\sigma)=0\,. \tag{2}$$

The parameter $\sigma\geq 0$ acts as an abstract "time" variable via the diffusion operator ∂_σ, an ordinary first-order derivative. Writing $\sigma=\ell^2\bar\sigma$ in terms of a fixed length scale ℓ and a dimensionless parameter $\bar\sigma$, Eq. (2) is recast in the form $(\partial_{\bar\sigma}-\ell^2\nabla_x^2)P=0$. The spatial generator ∇_x^2 is the Laplacian in the given metric background in Euclidean signature. The subscript x indicates its action on the x dependence of the heat kernel P, while x' is the initial point where diffusion starts. In translation-invariant spacetimes, P depends on the difference $x-x'$, but in fractional spaces with nontrivial measure this is no longer true; therefore we keep the notation $P(x,x',\sigma)$ separate from the often-employed $u(x-x',\sigma)$ [or $u(x,\sigma)$, fixing $x'=0$]. The diffusion equation is not completely specified without the set of initial conditions at $\sigma=0$. The choice $P(x,x',0)=\delta(x-x')$ describes diffusion of a point particle starting at $x=x'$. Extended shapes of the probe are possible (e.g., [8]), but the pointwise one allows to explore the local manifold structure of spacetime.

The solution P must be nonnegative for all x and x', Eq. (1), and normalized as $\int d^Dx\,\sqrt{g}P(x,x',\sigma)=1$ ($g=\det g_{\mu\nu}$). From the spatial trace of P, one gets the return probability $\mathcal{P}(\sigma)\,:=\,(\int d^Dx\,\sqrt{g})^{-1}\int d^Dx\,\sqrt{g}\,P(x,x,\sigma)$ and the spectral dimension

$$d_{\rm S}:=-2\frac{d\ln\mathcal{P}(\sigma)}{d\ln\sigma}\,. \tag{3}$$

When divergent, the volume prefactor in the definition of $\mathcal{P}$ can be regularized. In the case of a translation-invariant background, it cancels out with the position dependence in the numerator and $\mathcal{P}(\sigma)=u(0,s)$.

Ignoring curvature, the normalized solution of Eq. (2) is the Gaussian heat kernel $P(x,x',\sigma)=u_1(\Delta x,\sigma):=e^{-\Delta x^2/(4\sigma)}/(4\pi\sigma)^{D/2}$, where $\Delta x^2:=\sum_\mu|x_\mu-x'_\mu|^2$ is the Euclidean distance, $\mu=1,\ldots,D$. Clearly, $P>0$.

The return probability and spectral dimension read $\mathcal{P} \propto \sigma^{-D/2}$ and $d_{\rm S} = D$, respectively. There is no quantitative distinction between spectral and topological dimension. They are also equal to the Hausdorff dimension $d_{\rm H}$ of spacetime, determining the scaling law of the volume of a D-ball of radius R, $\mathcal{V}^{(D)} \propto R^{d_{\rm H}}$. Notice that, because ℓ is the only scale, it is not possible to define a hierarchy of scales and the geometry (and $d_{\rm S}$) is scale independent.

The ordinary diffusion equation (2) is associated with a *Wiener process* $B(\sigma)$, also known as *standard Brownian motion*. A Wiener process is such that (i) B is continuous in σ almost surely (i.e., with probability 1), (ii) $B(0) = 0$, and (iii) the increments of B are independent and governed by the Gaussian distribution u_1, so that $B(\sigma) - B(\sigma') \sim u_1(0, \sigma - \sigma')$ for $\sigma' < \sigma$.

Quantum geometry with fixed dimension. We now turn to quantum gravity. In the literature, it is common to assume the diffusion equation (2). The idea is that quantum geometry should modify either the initial condition $P(x, x', 0)$ [8] or the Laplacian $\nabla_x^2 \to \mathcal{K}_x$, or both. In the presence of one or more fundamental quantum scales ℓ_n (the Planck scale, or the label-dependent lengths of the simplices in a cellular complex), the operator $\mathcal{K}_x$ and/or the initial condition can introduce a complicated scale dependence which gives rise to a multiscale behavior. We maintain this attitude and modify the Laplacian, later introducing other changes to the diffusion equation.

It is instructive to specialize first to the case of fixed dimensionality (no scale hierarchy). We concentrate on the continuum formulation of fractional calculus, which guarantees anomalous (in particular, fractal) geometric properties of spacetimes [11, 12] and anomalous correlations in diffusion problems (e.g., [13]). We ignore curvature. The latter modifies the spectral properties of spacetime even in a classical setting, except in the UV limit $\sigma \to 0$. Quantum geometric effects, however, often modify spacetime globally even in the absence of curvature, which motivates the assumption (see also [3]).

For each direction, we replace ∂_x^2 with the operator

$$\mathcal{K}_{\gamma,\alpha} := -\frac{1}{\sqrt{v_\alpha(x)}} \frac{{}_\infty\partial_x^{2\gamma} + {}_\infty\bar{\partial}_x^{2\gamma}}{2\cos(\pi\gamma)} \left[\sqrt{v_\alpha(x)}\,\cdot\right], \qquad (4)$$

where $v_\alpha(x) = |x|^{\alpha-1}/\Gamma(\alpha)$ is the measure weight of the ambient space (the singularity in $x = 0$ is integrable and does not pose particular problems for the classical and quantum dynamics), $0 < \alpha \leq 1$ and $\gamma > 0$ are real parameters, and we make use of left and right Liouville-Caputo fractional derivatives (see [11, 12]). When $2\gamma = m$ is integer, ${}_\infty\partial^m = (-1)^m {}_\infty\bar{\partial}^m = \partial^m$. Definition (4) is such that, in a suitable domain, the operator $\mathcal{K}_{\gamma,\alpha}$ is self-adjoint and with eigenvalue $-|k|^{2\gamma}$ [11, 12].

We classify the stochastic and geometric properties associated with the diffusion equation

$$(\partial_\sigma - \mathcal{K}_{\gamma,\alpha})\, P = 0\,, \qquad (5)$$

with initial condition $P(x, x', 0) = [v_\alpha(x) v_\alpha(x')]^{-1/2} \times \delta(x - x')$. By the self-similarity of P, one can show that $d_{\rm S} = D\alpha/\gamma$ [12]. • When $\gamma = 1 = \alpha$, we recover ordinary diffusion and $d_{\rm S} = D$. • For $\gamma = 1 \neq \alpha$, this is ordinary Brownian motion but on a fractal spacetime with $d_{\rm S} = D\alpha$. • For $0 < \gamma < 1$ and $\alpha = 1$, we have a *Lévy process*. It is a Markovian process like Brownian motion (we recall that a process is Markovian if future states depend on the present state but not on past states), but characterized by a heavy-tailed distribution and "long jumps" connecting clusters of shorter steps. One has superdiffusion and $d_{\rm S} = D/\gamma > D = d_{\rm H}$. This does *not* correspond to a fractal spacetime ($d_{\rm S} \leq d_{\rm H}$ for fractals). • For $0 < \gamma, \alpha < 1$, one has a Lévy process on an anomalous spacetime (fractal if $\alpha \leq \gamma$). • When $\gamma > 1$, the solution of (5) is no longer nonnegative definite and the equation must be modified. In fact, one can include a source term, $(\partial_\sigma - \mathcal{K}_{\gamma,\alpha})P = \mathcal{S}(x, x', \sigma)$, which does not alter the spectral dimension. Hence, overlooking the check of (1) for the *Ansatz* (5) might result in the correct spectral dimension but a wrong diffusion equation.

Processes associated with nonhomogeneous equations may be non-Markovian even if they are meaningful in a probabilistic sense. An example is the quartic equation

$$\left(\partial_\sigma - \nabla_x^4\right) u(x, \sigma) = (\pi\sigma)^{-1/2} \nabla_x^2 u(x, 0)\,, \qquad (6)$$

with $\alpha = 1$ and source given by the initial condition. The solution gives the same $d_{\rm S} = D/2$ as the naive Eq. (5) with $\gamma = 2$, but the presence of the source guarantees that $u \geq 0$. Equation (6) governs an *iterated Brownian motion* (IBM) or Brownian-time Brownian motion [14]. Given two independent Wiener processes $B_{1,2}$, IBM is defined as $X_{\rm IBM}(\sigma) := B_1[|B_2(\sigma)|]$, where B_2 acts as a clock to B_1. Equation (6) can be regarded as the "iteration" of the fractional equation $(\partial_\sigma^{1/2} - \nabla_x^2)u = 0$, with *same* solution u. ∂_σ^β is the left Caputo derivative with lower terminal $\sigma' = 0$. In general, there exists a deep connection between higher-order diffusion equations with integer time, iterated stochastic processes, and diffusion equations with fractional time $(\partial_\sigma^\beta - \nabla_x^2)u = 0$, with $0 < \beta \leq 1$. The solution u is positive definite [15]. The process described by this equation (*fractional Brownian motion*) is subdiffusive: Due to a heavy tail in waiting times, in average it takes longer (with respect to Brownian motion) for the particle to cover a certain distance. Consequently, also Eq. (6) gives subdiffusion.

Extending the discussion to a nontrivial spacetime measure, the spectral dimension associated with the fractional diffusion equation $(\partial_\sigma^\beta - \mathcal{K}_{\gamma,\alpha})P = 0$ is $d_{\rm S} = (\beta/\gamma)d_{\rm H}$, where $d_{\rm H} = D\alpha$ [11, 12]. A fractal configuration is obtained whenever $\beta \leq \gamma$.

Multiscale quantum spacetimes. We now make a twofold conceptual step of relevance for quantum gravity. In stochastic and chaos theory, the adoption of a diffusion equation is motivated by phenomenology. Given a set of experiments evidencing some anomalous scaling laws, one proposes an ad-hoc diffusion equation reproducing those scalings. The theoretical model is then further tested against experiments. Or else, one defines the stochastic process underlying a certain physical system, and from

its probability distribution one infers the correct diffusion equation. For instance, IBM provides a stochastic description of diffusion in cracks [16]. Intuitively, it consists in a Brownian diffusion of a particle trapped in random fractal set (a crack) whose pattern resembles the graph of a Brownian motion. In quantum gravity, on the other hand, we do not have experiments but fragmentary knowledge such as the existence of anomalous scaling behaviors in the UV. This information determines the differential order of the operators in the diffusion equation, but it may be unable to fix the latter univocally. This means that dimensional flow in quantum geometry may be insensitive of the presence of source terms [see [3, 12]; Eq. (6) without source or with flipped sign in front of ∇^4 would still give the same $d_{\rm S}$], and we must resort to positivity of the diffusion solution to fix more details of the diffusion equation. In turn (this is the second part of the step), once we determine a reasonable diffusion equation with probabilistic interpretation, we can also find the stochastic process associated with that, thus physically characterizing quantum geometry.

Without further input from the theory except the behavior in the UV and in the IR, we can prescribe a sensible diffusion equation (with nonnegative solution P) reproducing the *whole* dimensional flow. This is achieved by applying the techniques of multiscale phenomena and multifractal geometry to the texture of spacetime itself [11, 12]. The generalization of the diffusion equation to a multiscale process is realized by summing over all possible values of α, β, and γ:

$$\sum\nolimits_n \left(\xi_n \partial_\sigma^{\beta_n} - \zeta_n \mathcal{K}_{\gamma_n,\alpha_n}\right) P(x,x',\sigma) = \mathcal{S}(x,x',\sigma)\,, \quad (7)$$

where ξ_n and ζ_n are dimensionful couplings which depend on the characteristic scales of the system. Typically, there is only a finite number N of terms in physical systems, so the sum representation (7) is realistic. The number $N-1$ of characteristic scales (hidden in ξ and ζ) determines the number N of plateaux (asymptotic regimes) in the profile of $d_{\rm S}$. The discrepancy between the number of scales and the number of regimes is due to the fact that a multiscale phenomenon is always defined by the relative size of the scales, not by an absolute hierarchy. This means that we can choose any of the N scales ℓ_n to represent the scale ℓ probed by a measurement. If we order the scales of the system as $\ell_1 < \ell_2 < \cdots < \ell_N$, we can take the largest as $\ell = \ell_N$. Thus, there are $N-1$ (not N) scales with the physical meaning of characteristic lengths. The spectral dimension is fixed when $N=1$; for $N=2$ (one scale), it has two asymptotic values $d_{\rm S} \sim d_{{\rm S}1,2}$, with a monotonic transient phase in between, in the regimes $\ell \ll \ell_1$ and $\ell \gg \ell_1$; for $N=3$, there will be an intermediate plateau where $d_{\rm S} \sim d_{{\rm S}3}$ between the limiting values $d_{{\rm S}1,2}$; and so on.

The first example is an interaction of Gaussian and anomalous dynamics which can describe certain turbulent media [17]. The diffusion equation is $(\partial_\sigma - \partial_x^2 - \zeta_1 \mathcal{K}_{\gamma,1})u = 0$, $u(x,0) = \delta(x)$, where $0 < \gamma < 1$ and we write the constant $\zeta_1 = \ell_1^{-2(1-\gamma)}$ in terms of a characteristic length. As the analytic solution shows, the transport is of Lévy type at large scales $\ell = k^{-1} \gg \ell_1$ ($d_{\rm S} \sim 1/\gamma > 1 = d_{\rm H}$) and normal at small scales $\ell \ll \ell_1$ ($d_{\rm S} \sim 1$). From the perspective of quantum spacetimes, this model is multiscale but not multifractal. Profiles of $d_{\rm S}$ overshooting the Hausdorff and topological dimensions appear also in lattice [5] and noncommutative geometries (last reference in [6]), with some caveats [12].

A second example is a fractional diffusion equation with two diffusion operators $\partial_\sigma^{\beta_{1,2}}$. To see its neat stochastic interpretation, we recall some results on the so-called telegraph processes ([18] and references therein). A telegraph process is defined as $V(\sigma) = V(0)\,(-1)^{\mathcal{N}(\sigma)}$, where $V(\sigma)$ is the velocity of a particle at time σ running on the real line, $V(0)$ is the initial velocity which is $\pm c$ with equal probability, and $\mathcal{N}$ is the cumulative number of events of a homogeneous Poisson process (the latter is a Lévy process) with rate $\lambda > 0$. The velocity of the particle flips direction at times obeying a Poisson distribution, hence the name "telegraph." The position of the particle at time σ is the integrated telegraph process $T(\sigma) = V(0)\int_0^s ds\,(-1)^{\mathcal{N}(s)}$ and its probability distribution obeys the telegraph equation $(\partial_\sigma^2 + 2\lambda\partial_\sigma - c^2\partial_x^2)u = 0$ with delta initial condition. We now consider a composite process called Brownian-time telegraph process or *fractional telegraph process*, $X_{\rm FTP}(\sigma) := T[|B(\sigma)|]$. It describes the motion of a particle that at time σ is located in the random spatial interval $(-\sigma|B(\sigma)|, \sigma|B(\sigma)|)$. This motion is governed by the diffusion equation $(\partial_\sigma + 2\lambda\partial_\sigma^{1/2} - c^2\partial_x^2)u = 0$. Other combinations of telegraph and Brownian processes are possible, leading to different diffusion equations. The solution of the fractional telegraph equation and its generalization $(\partial_\sigma^{2\beta} + 2\lambda\partial_\sigma^{\beta} - c^2\partial_x^2)u = 0$ is nonnegative and unique [18]. In the double limit $\lambda, c \to +\infty$, $\lambda/c^2 \to$ const, the stochastic process reduces to an IBM. Recasting these results in the language of multifractal spacetimes and extending to D dimensions, we set $[\sigma] = 0$, $c = \ell_*^2$ as the characteristic scale, and $\ell = \ell_*/(2\lambda)$ as the probed scale. In the limit $\ell \gg \ell_*$, diffusion in spacetime is Gaussian and described by a Brownian process ($d_{\rm S} \sim D$). At small scales $\ell \ll \ell_*$, on the other hand, one reaches a regime where diffusion is fractional and given by an IBM ($d_{\rm S} \sim D/2$). In between, diffusion in quantum spacetime obeys the law of a fractional telegraph process.

The monotonic profile $d_{\rm S}(\ell)$ of this single-scale spacetime can be plotted from the analytic form of the return probability. The probability distribution for more complicated multiscale spacetimes can be computed as well, but here we show how all these profiles are easily reproduced in the framework of multifractional geometry ($\alpha_n \neq 1$) when $\beta = 1 = \gamma$ and $\mathcal{S} = 0$. The coefficients ζ_n in (7) may be adjusted to give phenomenological profiles with different features, but here we argue that they have the natural form $\zeta_N = 1$ and

$$\zeta_1(\ell) = (\ell_1/\ell)^2\,, \qquad \zeta_n(\ell) = [\ell_n/(\ell - \ell_{n-1})]^2\,, \quad (8)$$

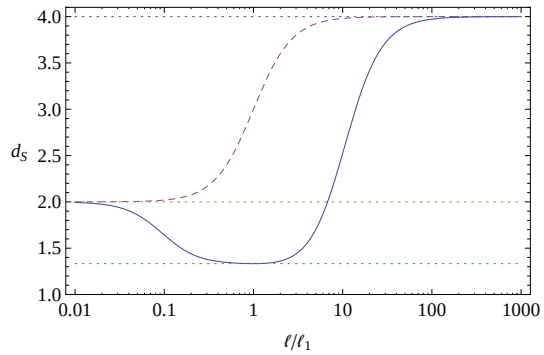

FIG. 1. The spectral dimension $d_{\rm S}(\ell)$ in $D=4$ for a multi-fractional model and normal diffusion ($\beta=1=\gamma$, $d_{\rm S}=d_{\rm H}$) with single scale (dashed curve) and two scales (solid curve).

where $\ell=\ell_N$. First, we notice that the Laplacians all have the same order 2, so the coefficients ζ_n all have the same scaling dimension and, in particular, we can always make them dimensionless. Write ζ_n as the ratio of some length scales, $\zeta_n=(l_{A,n}/l_{B,n})^q$. Without loss of generality, one can choose $q=2$ so that the spatial generator of the diffusion equation can be rendered dimensionless, in the form $\sum_n (l_{A,n})^2\mathcal{K}_{1,\alpha_n}$. Now, the nth term dominates over the others at scales $\ell\ll\ell_n$, so we could set $l_{A,n}=\ell_n$ and, tentatively, $l_{B,n}=\ell$. However, at scales smaller than ℓ_{n-1} the $(n-1)$th term takes the lead, so the smallest possible scale ℓ at which the nth term dominates is $\ell\sim\ell_{n-1}$. Then, the correct choice is $l_{B,n}=\ell-\ell_{n-1}$. In other words, the dimensional flow is always measured starting from the lowest or two scales ℓ_{n-1} to the next ℓ_n, and relatively to the latter, which sets a gauge for the rods. Since $\ell=\ell_N$ is the probed scale, $\zeta_N\equiv 1$.

We can plot the spectral dimension for any given profile $\alpha(\ell)$. Upgrading on [11], we motivate a realistic profile $\alpha(\ell)$ as an approximation of the sum in (7). Consider first the $N=2$ case with $\alpha_1\neq 1$ and $\alpha_2=1$. In one dimension, and by Eq. (4), $(\partial_x^2+\zeta_1\mathcal{K}^{\rm E}_{1,\alpha_1})P=\{(1+\zeta_1)\mathcal{K}^{\rm E}_{1,\alpha(\ell)}+\zeta_1(1-\alpha_1)^2/[4(1+\zeta_1)x^2]\}P$, where $\alpha_1(\ell):=[1+\zeta_1(\ell)\,\alpha_1]/[1+\zeta_1(\ell)]$. For both small and large ζ_1 the kinetic term in this expression dominates over the potential term, so the profile $\alpha_1(\ell)$ defines an effective fractional charge throughout the dimensional flow. With N coefficients α_n, $\alpha_N=1$, the effective charge reads

$$\alpha_{N-1}(\ell):=\frac{1+\sum_{n=1}^{N-1}\zeta_n(\ell)\,\alpha_n}{1+\sum_{n=1}^{N-1}\zeta_n(\ell)}\,. \tag{9}$$

This is nothing but the average $\langle\alpha\rangle$ of the coefficients α_n with respect to the weights ζ_n.

For two entries ($N=2$, $\alpha_1=2/D$, $\alpha_2=1$, one scale), dimensional flow is such that $d_{\rm S}\sim D$ in the IR and $d_{\rm S}\sim D\alpha_1=2$ in the UV, with no intermediate regime in between. This is the type of flow considered in [10, 11] and is shown in figure 1 (dashed curve), in agreement with the fractional-telegraph profile. A two-scale profile $d_{\rm S}(\ell)=4\alpha_2(\ell)$ with $\alpha_1=1/2$, $\alpha_2=1/3$ and $\ell_2=10\ell_1$ is also plotted (solid curve). At $\ell=0$, $d_{\rm S}=2$. At $\ell\sim\ell_1$, the spectral dimension acquires the minimum value $d_{\rm S}=4/3$. At scales $\ell\leq\ell_2$, the diffusion process corresponds to a recurrent random walk, where $d_{\rm S}<2$. Well above the larger critical scale, $\ell\gg\ell_2$, $d_{\rm S}$ hits the IR value ~ 4. Notably, this profile reproduces the dimensional flow of asymptotically-safe quantum gravity [3].

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