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Conformal anomaly for amplitudes in $\mathcal{N} = 6$ superconformal Chern–Simons theory

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Abstract
Scattering amplitudes in three-dimensional $\mathcal{N} = 6$ Chern–Simons theory are shown to be non-invariant with respect to the free representation of the $\mathfrak{osp}(6|4)$ symmetry generators. At tree and one-loop level these 'anomalous' terms occur only for non-generic, singular configurations of the external momenta and can be used to determine the form of the amplitudes. In particular we show that the symmetries predict that the one-loop six-point amplitude is non-vanishing and confirm this by means of an explicit calculation using generalized unitarity methods. We comment on the implications of this finding for any putative Wilson loop/amplitude duality in $\mathcal{N} = 6$ Chern–Simons theory.

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1. Introduction

A conformal field theory has no notion of distance. Consequently, two massless particles moving collinearly cannot be distinguished from each other in such a theory. The standard

Table 1. The conformal anomaly of scattering amplitudes depends on the signature and dimension. In four dimensions the holomorphic anomaly in (3, 1) signature is replaced by a sign-anomaly in (2, 2) signature. In three dimensions, the anomaly also takes the sign form.

Signature	3D	Signature	4D
(2,1)	$\frac{\partial}{\partial x}\,\mathrm{sgn}\,x = 2\delta(x)$ λ real	(2,2)	$\frac{\partial}{\partial x}\,\mathrm{sgn}\,x = 2\delta(x)$ $\lambda, \tilde{\lambda}$ real
–	–	(3,1)	$\frac{\partial}{\partial \bar{z}}\frac{1}{z} = \pi\delta^2(z)$ $\lambda, \quad \bar{\lambda}$ complex

formalism for scattering theory, however, distinguishes the different external particles of an amplitude—even if the particles have no mass. In the conformal four-dimensional $\mathcal{N} = 4$ super Yang–Mills (SYM) theory this is reflected in the fact that the standard, free representation of conformal symmetry on scattering amplitudes produces an anomaly for collinear momentum configurations. In fact, one finds no real quantum anomaly of the symmetry but rather an anomaly of the representation which can be cured by deformation terms [1, 2], see also [3] [9].

In four dimensions, this superficial violation of conformal symmetry is closely related to the holomorphic anomaly defined by the equation

$$\frac{\partial}{\partial \bar{z}}\frac{1}{z} = \pi\delta^2(z). \tag{1.1}$$

In (3, 1) signature, the massless momenta of scattering amplitudes are conveniently expressed in terms of two complex conjugate spinors λ and $\bar{\lambda}$. These take the place of z and $\bar{z}$ in the above equation such that the naive conformal generators (e.g. $\tilde{\mathfrak{S}} = \eta\partial/\partial\bar{\lambda}$) annihilate tree-level amplitudes only up to distributional terms. Switching to (2, 2) signature, the solution of the masslessness condition $p^2 = 0$ is given by two independent real spinors λ and $\tilde{\lambda}$ and the anomaly cannot be phrased in terms of the above complex equation anymore. In fact, the anomaly does not disappear in (2, 2) signature but it is harder to see [2]. Rewriting the amplitudes in this signature shows that the anomaly can be captured in terms of the singularity of a signum function with real argument x (corresponding to the real $\lambda, \tilde{\lambda}$):

$$\frac{\partial}{\partial x}\,\mathrm{sgn}\,x = 2\delta(x), \tag{1.2}$$

After all, the resulting additional contributions to the invariance equations following from symmetry seem to be essential for fixing the complete scattering matrix of $\mathcal{N} = 4$ SYM theory.

Certainly one can ask whether there is an analog of this anomaly of the conformal symmetry representation in dimension number different than 4. As the naive symmetry representation is still expected to be incompatible with the standard definition of scattering states, the existence of a similar phenomenon is plausible. In this paper we study superconformal three-dimensional $\mathcal{N} = 6$ Chern–Simons (SCS) theory, also called ABJM theory [4], where massless momenta are described by a single real spinor. Consequently, one cannot expect an anomaly to be of the form of the complex equation (1.1). We will show, however, that the anomaly in three dimensions takes the form of (1.2), cf table 1. As an application we predict a non-vanishing one-loop amplitude at six points and verify this result by an explicit unitarity construction. First, however, we give a brief motivation for the importance of the above anomalies followed by a short review of scattering amplitudes in ABJM theory.

[9] Here, and in the following, we consistently refer to the variation of the scattering amplitude with respect to the free representation of a symmetry generator as the anomaly. We maintain this usage even when the anomaly may in fact be absorbed into a deformation of the representation of the generator which preserves the symmetry algebra.

The emergence of the anomalies indicated above is particularly powerful in the context of planar scattering amplitudes, where, when combined with other recent developments, it may optimistically lead to an all-order understanding of these observables. Once more let us consider the case of $\mathcal{N} = 4$ SYM theory: its planar *S*-matrix is known to possess a dual-conformal symmetry, both at strong [5] and weak [6] coupling, which combines with the usual superconformal symmetry into a Yangian symmetry algebra [7]. Furthermore it has been understood as originating in the self-T-duality of the AdS/CFT-dual $AdS_5 \times S^5$ geometry [8]. Under this duality, scattering amplitudes are mapped into Wilson loops; specifically, it has been shown that MHV amplitudes are dual to bosonic Wilson loops [5, 9], while for superamplitudes the dual object is a generalized super-Wilson loop [10] (closely related objects are light-cone supercorrelation functions [11]). However, for the reasons mentioned, the discussed symmetries of scattering amplitudes are anomalous[10]. These anomalies, particularly those of the fermionic generators, and how they relate different tree-level amplitudes has been discussed in [1, 12, 13]. For MHV-amplitudes/bosonic-Wilson loops, understanding the conformal anomaly provides strong constraints at all orders in the coupling and allows the complete determination of the four- and five-point cases [14]. The anomalies of superamplitudes/super-Wilson loops have been studied at loop level in [13, 2, 15] and recently, by making use of these anomalies for the fermionic symmetries, all-order equations relating higher-loop superamplitudes to lower-loop ones have been found [16].

It is natural to ask whether similar results can be obtained for scattering amplitudes in three dimensions and particularly for planar amplitudes in $\mathcal{N} = 6$ SCS theory, though here the picture is still significantly less clear. While so far all attempts to consistently formulate a self-T-duality for the string background dual to the ABJM theory (cf [17]) have failed, there are strong indications pointing towards the existence of such a map. One of them is the discovery of Yangian symmetry of ABJM scattering amplitudes [18] and particularly its formal rewriting in terms of 'dual' coordinates [19]. Furthermore, at four points the two-loop scattering amplitude was shown to match the two-loop Wilson loop [20]. In fact, there is a remarkable congruence between the structure of the two-loop ABJM Wilson loop and the one-loop $\mathcal{N} = 4$ SYM Wilson loop, which extends to arbitrary number of edges [21] and which parallels a similar relation in the spectrum of planar anomalous dimensions persisting to all orders in the coupling. On the other hand, the analogy with the map in $\mathcal{N} = 4$ SYM theory is complicated due to the absence of an analogue of the four dimensional helicity classification of amplitudes: Before comparison to the Wilson loop, the MHV part of the scattering amplitude has to be stripped off. The lack of helicity and thus of an MHV scheme in three dimensions is therefore crucial for understanding a possible analogy[11]. It is known that all lightlike polygonal *n*-point Wilson loops vanish at one-loop order [22, 23]. Consequently it is interesting to study one-loop amplitudes at higher points to further understand any possible map. As we will see, the structure of the anomalous symmetries predicts that the one-loop six-point amplitude is non-zero, a fact which we confirm by an explicit generalized unitarity calculation[12]. This poses a puzzle as to what a possible Wilson loop/amplitude map could look like.

[10] For the dual Wilson loops the origin of the anomaly is conceptually different and arises from UV effects. The functional form of the anomalies, however, is closely related.

[11] To match the ABJM four-point Wilson loop and scattering amplitude at two loops, the tree-level part of the amplitude was stripped off. Its form is close to the one of the MHV four-point amplitude in four dimensions while the ABJM six-point tree-level amplitude resembles the four dimensional NMHV counterpart.

[12] After publishing this work on the arXiv, we were informed that also Simon Caron-Huot and Yu-tin Huang independently obtained a non-vanishing one-loop six-point ABJM amplitude.

After a review of scattering amplitudes in ABJM theory in section 2, we will discuss the origin and form of the anomaly for the superconformal symmetry, section 3. Applying the resulting anomaly vertex to the six-point amplitude at tree and one-loop level, section 4, we will see how this constrains the form of the amplitude and predicts a non-vanishing result at one loop. In section 5 we perform an explicit calculation of the one-loop six-point function using generalized unitarity methods and confirm the prediction of the symmetries. We close with a summary and discussion of some open questions.

2. Scattering amplitudes in ABJM theory

In this section, we briefly introduce the framework that is relevant to the study of scattering amplitudes in the $\mathcal{N} = 6$ supersymmetric Chern–Simons theory, or ABJM theory [4], which was developed in [25, 18, 19].

Fields and states

The matter content of ABJM theory comprises four complex scalar fields ϕ^A and four complex fermion fields ψ^a_A, $A = 1, \ldots, 4$, which transform in the $(\mathbf{N}, \bar{\mathbf{N}})$ representation of the $\mathrm{U}(N) \times \mathrm{U}(N)$ gauge group. The scalars ϕ^A form a fundamental multiplet of the internal $\mathrm{SU}(4)$ R-symmetry group, while the fermions ψ_A form an antifundamental multiplet. The Chern–Simons gauge fields $A_\mu, \hat{A}_\mu$ transforming in the $(\mathbf{ad}, \mathbf{1})$, $(\mathbf{1}, \mathbf{ad})$ representations of the gauge group have no freely propagating modes and thus cannot appear as external states in scattering amplitudes[13]. As both scalar and fermion particle numbers are conserved, this in particular implies that there are no scattering amplitudes for odd numbers of particles.

Unlike in four dimensions, there is no helicity degree of freedom for massless states in three dimensions. Hence, one-particle states are solely labeled by a massless momentum $p^\mu = \gamma^\mu_{ab}\lambda^a\lambda^b$, which is conveniently parametrized by a spacetime spinor λ using the real Dirac gamma matrices γ^μ. For momenta with positive energy $p^0 > 0$ in Minkowski signature, λ has to be real. For negative-energy momenta, λ has to be purely imaginary.

Superfields and superamplitudes

All free on-shell states $\phi^A(\lambda)$, $\psi_A(\lambda)$ can be combined in a single superfield $\Phi(\lambda, \eta)$ [18] with the help of a $\mathfrak{u}(3)$ Graßmann spinor η^A,

$$\Phi(\lambda, \eta) = \phi^4(\lambda) + \eta^A\psi_A(\lambda) + \tfrac{1}{2}\varepsilon_{ABC}\eta^A\eta^B\phi^C(\lambda) + \tfrac{1}{6}\varepsilon_{ABC}\eta^A\eta^B\eta^C\psi_4(\lambda). \tag{2.1}$$

This choice of superfield splits the internal R-symmetry into a manifest $\mathfrak{u}(3)$ and a non-manifest remainder, realized as multiplication and second-order differential operators in ηs. Its virtue is that the supersymmetry and superconformal generators take the simple form

$$\mathfrak{Q}^{aA} = \lambda^a\eta^A, \quad \mathfrak{Q}^a_A = \lambda^a\partial_A, \quad \mathfrak{S}^A_a = \eta^A\partial_a, \quad \mathfrak{S}_{aA} = \partial_a\partial_A, \tag{2.2}$$

where ∂_a, ∂_A denote derivatives with respect to λ^a, η^A. When splitting the matter fields into mutually conjugate components ϕ^A, $\bar{\phi}_A$, ψ_A, $\bar{\psi}^A$, it is convenient to use not the conjugate superfield $\bar{\Phi}(\bar{\lambda}, \bar{\eta})$ itself, but its Graßmann Fourier transform

$$\bar{\Phi}(\lambda, \eta) = \bar{\psi}^4(\lambda) + \eta^A\bar{\phi}_A(\lambda) + \tfrac{1}{2}\varepsilon_{ABC}\eta^A\eta^B\bar{\psi}^C(\lambda) + \tfrac{1}{6}\varepsilon_{ABC}\eta^A\eta^B\eta^C\bar{\phi}^4(\lambda). \tag{2.3}$$

[13] Nevertheless, the Chern–Simons zero-mode will turn out to play a significant role, see the discussion at the end of section 5 and in section 6.

With the help of the superfields Φ, $\bar{\Phi}$, scattering amplitudes for all possible combinations of n external states combine into a single superamplitude, from which individual component amplitudes can be extracted as coefficients of the appropriate η-monomials.

In the planar limit $N \to \infty$, scattering amplitudes can be decomposed into color-ordered amplitudes multiplied by planar color structures. The objects we study in this work are the color-ordered superamplitudes

$$A_n = A_n(\bar{\Lambda}_1, \Lambda_2, \bar{\Lambda}_3, \ldots, \bar{\Lambda}_{n-1}, \Lambda_n), \quad \Lambda_k = (\lambda_k, \eta_k), \tag{2.4}$$

where the ordering of the arguments is significant, and the bars signify that the respective Λ_ks parametrize conjugate fields $\bar{\Phi}$. The color decomposition requires that Φ and $\bar{\Phi}$ fields alternate, and implies invariance up to a sign under cyclic double-shifts,

$$A_n(\bar{\Lambda}_3, \ldots, \Lambda_n, \bar{\Lambda}_1, \Lambda_2) = (-1)^{n/2-1} A_n(\bar{\Lambda}_1, \Lambda_2, \bar{\Lambda}_3, \ldots, \Lambda_n). \tag{2.5}$$

By convention, conjugate superfields are put in odd arguments of the superamplitude. The sign is due to the fact that the conjugate field $\bar{\Phi}$ is fermionic. Consequently under the transformation, 'λ'-parity, or its supersymmetric generalization [26], $\Lambda \to -\Lambda$ the superamplitude transforms as

$$A_n(\Lambda_1, \ldots, -\Lambda_i, \ldots, \Lambda_n) = (-1)^i A_n(\Lambda_1, \ldots, \Lambda_i, \ldots, \Lambda_n). \tag{2.6}$$

The color-ordered amplitudes have another symmetry which is due to the reflection invariance of the fundamental vertices in the Lagrangian. This inversion symmetry is reflected in the following transformation behavior of the ℓ-loop amplitude[14]:

$$A_n^{(\ell)}(\bar{\Lambda}_1, \Lambda_2, \ldots, \bar{\Lambda}_{n-1}, \Lambda_n) = (-)^{n(n-2)/8+\ell} A_n^{(\ell)}(\bar{\Lambda}_1, \Lambda_n, \bar{\Lambda}_{n-1}, \ldots, \Lambda_2). \tag{2.7}$$

On-shell integration

Below, we will frequently need to integrate over complete sets of on-shell states. In the superfield language, such integrals take the simple form

$$\int \mathrm{d}\Lambda\, f(\bar{\Lambda})\, g(\mathrm{i}\Lambda), \quad \mathrm{d}\Lambda = \mathrm{d}^{2|3}\Lambda = \frac{1}{2}\,\mathrm{d}^2\lambda\,\mathrm{d}^3\eta, \tag{2.8}$$

where $\mathrm{i}\Lambda := (\mathrm{i}\lambda, \mathrm{i}\eta)$ switches the sign of both the momentum $\lambda^a\lambda^b$ and the supermomentum $\lambda^a\eta^A$ relative to Λ. The integration often needs to include both real and imaginary λ, that is the domain of integration for λ is $\mathbb{R}^2 \cup (\mathrm{i}\mathbb{R})^2$. The factor $1/2$ accounts for the double-counting of states due to the $\Lambda \to -\Lambda$ symmetry. By substituting $\Lambda \to \mathrm{i}\Lambda$, the integration over $(\mathrm{i}\mathbb{R})^2$ can be converted to an integration over $\mathbb{R}^2$ and vice versa:

$$\begin{aligned}\int \mathrm{d}\Lambda f(\bar{\Lambda}) g(\mathrm{i}\Lambda) &= -\mathrm{i}\int \mathrm{d}\Lambda f(\mathrm{i}\bar{\Lambda}) g(\Lambda) \\ &= \int_{\mathbb{R}} \mathrm{d}\Lambda\,(f(\bar{\Lambda})g(\mathrm{i}\Lambda) - \mathrm{i}f(\mathrm{i}\bar{\Lambda})g(\Lambda)) \\ &= \int_{\mathrm{i}\mathbb{R}} \mathrm{d}\Lambda\,(f(\bar{\Lambda})g(\mathrm{i}\Lambda) - \mathrm{i}f(\mathrm{i}\bar{\Lambda})g(\Lambda)).\end{aligned} \tag{2.9}$$

This assumes that $f(-\bar{\Lambda}) = -f(\bar{\Lambda})$ and $g(-\Lambda) = g(\Lambda)$, which is the case for f and g being scattering amplitudes, and will always be the case below. Note that the integration measure $\mathrm{d}\Lambda$ is fermionic and transforms according to $\mathrm{d}\Lambda \to (i^2/i^3)\,\mathrm{d}\Lambda = -\mathrm{i}\,\mathrm{d}\Lambda$ under $\Lambda \to \mathrm{i}\Lambda$.

[14] We thank Marco Bianchi, Matias Leoni, Andrea Mauri, Silvia Penati and Alberto Santambrogio for clarification of the loop dependence.

3. Anomaly vertex in three dimensions

In this section we first rewrite the four-point scattering amplitude of ABJM theory in such a way that its dependence on a specific sign function becomes explicit. The argument of this sign function is shown to be the spinor bracket of two neighboring external particles. The sign thus changes when two particles become collinear. We then show that this change of sign leads to an anomaly of the conformal symmetry. We explicitly act with the generator $\mathfrak{S}$ (2.2) on the four-point amplitude which yields an anomaly vertex supported on collinear momentum configurations.

Four-point amplitude

The four-point superamplitude of ABJM theory reads [18], see also [25],

$$A_4(\bar{1}, 2, \bar{3}, 4) = \frac{\delta^3(P)\,\delta^6(Q)}{\langle 12\rangle\langle 23\rangle}. \tag{3.1}$$

For positive energies p^0 (incoming particles), λ is real, while for negative energies (outgoing particles), λ is purely imaginary. Assuming that particles 1 and 2 carry the same energy sign, and particles 3 and 4 carry the opposite energy sign, we find

$$\begin{aligned} 1 &= |\langle 12\rangle| \int \mathrm{d}\alpha_3\,\mathrm{d}\beta_3\delta^2(\lambda_3 - \mathrm{i}\alpha_3\lambda_1 + \mathrm{i}\beta_3\lambda_2),\\ 1 &= |\langle 12\rangle| \int \mathrm{d}\alpha_4\,\mathrm{d}\beta_4\delta^2(\lambda_4 - \mathrm{i}\alpha_4\lambda_1 + \mathrm{i}\beta_4\lambda_2). \end{aligned} \tag{3.2}$$

Inserting these identities into the amplitude, the momentum delta function becomes

$$\begin{aligned} \delta^3(P) &= \delta^3\big(\lambda_1\lambda_1\big(1-\alpha_3^2-\alpha_4^2\big) + \lambda_2\lambda_2\big(1-\beta_3^2-\beta_4^2\big) + (\lambda_1\lambda_2+\lambda_2\lambda_1)(\alpha_3\beta_3+\alpha_4\beta_4)\big)\\ &= \frac{1}{|\langle 12\rangle|^3}\delta\big(1-\alpha_3^2-\alpha_4^2\big)\delta\big(1-\beta_3^2-\beta_4^2\big)\delta(\alpha_3\beta_3+\alpha_4\beta_4), \end{aligned} \tag{3.3}$$

where the last equality holds as long as λ_1 and λ_2 are linearly independent. The four-point amplitude thus can be written as[15]

$$\begin{aligned} A_4^{1,2\leftrightarrow 3,4}(\bar{1}, 2, \bar{3}, 4) = -\frac{\delta^6(Q)}{|\langle 12\rangle|\langle 12\rangle^2}\int \mathrm{d}\alpha_3\,\mathrm{d}\alpha_4\,\mathrm{d}\beta_3\,\mathrm{d}\beta_4\frac{1}{\mathrm{i}\alpha_3}\\ \cdot\delta\big(1-\alpha_3^2-\alpha_4^2\big)\delta\big(1-\beta_3^2-\beta_4^2\big)\delta(\alpha_3\beta_3+\alpha_4\beta_4)\\ \cdot\delta^2(\lambda_3 - \mathrm{i}\alpha_3\lambda_1 - \mathrm{i}\beta_3\lambda_2)\delta^2(\lambda_4 - \mathrm{i}\alpha_4\lambda_1 - \mathrm{i}\beta_4\lambda_2). \end{aligned} \tag{3.4}$$

Introducing polar coordinates

$$\alpha_3 = r_\alpha\sin\alpha, \quad \alpha_4 = r_\alpha\cos\alpha, \quad \beta_3 = r_\beta\sin\beta, \quad \beta_4 = r_\beta\cos\beta, \tag{3.5}$$

the radial integrations can be evaluated, leaving behind a Jacobi factor of $1/4$:

$$\begin{aligned} A_4^{1,2\leftrightarrow 3,4}(\bar{1}, 2, \bar{3}, 4) = -\frac{\delta^6(Q)}{|\langle 12\rangle|\langle 12\rangle^2}\int \mathrm{d}\alpha\,\mathrm{d}\beta\frac{1}{4\mathrm{i}\sin\alpha}\delta(\sin\alpha\sin\beta+\cos\alpha\cos\beta)\\ \cdot\delta^2(\lambda_3 - \mathrm{i}\sin\alpha\lambda_1 - \mathrm{i}\sin\beta\lambda_2)\delta^2(\lambda_4 - \mathrm{i}\cos\alpha\lambda_1 - \mathrm{i}\cos\beta\lambda_2). \end{aligned} \tag{3.6}$$

The first delta function localizes at $\beta = \alpha + s_1(2-s_2)\pi/2$, with $s_1, s_2 = \pm 1$, where

$$\sin\beta = s_1 s_2\cos\alpha, \qquad \cos\beta = -s_1 s_2\sin\alpha. \tag{3.7}$$

[15] The superscript $1,2\leftrightarrow 3,4$ signifies how the four particles are split into incoming (positive energy) and outgoing (negative energy) particles. Expressions for different energy distributions are given below.

Here we can collect the two signs into one which yields an overall factor of 2:

$$A_4^{1,2\leftrightarrow 3,4}(\bar{1},2,\bar{3},4) = -\frac{\delta^6(Q)}{|\langle 12\rangle|\langle 12\rangle^2}\sum_{s=\pm 1}\int_0^{2\pi} \mathrm{d}\alpha \frac{1}{2\mathrm{i}\sin\alpha} \cdot \delta^2(\lambda_3 - \mathrm{i}\sin\alpha\lambda_1 - \mathrm{i}s\cos\alpha\lambda_2)\delta^2(\lambda_4 - \mathrm{i}\cos\alpha\lambda_1 + \mathrm{i}s\sin\alpha\lambda_2). \tag{3.8}$$

Moving $\delta^6(Q)$ under the integral sign, contracting Q once with λ_3 and once with λ_4, and expanding $\lambda_{3,4}$ in terms of $\lambda_{1,2}$ shows that

$$\begin{aligned}\delta^6(Q) &= \langle 34\rangle^{-3}\delta^3(\langle 31\rangle\eta_1 + \langle 32\rangle\eta_2 + \langle 34\rangle\eta_4)\delta^3(\langle 41\rangle\eta_1 + \langle 42\rangle\eta_2 + \langle 43\rangle\eta_3)\\ &= -s\langle 12\rangle^3\delta^3(\eta_3 - \mathrm{i}\sin\alpha\eta_1 - \mathrm{i}s\cos\alpha\eta_2)\delta^3(\eta_4 - \mathrm{i}\cos\alpha\eta_1 + \mathrm{i}s\sin\alpha\eta_2).\end{aligned} \tag{3.9}$$

The four-point amplitude hence reads

$$A_4^{1,2\leftrightarrow 3,4}(\bar{1},2,\bar{3},4) = \mathrm{sgn}(\langle 12\rangle)\sum_{s=\pm 1}\int_0^{2\pi} \mathrm{d}\alpha \frac{s}{2\mathrm{i}\sin\alpha} \cdot \delta^{2|3}(\Lambda_3 - \mathrm{i}\sin\alpha\Lambda_1 - \mathrm{i}s\cos\alpha\Lambda_2)\delta^{2|3}(\Lambda_4 - \mathrm{i}\cos\alpha\Lambda_1 + \mathrm{i}s\sin\alpha\Lambda_2), \tag{3.10}$$

where $\Lambda_k = (\lambda_k, \eta_k)$. Reverting the direction of integration in one of the two terms under the sum shows that s accounts for a possible reflection in the rotation of $\Lambda_{1,2}$ into $\Lambda_{3,4}$:

$$A_4^{1,2\leftrightarrow 3,4}(\bar{1},2,\bar{3},4) = \mathrm{sgn}(\langle 12\rangle)\sum_{s=\pm 1}\int_0^{2\pi} \mathrm{d}\alpha \frac{1}{2\mathrm{i}\sin\alpha} \cdot \delta^{2|3}(\Lambda_3 - \mathrm{i}s(\sin\alpha\Lambda_1 + \cos\alpha\Lambda_2))\delta^{2|3}(\Lambda_4 - \mathrm{i}(\cos\alpha\Lambda_1 - \sin\alpha\Lambda_2)). \tag{3.11}$$

In the above derivation, it was assumed that particles 1 and 2 carry the same energy sign, and particles 3 and 4 carry the opposite energy sign. A similar analysis (carried out in the appendix) shows what the amplitude becomes when incoming/outgoing particles are distributed differently[16]:

$$\begin{aligned}A_4^{1,3\leftrightarrow 2,4}(\bar{1},2,\bar{3},4) = -\mathrm{i}\,\mathrm{sgn}_{\mathrm{c}}(\langle 12\rangle)\sum_{s_\alpha,s_\beta=\pm 1}\int_{-\infty}^{\infty} \mathrm{d}\alpha \frac{1}{4\sinh\alpha}\\ \cdot \delta^{2|3}(\Lambda_3 - s_\beta(s_\alpha\sinh\alpha\Lambda_1 + \mathrm{i}\cosh\alpha\Lambda_2))\\ \times \delta^{2|3}(\Lambda_4 - \mathrm{i}s_\alpha\cosh\alpha\Lambda_1 + \sinh\alpha\Lambda_2),\end{aligned} \tag{3.12}$$

$$\begin{aligned}A_4^{1,4\leftrightarrow 2,3}(\bar{1},2,\bar{3},4) = \mathrm{i}\,\mathrm{sgn}_{\mathrm{c}}(\langle 12\rangle)\sum_{s_\alpha,s_\beta=\pm 1}\int_{-\infty}^{\infty} \mathrm{d}\alpha \frac{s_\beta}{4\mathrm{i}\cosh\alpha}\\ \cdot \delta^{2|3}(\Lambda_3 - \mathrm{i}s_\alpha\cosh\alpha\Lambda_1 + \sinh\alpha\Lambda_2)\\ \times \delta^{2|3}(\Lambda_4 - s_\beta(s_\alpha\sinh\alpha\Lambda_1 + \mathrm{i}\cosh\alpha\Lambda_2)),\end{aligned} \tag{3.13}$$

Here and for the following it is helpful to introduce generalizations of the sign and absolute value functions to the real and imaginary axes:

$$\mathrm{sgn}_c(x) := \begin{cases}+1 & \text{for } x\in\mathbb{R}^+, \mathrm{i}\mathbb{R}^+,\\ -1 & \text{for } x\in\mathbb{R}^-, \mathrm{i}\mathbb{R}^-.\end{cases} \qquad |x|_{\mathrm{c}} := \begin{cases}|\mathrm{Re}\,x| & \text{for } x\in\mathbb{R},\\ \mathrm{i}|\mathrm{Im}\,x| & \text{for } x\in\mathrm{i}\mathbb{R}.\end{cases} \tag{3.14}$$

In conclusion, we can write the four-point amplitude as

$$A_4^{k_1,k_2\leftrightarrow k_3,k_4}(\bar{1},2,\bar{3},4) = \mathrm{sgn}_c(\langle 12\rangle)F^{k_1,k_2\leftrightarrow k_3,k_4}(\bar{1},2,\bar{3},4), \tag{3.15}$$

where F denotes a function whose explicit parametrization depends on the distribution of energies.

[16] In both cases, $\langle 12\rangle$ is purely imaginary.

Anomaly

Looking at the explicit form of (3.11), (3.12) and (3.13), it is immediate that the action of $\mathfrak{S}_a^A = \sum_k \eta_k^A \partial/\partial\lambda_k^a$ on the super delta functions produces terms of the form $x\,\delta(x)$, thus the function F in (3.15) is annihilated. The signum factor, however, produces a non-vanishing contribution whenever the momenta 1 and 2 are collinear:

$$\mathfrak{S}_a^A \operatorname{sgn}_c(\langle 12\rangle) = 2\varepsilon_{ab}\big(\eta_1^A\lambda_2^b - \eta_2^A\lambda_1^b\big)\ \delta_c(\langle 12\rangle). \tag{3.16}$$

Here we consistently define

$$\delta_c(x) := \begin{cases} \delta(\operatorname{Re} x) & \text{for } x \in \mathbb{R}, \\ -\mathrm{i}\delta(\operatorname{Im} x) & \text{for } x \in \mathrm{i}\mathbb{R}, \end{cases} \tag{3.17}$$

The anomaly vertex thus takes the form

$$\mathfrak{S}_a^A A_4^{k_1,k_2\leftrightarrow k_3,k_4}(\bar{1},2,\bar{3},4) = 2\varepsilon_{ab}\big(\eta_1^A\lambda_2^b - \eta_2^A\lambda_1^b\big)\ \delta_c(\langle 12\rangle)\, F^{k_1,k_2\leftrightarrow k_3,k_4}(\bar{1},2,\bar{3},4), \tag{3.18}$$

where F is anomaly free and on the support of $\delta_c(\langle 12\rangle)$, all four momenta are collinear. We believe this expression furnishes the building block for all anomaly contributions of higher-point and higher-loop amplitudes.

Let us evaluate the vertex for the configuration $(1,2\leftrightarrow 3,4)$ in order to obtain a more symmetric expression. It is convenient to parametrize the collinear particles in terms of the massless momentum $\lambda_{12}\lambda_{12} = \lambda_1\lambda_1 + \lambda_2\lambda_2$. To this end, we use the identity

$$\delta(\langle 12\rangle) = \int \mathrm{d}\Lambda_{12}\,\mathrm{d}\Lambda'\,\mathrm{d}\beta\delta^2(\lambda')\delta_1^{2|3}\delta_2^{2|3}, \tag{3.19}$$

with

$$\begin{aligned} \delta_1^{2|3} &= \delta^{2|3}(\Lambda_1 - (\sin\beta\Lambda_{12} + \cos\beta\Lambda')), \\ \delta_2^{2|3} &= \delta^{2|3}(\Lambda_2 - (\cos\beta\Lambda_{12} - \sin\beta\Lambda')), \end{aligned} \tag{3.20}$$

which, after shifting $\alpha \to \alpha - \beta$, turns the delta functions in (3.11) into

$$\begin{aligned} \delta_3^{2|3} &= \delta^{2|3}(\Lambda_3 - \mathrm{i}s(\sin\alpha\Lambda_{12} + \cos\alpha\Lambda')), \\ \delta_4^{2|3} &= \delta^{2|3}(\Lambda_4 - \mathrm{i}(\cos\alpha\Lambda_{12} - \sin\alpha\Lambda')). \end{aligned} \tag{3.21}$$

Using $\eta_1^A\lambda_2^b - \eta_2^A\lambda_1^b = \lambda_{12}^b\eta'^A$ under $\delta^2(\lambda')$, the anomaly vertex then reads

$$\begin{aligned} \mathfrak{S}_a^A A_4^{1,2\leftrightarrow 3,4}(\bar{1},2,\bar{3},4) = \sum_{s=\pm 1}\int \mathrm{d}\Lambda_{12}\mathrm{d}\Lambda'\mathrm{d}\alpha\,\mathrm{d}\beta\,\delta^2(\lambda') \\ \times \frac{1}{\mathrm{i}\sin(\alpha-\beta)}\varepsilon_{ab}\lambda_{12}^b\eta'^A\delta_1^{2|3}\delta_2^{2|3}\delta_3^{2|3}\delta_4^{2|3}. \end{aligned} \tag{3.22}$$

4. Conformal symmetry of the six-point amplitude

Let us now apply the conformal anomaly vertex to six-point amplitudes at tree level and one loop. The tree-level six-point amplitudes were first calculated by using the superconformal symmetries and explicit Feynman diagram calculations in [18]. Subsequently they were rederived from the orthogonal Graßmannian of [27], and given a perhaps more congenial form in [26] by means of the three-dimensional analogue of the BCFW recursion relations.

In order to generalize the above anomaly 4-vertex $\mathfrak{S}_4 = \mathfrak{S}A_4$ to higher-point and to loop amplitudes, we employ the following strategy. We first obtain the imaginary part of the respective amplitude using unitarity (the optical theorem), then determine the anomaly of the imaginary part, and finally find the anomaly of the amplitude by requiring that acting with the anomalous generator $\mathfrak{S}$ and taking the imaginary part commute, $\mathfrak{S}\operatorname{Im} A = \operatorname{Im}\mathfrak{S}A$.

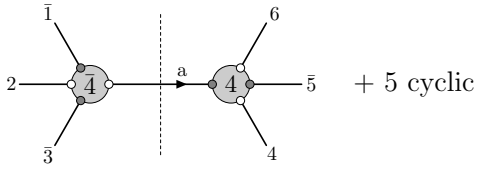

Figure 1. Imaginary part of the tree-level six-point amplitude.

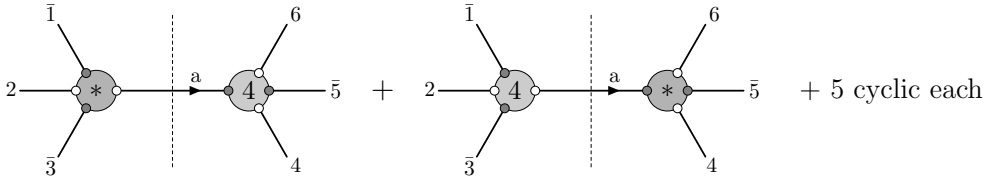

Figure 2. Anomaly of the imaginary part of the tree-level six-point amplitude.

Tree level

By the optical theorem, the imaginary part of the tree-level six-point amplitude in ABJM reads, see figure 1,

$$\begin{aligned} 2\,\mathrm{Im}\,A_6^{(0)}(\bar{1},2,\bar{3},4,\bar{5},6) &= \int_{\mathbb{R}} \mathrm{d}\Lambda_{\mathrm{a}} \bar{A}_4(\bar{1},2,\bar{3},\mathrm{i}\mathrm{a}) A_4(\bar{\mathrm{a}},4,\bar{5},6) \\ &\quad - \mathrm{i}\int_{\mathbb{R}} \mathrm{d}\Lambda_{\mathrm{a}} \bar{A}_4(\mathrm{i}\bar{\mathrm{a}},2,\bar{3},4) A_4(\bar{5},6,\bar{1},\mathrm{a}) + 4\ \text{cyclic}, \\ &= \int \mathrm{d}\Lambda_{\mathrm{a}} A_4(\bar{1},2,\bar{3},\mathrm{i}\mathrm{a}) A_4(\bar{\mathrm{a}},4,\bar{5},6) + 2\ \text{cyclic}, \end{aligned} \tag{4.1}$$

where the first two integrations run only over real λ_{a}, that is the internal momenta have a definite energy sign; see (2.8) for the definition of the measure $\mathrm{d}\Lambda$[17]. A bar over an amplitude denotes complex conjugation, but note that in fact the four-point amplitude is real, $\bar{A}_4 = A_4$. Furthermore note that the second term is real despite its imaginary prefactor. The amplitude $\bar{A}_4$ is an odd function of the (imaginary) first argument by construction. Both four-point amplitudes on the rhs are anomalous, and they will contribute to the anomaly of the six-point tree-level amplitude. That is their anomalies in the integral will give the imaginary part of the six-point anomaly, see figure 2. In fact, we do not expect further contributions: An off-shell propagator joining two four-point vertices yields a rational function. In four dimensions, the holomorphic anomaly at poles was responsible for anomalous transformations. Conversely, in three dimensions, the anomaly arises from derivatives of step functions.

A simple ansatz to reproduce the anomaly of the above imaginary part reads

$$\mathfrak{S}A_6^{(0)}(\bar{1},2,\bar{3},4,\bar{5},6) = -\frac{\mathrm{i}}{2}\int \mathrm{d}\Lambda_{\mathrm{a}} \mathfrak{S}_4(\bar{1},2,\bar{3},\mathrm{i}\mathrm{a}) A_4(\bar{\mathrm{a}},4,\bar{5},6) + 5\ \text{cyclic}, \tag{4.2}$$

which is graphically represented in figure 3, and where now the integration includes both real and imaginary λ_{a}. The minus sign on the right-hand side results from passing the

[17] Unfortunately, the prefactors of the various contributions are hard to come by. For example, reality requires a factor of $\pm\mathrm{i}$ in the second term in the above unitarity relation. Only for this (relative) prefactor two contributions will combine using (2.9) into an integral with undirected energy flow as desirable for tree level unitarity. The correct overall signs follow, ultimately, from evaluating statistics and color of the *S*-matrix operator carefully. In the figures here and below, we conveniently absorb such factors of ± 1, $\pm\mathrm{i}$ originating from the on-shell integration measure $\mathrm{d}\Lambda$ into a more intuitive definition of the diagram.

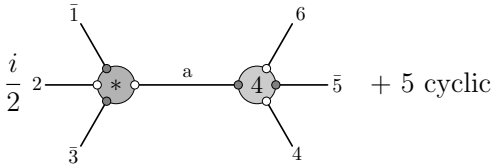

Figure 3. Anomaly of the tree-level six-point amplitude.

fermionic generator $\mathfrak{S}$ through the fermionic measure. The correctness of this expression can be confirmed by computing its imaginary part (by complex conjugation) and comparing to the anomaly of (4.1). Note that the four-point anomaly is real, $\bar{\mathfrak{S}}_4 = \mathfrak{S}_4$ (for a real choice of generator $\mathfrak{S}$). The anomaly is supported on configurations where three adjacent momenta are collinear.

One loop

At one loop, the imaginary part of the six-point amplitude expands to[18]

$$2\,\mathrm{Im}\,A_6^{(1)}(\bar{1},2,\bar{3},4,\bar{5},6) = -\mathrm{i}\int_{\mathbb{R}} \mathrm{d}\Lambda_\mathrm{a}\mathrm{d}\Lambda_\mathrm{b} A_4(\bar{1},2,\mathrm{i}\bar{\mathrm{a}},\mathrm{ib})A_6^{(0)}(\bar{\mathrm{b}},\mathrm{a},\bar{3},4,\bar{5},6) + 5\ \text{cyclic} + \text{c.c.} \tag{4.3}$$

These are the only contributions because all amplitudes have an even number of external particles[19]. Furthermore, a single-particle cut would split off a four-point one-loop amplitude which is known to vanish[20].

We can now act with a superconformal generator on (4.3) to find the anomaly of the imaginary part. The result is a sum of terms where the generator $\mathfrak{S}$ either acts on A_4, yielding $\mathfrak{S}A_4 = \mathfrak{S}_4$, or on $A_6^{(0)}$, yielding (4.2). The latter generates triangle-shaped diagrams as well as contributions where two 4-vertices are joined into a two-sided bubble with a third 4-vertex attached to one of the external legs. The triangle diagrams combine with their complex conjugates where the energy signs of internal lines are flipped. Collecting all the contributions, we find (see figure 4)

$$\begin{aligned}\mathfrak{S}2\,\mathrm{Im}\,A_6^{(1)} = &-\mathrm{i}\int_{\mathbb{R}} \mathrm{d}\Lambda_\mathrm{a}\mathrm{d}\Lambda_\mathrm{b}\mathfrak{S}_4(\bar{1},2,\mathrm{i}\bar{\mathrm{a}},\mathrm{ib})A_6^{(0)}(\bar{\mathrm{b}},\mathrm{a},\bar{3},4,\bar{5},6)\\ &-\frac{1}{2}\int_{\mathbb{R}} \mathrm{d}\Lambda_\mathrm{a}\mathrm{d}\Lambda_\mathrm{b}\int \mathrm{d}\Lambda_\mathrm{c}\mathfrak{S}_4(\bar{1},2,\mathrm{i}\bar{\mathrm{a}},\mathrm{ib})A_4(\bar{5},6,\bar{\mathrm{b}},\mathrm{ic})A_4(\bar{\mathrm{c}},\mathrm{a},\bar{3},4)\\ &-\frac{1}{2}\int_{\mathbb{R}} \mathrm{d}\Lambda_\mathrm{a}\mathrm{d}\Lambda_\mathrm{b}\int \mathrm{d}\Lambda_\mathrm{c}\mathfrak{S}_4(\bar{1},2,\mathrm{i}\bar{\mathrm{a}},\mathrm{ib})A_4(\bar{\mathrm{b}},\mathrm{a},\bar{3},\mathrm{ic})A_4(\bar{\mathrm{c}},4,\bar{5},6)\\ &-\frac{1}{2}\int_{\mathbb{R}} \mathrm{d}\Lambda_\mathrm{a}\mathrm{d}\Lambda_\mathrm{b}\int \mathrm{d}\Lambda_\mathrm{c}\mathfrak{S}_4(\bar{1},2,\mathrm{i}\bar{\mathrm{a}},\mathrm{ib})A_4(\bar{3},4,\bar{5},\mathrm{ic})A_4(\bar{\mathrm{c}},6,\bar{\mathrm{b}},\mathrm{a})\\ &+5\ \text{cyclic each} + \text{c.c.}\end{aligned} \tag{4.4}$$

[18] The prefactor of $-\mathrm{i}$ is related to the integral over one conjugate leg (see footnote 17). We are not certain whether the overall sign is correct, but it will be consistent with the derivations below.

[19] An important exception to this rule occurs if we consider the zero-mode of the Chern–Simons field. We will discuss this at the end of section 5 and in section 6. Corresponding singularities are supported on kinematical subspaces of greater codimensionality and are negligible for the considerations here.

[20] Indeed, the one-loop four-point amplitude, reduced to scalar integrals, can only get contributions from one- and two-mass triangles, bubbles and tadpoles. The one- and two-mass triangle integrals vanish in three-dimensions, and there are no bubble or tadpole diagrams in the three-dimensional, finite, superconformal theory.

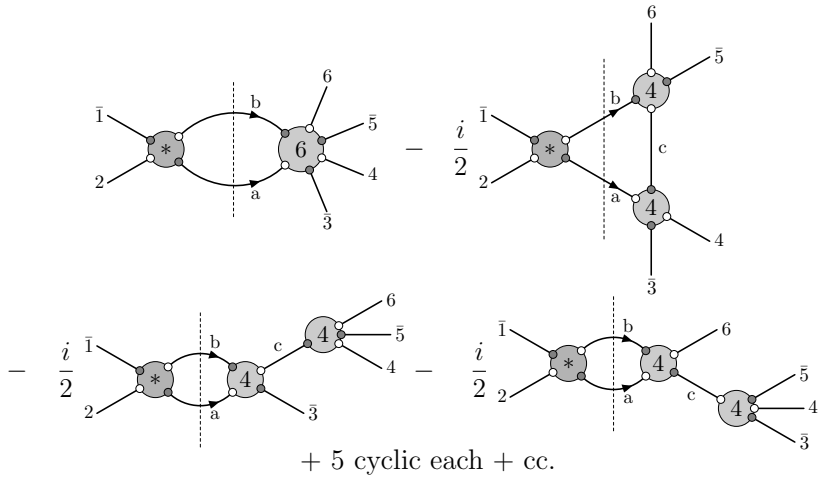

Figure 4. Anomaly of the imaginary part of the one-loop six-point amplitude.

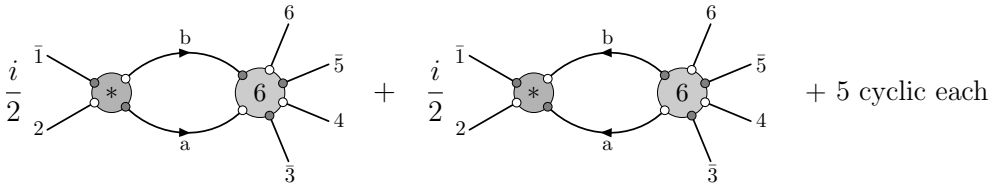

Figure 5. Anomaly of the one-loop six-point amplitude.

Here the latter three terms arise from the action of $\mathfrak{S}$ on $A_6^{(0)}$. They were obtained by combining with the complex conjugate contributions and using the vanishing of the one-loop four-point amplitude. Note that the third and the fourth term are of higher codimension. They require not only two, but three collinear momenta.

Now we can make an ansatz for the six-point one-loop anomaly $\mathfrak{S}A_6^{(1)}$ consisting of on-shell bubble and triangle integrals with six-vertices $A_6^{(0)}$, 4-vertices $A_4^{(0)}$, and anomaly vertices $\mathfrak{S}_4$. We compute the imaginary part of the one-loop anomaly ansatz making use of the complex conjugate of the six-point tree-level amplitude

$$\bar{A}_6^{(0)}(\bar{\mathrm{b}}, \mathrm{a}, \bar{3}, 4, \bar{5}, 6) = A_6^{(0)}(\bar{\mathrm{b}}, \mathrm{a}, \bar{3}, 4, \bar{5}, 6) - 2\mathrm{i}\,\mathrm{Im}\,A_6^{(0)}(\bar{\mathrm{b}}, \mathrm{a}, \bar{3}, 4, \bar{5}, 6), \quad (4.5)$$

where $\mathrm{Im}\,A_6^{(0)}$ is given in (4.1). We compare it to the anomaly (4.4) of the one-loop imaginary part, and we find that the following, relatively simple expression suffices:

$$\begin{aligned}\mathfrak{S}A_6^{(1)}(\bar{1}, 2, \bar{3}, 4, \bar{5}, 6) = \frac{1}{2}\int_{\mathbb{R}} \mathrm{d}\Lambda_\mathrm{a}\mathrm{d}\Lambda_\mathrm{b}\mathfrak{S}_4(\bar{1}, 2, \mathrm{i}\bar{\mathrm{a}}, \mathrm{ib})A_6^{(0)}(\bar{\mathrm{b}}, \mathrm{a}, \bar{3}, 4, \bar{5}, 6)\\ + \frac{1}{2}\int_{\mathbb{R}} \mathrm{d}\Lambda_\mathrm{a}\mathrm{d}\Lambda_\mathrm{b}\mathfrak{S}_4(\bar{1}, 2, \bar{\mathrm{a}}, \mathrm{b})A_6^{(0)}(\mathrm{i}\bar{\mathrm{b}}, \mathrm{ia}, \bar{3}, 4, \bar{5}, 6) + 5\text{ cyclic each.} \quad (4.6)\end{aligned}$$

The anomaly, see figure 5, is supported on configurations where two adjacent particles are collinear. In other words, for a generic configuration of particle momenta there is no anomaly at one loop.

Let us investigate the above integral for the one-loop six-point anomaly more explicitly. Making use of the anomaly vertex as induced by the four-point amplitude in the form of (3.11),

the integrals can be straightforwardly evaluated against the delta functions. The first term in (4.6) then becomes[21]

$$\mathcal{S}A_6^{(1)}(\bar{1},2,\bar{3},4,\bar{5},6)=\frac{1}{8}\varepsilon_{ab}\big(\eta_1^A\lambda_2^b-\eta_2^A\lambda_1^b\big)\,\delta_c(\langle 12\rangle)\sum_{s=\pm1}\int_0^{2\pi}\mathrm{d}\alpha\frac{1}{\mathrm{i}\sin(\alpha)}$$
$$\cdot A_6^{(0)}(\cos\alpha\Lambda_1-\sin\alpha\Lambda_2,s(\sin\alpha\Lambda_1+\cos\alpha\Lambda_2),\bar{3},4,\bar{5},6)+5\text{ cyclic}. \quad (4.7)$$

Substituting $y=\cos(\alpha)$, and using the Λ-parity (2.6) of $A_6^{(0)}$, this yields

$$\mathcal{S}A_6^{(1)}(\bar{1},2,\bar{3},4,\bar{5},6)=-\frac{\mathrm{i}}{2}\varepsilon_{ab}\big(\eta_1^A\lambda_2^b-\eta_2^A\lambda_1^b\big)\,\delta_c(\langle 12\rangle)\int_{-1}^{1}\mathrm{d}y\frac{1}{1-y^2}$$
$$\cdot A_6^{(0)}(y\Lambda_1-\sqrt{1-y^2}\Lambda_2,\sqrt{1-y^2}\Lambda_1+y\Lambda_2,\Lambda_3,\Lambda_4,\Lambda_5,\Lambda_6)+5\text{ cyclic}. \quad (4.8)$$

Now we make use of the explicit expression for the six-point amplitude introduced in [18]:

$$\mathcal{A}_6^{(0)}=\delta^3(P)\delta^6(Q)\big(f^+\delta^3(\alpha^+)+f^-\delta^3(\alpha^-)\big), \quad (4.9)$$

where

$$\alpha^{\pm,A}=\sum_{k=1}^{6}x_k^{\pm}\eta_k^A,\qquad \delta^3(\alpha^{\pm})=\prod_{A=1,2,3}\alpha^{\pm,A}, \quad (4.10)$$

and the coordinates $x_k^{\pm}$ can be chosen in accordance with the conditions in [18] to be

$$x_i^{\pm}=\frac{\mathrm{i}\varepsilon_{ijk}\langle jk\rangle}{2\sqrt{2}\sqrt{p_{6,1,2}^2}},\quad \mathrm{i},j,k=6,1,2,\qquad x_i^{\pm}=\frac{\pm\varepsilon_{ijk}\langle jk\rangle}{2\sqrt{2}\sqrt{p_{3,4,5}^2}},\quad \mathrm{i},j,k=3,4,5. \quad (4.11)$$

With the above choice of coordinates, the expressions for $\delta^3(P)$, $\delta^6(Q)$ and $\alpha^{\pm}$ do not change under the rotation

$$\Lambda_1\to\Lambda_a=y\Lambda_1-\sqrt{1-y^2}\Lambda_2,\qquad \Lambda_2\to\Lambda_b=\sqrt{1-y^2}\Lambda_1+y\Lambda_2. \quad (4.12)$$

A simple form for the functions $f^{\pm}$ can be found using the component amplitudes of [24]:

$$A_{6\psi}=A_6(\bar{\psi}^4,\psi_1,\bar{\psi}^2,\psi_4,\bar{\psi}^1,\psi_2)=0,\quad A_{6\phi}=A_6(\bar{\phi}_3,\phi^2,\bar{\phi}_1,\phi^3,\bar{\phi}_2,\phi^1)=1. \quad (4.13)$$

These yield

$$f^{\pm}=\frac{\mathrm{i}\sqrt{2}\sqrt{p_{3,4,5}^2}}{(\langle 2|p_{34}|5\rangle\mp\mathrm{i}\langle 61\rangle\langle 34\rangle)(\langle 3|p_{45}|6\rangle\mp\mathrm{i}\langle 12\rangle\langle 45\rangle)}. \quad (4.14)$$

The anomaly term (4.8) then becomes

$$\mathcal{S}A_6^{(1)}(\bar{1},2,\bar{3},4,\bar{5},6)=-\frac{\mathrm{i}}{2}\varepsilon_{ab}\big(\eta_1^A\lambda_2^b-\eta_2^A\lambda_1^b\big)\,\delta_c(\langle 12\rangle)\delta^3(P)\delta^6(Q)\big(i\sqrt{2}\sqrt{p_{3,4,5}^2}\big)$$
$$\times\big(I_{12}^+\delta^3(\alpha^+)+I_{12}^-\delta^3(\alpha^-)\big)+5\text{ cyclic}. \quad (4.15)$$

with the integrals given by

$$I_{12}^{\pm}=\int_{-1}^{1}\mathrm{d}y\,\frac{1}{1-y^2}\,\frac{1}{(\langle b|p_{34}|5\rangle\mp\mathrm{i}\langle 6a\rangle\langle 34\rangle)(\langle 3|p_{45}|6\rangle\mp\mathrm{i}\langle ab\rangle\langle 45\rangle)}. \quad (4.16)$$

Setting $\langle ab\rangle=0$ due to the presence of $\delta_c(\langle 12\rangle)$, the integrals simplify to

$$I_{12}^{\pm}=\frac{1}{\langle 3|p_{45}|6\rangle}\int_{-1}^{1}\mathrm{d}y\,\frac{1}{1-y^2}\,\frac{1}{(\langle b|p_{34}|5\rangle\mp\mathrm{i}\langle 6a\rangle\langle 34\rangle)}. \quad (4.17)$$

[21] Note that $\int\mathrm{d}\Lambda'\delta(\Lambda'-\Lambda)f(\Lambda')=1/2\,f(\Lambda)$ due to the definition of the measure (2.8).

The integral diverges at both integration boundaries, but taking a symmetric limit yields

$$I_{12}^{\pm} = \frac{1}{\langle 3|p_{45}|6\rangle} \frac{\mathrm{i}s_{\pm}\pi}{\langle 2|p_{34}|5\rangle \mp \mathrm{i}\langle 61\rangle\langle 34\rangle}, \tag{4.18}$$

where $s_{\pm}$ are sign factors that may depend on the external momenta, and whose explicit form we did not determine. Comparing this to (4.14) and keeping in mind that $\langle 12\rangle = 0$ under $\delta_c(\langle 12\rangle)$, the final result for (4.8) is

$$\begin{aligned}\mathfrak{S}A_6^{(1)}(\bar{1}, 2, \bar{3}, 4, \bar{5}, 6) = \frac{\pi}{2}\varepsilon_{ab}\big(\eta_1^A\lambda_2^b - \eta_2^A\lambda_1^b\big)\,\delta_c(\langle 12\rangle)\delta^3(P)\delta^6(Q)\\ \times\big(s_+ f^+\delta^3(\alpha^+) + s_- f^-\delta^3(\alpha^-)\big) + 5\text{ cyclic}.\end{aligned} \tag{4.19}$$

Notably, the evaluation of the above integrals leaves overall sign factors $s_{\pm}$ in this expression. Depending on the values of these sign factors, we notice that the anomaly of the one-loop amplitude either is proportional to the tree-level amplitude (4.9) itself, or to the tree-level amplitude with a flipped sign in front of f^-. As will become more transparent below, flipping the relative sign between f^+ and f^- in the tree-level amplitude amounts to a cyclic shift of the external particles. Hence we find

$$\begin{aligned}\mathfrak{S}_a^A A_6^{(1)}(\bar{1}, 2, \bar{3}, 4, \bar{5}, 6) = s\frac{\pi}{4}\big(\mathfrak{S}_a^A\,\mathrm{sgn}_{\mathrm{c}}\langle 12\rangle\big)\begin{cases}A_6^{(0)}(\bar{1}, 2, \bar{3}, 4, \bar{5}, 6) & \text{if} s_{\pm} = s\\ \mathrm{i}A_6^{(0)}(\bar{6}, 1, \bar{2}, 3, \bar{4}, 5) & \text{if} s_{\pm} = \pm s\end{cases}\\ + 5\text{ cyclic}.\end{aligned} \tag{4.20}$$

Discussion

The fact that the one-loop anomaly is singular can be used in connection with Yangian symmetry to gain some easy insights into the one-loop six-point amplitude. At six points there exist two independent Yangian (almost) invariant functions which we call $Y_{1,2}$[22]. They can be constructed by an explicit Feynman diagram calculation and their Yangian invariance checked [18], or by means of a contour integral over an orthogonal Graßmannian which is manifestly Yangian invariant [26]. Both approaches involve the two solutions of a quadratic equation with coefficients depending on the external momenta. In the Graßmannian approach, after choosing a particular patch, the Yangian invariants depend on the variables $c_{s\bar{r}}$, $\bar{r} = 1, 3, 5$, and $s = 2, 4, 6$, such that

$$\lambda_{\bar{r}} + \sum_s \lambda_s c_{s\bar{r}} = 0, \quad \text{and} \quad \sum_{\bar{r}} c_{s\bar{r}}c_{t\bar{r}} = \delta_{st}, \tag{4.21}$$

so that we label the two solutions $(c_{\pm}^*)_{s\bar{r}}$. The two Yangian (almost) invariants, in this approach, correspond to the Graßmannian integrand evaluated on these two solutions, schematically $Y_{1,2} \equiv Y(\{c_{\pm}^*\}_{s\bar{r}})$, and explicit expressions can be found in [26]. Significantly, in terms of the λ_is, these invariants are simply rational functions as all square roots can explicitly be performed.

We know that Yangian symmetry is unbroken at tree level except for a codimension-two anomaly[23]. Therefore the tree-level six-point amplitude must be some linear combination of the two invariants. We can fix which linear combination forms the tree amplitude by demanding the correct behavior of the amplitude under Λ-parity (2.6) with the odd-numbered legs being fermionic and the even-numbered legs being bosonic,

$$A_6^{(0)}(\bar{1}, 2, \bar{3}, 4, \bar{5}, 6) = c_6(Y_1 + Y_2)(\bar{1}, 2, \bar{3}, 4, \bar{5}, 6) \tag{4.22}$$

[22] They are almost invariants in the sense that they are only invariant up to distributional terms.
[23] As Yangian level-one generator we can use $\widehat{\mathfrak{P}}$ which is anomaly-free for finite contributions.

for some constant c_6 depending on the gauge coupling[24]. Here, the bars over the labels on the right-hand side merely signify that the function $(Y_1 + Y_2)(\Lambda_k)$ transforms odd under sign flips of the respective Λs. The other linear combination

$$(Y_1 - Y_2)(1, \bar{2}, 3, \bar{4}, 5, \bar{6}) \tag{4.23}$$

has exactly the opposite transformation property, as indicated by the distribution of bars on the labels, and thus cannot appear in the tree-level amplitude[25]. Notably, this linear combination again equals the tree-level amplitude when shifting all labels cyclically by one,

$$A_6^{(0)}(\bar{6}, 1, \bar{2}, 3, \bar{4}, 5) = \mathrm{i}c_6(Y_1 - Y_2)(1, \bar{2}, 3, \bar{4}, 5, \bar{6}). \tag{4.24}$$

At one-loop order, the loop momentum is still completely constrained by the four-point anomaly vertex (4.6), and hence the anomaly does not get smeared across all configurations of external momenta, but rather stays distributional, see figure 5. Consequently, also the one-loop six-point amplitude has to equal a linear combination of the tree-level Yangian (almost) invariants, with a prefactor that is constant at least locally. However, the support of the anomalies at one loop is different than at tree level. Most notably, the one-loop amplitude has codimension-one anomalies, which are absent at tree level and in the Yangian (almost) invariants. In conclusion, the one-loop amplitude has to be a linear combination

$$\begin{aligned} A_6^{(1)}(\bar{1}, 2, \bar{3}, 4, \bar{5}, 6) = {} & c_6^+(1, 2, 3, 4, 5, 6)(Y_1 + Y_2)(\bar{1}, 2, \bar{3}, 4, \bar{5}, 6) \\ & + c_6^-(\bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6})(Y_1 - Y_2)(1, \bar{2}, 3, \bar{4}, 5, \bar{6}) \end{aligned} \tag{4.25}$$

with coefficients $c_6^\pm$ that are locally constant, but discontinuous on the support of the codimension-one anomalies—that is they have to be piecewise constants that jump whenever two adjacent external particles become collinear. In order to maintain the correct statistics of the amplitude, c_6^+ has to transform bosonic in all labels, and c_6^- has to transform fermionic in all labels, as indicated by the bars over their labels. Thus the symmetries predict a non-vanishing result for the one-loop six-point amplitude.

Proposal summary

Let us summarize what we have found for the anomalies of the scattering amplitudes and outline a proposal for the anomalies at higher points and more loops. The variation of the four-point tree-level amplitude simply gives the distributional, inhomogeneous term found on the right-hand side of equation (3.22). That is, it is directly given by the anomaly vertex and has support only on the codimension-two surface[26]. For higher point amplitudes we attach the anomaly vertex to subamplitudes along a single internal leg. We saw this explicitly in the case of the six-point tree-level amplitude, see figure 3. At eight points and beyond there are additional possible configurations, where two anomaly vertex legs are attached to different subamplitudes, see figure 6. At ten points, there are in principle configurations where three vertex legs can be attached along internal lines to three different subamplitudes and at twelve points, all vertex legs can be attached to internal lines.

At one-loop level we also attach two anomaly vertex legs, but now both with the same energy and to the same subamplitude. We saw this for the six-point one-loop amplitude, see figure 5, where two legs were attached to a tree-level six-point amplitude. This gives

[24] The linear combination can also be fixed by demanding that the amplitude factorizes correctly into four-point amplitudes with fermionic odd-numbered legs. In the Graßmannian approach it follows naturally from the cyclic gauge choice.

[25] Phrased differently, $Y_1 \to (-1)^{k+1}Y_2$ and $Y_2 \to (-1)^{k+1}Y_1$ under $\Lambda_k \to -\Lambda_k$.

[26] Naively, this anomaly cannot be considered as a deformation of the generator acting on a lower point amplitude as there is no lower point amplitude. However, see the discussion in section 6.

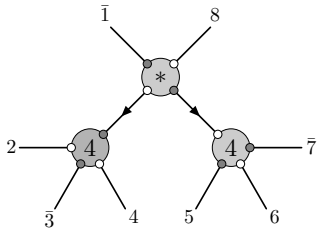

Figure 6. Two internal-leg contribution to tree-level eight-point anomaly.

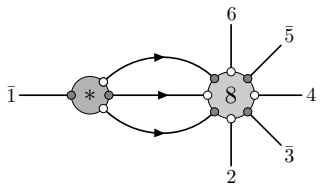

Figure 7. Proposed anomaly of two-loop six-point amplitude.

rise to an anomaly that is also purely distributional as it only occurs when two external legs are collinear. At higher loops more than two legs of the anomaly vertex can be attached to the same lower-loop subamplitude, see figure 7, with all legs having the same energy sign so it is necessary to extend the anomaly vertex into a different kinematic regime. Moreover, for these loop amplitudes this results in an anomaly for generic configurations of the external momenta. For example at two-loops, three anomaly vertex legs can be attached to the same tree-level amplitude and, because of the integrations, the resulting anomaly will have support for any value of the external momenta. This is analogous to what happens in $\mathcal{N} = 4$ SYM at one loop where the anomaly becomes 'smeared' by the loop integrations and so the amplitudes are anomalous with respect to the superconformal symmetries for generic states of the external particles.

5. One-loop six-point amplitude from unitarity

In this section we will apply the methods of generalized unitarity which have proved so useful in $\mathcal{N} = 4$ SYM [28] to $\mathcal{N} = 6$ SCS. In particular, we want to reconstruct the one-loop six-point superamplitude from tree-level amplitudes by evaluating the maximal cuts, i.e. triple cuts for the case of three dimensions. We assume[27] that an arbitrary n-point one-loop amplitude, $A_n^{(1)}$, can be written as a linear combination of scalar triangle diagrams, $\mathcal{I}_{3,i}$,

$$A_n^{(1)} = \sum_i \mathrm{d}_i \mathcal{I}_{3,i} \tag{5.1}$$

and thus we can use the maximal cuts to determine the coefficients d_i.

All on-shell amplitudes in $\mathcal{N} = 6$ SCS have an even number of legs, which implies that there are at least two external legs at each corner of the triangle. Expressions for the massive

[27] Using the standard arguments analogous to those in four dimensions, e.g. [29], it is possible to show that any three-dimensional CS matter one-loop amplitude can be written as a linear combination of triangle, bubbles and tadpoles. For a finite, superconformal, and indeed at least at weak coupling 'dual' superconformal, theory such as ABJM there will be no bubble or tadpole scalar integrals and we can use the scalar triangles as a basis.

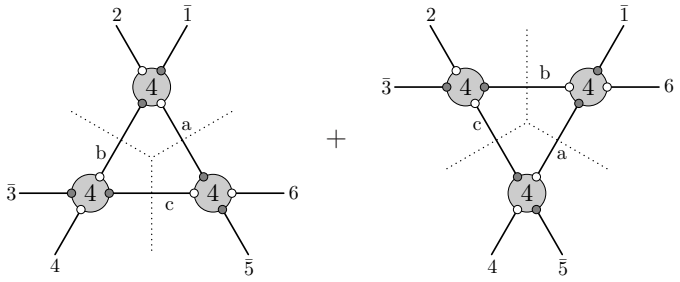

Figure 8. Maximal cuts for the one-loop six-point amplitude in terms of massive triangles.

triangles in an arbitrary number of dimensions can be found in [30]. For external momenta K_1, K_2, K_3 in $D = 3 - 2\epsilon$ the triangle integral evaluates to[28]

$$\begin{aligned}\mathcal{I}_3 &= c_N \int \frac{\mathrm{d}^3\ell}{(-\ell^2 + \mathrm{i}\epsilon)(-(\ell + K_1)^2 + \mathrm{i}\epsilon)(-(\ell + K_1 + K_2)^2 + \mathrm{i}\epsilon)} \\ &= -\frac{\mathrm{i}\pi}{2} \frac{1}{\sqrt{K_1^2 - \mathrm{i}\epsilon}\sqrt{K_2^2 - \mathrm{i}\epsilon}\sqrt{K_3^2 - \mathrm{i}\epsilon}}.\end{aligned} \tag{5.2}$$

Obviously, the minimal number of external legs is six implying that the four-point amplitude is uncorrected at one loop. For six points there are two relevant massive triangle diagrams, see figure 8,

$$A_6^{(1)}(\bar{1}, 2, \bar{3}, 4, \bar{5}, 6) = d_1\mathcal{I}_{3,1} + d_2\mathcal{I}_{3,2}. \tag{5.3}$$

The integral $\mathcal{I}_{3,1}$ has external momenta

$$K_1 = p_1 + p_2, \quad K_2 = p_3 + p_4, \quad K_3 = p_5 + p_6 \tag{5.4}$$

and $\mathcal{I}_{3,2}$ correspondingly

$$K_1 = p_6 + p_1, \quad K_2 = p_2 + p_3, \quad K_3 = p_4 + p_5. \tag{5.5}$$

The triple cuts correspond to putting all internal propagators on-shell and one must sum over all such momentum configurations. For six points the sum is thus over products of tree-level four-point amplitudes, where the sum over internal states can be done by performing Graßmann integrations over the internal legs as in (2.8).

The coefficient of the first integral, $\mathcal{I}_{3,1}$, is given by the left-hand cut in figure 8,

$$d_1 = \frac{1}{2}\sum_{\mathrm{sol}} \int \prod_{i=\mathrm{a,b,c}} d^{0|3}\eta_i \, A_4(\bar{1}, 2, \mathrm{i}\bar{\mathrm{b}}, \mathrm{a}) A_4(\bar{3}, 4, \mathrm{i}\bar{\mathrm{c}}, \mathrm{b}) A_4(\bar{5}, 6, \mathrm{i}\bar{\mathrm{a}}, \mathrm{c}). \tag{5.6}$$

[28] For the scalar integral we choose the three-dimensional normalization to be $c_N = \frac{4\pi}{(2\pi)^3}$ corresponding to the usual momentum measure factor and an additional factor of 4π (cf normalization of four-dimensional scalar box integrals in [31]). This normalization is consistent with earlier sections.

The on-shell loop momenta, ℓ_i, are completely fixed by the delta-functions from the cut propagators so there is no remaining integration but rather a sum over the two solutions to the equations[29]

$$\ell_{\rm a}^2 = \ell^2 = 0, \quad \ell_{\rm b}^2 = (\ell + K_1)^2 = 0, \quad \ell_{\rm c}^2 = (\ell + K_1 + K_2)^2 = 0. \tag{5.7}$$

We now turn our attention to the Graßmann integration over the delta-functions appearing in the tree-level four point amplitudes

$$\int \prod_{\rm i=a,b,c} d^{0|3}\eta_i \delta^{0|6}(\Theta_{13} - \lambda_{\rm b}\eta_{\rm b} + \lambda_{\rm a}\eta_{\rm a})\delta^{0|6}(\Theta_{35} - \lambda_{\rm c}\eta_{\rm c} + \lambda_{\rm b}\eta_{\rm b})\delta^{0|6}(\Theta_{51} - \lambda_{\rm a}\eta_{\rm a} + \lambda_{\rm c}\eta_{\rm c})$$
$$= \frac{\delta^{0|6}(Q)}{\langle {\rm cb}\rangle^3}\delta^{0|3}(\langle {\rm a}|x_{13}x_{35}|\Theta_{51}\rangle + \langle\Theta_{13}|x_{35}x_{51}|{\rm a}\rangle). \tag{5.8}$$

In this equation we have pulled out an overall factor, $\delta^{(6)}(Q)$, of the total supermomentum Q and we have repeatedly used Schouten's identity which accounts for the fact that λ_a and $\lambda_{b,c}$ appear with different weights. Finally we have used the notation, $x_{jk} = \sum_{i=j}^{k-1} p_i$ and $\langle\Theta_{jk}| = \sum_{i=j}^{k-1}\eta_i\langle i|$. Thus, including the denominator factors from the four point amplitudes, we find the coefficient for the cut

$$d_1 = \frac{\rm i}{2}\sum_{s=\pm}\frac{\delta^3(P)\delta^{0|6}(Q)}{\langle {\rm bc}\rangle^3}\frac{\delta^{0|3}(\langle {\rm a}|x_{13}x_{35}|\Theta_{51}\rangle + \langle\Theta_{13}|x_{35}x_{51}|{\rm a}\rangle)}{\langle 12\rangle\langle 34\rangle\langle 56\rangle\langle {\rm b}2\rangle\langle {\rm c}4\rangle\langle {\rm a}6\rangle}. \tag{5.9}$$

The complete contribution to the one-loop six-point amplitude from this scalar amplitude is thus

$$d_1\mathcal{I}_{3,1} = -\frac{{\rm i}\pi}{2}\frac{d_1}{\sqrt{-\langle 12\rangle^2 - {\rm i}\epsilon}\sqrt{-\langle 34\rangle^2 - {\rm i}\epsilon}\sqrt{-\langle 56\rangle^2 - {\rm i}\epsilon}}. \tag{5.10}$$

We can evaluate the square roots carefully to find

$$\sqrt{-x - {\rm i}\epsilon} = -{\rm i}|x|_{\rm c} = -{\rm i}x\,{\rm sgn}_{\rm c}(x) \tag{5.11}$$

where the generalizations of the absolute value and sign functions were defined in (3.14). We end up with

$$d_1\mathcal{I}_{3,1} = -\frac{\pi}{2}\frac{d_1}{\langle 12\rangle\langle 34\rangle\langle 56\rangle}\,{\rm sgn}_{\rm c}\langle 12\rangle\,{\rm sgn}_{\rm c}\langle 34\rangle\,{\rm sgn}_{\rm c}\langle 56\rangle, \tag{5.12}$$

which is proportional to the shifted tree-level scattering amplitude (4.24) up to some sign factors[30],

$$d_1\mathcal{I}_{3,1} = \frac{\rm i}{4}\pi A_6^{(0)}(\bar 6, 1, \bar 2, 3, \bar 4, 5)\,{\rm sgn}_{\rm c}\langle 12\rangle\,{\rm sgn}_{\rm c}\langle 34\rangle\,{\rm sgn}_{\rm c}\langle 56\rangle. \tag{5.13}$$

In fact, and as discussed in section 4, the tree-level six-point amplitude of [18] is the sum of two terms Y_1, Y_2 related by $\Lambda_k \to -\Lambda_k$ parity transformations. These two terms are exactly the $s = \pm 1$ terms appearing in the cut. Here it is important to note that the shifted tree-level amplitude $A_6^{(0)}(\bar 6, 1, \bar 2, 3, \bar 4, 5)$ has the opposite assignment of conjugate particles compared to the one-loop amplitude $A_6^{(1)}(\bar 1, 2, \bar 3, 4, \bar 5, 6)$. Superficially this leads to the wrong sign under

[29] One way to determine ℓ is to choose $\ell^\mu = \alpha K_1^\mu + \beta K_2^\mu + \gamma K_\times^\mu$ where $K_\times^\mu = \epsilon^{\mu\nu\rho}K_{1\nu}K_{2\rho}$ and use (5.7) to determine α, β and γ. One finds two solutions

$$\alpha = \frac{(K_1\cdot K_2)K_2^2 - K_1^2K_2^2 + 2(K_1\cdot K_2)^2}{2K_\times^2}, \quad \beta = \frac{K_1^2(K_1\cdot K_2 + K_2^2)}{2K_\times^2}, \quad \gamma = s\frac{\sqrt{K_1^2K_2^2K_3^2}}{2K_\times^2}.$$

where $s = \pm$ enumerates the two solutions. Alternatively, one can directly solve the equations for the corresponding λs.

[30] This comparison is done by taking specific values for the external momenta and evaluating various component amplitudes numerically.

any of the transformations $\Lambda_k \to -\Lambda_k$, which is compensated by the sign functions involving all of the external λ_k.

The scalar triangle integral $\mathcal{I}_{3,2}$ is captured by the right-hand cut in figure 8,

$$d_2 = \frac{1}{2}\sum_{\text{sol}}\int \prod_{i=\text{a,b,c}} d^{0|3}\eta_i A_4(\bar{1}, \text{b}, \text{i}\bar{\text{a}}, 6)A_4(\bar{3}, \text{c}, \text{i}\bar{\text{b}}, 2)A_4(\bar{5}, \text{a}, \text{i}\bar{\text{c}}, 4). \tag{5.14}$$

The calculation is identical to the previous case and the result is

$$d_2 = \frac{\text{i}}{2}\sum_{s=\pm}\frac{\delta^3(P)\delta^{0|6}(Q)}{\langle \text{cb}\rangle^3}\frac{\delta^{0|3}\big(\langle \text{a}|x_{62}x_{24}|\Theta_{46}\rangle + \langle\Theta_{62}|x_{24}x_{46}|\text{a}\rangle\big)}{\langle 61\rangle\langle 32\rangle\langle 54\rangle\langle 6\text{a}\rangle\langle 2\text{b}\rangle\langle 4\text{c}\rangle}, \tag{5.15}$$

where the on-shell loop momenta, $\ell_i^2 = 0$, are related by $\ell_\text{b} = \ell_\text{a} - K_1$ and $\ell_\text{c} = \ell_\text{b} - K_2$. Expanding the square roots in the triangle integral we find

$$d_2\mathcal{I}_{3,2} = \frac{\text{i}}{4}\pi A_6^{(0)}(\bar{6}, 1, \bar{2}, 3, \bar{4}, 5)\ \text{sgn}_\text{c}\langle 61\rangle\ \text{sgn}_\text{c}\langle 23\rangle\ \text{sgn}_\text{c}\langle 45\rangle, \tag{5.16}$$

Altogether our result for the one-loop six-point amplitudes reads

$$A_6^{(1)}(\bar{1}, 2, \bar{3}, 4, \bar{5}, 6) = \frac{\text{i}\pi}{4}c_6(\bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6})A_6^{(0)}(\bar{6}, 1, \bar{2}, 3, \bar{4}, 5) \tag{5.17}$$

with the piecewise constant combination of sign functions

$$c_6(\bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}) = \text{sgn}_\text{c}\langle 12\rangle\ \text{sgn}_\text{c}\langle 34\rangle\ \text{sgn}_\text{c}\langle 56\rangle + \text{sgn}_\text{c}\langle 61\rangle\ \text{sgn}_\text{c}\langle 23\rangle\ \text{sgn}_\text{c}\langle 45\rangle. \tag{5.18}$$

Here the bars over the labels indicate that c_6 transforms odd under sign flips of any λ_k. The relative plus sign between the two products of sgn_c functions follows from the calculation, however we can also understand it from the symmetries of the amplitudes. As previously described (2.7), the color ordered six-point superamplitudes are odd/even functions under an inversion of the color ordering, $A(\bar{1}, 2, \bar{3}, 4, \bar{5}, 6) = \mp A(\bar{1}, 6, \bar{5}, 4, \bar{3}, 2)$. While the tree-level amplitude and its cyclically shifted version are odd, the one-loop amplitude is even under this transformation. Thus the piecewise constant, c_6, must be an odd function under the inversion map, and this requires the relative plus sign between the two terms.

To conclude, let us analyze the cuts of the above expression. In this regard the near equality of tree and loop level amplitudes begs for an explanation. How can the one-loop result have the same set of discontinuities as the corresponding one at tree level when the cuts are obviously different? In particular, the pole in the three-particle channel at tree level originates from the splitting into two four-particle trees (4.1). The one-loop result has the same pole but no apparent splitting into subamplitudes. Taking a closer look at the origin of the discontinuity one finds that it requires the momentum transfer in one of the corners of the triangle to be zero. At this point, the four-point amplitude has a pole which is responsible for the three-particle pole. This pole corresponds to the zero-mode of the Chern–Simons gauge field. So indeed there is a physical cut for the one-loop six-point amplitude in the three-particle channel, see figure 9[31]. Somewhat surprisingly, this pole agrees precisely with the pole of the cyclically shifted tree-level amplitude up to some sign factors. The other relevant cuts are in two-particle channels. These are the natural cuts at the one-loop level but are not present at tree level and thus they must be associated to the additional sign factors which we recall originate in square roots of 'masses', $\sqrt{-m^2} = \sqrt{(p_i + p_{i+1})^2}$, occurring in the scalar triangles, whose branch cuts are just next to the real axis when the two inflowing energies are aligned (positive m^2).

[31] It is worth pointing out that unitarity in the original meaning of the word is different from the cutting rules of generalized unitarity: one would not expect the non-propagating Chern–Simons field to appear in unitarity cuts. This means that the above mentioned poles are to be evaluated in a principal value prescription. In this case the zero-mode does not appear in the imaginary part of amplitudes: $\text{Im}\,F(x) = 0$ for $F(x) = 1/x$. Conversely, in generalized unitarity the zero-mode is detectable by the cutting rule $F(x + \text{i}\epsilon) - F(x - \text{i}\epsilon) = -2\pi\text{i}\delta(x)$, and needs to be taken into consideration.

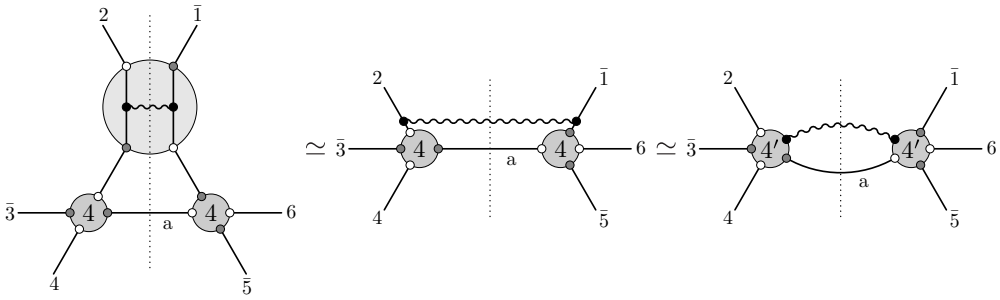

Figure 9. Cut of the one-loop six-point amplitude in the 123 channel.

It is also quite clear that the sign factors in c_6 give rise to the correct codimension-one superconformal anomalies discussed in section 4. The superconformal variations act as derivatives which turn the sign factors into delta functions supported on collinear configurations of any two adjacent particles as described previously.

6. Conclusions and outlook

In this paper we have shown that scattering amplitudes in three-dimensional $\mathcal{N} = 6$ SCS theory give rise to an anomaly of the (super)conformal symmetry in a fashion similar to $\mathcal{N} = 4$ SYM theory. As for the four-dimensional theory in $(2, 2)$ signature [2], here the anomaly arises from sign functions of spinor brackets. These sign factors emerge when rewriting the four-point amplitude by making use of the different scaling behavior of bosonic and fermionic delta functions, schematically (cf section 3):

$$\frac{1}{x^2}\delta^2(x\,\mathrm{bos})\delta^3(x\,\mathrm{ferm}) = \frac{x}{|x|}\delta^2(\mathrm{bos})\delta^3(\mathrm{ferm}) = \mathrm{sgn}\,x\,\delta^2(\mathrm{bos})\delta^3(\mathrm{ferm}), \quad (6.1)$$

where x represents the spinor brackets. While in four dimensions the anomaly can be captured in terms of a vertex with three legs, we find a corresponding 4-vertex $\mathfrak{S}_4 = \mathfrak{S}A_4$ with support on collinear momentum configurations. We have employed the anomaly to predict the non-vanishing of the one-loop six-point amplitude: Firstly, there are two Yangian invariants (up to anomalous contributions) whose linear combinations furnish the tree-level and one-loop six-point amplitude. Considering the different anomalies of the tree and one-loop term as well as discrete symmetries of physical amplitudes, shows that the proportionality factor translating between them is a non-trivial function of the external momenta. Consequently, the one-loop six-point amplitude is non-trivial[32] and proportional to the tree-level contribution cyclically shifted: $A_6^{(1)} = c_6 A_6^{(0)}(\{\mathrm{i} \to \mathrm{i}-1\})$. We have confirmed this result by a unitarity construction of the one-loop amplitude using a triple cut. It is important to note the different structure of the anomaly of the six-point amplitude at loop level when compared to $\mathcal{N} = 4$ SYM theory. While in four dimensions the anomaly is only distributional at tree-level and gets smeared in the loop integration at one loop, the three-dimensional anomaly is still distributional at one loop and only gets smeared at two loops. As a consequence, the one-loop six-point amplitude obeys the same symmetry constraints as the tree-level expression up to distributional terms.

The non-vanishing of the one-loop six-point amplitude is particularly interesting in the light of a possible duality between scattering amplitudes and Wilson loops in ABJM theory.

[32] Note that a non-vanishing result for the one-loop six-point amplitude was also obtained by an independent Feynman calculation in [24].

As the lightlike hexagon Wilson loop is known to vanish at one-loop order [22, 23], the non-trivial one-loop six-point amplitude poses the puzzle of what a possible map could look like. A simple solution would be that there is no self-T-duality in ABJM theory and thus no reason for Wilson loops and scattering amplitudes to match. In order to look for further hints for either outcome one could start by stripping off the tree-level contribution and try to compare only scalar loop corrections. The most interesting question seems to be: can one define a hexagon Wilson loop that matches our one-loop result of the six-point amplitude? Relatedly, it may be useful to find the correct analog of the super-Wilson loop [10, 15, 32] in three dimensions to match with superamplitudes. A similar question occurs when we wish to consider Wilson loops with an odd number of edges, which *a priori* exist and are non-trivial, though they will be generically complex [22], while there are only amplitudes for even numbers of external legs if we allow only on-shell scalars and fermions as external particles.

An important point we have not discussed here is whether the breaking of conformal symmetry can be cured by deformation of the free representation of $\mathfrak{osp}(6|4)$ on amplitudes. This seems very plausible, in particular with regard to analogous considerations in $\mathcal{N}=4$ SYM theory where the conformal generators of the $\mathfrak{psu}(2,2|4)$ representation were deformed to compensate for the anomaly. This is particularly desirable since it renders the scattering matrix an exact symmetry invariant and thereby recursively relates amplitudes with different numbers of legs to each other, e.g. $\tilde{\mathfrak{S}}A_n^{\text{4d-MHV}} + \tilde{\mathfrak{S}}_3^+ A_{n-1}^{\text{4d-MHV}} = 0$. Here $\tilde{\mathfrak{S}}_3^+$ denotes the deformation of the generator $\tilde{\mathfrak{S}}$ corresponding to the three-point anomaly vertex in four dimensions. The starting point of the four-dimensional recursion is $\tilde{\mathfrak{S}}A_4^{\text{4d}} = 0$ being consistent with a three-point amplitude that vanishes for physical kinematics. Straightforwardly translating this recursion to ABJM theory with only even-point amplitudes yields

$$\mathfrak{S}A_n + \mathfrak{S}_4^+ A_{n-2} \stackrel{?}{=} 0. \tag{6.2}$$

Here, however, $\mathfrak{S}A_4$ is non-vanishing as discussed above and the inductive symmetry would require a non-trivial two-point invariant with $\mathfrak{S}A_2 = 0$ as a starting point. Notably, we can construct a two-point $\mathfrak{osp}(6|4)$ invariant that renders (6.2) correct for $n=4$. It takes the form $A_2 = \delta^{2|3}(\Lambda_1 \pm \mathrm{i}\Lambda_2)$. Provided the algebra of deformed generators still holds, this gives hope for a construction similar to four dimensions. But is the two-point invariant really part of the ABJM scattering matrix?

Taking a closer look at the interaction Hamiltonian that induces the scattering matrix, the vertex with the lowest number of points has three legs and includes the Chern–Simons field. The Chern–Simons field, however, is not dynamical and should thus not appear as an external particle in scattering amplitudes. Could the three-vertex still give rise to a non-vanishing two-point invariant in the scattering matrix such that $A_2 = A_3|_{p_{\text{CS}}=0}$? At first sight it seems not clear how to technically investigate this point. The Chern–Simons field has no on-shell degrees of freedom and is thus not captured by the ordinary on-shell superspace formulation of scattering amplitudes. Considering the scattering matrix in terms of oscillators corresponding to field excitations, the same problem arises. While creation and annihilation operators live on the forward and backward mass shell, respectively, *a priori* neither choice seems appropriate to describe the Chern–Simons field. Can one still introduce a corresponding oscillator? What commutation relations would this imply? Notably, the zero-mode of the Chern–Simons field already played a special role in explaining the discontinuity of the one-loop six-point amplitude in section 5 (cf figure 9). Its generic role for the generalized unitarity construction of scattering amplitudes seems yet unclear. A deeper understanding of these issues appears to have the potential for new insights into the structure of Chern–Simons theories in general.

Obviously, it is very tempting to extend our results for the one-loop six-point amplitude to higher numbers of loops and legs. In particular this should shed light on the general structure

of one-loop amplitudes in ABJM theory which is of great importance for a possible duality to Wilson loops. A starting point could be the investigation of constraints on generic one-loop amplitudes imposed by symmetry and the form of the anomaly. It would also be very interesting to see whether a similar anomaly arises in dimensions greater than 4.

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Note added. As this work was being completed we were informed by the authors of [24] that they had obtained related results for one-loop amplitudes in ABJM and in particular the same result for the one-loop six-point amplitude. This work has appeared concurrently on the arXiv.

Appendix. Mixed energy signs

Distributing positive/negative energies (incoming/outgoing particles) in different ways, the analysis of section 3 gets slightly modified.

Same sign on 1 and 3

If particles 1 and 3 carry the same energy sign, and particles 2 and 4 carry the opposite energy sign, the identities (3.2) have to be modified by appropriate factors of i:

$$\begin{aligned} 1 &= \mathrm{i}|\langle 12\rangle| \int \mathrm{d}\alpha_3\, \mathrm{d}\beta_3 \delta^2(\lambda_3 - \alpha_3\lambda_1 - \mathrm{i}\beta_3\lambda_2), \\ 1 &= \mathrm{i}|\langle 12\rangle| \int \mathrm{d}\alpha_4\, \mathrm{d}\beta_4 \delta^2(\lambda_4 - \mathrm{i}\alpha_4\lambda_1 - \beta_4\lambda_2). \end{aligned} \tag{A.1}$$

The momentum conservation delta function becomes

$$\begin{aligned} \delta^3(P) &= \delta^3\big(\lambda_1\lambda_1\big(1+\alpha_3^2-\alpha_4^2\big) + \lambda_2\lambda_2\big(1-\beta_3^2+\beta_4^2\big) + \mathrm{i}(\lambda_1\lambda_2+\lambda_2\lambda_1)(\alpha_3\beta_3+\alpha_4\beta_4)\big) \\ &= \frac{1}{|\langle 12\rangle|^3}\delta\big(1+\alpha_3^2-\alpha_4^2\big)\delta\big(1-\beta_3^2+\beta_4^2\big)\delta(\alpha_3\beta_3+\alpha_4\beta_4), \end{aligned} \tag{A.2}$$

such that the amplitude reads

$$\begin{aligned} A_4^{1,3\leftrightarrow 2,4}(\bar{1},2,\bar{3},4) = \frac{\delta^6(Q)}{|\langle 12\rangle|\langle 12\rangle^2} \int \mathrm{d}\alpha_3\, \mathrm{d}\alpha_4\, \mathrm{d}\beta_3\, \mathrm{d}\beta_4 \frac{1}{\alpha_3} \\ \cdot \delta\big(1+\alpha_3^2-\alpha_4^2\big)\delta\big(1-\beta_3^2+\beta_4^2\big)\delta(\alpha_3\beta_3+\alpha_4\beta_4) \\ \cdot \delta^2(\lambda_3 - \alpha_3\lambda_1 - \mathrm{i}\beta_3\lambda_2)\delta^2(\lambda_4 - \mathrm{i}\alpha_4\lambda_1 - \beta_4\lambda_2). \end{aligned} \tag{A.3}$$

The first two delta functions are each supported on a pair of hyperbolas in (α_3, α_4) and (β_3, β_4) space. Using the parametrization

$$\alpha_3 = r_\alpha \sinh\alpha, \qquad \alpha_4 = r_\alpha \cosh\alpha, \qquad \beta_3 = r_\beta \cosh\beta, \qquad \beta_4 = r_\beta \sinh\beta, \tag{A.4}$$

where the radial variables r_α, r_β take all real values, the radial integrals localize at $r_\alpha = \pm 1$, $r_\beta = \pm 1$:

$$A_4^{1,3\leftrightarrow 2,4}(\bar{1},2,\bar{3},4) = \frac{\delta^6(Q)}{|\langle 12\rangle|\langle 12\rangle^2}\sum_{s_\alpha,s_\beta=\pm 1}\int \mathrm{d}\alpha\,\mathrm{d}\beta\frac{s_\alpha}{4\sinh\alpha}\delta(\sinh(\alpha+\beta))$$
$$\cdot\,\delta^2(\lambda_3 - s_\alpha\sinh\alpha\lambda_1 - \mathrm{i}s_\beta\cosh\beta\lambda_2)$$
$$\times\,\delta^2(\lambda_4 - \mathrm{i}s_\alpha\cosh\alpha\lambda_1 - s_\beta\sinh\beta\lambda_2). \tag{A.5}$$

The first delta function localizes the beta integral at $\beta = -\alpha$, thus

$$A_4^{1,3\leftrightarrow 2,4}(\bar{1},2,\bar{3},4) = \frac{\delta^6(Q)}{|\langle 12\rangle|\langle 12\rangle^2}\sum_{s_\alpha,s_\beta=\pm 1}\int_{-\infty}^{\infty}\mathrm{d}\alpha\frac{s_\alpha}{4\sinh\alpha}$$
$$\cdot\,\delta^2(\lambda_3 - s_\alpha\sinh\alpha\lambda_1 - \mathrm{i}s_\beta\cosh\alpha\lambda_2)$$
$$\times\,\delta^2(\lambda_4 - \mathrm{i}s_\alpha\cosh\alpha\lambda_1 + s_\beta\sinh\alpha\lambda_2). \tag{A.6}$$

Moving $\delta^6(Q)$ under the integral sign, contracting Q once with λ_3 and once with λ_4, and expanding $\lambda_{3,4}$ in terms of $\lambda_{1,2}$ shows that

$$\begin{aligned}\delta^6(Q) &= \langle 34\rangle^{-3}\delta^3(\langle 31\rangle\eta_1 + \langle 32\rangle\eta_2 + \langle 34\rangle\eta_4)\delta^3(\langle 41\rangle\eta_1 + \langle 42\rangle\eta_2 + \langle 43\rangle\eta_3)\\ &= -s_\alpha s_\beta\langle 12\rangle^3\delta^3(\eta_3 - s_\alpha\sinh\alpha\eta_1 - \mathrm{i}s_\beta\cosh\alpha\eta_2)\\ &\quad\cdot\,\delta^3(\eta_4 - \mathrm{i}s_\alpha\cosh\alpha\eta_1 + s_\beta\sinh\alpha\eta_2).\end{aligned} \tag{A.7}$$

The four-point amplitude hence reads

$$A_4^{1,3\leftrightarrow 2,4}(\bar{1},2,\bar{3},4) = -\mathrm{i}\,\mathrm{sgn}_{\mathrm{c}}\langle 12\rangle\sum_{s_\alpha,s_\beta=\pm 1}\int_{-\infty}^{\infty}\mathrm{d}\alpha\frac{s_\beta}{4\sinh\alpha}$$
$$\cdot\,\delta^{2|3}(\Lambda_3 - s_\alpha\sinh\alpha\Lambda_1 - \mathrm{i}s_\beta\cosh\alpha\Lambda_2)$$
$$\times\,\delta^{2|3}(\Lambda_4 - \mathrm{i}s_\alpha\cosh\alpha\Lambda_1 + s_\beta\sinh\alpha\Lambda_2), \tag{A.8}$$

where again $\Lambda = (\lambda,\eta)$. Reverting the direction of integration in the terms with $s_\beta = -1$ gives (3.12):

$$A_4^{1,3\leftrightarrow 2,4}(\bar{1},2,\bar{3},4) = -\mathrm{i}\,\mathrm{sgn}_{\mathrm{c}}\langle 12\rangle\sum_{s_\alpha,s_\beta=\pm 1}\int_{-\infty}^{\infty}\mathrm{d}\alpha\frac{1}{4\sinh\alpha}$$
$$\cdot\,\delta^{2|3}(\Lambda_3 - s_\beta(s_\alpha\sinh\alpha\Lambda_1 + \mathrm{i}\cosh\alpha\Lambda_2))$$
$$\times\,\delta^{2|3}(\Lambda_4 - \mathrm{i}s_\alpha\cosh\alpha\Lambda_1 + \sinh\alpha\Lambda_2). \tag{A.9}$$

Alternatively, reverting the direction of integration in the terms with $s_\alpha = -1$ results in

$$A_4^{1,3\leftrightarrow 2,4}(\bar{1},2,\bar{3},4) = -\mathrm{i}\,\mathrm{sgn}_{\mathrm{c}}\langle 12\rangle\sum_{s_\alpha,s_\beta=\pm 1}\int_{-\infty}^{\infty}\mathrm{d}\alpha\frac{s_\alpha s_\beta}{4\sinh\alpha}$$
$$\cdot\,\delta^{2|3}(\Lambda_3 - \sinh\alpha\Lambda_1 - \mathrm{i}s_\beta\cosh\alpha\Lambda_2)$$
$$\times\,\delta^{2|3}(\Lambda_4 - s_\alpha(\mathrm{i}\cosh\alpha\Lambda_1 - s_\beta\sinh\alpha\Lambda_2)). \tag{A.10}$$

Same sign on 1 and 4

If the momenta of particles 1 and 4 carry the same energy sign (opposed to 1 and 3), then the previous derivation up to (A.8) works exactly the same, up to the following substitutions:

$$\alpha_3 \to \mathrm{i}\alpha_3, \qquad \alpha_4 \to -\mathrm{i}\alpha_4, \qquad \beta_3 \to -\mathrm{i}\beta_3, \qquad \beta_4 \to \mathrm{i}\beta_4, \tag{A.11}$$

and accordingly

$$\sinh\alpha \to \mathrm{i}\cosh\alpha, \quad \cosh\alpha \to -\mathrm{i}\sinh\alpha, \quad \sinh\beta \to \mathrm{i}\cosh\beta, \quad \cosh\beta \to -\mathrm{i}\sinh\beta. \tag{A.12}$$

Hence the amplitude in this case reads

$$\begin{aligned} A_4^{1,4\leftrightarrow 2,3}(\bar{1},2,\bar{3},4) = -\mathrm{i}\,\mathrm{sgn}_{\mathrm{c}}\langle 12\rangle \sum_{s_\alpha,s_\beta=\pm 1}\int_{-\infty}^{\infty}\mathrm{d}\alpha\,\frac{s_\beta}{4\mathrm{i}\cosh\alpha} \\ \cdot\,\delta^{2|3}(\Lambda_3 - \mathrm{i}s_\alpha\cosh\alpha\Lambda_1 - s_\beta\sinh\alpha\Lambda_2) \\ \times\,\delta^{2|3}(\Lambda_4 - s_\alpha\sinh\alpha\Lambda_1 + \mathrm{i}s_\beta\cosh\alpha\Lambda_2). \end{aligned} \tag{A.13}$$

Substituting $s_\beta \to -s_\beta$ and subsequently reverting the direction of integration when $s_\beta = -1$ yields (3.13)

$$\begin{aligned} A_4^{1,4\leftrightarrow 2,3}(\bar{1},2,\bar{3},4) = \mathrm{i}\,\mathrm{sgn}_{\mathrm{c}}\langle 12\rangle \sum_{s_\alpha,s_\beta=\pm 1}\int_{-\infty}^{\infty}\mathrm{d}\alpha\,\frac{s_\beta}{4\mathrm{i}\cosh\alpha} \\ \cdot\,\delta^{2|3}(\Lambda_3 - \mathrm{i}s_\alpha\cosh\alpha\Lambda_1 + \sinh\alpha\Lambda_2) \\ \times\,\delta^{2|3}(\Lambda_4 - s_\beta(s_\alpha\sinh\alpha\Lambda_1 + \mathrm{i}\cosh\alpha\Lambda_2)). \end{aligned} \tag{A.14}$$

Alternatively, substituting $s_\beta \to -s_\beta$ and subsequently reverting the direction of integration when $s_\alpha = -1$ gives

$$\begin{aligned} A_4^{1,4\leftrightarrow 2,3}(\bar{1},2,\bar{3},4) = \mathrm{i}\,\mathrm{sgn}_{\mathrm{c}}\langle 12\rangle \sum_{s_\alpha,s_\beta=\pm 1}\int_{-\infty}^{\infty}\mathrm{d}\alpha\,\frac{s_\beta}{4\mathrm{i}\cosh\alpha} \\ \cdot\,\delta^{2|3}(\Lambda_3 - s_\alpha(\mathrm{i}\cosh\alpha\Lambda_1 - s_\beta\sinh\alpha\Lambda_2)) \\ \times\,\delta^{2|3}(\Lambda_4 - \sinh\alpha\Lambda_1 - \mathrm{i}s_\beta\cosh\alpha\Lambda_2). \end{aligned} \tag{A.15}$$

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