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SUPPLEMENT TO "MULTISCALE METHODS FOR SHAPE CONSTRAINTS IN DECONVOLUTION: CONFIDENCE STATEMENTS FOR QUALITATIVE FEATURES"

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The supplementary document contains additional technical results and provides proofs for the theorems in "Multiscale methods for shape constraints in deconvolution: confidence statements for qualitative features".

Appendix A Proofs of the main theorems. Throughout the appendix, let

$$w_h = \frac{\sqrt{\frac{1}{2}\log\frac{\nu}{h}}}{\log\log\frac{\nu}{h}}, \quad \widetilde{w}_h = \frac{\log\frac{\nu}{h}}{\log\log\frac{\nu}{h}}.$$

Furthermore, we often use the normalized differential $đ\xi := (2\pi)^{-1}d\xi$

Proof of Theorem 1. In a first step we study convergence of the statistic

$$T_n^{(1)} = \sup_{(t,h)\in B_n} w_h \frac{\left|T_{t,h} - \mathbb{E}T_{t,h}\right|}{V_{t,h}\ \sqrt{g(t)}} - \widetilde{w}_h.$$

Note that $T_n^{(1)}$ is the same as T_n, but $\widehat{g}_n$ is replaced by the true density g. We show that there exists a (two-sided) Brownian motion W, such that with

$$T_n^{(2)}(W) := \sup_{(t,h)\in B_n} w_h \frac{\left|\int \psi_{t,h}(s)\sqrt{g(s)}dW_s\right|}{V_{t,h}\ \sqrt{g(t)}} - \widetilde{w}_h,$$

we have

$$(1) \qquad \sup_{G\in\mathcal{G}_{c,C,q}} \left|T_n^{(1)} - T_n^{(2)}(W)\right| = o_P(r_n).$$

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The main argument is based on the standard version of KMT (cf. [44]). This is a fairly classical result, but has never been used to describe the asymptotic distribution of a multiscale statistic, the only exception being Walther [46]. In order to state the result, let us define a Brownian bridge on the index set $[0,1]$ as a centered Gaussian process $(B(f))_{\{f\in\mathcal{F}\}}$, $\mathcal{F}\subset L^2([0,1])$ with covariance structure

$$\operatorname{Cov}\big(B(f),B(g)\big)=\langle f,g\rangle-\langle f,1\rangle\langle g,1\rangle.$$

For $\mathcal{F}_0:=\{x\mapsto \mathbb{I}_{[0,s]}(x):s\in[0,1]\}$, the process $(B(f))_{\{f\in\mathcal{F}_0\}}$ coincides with the classical definition of a Brownian bridge. If $U_i\sim\mathcal{U}[0,1]$, i.i.d., the uniform empirical process on the function class $\mathcal{F}$ is defined as

$$\mathbb{U}_n(f)=\sqrt{n}\,\Big(\frac{1}{n}\sum_{i=1}^n f(U_i)-\int f(x)dx\Big),\quad f\in\mathcal{F}.$$

In particular

$$T_{t,h}-\mathbb{E}\,T_{t,h}=\mathbb{U}_n\big(\psi_{t,h}\circ G^{-1}\big),$$

where G^{-1} denotes the quantile function of Y. For convenience, we restate the celebrated KMT inequality for the uniform empirical process.

THEOREM 1 (KMT on $[0,1]$, cf. [44]). *There exist versions of $\mathbb{U}_n$ and a Brownian bridge B such that for all x*

$$\mathbb{P}\Big(\sup_{f\in\mathcal{F}_0}\big|\mathbb{U}_n(f)-B(f)\big|>n^{-1/2}(x+C\log n)\Big)<Ke^{-\lambda x},$$

where $C,K,\lambda>0$ are universal constants.

However, we need a functional version of KMT. We shall prove this by using the theorem above in combination with a result due to Koltchinskii [43], (Theorem 11.4, p. 112) stating that the supremum over a function class $\mathcal{F}$ behaves as the supremum over the symmetric convex hull $\overline{\mathrm{sc}}(\mathcal{F})$, defined by

$$\overline{\mathrm{sc}}(\mathcal{F}):=\Big\{\sum_{i=1}^\infty\lambda_i f_i: f_i\in\mathcal{F},\lambda_i\in[-1,1],\sum_{i=1}^\infty|\lambda_i|\le 1\Big\}.$$

THEOREM 2. *Assume there exists a version B of a Brownian bridge, such that for a sequence $(\widetilde{\delta}_n)_n$ tending to 0,*

$$\mathbb{P}^*\Big(\sup_{f\in\mathcal{F}}|\mathbb{U}_n(f)-B(f)|\ge\widetilde{\delta}_n(x+C\log n)\Big)\le Ke^{-\lambda x},$$

where $C, K, \lambda > 0$ are constants depending only on $\mathcal{F}$. Then, there exists a version $\widetilde{B}$ of a Brownian bridge, such that

$$\mathbb{P}^*\Big(\sup_{f\in\overline{\mathrm{sc}}(\mathcal{F})} |\mathbb{U}_n(f) - \widetilde{B}(f)| \geq \widetilde{\delta}_n(x + C'\log n)\Big) \leq K'e^{-\lambda' x}$$

for constants $C', K', \lambda' > 0$.

In Theorem 2, $\mathbb{P}^\star$ refers to the outer measure, however, for the function class considered in this paper, we have measurability of the corresponding event and hence may replace $\mathbb{P}^\star$ by $\mathbb{P}$. It is well-known (cf. Giné *et al.* [17], p. 172) that

$$\big\{\rho \;\big|\; \rho : \mathbb{R} \to \mathbb{R},\ \operatorname{supp}\rho \subset [0,1],\ \rho(1) = 0,\ \mathrm{TV}(\rho) \leq 1\big\} \subset \overline{\mathrm{sc}}(\mathcal{F}_0). \tag{2}$$

Now, assume that $\rho : \mathbb{R} \to \mathbb{R}$ is such that $\mathrm{TV}(\rho) + 3|\rho(1)| \leq 1$. Define $\widetilde{\rho} = (\rho - \rho(1)\mathbb{I}_{[0,1]})/(1 - |\rho(1)|)$ and observe that $\mathrm{TV}(\widetilde{\rho}) \leq 1$ and $\widetilde{\rho}(1) = 0$. By (2) there exists $\lambda_1, \lambda_2, \ldots \in \mathbb{R}$ and $t_1, t_2, \ldots \in [0,1]$ such that $\widetilde{\rho} = \sum \lambda_i \mathbb{I}_{[0,t_i]}$ and $\sum |\lambda_i| \leq 1$. Therefore, $\rho = (1 - |\rho(1)|)\widetilde{\rho} + \rho(1)\mathbb{I}_{[0,1]}$ can be written as linear combination of indicator functions, such that the sum of the absolute values of weights is bounded by 1. This shows

$$\big\{\rho \;\big|\; \rho : \mathbb{R} \to \mathbb{R},\ \operatorname{supp}\rho \subset [0,1],\ \mathrm{TV}(\rho) + 3|\rho(1)| \leq 1\big\} \subset \overline{\mathrm{sc}}(\mathcal{F}_0).$$

Since $\mathrm{TV}(\psi_{t,h} \circ G^{-1}) \leq \mathrm{TV}(\psi_{t,h})$ it follows by Assumption 1 (ii) that the function class

$$\mathcal{F}_n := \Big\{C_\star V_{t,h}^{-1}\sqrt{h}\ \psi_{t,h} \circ G^{-1} : (t,h) \in B_n,\ G \in \mathcal{G}_{c,C,q}\Big\}$$

is a subset of $\overline{\mathrm{sc}}(\mathcal{F}_0)$ for sufficiently small constant $C_\star$. Combining Theorems 1 and 2 shows for $\widetilde{\delta}_n = n^{-1/2}$ that there are constants C', K', λ' and a Brownian bridge $(B(f))_{f\in\overline{\mathrm{sc}}(\mathcal{F}_0)}$ such that for $x > 0$, the probability of

$$\Big\{\sup_{(t,h)\in B_n,\ G\in\mathcal{G}} C_\star \frac{\sqrt{h}\big|\mathbb{U}_n\big(\psi_{t,h}\circ G^{-1}\big) - B\big(\psi_{t,h}\circ G^{-1}\big)\big|}{V_{t,h}} \geq \frac{1}{\sqrt{n}}(x + C'\log n)\Big\}$$

is bounded by $K'e^{-\lambda' x}$. Due to Lemma B.11 (i) and $l_n \geq \nu/n$ for sufficiently large n, we have that $w_{l_n} \leq w_{\nu/n}$. This readily implies with $x = \log n$ that

$$\sup_{(t,h)\in B_n,\ G\in\mathcal{G}} w_h \frac{\Big|\big|T_{t,h} - \mathbb{E}\,T_{t,h}\big| - \big|B\big(\psi_{t,h}\circ G^{-1}\big)\big|\Big|}{V_{t,h}\ \sqrt{g(t)}} = O_P\Big(\frac{1}{\sqrt{l_n n}} w_{\nu/n}\log n\Big).$$

Now, let us introduce the (general) Brownian motion $W(f)$ as a centered Gaussian process with covariance $\mathbb{E}[W(f)W(g)] = \langle f, g\rangle$. In particular, $W(f) = B(f) + (\int f)\xi$, $\xi \sim \mathcal{N}(0,1)$ and independent of B, defines a Brownian motion and hence there exists a version of $(W(f))_{f\in\overline{\mathrm{sc}}(\mathcal{F}_0)}$ such that $B(f) = W(f) - (\int f)W(1)$. We have

$$\sup_{(t,h)\in B_n,\ G\in\mathcal{G}} w_h \frac{\left|\int \psi_{t,h}(u)\ dG(u)\right|}{V_{t,h}\ \sqrt{g(t)}} \leq c^{-1} \sup_{(t,h)\in B_n,\ G\in\mathcal{G}} w_h \frac{\|\psi_{t,h}\|_1}{V_{t,h}\ \sqrt{g(t)}}$$
$$\lesssim \sup_{h\in[l_n,u_n]} w_h h^{1/2} \leq w_{u_n} u_n^{1/2},$$

where the second inequality follows from Assumption 1 (ii) and the last inequality from Lemma B.11 (ii). This implies further

$$\mathbb{E}\Big[\Big\|\frac{w_h}{V_{t,h}\ \sqrt{g(t)}}\Big[\big|B\big(\psi_{t,h}\circ G^{-1}\big)\big| - \big|W\big(\psi_{t,h}\circ G^{-1}\big)\big|\Big]\Big\|_{\mathcal{F}_n}\Big] = O(w_{u_n}u_n^{1/2}),$$

and therefore

$$\sup_{G\in\mathcal{G}}\Big|T_n^{(1)} - \sup_{(t,h)\in B_n} w_h \frac{\big|W\big(\psi_{t,h}\circ G^{-1}\big)\big|}{V_{t,h}\ \sqrt{g(t)}} - \widetilde{w}_h\Big| = O_P\Big(\frac{w_{1/n}\log n}{\sqrt{l_n n}} + w_{u_n}u_n^{1/2}\Big),$$

and

$$\sup_{G\in\mathcal{G}}\Big|T_n^{(1)} - T_n^{(2)}(W)\Big| = O_P(l_n^{-1/2}n^{-1/2}w_{1/n}\log n + w_{u_n}u_n^{1/2}).$$

In the last equality we used that $(W_t^{(1)})_{t\in[0,1]} = (W(\mathbb{I}_{[0,t]}(\cdot)))_{t\in[0,1]}$ and

$$(W_t)_{t\in\mathbb{R}} = \Big(\int_0^t \frac{\mathbb{I}_{\{g>0\}}(s)}{\sqrt{g(s)}} dW^{(1)}_{G(s)}\Big)_{t\in\mathbb{R}}$$

are (two-sided) standard Brownian motions, proving $W(\psi_{t,h}\circ G^{-1}) = \int \psi_{t,h}(s)\sqrt{g(s)}dW_s$ and hence (1). Further note that Assumption 1 (iii) together with Lemma B.10 shows that

$$\sup_{G\in\mathcal{G}}\Big|T_n^{(2)}(W) - \sup_{(t,h)\in B_n} w_h\frac{\left|\int\psi_{t,h}(s)dW_s\right|}{V_{t,h}} - \widetilde{w}_h\Big| = O_P(\kappa_n).$$

In a final step let us show that (13) is almost surely bounded. In order to establish the result, we use Theorem 6.1 and Remark 1 of Dümbgen and Spokoiny [12]. We set $\rho\big((t,h),(t',h')\big) = (|t-t'| + |h-h'|)^{1/2}$. Further, let $X(t,h) = \sqrt{h}V_{t,h}^{-1}\int\psi_{t,h}(s)dW_s$ and $\sigma(t,h) = h^{1/2}$.

By assumption, X has continuous sample paths on $\mathcal{T}$ and obviously, for all $(t,h),(t',h') \in \mathcal{T}$,

$$\sigma^2(t,h) \leq \sigma^2(t',h') + \rho^2((t,h),(t',h')).$$

Let $Z \sim \mathcal{N}(0,1)$. Since $X(t,h)$ is a Gaussian process and $V_{t,h} \geq \|\phi_{t,h}\|_2$, $\mathbb{P}(X(t,h) > \sigma(t,h)\eta) \leq \mathbb{P}(Z > \eta) \leq \exp(-\eta^2/2)$ for any $\eta > 0$. Further, denote by

$$A_{t,t',h,h'} := \left\| \frac{\psi_{t,h}\sqrt{h}}{V_{t,h}} - \frac{\psi_{t',h'}\sqrt{h}'}{V_{t',h'}} \right\|_2 . \tag{3}$$

Because of $\mathbb{P}(\left|X(t,h) - X(t',h')\right| \geq A_{t,t',h,h'}\eta\Big) \leq 2\exp\big(-\eta^2/2\big)$ we have by Lemma B.6 for a universal constant $K > 0$,

$$\mathbb{P}\Big(\left|X(t,h) - X(t',h')\right| \geq \rho((t,h),(t',h'))\eta\Big) \leq 2\exp\big(-\eta^2/(2K^2)\big).$$

Finally, we can bound the entropy $\mathcal{N}((\delta u)^{1/2}, \{(t,h) \in \mathcal{T} : h \leq \delta\})$ similarly as in [12], p. 145. Therefore, application of Remark 1 in [12] shows that

$$S := \sup_{(t,h)\in\mathcal{T}} \frac{\sqrt{\frac{1}{2}\log\frac{e}{h}}\ \left|\int \psi_{t,h}(s)dW_s\right|}{\log\big(e\log\frac{e}{h}\big)\ V_{t,h}} - \frac{\sqrt{\log(\frac{1}{h})\log(\frac{e}{h})}}{\log\big(e\log\frac{e}{h}\big)}$$

is almost surely bounded from above. Define

$$S' := \sup_{(t,h)\in\mathcal{T}} \frac{\sqrt{\frac{1}{2}\log\frac{\nu}{h}}\ \left|\int \psi_{t,h}(s)dW_s\right|}{\log\log\frac{\nu}{h}\ V_{t,h}} - \frac{\sqrt{\log(\frac{1}{h})\log(\frac{\nu}{h})}}{\log\log\frac{\nu}{h}}.$$

If $e < \nu \leq e^e$, then

$$\log\log\tfrac{\nu}{h} = \log\Big(\tfrac{\log\nu}{e}\log\tfrac{e^e}{h^{e/\log\nu}}\Big) \geq \log\log\nu - 1 + \log\big(e\log\tfrac{e}{h}\big)$$

implies

$$\frac{\log\big(e\log\frac{e}{h}\big)}{\log\log\frac{\nu}{h}} \leq \frac{1}{\log\log\nu} + 1.$$

Furthermore, $\log\nu/h \leq (\log\nu)(\log e/h)$. Suppose now that $S' > 0$ (otherwise S' is bounded from below by 0). Then, $S' \lesssim S$ and hence S' is almost surely bounded. Finally,

$$\sqrt{\log\tfrac{\nu}{h}}\left|\sqrt{\log\tfrac{1}{h}} - \sqrt{\log\tfrac{\nu}{h}}\right| \leq \log\nu.$$

Therefore, (13) holds, i.e.

$$\sup_{(t,h)\in\mathcal{T}} w_h \frac{\left|\int \psi_{t,h}(s)dW_s\right|}{V_{t,h}} - \widetilde{w}_h$$

is almost surely bounded.

In the last step, it remains to prove that $\sup_{G\in\mathcal{G}_{c,C,q}} |T_n - T_n^{(1)}| = O_P(\sup_{G\in\mathcal{G}} \|\widehat{g}_n - g\|_\infty \log n/\log\log n)$. For sufficiently large n and because $G\in\mathcal{G}$, $\widehat{g}_n \geq c/2$ for all $t\in[0,1]$. Therefore using Lemma B.11 (i),

$$\begin{aligned}\sup_{G\in\mathcal{G}} |T_n - T_n^{(1)}| &\leq \sup_{(t,h)\in B_n,\ G\in\mathcal{G}} w_h \frac{\left|T_{t,h} - \mathbb{E}[T_{t,h}]\right|}{V_{t,h}\ \sqrt{g(t)}} \frac{\sup_{G\in\mathcal{G}} \left\|\widehat{g}_n - g\right\|_\infty}{\widehat{g}_n(t)} \\ &\leq \frac{2\sup_{G\in\mathcal{G}} \left\|\widehat{g}_n - g\right\|_\infty}{c} \sup_{(t,h)\in B_n,\ G\in\mathcal{G}} w_h \frac{\left|T_{t,h} - \mathbb{E}[T_{t,h}]\right|}{V_{t,h}\ \sqrt{g(t)}} \\ &\leq \frac{2\sup_{G\in\mathcal{G}} \left\|\widehat{g}_n - g\right\|_\infty}{c} (T_n^{(1)} + \sup_{h\in[l_n,u_n]} \widetilde{w}_h) \\ &\leq \frac{2\sup_{G\in\mathcal{G}} \left\|\widehat{g}_n - g\right\|_\infty}{c} \big(T_n^{(1)} + O(\frac{\log n}{\log\log n})\big). \qquad (4)\end{aligned}$$

Since $T_n^{(1)}$ is a.s. bounded by Theorem 1, the result follows. □

REMARK 1. *Next, we give a proof of Theorem 2. In fact we proof a slightly stronger version, which does not necessarily require the symbol a to be elliptic and* $V_{t,h} = \|v_{t,h}\|_2$. *It is only assumed that*

(i) $V_{t,h} \geq \|v_{t,h}\|_2$,
(ii) there exists constants c_V, C_V *with* $0 < c_V \leq h^{m+r-1/2} V_{t,h} \leq C_V < \infty$
(iii) for all $(t,h),(t',h')\in\mathcal{T}$ *and whenever* $h\leq h'$ *it holds that* $h^{m+r}|V_{t,h} - V_{t',h'}| \leq C_V(|t-t'|+|h-h'|)^{1/2}$.

As a special case these conditions are satisfied for $V_{t,h} = \|v_{t,h}\|_2$ *and* $\mathrm{op}(a)$ *elliptic. This follows directly from Lemmas B.3 and B.5.*

PROOF OF THEOREM 2. In order to prove the statements it is sufficient to check the conditions of Theorem 1. For $h>0$ define the symbol

$$a^\star_{t,h}(x,\xi) := h^{\overline{m}} a^\star(xh + t, h^{-1}\xi). \qquad (5)$$

Under the imposed conditions and by Remark B.1 we may apply Lemma B.2 for $\mathfrak{a}^{(t,h)} = a^\star_{t,h}$ and therefore, uniformly over $(t,h)\in\mathcal{T}$ and $u,u'\in\mathbb{R}$,

(I) $|v_{t,h}(u)| \lesssim h^{-m-r} \min\big(1, \frac{h^2}{(u-t)^2}\big)$.

(II) $|v_{t,h}(u) - v_{t,h}(u')| \lesssim h^{-m-r-1}|u-u'|$ and if $u, u' \neq t$,

$$|v_{t,h}(u) - v_{t,h}(u')| \lesssim h^{1-m-r}\frac{|u-u'|}{|u'-t|\;|u-t|} = h^{1-m-r}\Big|\int_{u'}^{u}\frac{1}{(x-t)^2}dx\Big|.$$

Using (I), we obtain $\|v_{t,h}\|_\infty \lesssim h^{-m-r}$ and $\|v_{t,h}\|_1 \lesssim h^{1-m-r}$. In order to show that the total variation is of the right order, let us decompose $v_{t,h}$ further into $v_{t,h}^{(1)} = v_{t,h}\mathbb{I}_{[t-h,t+h]}$ and $v_{t,h}^{(2)} = v_{t,h} - v_{t,h}^{(1)}$. By (II), $\mathrm{TV}(v_{t,h}^{(1)}) \lesssim h^{-m-r}$ and

$$\mathrm{TV}(v_{t,h}^{(2)}) \lesssim h^{-m-r} + h^{1-m-r}\int_{t+h}^{\infty}\frac{1}{(x-t)^2}dx \lesssim h^{-m-r}.$$

Since $\mathrm{TV}(v_{t,h}) \leq \mathrm{TV}(v_{t,h}^{(1)}) + \mathrm{TV}(v_{t,h}^{(2)}) \lesssim h^{-m-r}$, this shows together with Remark 1 that part (ii) of Assumption 1 is satisfied.

Next, we verify Assumption 1, (iii) with $\kappa_n = \sup_{(t,h)\in B_n} w_h h^{1/2}\log(1/h) \lesssim u_n^{1/2}\log^{3/2} n$ (cf. Lemma B.11, (ii)), i.e. we show

$$\sup_{(t,h)\in B_n,\ G\in\mathcal{G}} w_h \frac{\mathrm{TV}\left(v_{t,h}(\cdot)[\sqrt{g(\cdot)} - \sqrt{g(t)}]\langle\cdot\rangle^\alpha\right)}{V_{t,h}} \lesssim u_n^{1/2}\log^{3/2} n.$$

By Lemma B.12, we see that this holds for $v_{t,h}$ replaced by $v_{t,h}^{(1)}$. Therefore, it remains to prove the statement for $v_{t,h}^{(2)}$. Let us decompose $v_{t,h}^{(2)}$ further into $v_{t,h}^{(2,1)} = v_{t,h}\mathbb{I}_{[t-1,t+1]\cap[t-h,t+h]^c}$ and $v_{t,h}^{(2,2)} = v_{t,h}^{(2)} - v_{t,h}^{(2,1)} = v_{t,h}\mathbb{I}_{[t-1,t+1]^c}$. For the remaining part, let u, u' be such that $|u-t| \geq |u'-t| \geq h$. We have

$$\begin{aligned}\mathrm{TV}\left(v_{t,h}^{(2,1)}(\cdot)\big[\sqrt{g(\cdot)} - \sqrt{g(t)}\big]\langle\cdot\rangle^\alpha\right) &\lesssim \big\|v_{t,h}^{(2,1)}(\cdot)\big[\sqrt{g(\cdot)} - \sqrt{g(t)}\big]\big\|_\infty \\ &\quad + \mathrm{TV}\left(v_{t,h}^{(2,1)}(\cdot)\big[\sqrt{g(\cdot)} - \sqrt{g(t)}\big]\right).\end{aligned} \tag{6}$$

Using (I) and (II) together with the properties of the class $\mathcal{G}$ we can bound the variation $\big|v_{t,h}^{(2,1)}(u)\big[\sqrt{g(u)} - \sqrt{g(t)}\big] - v_{t,h}^{(2,1)}(u')\big[\sqrt{g(u')} - \sqrt{g(t)}\big]\big|$ by

$$\begin{aligned}&\big|v_{t,h}^{(2,1)}(u) - v_{t,h}^{(2,1)}(u')\big| \cdot \big|\sqrt{g(u')} - \sqrt{g(t)}\big| + \big|v_{t,h}^{(2,1)}(u)\big| \cdot \big|\sqrt{g(u)} - \sqrt{g(u')}\big| \\ &\lesssim h^{1-m-r}\tfrac{|u-u'|}{|u-t|} + h^{2-m-r}\tfrac{|u-u'|}{|u-t|^2} \lesssim h^{1-m-r}\tfrac{|u-u'|}{|u-t|} \leq h^{1-m-r}\Big|\int_{u'}^{u}\tfrac{1}{|x-t|}dx\Big|.\end{aligned}$$

Due to $h \geq l_n \gtrsim 1/n$ this yields

$$\begin{aligned}\mathrm{TV}\left(v_{t,h}^{(2,1)}(\cdot)[\sqrt{g(\cdot)} - \sqrt{g(t)}]\right) &\lesssim h^{1-m-r} + h^{1-m-r}\int_{t+h}^{t+1}\frac{du}{|u-t|} \\ &\lesssim h^{1-m-r}\log\tfrac{1}{h} \lesssim h^{1-m-r}\log n\end{aligned}$$

and with (6) also

$$\mathrm{TV}\left(v_{t,h}^{(2,1)}(\cdot)\big[\sqrt{g(\cdot)}-\sqrt{g(t)}\big]\langle\cdot\rangle^{\alpha}\right)\lesssim h^{1-m-r}\log n. \tag{7}$$

Finally, let us address the total variation term involving $v_{t,h}^{(2,2)}$. Given $\mathcal{G}_{c,C,q}$ we can choose α such that $\alpha>1/2$ and $\alpha+q<1$ (recall that $0\le q<1/2$). By Lemma B.7, we find that

$$\big|v_{t,h}^{(2,2)}(u)\langle u\rangle^{\alpha}-v_{t,h}^{(2,2)}(u')\langle u'\rangle^{\alpha}\big|\lesssim h^{1-m-r}\Big|\int_{u'}^{u}\frac{1}{(x-t)^{2-\alpha}}+\frac{1}{(x-t)^{2}}dx\Big|.$$

Moreover

$$\begin{aligned}\langle u\rangle^{\alpha}(1+|u'|+|u|)^{q}&\le(1+|u'|+|u|)^{q+\alpha}\\&\le(3+2|u-t|)^{q+\alpha}\le 3+2|u-t|^{q+\alpha}\end{aligned}$$

and thus

$$\big|v_{t,h}^{(2,2)}(u)\langle u\rangle^{\alpha}\big|\;\big|\sqrt{g(u)}-\sqrt{g(u')}\big|\lesssim h^{2-m-r}\frac{|u-t|^{q+\alpha}+1}{|u-t|^{2}}|u-u'|.$$

This allows us to bound the variation by

$$\begin{aligned}&\big|v_{t,h}^{(2,2)}(u)\big[\sqrt{g(u)}-\sqrt{g(t)}\big]\langle u\rangle^{\alpha}-v_{t,h}^{(2,2)}(u')\big[\sqrt{g(u')}-\sqrt{g(t)}\big]\langle u'\rangle^{\alpha}\big|\\&\le\big|v_{t,h}^{(2,2)}(u)\langle u\rangle^{\alpha}\big|\;\big|\sqrt{g(u)}-\sqrt{g(u')}\big|+\frac{2}{\sqrt{c}}\big|v_{t,h}^{(2,2)}(u)\langle u\rangle^{\alpha}-v_{t,h}^{(2,2)}(u')\langle u'\rangle^{\alpha}\big|\\&\lesssim h^{1-m-r}\Big|\int_{u'}^{u}\frac{1}{(x-t)^{2-q-\alpha}}+\frac{1}{(x-t)^{2-\alpha}}+\frac{1}{(x-t)^{2}}\;dx\;\Big|\end{aligned}$$

and therefore we conclude that

$$\begin{aligned}&\mathrm{TV}\left(v_{t,h}^{(2,2)}(\cdot)\big[\sqrt{g(\cdot)}-\sqrt{g(t)}\big]\langle\cdot\rangle^{\alpha}\right)\\&\lesssim h^{1-m-r}+h^{1-m-r}\int_{t+1}^{\infty}\frac{1}{(x-t)^{2-q-\alpha}}+\frac{1}{(x-t)^{2-\alpha}}+\frac{1}{(x-t)^{2}}dx\\&\le h^{1-m-r}.\end{aligned}$$

Together with the bound for $v_{t,h}^{(1)}$ and (7) this yields Assumption 1, (iii).

Finally, Assumption 1 (iv) follows from Lemma B.5 and Remark 1 due to $\phi\in H^{\lceil r+m\rceil}\cap H^{r+m+1/2}$, $\operatorname{supp}\phi\subset[0,1]$ and $\phi\in\mathrm{TV}(D^{\lceil r+m\rceil}\phi)<\infty$. This shows that Assumption 1 holds for $(v_{t,h},V_{t,h})$.

In the next step, we verify that $(t,h)\mapsto X(t,h)=\sqrt{h}V_{t,h}^{-1}\int v_{t,h}(s)dW_s$ has continuous sample paths. Note that in view of Lemma B.10, it is sufficient to show that there is an α with $1/2<\alpha<1$ such that

$$\mathrm{TV}\left(\left(\sqrt{h}V_{t,h}^{-1}v_{t,h}-\sqrt{h'}V_{t',h'}^{-1}v_{t',h'}\right)\langle\cdot\rangle^{\alpha}\right)\to 0,$$

whenever $(t', h') \to (t, h)$ on the space $\mathcal{T}$. Since Assumption 1 (iv) holds, we have

$$\big|\sqrt{h} V_{t,h}^{-1} - \sqrt{h'} V_{t',h'}^{-1}\big| \le \frac{\sqrt{|h-h'|}}{V_{t,h}} + V_{t,h}^{-1} \frac{\sqrt{h'}|V_{t',h'} - V_{t,h}|}{V_{t',h'}} \to 0,$$

for $(t', h') \to (t, h)$. By Lemma B.7, $\mathrm{TV}(v_{t,h}(\cdot)\langle\cdot\rangle^{\alpha}) < \infty$. Therefore, it is sufficient to show that

$$\mathrm{TV}\big((v_{t,h} - v_{t',h'})\langle\cdot\rangle^{\alpha}\big) \to 0, \quad \text{whenever } (t', h') \to (t, h). \tag{8}$$

Using (18), we obtain

$$\begin{aligned}
&(K_{t,h}^{\gamma,\overline{m}} a_{t,h}^{\star})(u) \\
&= v_{t,h} - v_{t',h'} \\
&= h^{-\overline{m}} \int \lambda_{\gamma}^{\mu}\big(\tfrac{s}{h}\big) \mathcal{F}\big(\operatorname{Op}(a_{t,h}^{\star})(\phi - \phi \circ S_{t',h'} \circ S_{t,h}^{-1})\big)(s) e^{is(u-t)/h} \,đs
\end{aligned}$$

and by Remark B.1, we can apply Lemma B.2 again (here ϕ should be replaced by $\phi - \phi \circ S_{t',h'} \circ S_{t,h}^{-1}$). In order to verify (8), observe that by Lemma B.7 it is enough to show $\|\phi - \phi \circ S_{t',h'} \circ S_{t,h}^{-1}\|_{H_4^{\overline{q}}} \to 0$ for some $\overline{q} > r + m + 3/2$ whenever $(t', h') \to (t, h)$ in $\mathcal{T}$. Note that

$$\begin{aligned}
&\big\|\phi - \phi \circ S_{t',h'} \circ S_{t,h}^{-1}\big\|_{H_4^{\overline{q}}}^2 \\
&= \tfrac{1}{h} \sum_{j=0}^{4} \int \langle s\rangle^{2\overline{q}} \Big|\mathcal{F}\big((x^j\phi) \circ S_{t,h}\big)(s) - \mathcal{F}\big((S_{t,h}(\cdot))^j(\phi \circ S_{t',h'})\big)(s)\Big|^2 ds \\
&\le \frac{2}{h} \sum_{j=0}^{4} \big\|(x^j\phi) \circ S_{t,h} - (x^j\phi) \circ S_{t',h'}\big\|_{H^{\overline{q}}}^2 \\
&\quad + \int \langle s\rangle^{2\overline{q}} \Big|\mathcal{F}\big(\big[(S_{t',h'}(\cdot))^j - (S_{t,h}(\cdot))^j\big](\phi \circ S_{t',h'})\big)(s)\Big|^2 ds
\end{aligned} \tag{9}$$

with $(S_{t,h}(\cdot))^j := \big(\frac{\cdot - t}{h}\big)^j$. For real numbers a, b we have the identity $a^j - b^j = \sum_{\ell=1}^{k} \binom{k}{\ell} b^{k-\ell}(a-b)^{\ell}$. Moreover, we can apply Lemma B.5 for $\overline{q}$ with $m + r + 3/2 < \overline{q} < \lfloor r + m + 5/2 \rfloor$ (and such a $\overline{q}$ clearly exists). Thus, with $a = S_{t,h}(\cdot),\ b = S_{t',h'}(\cdot)$ and $S_{t,h} - S_{t',h'} = (h/h' - 1)S_{t',h'} - (t' - t)/h$ the r.h.s. of (9) converges to zero if $(t', h') \to (t, h)$.

□

Proof of Theorem 3. By assumption, $p_R(x, \xi) = a_R(x, \xi)|\xi|^{\gamma_1}\iota_{\xi}^{\mu_1}$ with $a_R \in S^{\overline{m}_1}$ and $\overline{m}_1 + \gamma_1 = m'$. Recall that $p_P(x, \xi) = a_P(x)|\xi|^m\iota_{\xi}^{\mu}$. Since a_P

is real-valued, $\mathrm{Op}(a_P)$ is self-adjoint. Taking the adjoint is a linear operator and therefore arguing as in (18) yields

$$\begin{aligned}\mathcal{F}\big(\operatorname{op}(p)^\star(\phi\circ S_{t,h})\big)(s) =&|s|^m\iota_s^{-\mu}\mathcal{F}\big(a_P(\phi\circ S_{t,h})\big)(s)\\ &+|s|^{\gamma_1}\iota_s^{-\mu_1}\mathcal{F}\big(\operatorname{Op}(a_R^\star)(\phi\circ S_{t,h})\big)(s).\end{aligned}$$

Decompose $v_{t,h}=v_{t,h}^{(1)}+v_{t,h}^{(2)}$ with

$$\begin{aligned}v_{t,h}^{(1)}(u)&:=\int\lambda_m^\mu(s)\mathcal{F}\big(a_P(\phi\circ S_{t,h})\big)(s)e^{isu}đs\\ &=\int\lambda_m^\mu\big(\tfrac{s}{h}\big)\mathcal{F}\big(a_P(\cdot h+t)\phi\big)(s)e^{is(u-t)/h}đs\\ v_{t,h}^{(2)}(u)&:=\int\lambda_{\gamma_1}^{\mu_1}(s)\mathcal{F}\big(\operatorname{Op}(a_R^\star)(\phi\circ S_{t,h})\big)(s)e^{isu}đs\\ &=h^{-\overline{m}_1}\int\lambda_{\gamma_1}^{\mu_1}\big(\tfrac{s}{h}\big)\mathcal{F}\big(\operatorname{Op}(a_{t,h}^{(1)})\phi\big)(s)e^{is(u-t)/h}đs\end{aligned}$$

using similar arguments as in (18) and $a_{t,h}^{(1)}(x,\xi):=h^{\overline{m}_1}a_R^\star(xh+t,h^{-1}\xi)$. For $j=1,2$ we denote by $T_{t,h}^{(j)}$ and $T_n^{P,(j)}$ the statistics $T_{t,h}$ and T_n^P with $v_{t,h}$ replaced by $v_{t,h}^{(j)}$, $j=1,2$, respectively. Recall the definitions of σ and τ and set

$$\begin{aligned}v_{t,h}^P(u)&:=Aa_P(t)\int|s|^{r+m}\iota_s^{-\rho-\mu}\mathcal{F}(\phi\circ S_{t,h})(s)e^{isu}đs\\ &=Ah^{-r-m}a_P(t)\int|s|^{r+m}\iota_s^{-\rho-\mu}\mathcal{F}(\phi)(s)e^{is(u-t)/h}đs\\ &=Aa_P(t)D_+^\sigma D_-^\tau\phi\big(\tfrac{u-t}{h}\big).\end{aligned}\tag{10}$$

Further let

$$V_{t,h}^P:=\|v_{t,h}^P\|_2=|Aa_P(t)|\big\|D_+^{r+m}\phi((\cdot-t)/h)\big\|_2=h^{1/2-r-m}|Aa_P(t)|\big\|D_+^{r+m}\phi\big\|_2,$$

and

$$T_n^{P,(1),\infty}(W):=\sup_{(t,h)\in B_n}w_h\left(\frac{\big|\int\operatorname{Re}v_{t,h}^{(1)}(s)dW_s\big|}{V_{t,h}^P}-\sqrt{2\log\tfrac{\nu}{h}}\right).$$

Note that for the approximation of T_n^P we can write

$$T_n^{P,\infty}(W)=\sup_{(t,h)\in B_n}w_h\left(\frac{\big|\int\operatorname{Re}v_{t,h}^{P}(s)dW_s\big|}{V_{t,h}^P}-\sqrt{2\log\tfrac{\nu}{h}}\right).$$

Since $|T_n^P - T_n^{P,\infty}(W)| \leq |T_n^P - T_n^{P,(1)}| + |T_n^{P,(1)} - T_n^{P,(1),\infty}(W)| + |T_n^{P,(1),\infty}(W) - T_n^{P,\infty}(W)|$ it is sufficient to show that there exists a Brownian motion W such that the terms on the right hand side converge to zero in probability. This will be done separately, and proofs for the single terms are denoted by $(I), (II)$ and (III). From (II) and (III) we will be able to conclude the boundedness of the approximating statistic.

(I): It is easy to see that for a constant K, $\|v_{t,h}^{(2)}\|_2 \leq Kh^{1/2-m'-r} =: V_{t,h}^R$. By Remark 1 and

$$\left|T_n^P - T_n^{P,(1)}\right| \leq \sup_{h\in[l_n,u_n]} \frac{V_{t,h}^R}{V_{t,h}^P}\Big(\sup_{(t,h)\in B_n} w_h\Big(\frac{|T_{t,h}^{(2)} - \mathbb{E}T_{t,h}^{(2)}|}{\sqrt{\widehat{g}_n(t)}\ V_{t,h}^R} - \sqrt{2\log\left(\tfrac{\nu}{h}\right)}\Big) + \sup_{h\in[l_n,u_n]} w_h\sqrt{2\log\left(\tfrac{\nu}{h}\right)}\Big),$$

we can apply Theorem 2 where m should be replaced by m', of course. Because of $u_n^{m-m'}\log n \to 0$, (I) is proved.

(II): We show that there is a Brownian motion W such that $|T_n^{P,(1)} - T_n^{P,(1),\infty}(W)| \leq |T_n^{P,(1)} - \widetilde{T}_n^{(1)}| + |\widetilde{T}_n^{(1)} - \widetilde{T}_n^{(1),\infty}(W)| + |\widetilde{T}_n^{(1),\infty}(W) - T_n^{P,(1),\infty}(W)| = o_P(1)$ with

$$\widetilde{T}_n^{(1)} := \sup_{(t,h)\in B_n} w_h\Big(\frac{\left|T_{t,h}^{(1)} - \mathbb{E}T_{t,h}^{(1)}\right|}{\sqrt{\widehat{g}_n(t)}\ \|v_{t,h}^{(1)}\|_2} - \sqrt{2\log\left(\tfrac{\nu}{h}\right)}\Big)$$

and

$$\widetilde{T}_n^{(1),\infty}(W) := \sup_{(t,h)\in B_n} w_h\Big(\frac{\left|\int \operatorname{Re} v_{t,h}^{(1)}(s)dW_s\right|}{\sqrt{\widehat{g}_n(t)}\ \|v_{t,h}^{(1)}\|_2} - \sqrt{2\log\left(\tfrac{\nu}{h}\right)}\Big).$$

Since by Assumption 4, $a_p \in S^0$ is elliptic and $p_P \in \underline{S}^m$, we find that $|\widetilde{T}_n^{(1)} - \widetilde{T}_n^{(1),\infty}(W)| = o_P(1)$ and

$$\widetilde{T}_n^{(1),\infty}(W) \leq \sup_{(t,h)\in\mathcal{T}} w_h\Big(\frac{\left|\int \operatorname{Re} v_{t,h}^{(1)}(s)dW_s\right|}{\sqrt{\widehat{g}_n(t)}\ \|v_{t,h}^{(1)}\|_2} - \sqrt{2\log\left(\tfrac{\nu}{h}\right)}\Big) < \infty \quad \text{a.s.} \tag{11}$$

by applying Theorem 2. Moreover, similar as in (4) and using $w_h\sqrt{2\log\left(\frac{\nu}{h}\right)} \geq 1$,

$$\sup_{G\in\mathcal{G}}\left|T_n^{P,(1)} - \widetilde{T}_n^{(1)}\right| \leq \sup_{(t,h)\in B_n} w_h\sqrt{2\log\left(\tfrac{\nu}{h}\right)}\frac{\left|V_{t,h}^P - \|v_{t,h}^{(1)}\|_2\right|}{V_{t,h}^P}\Big(1 + \sup_{G\in\mathcal{G}}\widetilde{T}_n^{(1)}\Big)$$

and

$$\begin{aligned}&\big|\widetilde{T}_n^{(1),\infty}(W) - T_n^{P,(1),\infty}(W)\big|\\ &\leq \sup_{(t,h)\in B_n} w_h\sqrt{2\log\left(\tfrac{\nu}{h}\right)}\frac{\big|V_{t,h}^P - \|v_{t,h}^{(1)}\|_2\big|}{V_{t,h}^P}\Big(1+\widetilde{T}_n^{(1),\infty}(W)\Big).\end{aligned}$$

To finish the proof for (II) it remains to verify

$$\sup_{(t,h)\in B_n} w_h\sqrt{2\log\left(\tfrac{\nu}{h}\right)}\frac{\|v_{t,h}^P - v_{t,h}^{(1)}\|_2}{V_{t,h}^P} = o(1), \tag{12}$$

which will be done below.

(III): By Lemma B.10, we obtain $|T_n^{P,(1),\infty} - T_n^{P,\infty}| = o_P(1)$ if for some $\alpha > 1/2$,

$$\sup_{(t,h)\in B_n} w_h \frac{\mathrm{TV}\left((v_{t,h}^P - v_{t,h}^{(1)})\langle\cdot\rangle^\alpha\right)}{V_{t,h}^P} = o(1). \tag{13}$$

Let χ be a cut function, i.e. $\chi\in\mathcal{S}$ (the Schwartz space), $\chi(x)=1$ for $x\in[-1,1]$ and $\chi(x)=0$ for $x\in(-\infty,-2]\cup[2,\infty)$ and define $p_{t,h}^{(1)}(x,\xi) = h^{-1}\chi(x)(a_P(xh+t)-a_P(t))$ and $p_{t,h}^{(2)}(x,\xi) = (xh)^{-1}(1-\chi(x))(a_P(xh+t)-a_P(t))$. Then, $p_{t,h}^{(1)}, p_{t,h}^{(2)} \in S^0$ and $\big(a_P(\cdot h+t)-a_P(t)\big)\phi = h\,\mathrm{Op}(p_{t,h}^{(1)})\phi + h\,\mathrm{Op}(p_{t,h}^{(2)})(x\phi)$. Define the function

$$d_{t,h} := \int e^{is(\cdot - t)/h}\Big(\frac{1}{\mathcal{F}(f_\epsilon)(-\frac{s}{h})} - A\iota_s^{-\rho}\big|\tfrac{s}{h}\big|^r\Big)\iota_s^{-\mu}|s|^m\mathcal{F}(\phi)(s)\,đs \tag{14}$$

and note that

$$\|d_{t,h}\|_2^2 \lesssim h^{1+2m}\int \big\langle\tfrac{s}{h}\big\rangle^{2r+2m-2\beta_0}\big|\mathcal{F}(\phi)(s)\big|^2 ds \lesssim h^{1+2\beta_0^\star - 2r}\|\phi\|_{H^{r+m}}^2$$

with $\beta_0^\star := \beta_0\wedge(m+r)$. Using (17), we have now the decomposition

$$v_{t,h}^{(1)} - v_{t,h}^P = hK_{t,h}^{m,0}p_{t,h}^{(1)} + hK_{t,h}^{m,0}p_{t,h}^{(2)} + a_P(t)h^{-m}d_{t,h}, \tag{15}$$

where ϕ needs to be replaced by $x\phi$ in the second term of the right hand side. By assumption there exists $q > m+r+3/2$ such that $\phi\in H_5^q$. Since the assumptions on $p_{t,h}^{(1)}$ and $p_{t,h}^{(2)}$ of Lemma B.2 can be easily verified, we may apply Lemma B.2 to the first two terms on the right hand side of (15). This yields together with Lemmas B.7, B.8, and B.9, uniformly over $(t,h) \in \mathcal{T}$,

$$\begin{aligned}
&\mathrm{TV}\left((v_{t,h}^{P} - v_{t,h}^{(1)})\langle\cdot\rangle^{\alpha}\right) \\
&\leq \mathrm{TV}\left(\left(hK_{t,h}^{m,0}p_{t,h}^{(1)} + hK_{t,h}^{m,0}p_{t,h}^{(2)} + a_P(t)h^{-m}d_{t,h}\right)\langle\cdot\rangle^{\alpha}\mathbb{I}_{[t-1,t+1]}\right) \\
&\quad + \mathrm{TV}\left(v_{t,h}^{P}\langle\cdot\rangle^{\alpha}\mathbb{I}_{\mathbb{R}\setminus[t-1,t+1]}\right) + \mathrm{TV}\left(v_{t,h}^{(1)}\langle\cdot\rangle^{\alpha}\mathbb{I}_{\mathbb{R}\setminus[t-1,t+1]}\right) \\
&\lesssim h^{1-m-r} + h^{\beta_0^\star - m - r} + h^{1-r-m}.
\end{aligned}$$

Since $m+r > 1/2$ this implies (13). From the decomposition (15) we obtain further $\|v_{t,h}^{P} - v_{t,h}^{(1)}\|_2 \lesssim h^{3/2-m-r} + h^{1/2+\beta_0^\star-m-r}$ and this shows (12). Thus, the first part of the theorem is proved.

Finally with Lemma B.10 it is easy to check that (11) implies that (27) is bounded since (12) and (13) also hold with B_n and $o(1)$ replaced by $\mathcal{T}$ and $O(1)$, respectively. □

Appendix B Technical results for the proofs of the main theorems. We have the following uniform and continuous embedding of Sobolev spaces.

LEMMA B.1. *Let $\mathcal{P} \subset S^m$ be a symbol class of pseudo-differential operators. Suppose further that for $\alpha \in \{0,1\}$, $k \in \mathbb{N}$ and finite constants C_k, depending on k only,*

$$\sup_{p\in\mathcal{P}} |\partial_x^k \partial_\xi^\alpha p(x,\xi)| \leq C_k(1+|\xi|)^m, \quad \forall x,\xi \in \mathbb{R}.$$

Then, for any $s \in \mathbb{R}$, there exists a finite constant C, depending only on s, m and $\max_{k\leq 1+2|s|+2|m|} C_k$, such that

$$\sup_{p\in\mathcal{P}} \|\operatorname{Op}(p)\phi\|_{H^{s-m}} \leq C\|\phi\|_{H^s}, \quad \textit{for all} \ \ \phi \in H^s.$$

PROOF. This proof requires some subtle technicalities, appearing in the theory of pseudo-differential operators. By Theorem 2 in Hwang [42], there exists a universal constant C_1, such that for any symbol $a \in S^0$,

$$\|\operatorname{Op}(a)u\|_2 \leq C_1 \max_{\alpha,\beta\in\{0,1\}} \left\|\partial_x^\beta \partial_\xi^\alpha a(x,\xi)\right\|_{L^\infty(\mathbb{R}^2)} \|u\|_2, \quad \text{for all } u \in L^2. \tag{16}$$

For $r \in \mathbb{R}$ denote by $\operatorname{Op}(\langle\xi\rangle^r)$ the pseudo-differential operator with symbol $(x,\xi) \mapsto \langle\xi\rangle^r$. It is well-known that this symbol is in S^r. Throughout the remaining proof let

$$C = C(s, m, \max_{k\leq 1+2|s|+2|m|} C_k)$$

denote a finite but unspecified constant which may even change from line to line. In order to prove the result it is sufficient to show that

$$\sup_{p\in\mathcal{P}} \big\| \operatorname{Op}(\langle\xi\rangle^{s-m}) \circ \operatorname{Op}(p) \circ \operatorname{Op}(\langle\xi\rangle^{-s})\psi\big\|_2 \leq C\|\psi\|_2, \quad \text{for all } \psi \in L^2$$

(set $\phi = \langle D\rangle^{-s}\psi$). The composition of two operators with symbols in S^{m_1} and S^{m_2}, respectively, is again a pseudo-differential operator and its symbol is in $S^{m_1+m_2}$. Therefore, the operator $A : \mathcal{P} \to S^0$, mapping $p \in \mathcal{P}$ to the symbol of $\operatorname{Op}(\langle\xi\rangle^{s-m}) \circ \operatorname{Op}(p) \circ \operatorname{Op}(\langle\xi\rangle^{-s})$ (which is in S^0), is well-defined. With (16) the lemma is proved, once we have established that

$$\sup_{p\in\mathcal{P}} \max_{\alpha,\beta\in\{0,1\}} \big\|\partial_x^\beta \partial_\xi^\alpha Ap(x,\xi)\big\|_{L^\infty(\mathbb{R}^2)} \leq C < \infty.$$

It is not difficult to see that $\operatorname{Op}(p) \circ \operatorname{Op}(\langle\xi\rangle^{-s}) = \operatorname{Op}(p\langle\xi\rangle^{-s})$. By Theorem 4.1 in [40], $Ap = \langle\xi\rangle^{s-m}\#(p\langle\xi\rangle^{-s})$, where $\#$ denotes the Leibniz product, i.e. for $p^{(1)} \in S^{m_1}$ and $p^{(2)} \in S^{m_2}$, $p^{(1)}\#p^{(2)}$ can be written as an oscillatory integral (cf. [40, 47]), that is

$$\begin{aligned}
\big(p^{(1)}\#p^{(2)}\big)(x,\xi) &:= \operatorname{Os} - \int\int e^{-iy\eta} p^{(1)}(x,\xi+\eta)p^{(2)}(x+y,\xi)dy\,d\!\!\bar{}\,\eta \\
&:= \lim_{\epsilon\to 0} \int\int \chi(\epsilon y, \epsilon\eta) e^{-iy\eta} p^{(1)}(x,\xi+\eta)p^{(2)}(x+y,\xi)dy\,d\!\!\bar{}\,\eta,
\end{aligned}$$

for any χ in the Schwartz space of rapidly decreasing functions on $\mathbb{R}^2$ with $\chi(0,0) = 1$. Further for $a \in S^m$ and arbitrary $l \in \mathbb{N}$, $2l > 1 + m$,

$$\begin{aligned}
&\operatorname{Os} - \int\int e^{-iy\eta} a(y,\eta)dy\,d\!\!\bar{}\,\eta \\
&= \int\int e^{-iy\eta} \langle y\rangle^{-2}(1-\partial_\eta^2)\big[\langle\eta\rangle^{-2l}(1-\partial_y^2)^l a(y,\eta)\big]dy\,d\!\!\bar{}\,\eta
\end{aligned}$$

and the integrand on the r.h.s. is in L^1 (cf. [47], p.235). This can be also used to show that differentiation and integration commute for oscillatory integrals,

$$\partial_x^\alpha \partial_\xi^\beta \operatorname{Os} - \int\int e^{-iy\eta} a(x,y,\xi,\eta)dy\,d\!\!\bar{}\,\eta = \operatorname{Os} - \int\int e^{-iy\eta} \partial_x^\alpha \partial_\xi^\beta a(x,y,\xi,\eta)dy\,d\!\!\bar{}\,\eta.$$

Using Peetre's inequality, i.e. $\langle\xi+\eta\rangle^s \leq 2^{|s|}\langle\xi\rangle^{|s|}\langle\eta\rangle^s$, we see that for $\alpha,\beta \in \{0,1\}$, $p \in \mathcal{P}$, and (x,ξ) fixed, the function $(y,\eta) \mapsto \partial_x^\beta \partial_\xi^\alpha \langle\xi+\eta\rangle^{s-m} p(x+y,\xi)\langle\xi\rangle^{-s}$ defines a symbol in S^{s-m}. Hence, for $\ell \in \mathbb{N}$, $1 < 2\ell - |s-m| \leq 2$, $\alpha,\beta \in \{0,1\}$, $p \in \mathcal{P}$, we can rewrite $\partial_x^\beta \partial_\xi^\alpha Ap(x,\xi)$ as

$$\int\int e^{-iy\eta}\langle y\rangle^{-2}(1-\partial_\eta^2)\big[\langle\eta\rangle^{-2\ell}(1-\partial_y^2)^\ell \partial_x^\beta \partial_\xi^\alpha \langle\xi+\eta\rangle^{s-m} p(x+y,\xi)\langle\xi\rangle^{-s}\big]dy\,d\!\!\bar{}\,\eta.$$

With the imposed uniform bound on $\partial_x^k \partial_\xi^\alpha p(x,\xi)$ we obtain, treating the cases $\alpha = 0$ and $\alpha = 1$ separately,

$$
\begin{aligned}
&\sup_{p\in\mathcal{P}} \left|\partial_x^\beta \partial_\xi^\alpha Ap(x,\xi)\right| \\
&\leq C\langle\xi\rangle^{m-s}\Big[\int \left|(1-\partial_\eta^2)\langle\eta\rangle^{-2\ell}\langle\xi+\eta\rangle^{s-m}\right|d\eta \\
&\qquad\qquad + \int \left|(1-\partial_\eta^2)\langle\eta\rangle^{-2\ell}\partial_\xi\langle\xi+\eta\rangle^{s-m}\right|d\eta\Big] \\
&\leq C + C\langle\xi\rangle^{m-s}\Big[\int \left|\partial_\eta^2\langle\eta\rangle^{-2\ell}\langle\xi+\eta\rangle^{s-m}\right|d\eta + \int \left|\partial_\eta^2\langle\eta\rangle^{-2\ell}\partial_\xi\langle\xi+\eta\rangle^{s-m}\right|d\eta\Big]
\end{aligned}
$$

using Peetre's inequality again and $2\ell > 1+|s-m|$ for the second estimate. Since $\langle\xi\rangle^q \in S^q$ for $q\in\mathbb{R}$, it follows that $|\partial_\xi^\alpha\langle\xi\rangle^q| \lesssim \langle\xi\rangle^{q-\alpha}$, and since $\langle.\rangle \geq 1$,

$$
\partial_\eta^2\langle\eta\rangle^{-2\ell}\langle\xi+\eta\rangle^{s-m} \lesssim \sum_{k=0}^{2}\langle\eta\rangle^{-2\ell-k}\langle\xi+\eta\rangle^{s-m-2+k} \lesssim \langle\eta\rangle^{-2\ell}\langle\xi+\eta\rangle^{s-m}.
$$

Similar for the second term. Application of Peetre's inequality as above completes the proof. $\square$

Note that for bounded intervals $[a,b]$, partial integration holds $\int_a^b f'g = fg|_a^b - \int_a^b fg'$ whenever f and g are absolute continuous on $[a,b]$. As a direct consequence, we have $\int_{\mathbb{R}} f'g = -\int_{\mathbb{R}} fg'$ if f' and g' exist and $fg, f'g, fg' \in L^1$.

In order to formulate the key estimate for proving Theorems 2 and 3, let us introduce for fixed ϕ a generic symbol $\mathfrak{a}^{(t,h)} \in S^{\overline{m}}$ and $\lambda = \lambda_\gamma^\mu$ as in (21):

$$
(K_{t,h}^{\gamma,\overline{m}}\mathfrak{a}^{(t,h)})(u) = h^{-\overline{m}}\int \lambda\big(\tfrac{s}{h}\big)\mathcal{F}\big(\operatorname{Op}(\mathfrak{a}^{(t,h)})\phi\big)(s)e^{is(u-t)/h}\,đs. \tag{17}
$$

From the context it will be always clear which ϕ the operator $K_{t,h}^{\gamma,\overline{m}}\mathfrak{a}^{(t,h)}$ refers to. To simplify the expressions we do not indicate the dependence on ϕ and f_ϵ explicitly.

Remark B.1. *Recall* (5) *and note that if $a \in S^{\overline{m}}$ then also $a_{t,h}^\star \in S^{\overline{m}}$. Due to*

$$
\big(\operatorname{Op}(a_{t,h}^\star)\phi\big)\circ S_{t,h} = h^{-\overline{m}}\operatorname{Op}(a^\star)(\phi\circ S_{t,h})
$$

we obtain for $v_{t,h}$ in (20) the representation,

$$
v_{t,h}(u) = h^{-\overline{m}}\int \lambda_\gamma^\mu\big(\tfrac{s}{h}\big)\mathcal{F}\big(\operatorname{Op}(a_{t,h}^\star)\phi\big)(s)e^{is(u-t)/h}\,đs = \big(K_{t,h}^{\gamma,\overline{m}}a_{t,h}^\star\big)(u). \tag{18}
$$

LEMMA B.2. *For $\mathfrak{a}^{(t,h)} \in S^{\overline{m}}$ and $\gamma + \overline{m} = m$ let $K_{t,h}^{\gamma,\overline{m}}\mathfrak{a}^{(t,h)}$ be as defined in (17). Work under Assumption 2 and suppose that*

(i) $\phi \in H_4^q$ *with* $q > m + r + 3/2$,
(ii) $\gamma \in \{0\} \cup [1, \infty)$, *and*
(iii) for $k \in \mathbb{N}$, $\alpha \in \{0, 1, \ldots, 5\}$, *there exist finite constants* C_k *such that*

$$\sup_{(t,h)\in\mathcal{T}} \left|\partial_x^k \partial_\xi^\alpha \mathfrak{a}^{(t,h)}(x,\xi)\right| \le C_k(1+|\xi|)^{\overline{m}}, \quad \text{for all } x, \xi \in \mathbb{R}.$$

Then, there exists a constant $C = C(q, r, \gamma, \overline{m}, C_l, C_u, \max_{k\le 4q} C_k)$ *(C_l and C_u as in Assumption 2) such that for* $(t,h) \in \mathcal{T}$,

(i) $|(K_{t,h}^{\gamma,\overline{m}}\mathfrak{a}^{(t,h)})(u)| \le C\|\phi\|_{H_4^q} h^{-m-r} \min\left(1, \frac{h^2}{(u-t)^2}\right)$,
(ii) $|(K_{t,h}^{\gamma,\overline{m}}\mathfrak{a}^{(t,h)})(u) - (K_{t,h}^{\gamma,\overline{m}}\mathfrak{a}^{(t,h)})(u')| \le C\|\phi\|_{H_4^q} h^{-m-r-1}|u-u'|$ *and for* $u, u' \neq t$,

$$\begin{aligned}|(K_{t,h}^{\gamma,\overline{m}}\mathfrak{a}^{(t,h)})(u) - (K_{t,h}^{\gamma,\overline{m}}\mathfrak{a}^{(t,h)})(u')| &\le C\|\phi\|_{H_4^q} \frac{h^{1-m-r}|u-u'|}{|u'-t|\ |u-t|} \\ &= C\|\phi\|_{H_4^q} h^{1-m-r}\Big|\int_{u'}^{u} \frac{1}{(x-t)^2}dx\Big|.\end{aligned}$$

PROOF. During this proof, $C = C(q, r, \gamma, \overline{m}, C_l, C_u, \max_{k\le 4q} C_k)$ denotes an unspecified constant which may change in every line. The proof relies essentially on the well-known commutator relation for pseudo-differential operators, $[x, \mathrm{Op}(p)] = i\,\mathrm{Op}(\partial_\xi p)$, with $\partial_\xi p : (x,\xi) \mapsto \partial_\xi p(x,\xi)$ (cf. Theorem 18.1.6 in [24]). By induction for $k \in \mathbb{N}$,

$$x^k \,\mathrm{Op}(\mathfrak{a}^{(t,h)}) = \sum_{r=0}^{k} \binom{k}{r} i^r \,\mathrm{Op}\left(\partial_\xi^r \mathfrak{a}^{(t,h)}\right) x^{k-r}. \tag{19}$$

As a preliminary result, let us show that for $k = 0, 1, 2$ the L^1-norms of

$$\langle s\rangle\ D_s^k\ \lambda\left(\tfrac{s}{h}\right)\mathcal{F}(\mathrm{Op}(\mathfrak{a}^{(t,h)})\phi)(s), \tag{20}$$

are bounded by $C\|\phi\|_{H_2^q} h^{-r-\gamma}$. Using Assumption 2 and Lemma B.1 this follows immediately for $k = 0$ and $q > r + m + 3/2$ by

$$\begin{aligned}\int \left|\langle s\rangle\ \lambda\left(\tfrac{s}{h}\right)\mathcal{F}(\mathrm{Op}(\mathfrak{a}^{(t,h)})\phi)(s)\right| ds &\le C_l^{-1} h^{-r-\gamma}\left\|\langle\cdot\rangle^{1+r+\gamma}\ \mathcal{F}(\mathrm{Op}(\mathfrak{a}^{(t,h)})\phi)\right\|_1 \\ &\le C h^{-r-\gamma}\left\|\mathrm{Op}(\mathfrak{a}^{(t,h)})\phi\right\|_{H^{q-\overline{m}}} \\ &\le C h^{-r-\gamma}\|\phi\|_{H^q}.\end{aligned} \tag{21}$$

Now, $\mathfrak{a}^{(t,h)} \in S^{\overline{m}}$ implies that for $k \in \mathbb{N}$, $\partial_\xi^k \mathfrak{a}^{(t,h)} \in S^{\overline{m}-k} \subset S^{\overline{m}}$. Since by (19), Assumptions (i) and (iii), and Lemma B.1,

$$\|\langle x\rangle^2 \operatorname{Op}(\mathfrak{a}^{(t,h)})\phi\|_1 \lesssim \|(1+x^4) \operatorname{Op}(\mathfrak{a}^{(t,h)})\phi\|_2 \leq C\|\phi\|_{H_4^{\overline{m}}} < \infty, \tag{22}$$

we obtain for $j \in \{1,2\}$,

$$D_s^j \mathcal{F}(\operatorname{Op}(\mathfrak{a}^{(t,h)})\phi) = (-i)^j \mathcal{F}(x^j \operatorname{Op}(\mathfrak{a}^{(t,h)})\phi)(s)$$

by interchanging differentiation and integration. Explicit calculations thus show

$$\begin{aligned} D_s\lambda\big(\tfrac{s}{h}\big)\mathcal{F}\big(\operatorname{Op}(\mathfrak{a}^{(t,h)})\phi\big)(s) &= \big(D_s\lambda\big(\tfrac{s}{h}\big)\big)\mathcal{F}\big(\operatorname{Op}(\mathfrak{a}^{(t,h)})\phi\big)(s) \\ &\quad - i\lambda\big(\tfrac{s}{h}\big)\mathcal{F}\big(x\operatorname{Op}(\mathfrak{a}^{(t,h)})\phi\big)(s) \end{aligned}$$

and

$$\begin{aligned} D_s^2\lambda\big(\tfrac{s}{h}\big)\mathcal{F}\big(\operatorname{Op}(\mathfrak{a}^{(t,h)})\phi\big)(s) &= \big(D_s^2\lambda\big(\tfrac{s}{h}\big)\big)\mathcal{F}\big(\operatorname{Op}(\mathfrak{a}^{(t,h)})\phi\big)(s) \\ &\quad - 2i\big(D_s\lambda\big(\tfrac{s}{h}\big)\big)\mathcal{F}\big(x\operatorname{Op}(\mathfrak{a}^{(t,h)})\phi\big)(s) \\ &\quad - \lambda\big(\tfrac{s}{h}\big)\mathcal{F}\big(x^2\operatorname{Op}(\mathfrak{a}^{(t,h)})\phi\big)(s). \end{aligned} \tag{23}$$

To finish the proof of (20) let us distinguish two cases, namely (I) $\gamma \in \{0\} \cup [2,\infty)$ and (II) $\gamma \in (1,2)$.

(I): For $k = 0,1,2$, $s \neq 0$, we see by elementary calculations, $\big|\langle s\rangle D_s^k \lambda\big(\frac{s}{h}\big)\big| \leq Ch^{-r-\gamma}\langle s\rangle^{r+\gamma+1}$. Using (19) and arguing similar as for (21) we obtain (replacing ϕ by $x\phi$ or $x^2\phi$ if necessary) bounds of the L^1-norms which are of the correct order $\|\phi\|_{H_4^q} h^{-r-\gamma}$.

(II): In principal we use the same arguments as in (I) but a singularity appears by expanding the first term on the r.h.s. of (23). In fact, it is sufficient to show that

$$\begin{aligned} &\int_{-1}^{1} \Big| \frac{D_s^2 \big|\frac{s}{h}\big|^\gamma \iota_s^{-\mu}}{\mathcal{F}(f_\epsilon)\big(-\frac{s}{h}\big)} \mathcal{F}\big(\operatorname{Op}(\mathfrak{a}^{(t,h)})\phi\big)(s)\Big| ds \\ &\quad \leq C_l h^{-r-\gamma} \big\|\mathcal{F}\big(\operatorname{Op}(\mathfrak{a}^{(t,h)})\phi\big)\big\|_\infty \int_1^1 |s|^{\gamma-2} ds \\ &\quad \lesssim C_l h^{-r-\gamma} \big\|\operatorname{Op}(\mathfrak{a}^{(t,h)})\phi\big\|_1 \leq Ch^{-r-\gamma}\|\phi\|_{H_4^{\overline{m}}}, \end{aligned}$$

where the last inequality follows from (22). Since this has the right order $h^{-r-\gamma}\|\phi\|_{H_4^q}$, (20) follows for $\gamma \in (1,2)$.

Together (I) and (II) prove (20). Hence, we can apply partial integration twice and obtain for $t \neq u$,

$$(K_{t,h}^{\gamma,\overline{m}}\mathfrak{a}^{(t,h)})(u) = -\frac{h^{2-\overline{m}}}{(u-t)^2}\int e^{is(u-t)/h}\, D_s^2\lambda\big(\tfrac{s}{h}\big)\mathcal{F}\big(\operatorname{Op}(\mathfrak{a}^{(t,h)})\phi\big)(s)\,đs \tag{24}$$

and similarly, first interchanging integration and differentiation,

$$\begin{aligned} D_u(K_{t,h}^{\gamma,\overline{m}}\mathfrak{a}^{(t,h)})(u) &= ih^{-\overline{m}-1}\int e^{is(u-t)/h}s\lambda\big(\tfrac{s}{h}\big)\mathcal{F}\big(\operatorname{Op}(\mathfrak{a}^{(t,h)})\phi\big)(s)\,đs \\ &= -\frac{ih^{1-\overline{m}}}{(u-t)^2}\int e^{is(u-t)/h}D_s^2 s\lambda\big(\tfrac{s}{h}\big)\mathcal{F}\big(\operatorname{Op}(\mathfrak{a}^{(t,h)})\phi\big)(s)\,đs \end{aligned} \tag{25}$$

(i): The estimates $|(K_{t,h}^{\gamma,\overline{m}}\mathfrak{a}^{(t,h)})(u)| \leq C\|\phi\|_{H_4^q}h^{-m-r}$ and $|(K_{t,h}^{\gamma,\overline{m}}\mathfrak{a}^{(t,h)})(u)| \leq C\|\phi\|_{H_4^q}h^{2-m-r}/(u-t)^2$ follow directly from (21) as well as (24) together with the L^1 bound of (20) for $k=2$.

(ii): To prove $|(K_{t,h}^{\gamma,\overline{m}}\mathfrak{a}^{(t,h)})(u) - (K_{t,h}^{\gamma,\overline{m}}\mathfrak{a}^{(t,h)})(u')| \leq C\|\phi\|_{H_4^q}h^{-m-r-1}|u-u'|$ it is enough to note that $|e^{ix}-e^{iy}| \leq |x-y|$. The result then follows from (21) again. For the second bound, see (25). The estimate for the L^1-norm of (20) with $k=2$ completes the proof. $\square$

LEMMA B.3. *Work under the assumptions of Theorem 2. If $v_{t,h}$ is given as in (20), then,*

$$\|v_{t,h}\|_2 \gtrsim h^{1/2-m-r}.$$

PROOF. We only discuss the case $\gamma > 0$. If $\gamma = 0$ the proof can be done similarly. It follows from the definition that

$$\begin{aligned} \|v_{t,h}\|_2^2 = \int \frac{1+|s|^{2\gamma}}{|\mathcal{F}(f_\epsilon)(-s)|^2}\big|\mathcal{F}\big(\operatorname{Op}(a^\star)(\phi\circ S_{t,h})\big)(s)\big|^2 ds \\ - \Big\|\frac{\mathcal{F}\big(\operatorname{Op}(a^\star)(\phi\circ S_{t,h})\big)}{\mathcal{F}(f_\epsilon)(-\cdot)}\Big\|_2^2. \end{aligned}$$

Since the adjoint is given by $a^\star(x,\xi) = e^{\partial_x\partial_\xi}\overline{a}(x,\xi)$ in the sense of asymptotic summation, it follows immediately that $a^\star(x,\xi) = \overline{a}(x,\xi) + r(x,\xi)$ with $r \in S^{\overline{m}-1}$. From this we conclude that $\operatorname{Op}(a^\star)$ is an elliptic pseudo-differential operator. Because of $a^\star \in S^{\overline{m}}$ and ellipticity there exists a so called left parametrix $(a^\star)^{-1} \in S^{-\overline{m}}$ such that $\operatorname{Op}((a^\star)^{-1})\operatorname{Op}(a^\star) = 1 + \operatorname{Op}(a')$ and $a' \in S^{-\infty}$, where $S^{-\infty} = \bigcap_m S^m$ (cf. Theorem 18.1.9 in Hörmander [24]). In particular, $a' \in S^{-1}$. Moreover, $\operatorname{Op}((a^\star)^{-1}) : H^{r+\gamma} \to H^{r+m}$ is a continuous and linear and therefore bounded operator (cf. Lemma B.1). Introduce the function $Q = (\cdot \vee 0)^2$. Furthermore, by convexity, $1 + |s|^{2\gamma} \geq 2^{-\gamma}\langle s\rangle^{2\gamma}$ and there exists a finite constant $c > 0$ such that

$$
\begin{aligned}
&\int \frac{1+|s|^{2\gamma}}{|\mathcal{F}(f_\epsilon)(-s)|^2} \big|\mathcal{F}\big(\operatorname{Op}(a^\star)(\phi \circ S_{t,h})\big)(s)\big|^2 ds \\
&\quad \geq 2^{-\gamma} C_l^2 \|\operatorname{Op}(a^\star)(\phi \circ S_{t,h})\|_{H^{r+\gamma}}^2 \\
&\quad \gtrsim \|\operatorname{Op}((a^\star)^{-1}) \operatorname{Op}(a^\star)(\phi \circ S_{t,h})\|_{H^{r+m}}^2 \\
&\quad = \|(1+\operatorname{Op}(a'))(\phi \circ S_{t,h})\|_{H^{r+m}}^2 \\
&\quad \geq Q\big(\|\phi \circ S_{t,h}\|_{H^{r+m}} - \|\operatorname{Op}(a')(\phi \circ S_{t,h})\|_{H^{r+m}}\big) \\
&\quad \geq Q\big(\|\phi \circ S_{t,h}\|_{H^{r+m}} - c\|\phi \circ S_{t,h}\|_{H^{r+m-1}}\big) \\
&\quad \geq h \int \big(1 + \big|\tfrac{s}{h}\big|^2\big)^{m+r} \big|\mathcal{F}(\phi)(s)\big|^2 ds + O(h^{2(1-r-m)}) \\
&\quad \geq h^{1-2(r+m)} \int |s|^{2m+2r} \big|\mathcal{F}(\phi)(s)\big|^2 ds + O(h^{2(1-r-m)}).
\end{aligned}
$$

On the other hand, we see immediately that

$$
\begin{aligned}
\Big\| \frac{\mathcal{F}\big(\operatorname{Op}(a^\star)(\phi \circ S_{t,h})\big)}{\mathcal{F}(f_\epsilon)(-\cdot)} \Big\|_2^2 &\lesssim \|\operatorname{Op}(a^\star)(\phi \circ S_{t,h})\|_{H^r}^2 \\
&\lesssim \|\phi \circ S_{t,h}\|_{H^{r+\overline{m}}}^2 \lesssim h^{1-2(r+\overline{m})}.
\end{aligned}
$$

Since $\phi \in L^2$ and h tends to zero the claim follows. ☐

Lemma B.4 (Dümbgen, Spokoiny [12]. *, p.145] Suppose that* $\operatorname{supp}\psi \subset [0,1]$ *and* $\operatorname{TV}(\psi) < \infty$. *If* $(t,h), (t',h') \in \mathcal{T}$, *then*

$$
\big\|\psi\big(\tfrac{\cdot - t}{h}\big) - \psi\big(\tfrac{\cdot - t'}{h'}\big)\big\|_2^2 \leq 2\operatorname{TV}(\psi)^2\big(|h-h'| + |t-t'|\big).
$$

Let $\lceil x \rceil$ be the smallest integer which is not smaller than x.

Lemma B.5. *Let* $0 \leq \ell \leq 1/2$ *and* $q \geq 0$. *Assume that* $\phi \in H^{\lceil q \rceil} \cap H^{q+\ell}$, $\operatorname{supp}\phi \subset [0,1]$ *and* $\operatorname{TV}(D^{\lceil q \rceil}\phi) < \infty$. *Then, for* $h \leq h'$,

$$
\|\phi \circ S_{t,h} - \phi \circ S_{t',h'}\|_{H^q} \lesssim h^{-q}\sqrt{|t-t'|^{2\ell} + |h'-h|}.
$$

In particular, for $\phi \in H^{\lceil r+m \rceil} \cap H^{r+m+1/2}$, $\operatorname{supp}\phi \subset [0,1]$ *and* $\operatorname{TV}(D^{\lceil r+m \rceil}\phi) < \infty$, $h \leq h'$,

$$
\|v_{t,h} - v_{t',h'}\|_2 \lesssim h^{-r-m}\sqrt{|t-t'| + |h'-h|}.
$$

PROOF. Since

$$\begin{aligned}&\big\|\phi\circ S_{t,h}-\phi\circ S_{t',h'}\big\|_{H^q}^2\\&\quad\lesssim\int\langle s\rangle^{2q}\big|1-e^{is(t-t')}\big|^2\big|\mathcal{F}\big(\phi\big(\tfrac{\cdot}{h}\big)\big)(s)\big|^2ds+\big\|\phi\big(\tfrac{\cdot}{h}\big)-\phi\big(\tfrac{\cdot}{h'}\big)\big\|_{H^q}^2\end{aligned}$$

and $|1-e^{is(t-t')}|\le 2\min(|s||t-t'|,1)\le 2\min(|s|^\ell|t-t'|^\ell,1)\le 2|s|^\ell|t-t'|^\ell$, we obtain

$$\big\|\phi\circ S_{t,h}-\phi\circ S_{t',h'}\big\|_{H^q}^2\lesssim|t-t'|^{2\ell}h^{1-2q-2\ell}+\big\|\phi\big(\tfrac{\cdot}{h}\big)-\phi\big(\tfrac{\cdot}{h'}\big)\big\|_{H^q}^2$$

(note that $\phi\in H^{q+\ell}$). Set $k=\lceil q\rceil$. Then

$$\begin{aligned}\big\|\phi\big(\tfrac{\cdot}{h}\big)-\phi\big(\tfrac{\cdot}{h'}\big)\big\|_{H^q}^2&\lesssim h^{1-2q}\big\|\phi-\phi\big(\tfrac{h}{h'}\cdot\big)\big\|_{H^q}^2\\&\lesssim h^{1-2q}\big\|\phi-\phi\big(\tfrac{h}{h'}\cdot\big)\big\|_2^2+h^{1-2q}\big\|D^k\big(\phi-\phi\big(\tfrac{h}{h'}\cdot\big)\big)\big\|_2^2.\end{aligned}$$

For $j\in\{0,k\}$,

$$\begin{aligned}\big\|D^j\big(\phi-\phi\big(\tfrac{h}{h'}\cdot\big)\big)\big\|_2^2&\le 2\big\|\phi^{(j)}-\phi^{(j)}\big(\tfrac{h}{h'}\cdot\big)\big\|_2^2+2\big(1-\big(\tfrac{h}{h'}\big)^j\big)^2\big\|\phi^{(j)}\big(\tfrac{h}{h'}\cdot\big)\big\|_2^2\\&\lesssim h^{-1}\big\|\phi^{(j)}\big(\tfrac{\cdot}{h}\big)-\phi^{(j)}\big(\tfrac{\cdot}{h'}\big)\big\|_2^2+|h'-h|\ h^{-1}\|\phi^{(j)}\|_2^2.\end{aligned}$$

Now, application of Lemma B.4 completes the proof for the first part. The second claim follows from

$$\begin{aligned}\|v_{t,h}-v_{t',h'}\|_2^2&=\int|\lambda(s)|^2\big|\mathcal{F}\big(\operatorname{Op}(a^\star)(\phi\circ S_{t,h}-\phi\circ S_{t',h'}))\big)(s)\big|^2ds\\&\lesssim\big\|\phi\circ S_{t,h}-\phi\circ S_{t',h'}\big\|_{H^{r+m}}^2.\end{aligned}$$

$\square$

LEMMA B.6. *Let $A_{t,t',h,h'}$ be defined as in (3) and work under Assumption 1. Then, for a global constant $K>0$,*

$$A_{t,t',h,h'}\le K\sqrt{|t-t'|+|h-h'|}.$$

PROOF. Without loss of generality, assume that for fixed (t,h), $V_{t,h}\ge V_{t',h'}$. We can write

$$\begin{aligned}A_{t,t',h,h'}&\le\frac{\|\psi_{t,h}\sqrt{h}-\psi_{t',h'}\sqrt{h'}\|_2}{V_{t,h}}+\sqrt{h'}\|\psi_{t',h'}\|_2\Big|\frac{1}{V_{t,h}}-\frac{1}{V_{t',h'}}\Big|\\&\le\frac{\|\psi_{t,h}\sqrt{h}-\psi_{t',h'}\sqrt{h'}\|_2}{V_{t,h}}+\sqrt{h'}\frac{|V_{t,h}-V_{t',h'}|}{V_{t,h}}.\end{aligned}$$

By triangle inequality, $\|\psi_{t,h}\sqrt{h} - \psi_{t',h'}\sqrt{h'}\|_2 \leq \sqrt{h'}\|\psi_{t,h} - \psi_{t',h'}\|_2 + |\sqrt{h} - \sqrt{h'}|\ \|\psi_{t,h}\|_2$. Thus,

$$A_{t,t',h,h'} \leq \frac{\sqrt{h'}}{V_{t,h}}\Big(\|\psi_{t,h} - \psi_{t',h'}\|_2 + |V_{t,h} - V_{t',h'}|\Big) + \sqrt{|h-h'|}.$$

If $h' \leq h$, then the result follows by Assumption 1 (iv) and some elementary computations. Otherwise we can estimate $\sqrt{h'} \leq \sqrt{|h-h'|} + \sqrt{h}$ and so

$$A_{t,t',h,h'} \leq \frac{\sqrt{h}}{V_{t,h}}\Big(\|\psi_{t,h} - \psi_{t',h'}\|_2 + |V_{t,h} - V_{t',h'}|\Big) + 5\sqrt{|h-h'|}.$$

$\square$

REMARK 2. *For the proofs of the subsequent lemmas, we make often use of elementary facts related to the function $\langle\cdot\rangle^\alpha \in S^\alpha$ with $0 < \alpha < 1$. Note that for $t \in [0,1]$, $D_u\langle u\rangle^\alpha \leq \alpha\langle u\rangle^{\alpha-1} \in S^{\alpha-1}$, $D_u\langle u\rangle^\alpha \leq \alpha$,*

$$\langle u\rangle^\alpha \leq \frac{1}{2}(1+|u|^\alpha) \leq 1 + |u-t|^\alpha, \quad \textit{and} \quad \langle u\rangle^{\alpha-1} \leq 2|u-t|^{\alpha-1}, \tag{26}$$

where the last inequality follows from $|u-t|^{1-\alpha}\langle u\rangle^{\alpha-1} \leq |u|^{1-\alpha}\langle u\rangle^{\alpha-1} + 1 \leq 2$.

LEMMA B.7. *For $(t,h) \in \mathcal{T}$ let $r_{t,h}$ be a function satisfying the conclusions of Lemma B.2 for r, m and ϕ. Assume $1/2 < \alpha < 1$. Then, there exists a constant K independent of $(t,h) \in \mathcal{T}$ and ϕ such that*

$$\big|r_{t,h}(u)\langle u\rangle^\alpha - r_{t,h}(u')\langle u'\rangle^\alpha\big| \leq K\|\phi\|_{H_4^q} h^{1-m-r}\Big|\int_{u'}^{u} \frac{1}{(x-t)^{2-\alpha}} + \frac{1}{(x-t)^2}dx\Big|,$$

for all $u, u' \neq t$ and

$$\mathrm{TV}\left(r_{t,h}\langle\cdot\rangle^\alpha \mathbb{I}_{[t-1,t+1]}\right) \leq K\|\phi\|_{H_4^q} h^{-m-r},$$
$$\mathrm{TV}\left(r_{t,h}\langle\cdot\rangle^\alpha \mathbb{I}_{\mathbb{R}\setminus[t-1,t+1]}\right) \leq K\|\phi\|_{H_4^q} h^{1-m-r}.$$

PROOF. Let C be as in Lemma B.2. In this proof $K = K(\alpha, C)$ denotes a generic constant which may change from line to line. Without loss of generality, we may assume that $|u-t| \geq |u'-t|$. Furthermore, the bound is trivial if $u' \leq t \leq u$ or $u \leq t \leq u'$. Therefore, let us assume further that $u \geq u' > t$ (the case $u \leq u' < t$ can be treated similarly). Together with the conclusions from Lemma B.2 and Remark 2 this shows that

$$\begin{aligned}&\big|r_{t,h}(u)\langle u\rangle^{\alpha}-r_{t,h}(u')\langle u'\rangle^{\alpha}\big|\\&\leq \big|r_{t,h}(u)\big|\,\big|\langle u\rangle^{\alpha}-\langle u'\rangle^{\alpha}\big|+\langle u'\rangle^{\alpha}\big|r_{t,h}(u)-r_{t,h}(u')\big|\\&\leq K\|\phi\|_{H_4^q}\Big[h^{2-m-r}\frac{1}{(u-t)^2}+h^{1-m-r}\frac{|u'-t|^{\alpha}+1}{|u'-t|\;|u-t|}\Big]|u-u'|.\end{aligned}$$

Clearly, the second term in the bracket dominates uniformly over $h\in(0,1]$. By Taylor expansion

$$\begin{aligned}\frac{|u-u'|}{|u'-t|^{1-\alpha}\;|u-t|}&=\frac{u-u'}{(u-t)^{\alpha}(u'-t)^{1-\alpha}(u-t)^{1-\alpha}}\\&\leq\frac{(u-t)^{1-\alpha}-(u'-t)^{1-\alpha}}{(1-\alpha)(u'-t)^{1-\alpha}(u-t)^{1-\alpha}}=\int_{u'}^{u}\frac{1}{(x-t)^{2-\alpha}}dx.\end{aligned}$$

Hence,

$$\frac{1}{|u'-t|\;|u-t|}|u-u'|=\Big|\int_{u'}^{u}\frac{1}{(x-t)^{2}}dx\Big|$$

completes the proof for the first part. For the second part decompose $r_{t,h}\mathbb{I}_{[t-1,t+1]}$ in $r_{t,h}^{(1)}=r_{t,h}\mathbb{I}_{[t-h,t+h]}$ and $r_{t,h}^{(2)}=r_{t,h}\mathbb{I}_{[t-1,t+1]}-r_{t,h}^{(1)}$. Observe that the conclusions of Lemma B.2 imply

$$\begin{aligned}\mathrm{TV}\left(r_{t,h}^{(1)}\langle\cdot\rangle^{\alpha}\right)&\leq\|\langle\cdot\rangle^{\alpha}\mathbb{I}_{[t-h,t+h]}\|_\infty\,\mathrm{TV}(r_{t,h}^{(1)})+\mathrm{TV}\left(\langle\cdot\rangle^{\alpha}\mathbb{I}_{[t-h,t+h]}\right)\|r_{t,h}^{(1)}\|_\infty\\&\leq K\|\phi\|_{H_4^q}h^{-m-r}.\end{aligned}$$

By using the first part of the lemma, we conclude that uniformly in $(t,h)\in\mathcal{T}$,

$$\begin{aligned}\mathrm{TV}\left(r_{t,h}\langle\cdot\rangle^{\alpha}\mathbb{I}_{[t-1,t+1]}\right)&\leq\mathrm{TV}\left(r_{t,h}^{(1)}\langle\cdot\rangle^{\alpha}\right)+\mathrm{TV}\left(r_{t,h}^{(2)}\langle\cdot\rangle^{\alpha}\right)\\&\lesssim K\|\phi\|_{H_4^q}(h^{-m-r}+h^{-m-r})\end{aligned}$$

and also $\mathrm{TV}\left(r_{t,h}\langle\cdot\rangle^{\alpha}\mathbb{I}_{\mathbb{R}\setminus[t-1,t+1]}\right)\leq K\|\phi\|_{H_4^q}h^{1-m-r}$. □

LEMMA B.8. *Work under Assumptions 2 and 3 and suppose that* $m+r>1/2$, $\langle x\rangle\phi\in L^1$, *and* $\phi\in H_1^{m+r+1}$. *Let* $d_{t,h}$ *be as defined in* (14). *Then, there exists a constant* K *independent of* $(t,h)\in\mathcal{T}$, *such that for* $1/2<\alpha<1$,

$$\mathrm{TV}(d_{t,h}\langle\cdot\rangle^{\alpha}\mathbb{I}_{[t-1,t+1]})\leq Kh^{\beta_0\wedge(m+r)-r}\log\left(\tfrac{1}{h}\right).$$

PROOF. For convenience let $\beta_0^\star := \beta_0 \wedge (m+r)$ and substitute $s \mapsto -s$ in (14), i.e.

$$d_{t,h}(u) := \int e^{-is(u-t)/h}\Big(\frac{1}{\mathcal{F}(f_\epsilon)(\frac{s}{h})} - A\iota_s^\rho\big|\tfrac{s}{h}\big|^r\Big)\iota_s^\mu |s|^m \mathcal{F}(\phi)(-s) đs.$$

Define

$$F_h(s) := \frac{1}{\mathcal{F}(f_\epsilon)(\frac{s}{h})} - A\iota_s^\rho\big|\tfrac{s}{h}\big|^r.$$

By Assumptions 2 and 3, we can bound the L^1-norm of

$$s \mapsto \langle s\rangle F_h(s)\iota_s^\mu |s|^m \mathcal{F}(\phi)(-s) \tag{27}$$

uniformly in (t,h) by $\int \langle s\rangle \langle \frac{s}{h}\rangle^{r-\beta_0}|s|^m\big|\mathcal{F}(\phi)(-s)\big|ds$. Bounding $\langle \frac{s}{h}\rangle^{r-\beta_0}$ by $\langle \frac{s}{h}\rangle^{r-\beta_0^\star}$ and considering the cases $r \le \beta_0^\star$ and $r > \beta_0^\star$ separately, we find $h^{\beta_0^\star - r}\int \langle s\rangle^{1+r+m-\beta_0^\star}|\mathcal{F}(\phi)(-s)|ds \lesssim h^{\beta_0^\star-r}\|\phi\|_{H^{r+m+1}}$ as an upper bound for (27), uniformly in $(t,h) \in \mathcal{T}$. Furthermore,

$$D_sF_h(s) = -\frac{D_s\mathcal{F}(f_\epsilon)(\frac{s}{h})}{\big(\mathcal{F}(f_\epsilon)(\frac{s}{h})\big)^2} - Ari\iota_s^{\rho-1}h^{-1}\big|\tfrac{s}{h}\big|^{r-1}$$

and by Assumptions 2 and 3,

$$\begin{aligned}\Big|sD_sF_h(s)\Big| &\le \big|sD_s\mathcal{F}(f_\epsilon)(\tfrac{s}{h})\big|\Big|A^2\iota_s^{2\rho}\big|\tfrac{s}{h}\big|^{2r} - \frac{1}{\big(\mathcal{F}(f_\epsilon)(\frac{s}{h})\big)^2}\Big| \\ &\quad + |A|r\big|\tfrac{s}{h}\big|^r\Big| - A(ri)^{-1}\iota_s^{\rho+1}h\big|\tfrac{s}{h}\big|^{r+1}D_s\mathcal{F}(f_\epsilon)\big(\tfrac{s}{h}\big) - 1\Big| \\ &\lesssim \Big(\big|\tfrac{s}{h}\big|\big\langle\tfrac{s}{h}\big\rangle^{r-1} + \big|\tfrac{s}{h}\big|^r\Big)\big\langle\tfrac{s}{h}\big\rangle^{-\beta_0} \le 2\big\langle\tfrac{s}{h}\big\rangle^{r-\beta_0^\star}.\end{aligned}$$

Similarly as above, we can conclude that the L^1-norm of

$$s \mapsto D_s sF_h(s)\iota_s^\mu|s|^m\mathcal{F}(\phi)(-s)$$

is bounded by const.$\times h^{\beta_0^\star-r}\|\phi\|_{H_1^{r+m+1}}$, uniformly over all $(t,h) \in \mathcal{T}$. Therefore, we have by interchanging differentiation and integration first and partial integration,

$$\begin{aligned}D_ud_{t,h}(u) &= \frac{-i}{h}\int se^{-is(u-t)/h}F_h(s)\iota_s^\mu|s|^m\mathcal{F}(\phi)(-s)đs \\ &= \frac{-1}{u-t}\int e^{-is(u-t)/h}D_ssF_h(s)\iota_s^\mu|s|^m\mathcal{F}(\phi)(-s)đs\end{aligned}$$

and the second equality holds for $u \neq t$. Together with (27) this shows that $|d_{t,h}(u)| \lesssim h^{\beta_0^\star - r}$ and $|D_u d_{t,h}(u)| \lesssim h^{\beta_0^\star - r-1}\min(1, h/|u-t|)$. Using Remark 2 we find for the sets $A_{t,h}^{(1)} := [t-h, t+h]$ and $A_{t,h}^{(2)} := [t-1, t+1] \setminus A_{t,h}^{(1)}$,

$$\begin{aligned}\mathrm{TV}(d_{t,h}\mathbb{I}_{[t-1,t+1]}) &\leq 2\|d_{t,h}\|_\infty + \int_{A_{t,h}^{(1)}} |D_u d_{t,h}(u)|du + \int_{A_{t,h}^{(2)}} |D_u d_{t,h}(u)|du \\ &\lesssim h^{\beta_0^\star - r}\log\left(\tfrac{1}{h}\right).\end{aligned}$$

Thus, $\mathrm{TV}(d_{t,h}\langle\cdot\rangle^\alpha \mathbb{I}_{[t-1,t+1]}) \lesssim \|d_{t,h}\|_\infty + \mathrm{TV}(d_{t,h}\mathbb{I}_{[t-1,t+1]}) \lesssim h^{\beta_0^\star - r}\log\left(\tfrac{1}{h}\right)$. □

LEMMA B.9. *Work under the assumptions of Theorem 3 and let $v_{t,h}^P$ be defined as in (10). Then, for $1/2 < \alpha < 1$,*

$$\mathrm{TV}(v_{t,h}^P \langle\cdot\rangle^\alpha \mathbb{I}_{\mathbb{R}\setminus[t-1,t+1]}) \leq K h^{1-r-m},$$

where the constant K does not depend on (t, h).

PROOF. The proof uses essentially the same arguments as the proof of Lemma B.2. Let $q := \lfloor r + m + 5/2 \rfloor$ and recall that by assumption $\langle x\rangle^2\phi \in L^1$. Decomposing the L^1-norm on $\mathbb{R}$ into $L^1([-1,1])$ and $L^1(\mathbb{R}\setminus[-1,1])$, using Cauchy-Schwarz inequality, and $\|\mathcal{F}(\phi)\|_\infty \leq \|\phi\|_1$, we see that for $j \in \{0, 1\}$, the L^1-norm of $s \mapsto D_s^j |s|^{r+m}\iota_s^{-\rho-\mu}\mathcal{F}(\phi)(s)$ is bounded by const. $\times$ $(\|\phi\|_{H_1^q} + \|\phi\|_1)$. Similarly, for $k \in \{0, 1, 2\}$ the L^1-norms of $s \mapsto D_s^k |s|^{r+m+1}\iota_s^{-\rho-\mu+1}\mathcal{F}(\phi)(s)$ are bounded by a multiple of $\|\phi\|_{H_2^q} + \|\phi\|_1$. Hence we have

$$v_{t,h}^P(u) = \frac{Ah^{1-r-m} i a_P(t)}{u-t}\int e^{is(u-t)/h} D_s |s|^{r+m}\iota_s^{-\rho-\mu}\mathcal{F}(\phi)(s) đs$$

and

$$D_u v_{t,h}^P(u) = \frac{-Ah^{1-r-m} a_P(t)}{(u-t)^2}\int e^{is(u-t)/h} D_s^2 |s|^{r+m+1}\iota_s^{-\rho-\mu+1}\mathcal{F}(\phi)(s) đs.$$

Together with Remark 2 this shows that

$$\begin{aligned}\mathrm{TV}\left(v_{t,h}^P\langle\cdot\rangle^\alpha \mathbb{I}_{[t+1,\infty)}\right) &\leq \|v_{t,h}^P\langle\cdot\rangle^\alpha \mathbb{I}_{[t+1,\infty)}\|_\infty + \int_{t+1}^\infty |D_u v_{t,h}^P(u)\langle\cdot\rangle^\alpha| du \\ &\lesssim h^{1-r-m} + \int_{t+1}^\infty \frac{h^{1-r-m}}{|u-t|^{2-\alpha}} + \frac{h^{1-r-m}}{|u-t|^2} du \lesssim h^{1-r-m}.\end{aligned}$$

Similarly we can bound the total variation on $(-\infty, t-1]$. □

The next lemma extends a well-known bound for functions with compact support to general càdlàg functions. We found this result useful for estimating the supremum over a Gaussian process if entropy bounds are difficult.

LEMMA B.10. *Let $(W_t)_{t\in\mathbb{R}}$ denote a two-sided Brownian motion. For a class of real-valued càdlàg functions $\mathcal{F}$ and any $\alpha > 1/2$ there exists a constant C_α such that*

$$\sup_{f\in\mathcal{F}} \Big| \int f(s) dW_s \Big| \leq C_\alpha \sup_{s\in[0,1]} |\overline{W}_s| \sup_{f\in\mathcal{F}} \mathrm{TV}(\langle\cdot\rangle^\alpha f),$$

where $\overline{W}$ is a standard Brownian motion on the same probability space.

PROOF. The proof consists of two steps. First suppose that $\bigcup_{f\in\mathcal{F}} \operatorname{supp} f \subset [0,1]$ and assume that the f are of bounded variation. Then, for any $f \in \mathcal{F}$, there exists a function q_f with $\|q_f\|_\infty \leq \mathrm{TV}(f)$ and a probability measure P_f with $P_f[0,1[= 1$, such that $f(u) = \int_{[0,u]} q_f(u) P_f(du)$ for all $u \in \mathbb{R}$, because f is càdlàg and thus $f(1) = 0$. With probability one,

$$\sup_{f\in\mathcal{F}} \Big| \int f(s) dW_s \Big| = \sup_{f\in\mathcal{F}} \Big| \int W_s q_f(s) P_f(ds) \Big| \leq \sup_{s\in[0,1]} |W_s| \sup_{f\in\mathcal{F}} \mathrm{TV}(f).$$

Now let us consider the general case. If $C_\alpha := \|\langle\cdot\rangle^{-\alpha}\|_2$ then $h(s) = C_\alpha^{-2} \langle s\rangle^{-2\alpha}$ is a density of a random variable. Let H be the corresponding distribution function. Note that

$$(\overline{W}_t)_{t\in[0,1]} = \Big(\int_0^t \sqrt{h(H^{-1}(s))} dW_{H^{-1}(s)} \Big)_{t\in[0,1]}$$

is a standard Brownian motion satisfying $d\overline{W}_{H(s)} = \sqrt{h(s)} dW_s$ and thus with $Af = \langle\cdot\rangle^\alpha f$,

$$\begin{aligned}\sup_{f\in\mathcal{F}} \Big| \int f(s) dW_s \Big| &= C_\alpha \sup_{f\in\mathcal{F}} \Big| \int Af(s) d\overline{W}_{H(s)} \Big| \\ &= C_\alpha \sup_{f\in\mathcal{F}} \Big| \int_0^1 Af(H^{-1}(s)) d\overline{W}_s \Big|.\end{aligned}$$

Since $\mathrm{TV}(Af \circ H^{-1}) = \mathrm{TV}(Af)$ the result follows from the first part. □

In the next lemma, we study monotonicity properties of the calibration weights w_h.

LEMMA B.11. *For $h \in (0,1]$ and $\nu > e$ let $w_h := \sqrt{2^{-1}\log(\nu/h)}/\log\log(\nu/h)$. Then*

(i) $h \mapsto w_h$ is strictly decreasing on $\big(0, \nu\exp(e^{-2})\big]$, and
(ii) $h \mapsto w_h h^{1/2}$ is strictly increasing on $(0,1]$.

PROOF. With $x = x(h) := \log\log(\nu/h) > 0$, we have $\log w_h = -\log(2)/2 + x/2 - \log x$. Since the derivative of this w.r.t. x equals $1/2 - 1/x$ and is strictly positive for $x > 2$, we conclude that $\log w_h$ is strictly increasing for $x(h) \geq 2$, i.e. $h \leq \nu\exp(e^{-2})$. Moreover, $\log(w_h h^{1/2}) = \log(\nu/2)/2 + x/2 - \log x - e^x/2$, and the derivative of this w.r.t. $x > 0$ equals $1/2 - 1/x - e^x/2 < 0$. Thus, $w_h h^{1/2}$ is strictly increasing in $h \in (0,1]$. $\square$

LEMMA B.12. *Condition (iii) in Assumption 1 is fulfilled with $\kappa_n = w_{u_n} u_n^{1/2}$ whenever Condition (ii) of Assumption 1 holds, and for all $(t,h) \in B_n$, $\operatorname{supp}\psi_{t,h} \subset [t-h, t+h]$.*

PROOF. Let $1/2 < \alpha < 1$. Then $\langle\cdot\rangle^\alpha : \mathbb{R} \to \mathbb{R}$ is Lipschitz. Recall that $\mathrm{TV}(fg) \leq \|f\|_\infty \mathrm{TV}(g) + \|g\|_\infty \mathrm{TV}(f)$. Since $\bigcup_{(t,h)\in B_n} \operatorname{supp}\psi_{t,h} \subset [-1,2]$ is bounded and contains the support of all functions $s \mapsto \psi_{t,h}(s)\big[\sqrt{g(s)} - \sqrt{g(t)}\big]\langle s\rangle^\alpha$ (indexed in $(t,h) \in B_n$), we obtain uniformly over $(t,h) \in B_n$ and $G \in \mathcal{G}$,

$$\begin{aligned}
&\mathrm{TV}\Big(\psi_{t,h}(\cdot)\big[\sqrt{g(\cdot)} - \sqrt{g(t)}\big]\langle\cdot\rangle^\alpha\Big) \\
&\quad\lesssim \big\|\psi_{t,h}(\cdot)\big[\sqrt{g(\cdot)} - \sqrt{g(t)}\big]\big\|_\infty + \mathrm{TV}\Big(\psi_{t,h}(\cdot)\big[\sqrt{g(\cdot)} - \sqrt{g(t)}\big]\Big)
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\mathrm{TV}\Big(\psi_{t,h}(\cdot)\big[\sqrt{g(\cdot)} - \sqrt{g(t)}\big]\Big) &\leq \|\psi_{t,h}\|_\infty \mathrm{TV}\big(\big[\sqrt{g(\cdot)} - \sqrt{g(t)}\big]\mathbb{I}_{[t-h,t+h]}(\cdot)\big) \\
&\quad + \mathrm{TV}\left(\psi_{t,h}\right)\big\|\big[\sqrt{g(\cdot)} - \sqrt{g(t)}\big]\mathbb{I}_{[t-h,t+h]}(\cdot)\big\|_\infty \\
&\lesssim V_{t,h} h^{1/2},
\end{aligned}$$

where the last inequality follows from Assumption 1 (ii) as well as the properties of $\mathcal{G}$. With Lemma B.11 (ii) the result follows. $\square$

Appendix C Further results on multiscale statistics. The following result shows that multiscale statistics computed over sufficiently rich index sets B_n are also bounded from below.

LEMMA C.1. *Assume that* $K_n \to \infty$, $\psi_{t,h} = \psi\big(\frac{\cdot - t}{h}\big)$ *and* $V_{t,h} = \|\psi_{t,h}\|_2 = \sqrt{h}\|\psi\|_2$. *Suppose that* $\lim_{j\to\infty} \log(j)|\int \psi(s-j)\psi(s)ds| \to 0$. *Then, with* w_h *and* $B^\circ_{K_n}$ *as defined in (10) and (14), respectively,*

$$\sup_{(t,h)\in B^\circ_{K_n}} w_h \left(\frac{|\int \psi_{t,h}(s)dW_s|}{\|\psi_{t,h}\|_2} - \sqrt{2\log\frac{\nu}{h}} \right) \to -\frac{1}{4}, \quad \textit{in probability.}$$

PROOF. Write $K := K_n$ and let $\xi_j := \|\psi_{t,h}\|_2^{-1} \int \psi_{j/K,1/K}(s)dW_s$ for $j = 0, \ldots, K-1$. Now, $(\xi_j)_j$ is a stationary sequence of centered and standardized normal random variables. In particular the distribution of $(\xi_j)_j$ does not depend on K and the covariance decays by assumption at a faster rate than logarithmically. By Theorem 4.3.3 (ii) in [45] the maximum behaves as the maximum of K independent standard normal r.v., i.e.

$$\mathbb{P}\big(\max(\xi_1,\ldots,\xi_K) \le a_K + b_K t\big) \to \exp\big(-e^{-t}\big), \quad \text{for } t\in\mathbb{R} \text{ and } K\to\infty,$$

where

$$b_K := \frac{1}{\sqrt{2\log K}}, \quad \text{and} \quad a_K = \sqrt{2\log K} - \frac{\log\log K + \log(4\pi)}{\sqrt{8\log K}}.$$

Using the tail-equivalence criterion (cf. [41], Proposition 3.3.28), we obtain further

$$\lim_{K\to\infty} \mathbb{P}\big(\max(|\xi_1|,\ldots,|\xi_K|) \le a_K + b_K(t+\log 2)\big) = \exp\big(-e^{-t}\big), \quad \text{for } t\in\mathbb{R}.$$

Note that $T_n^\circ := \sup_{(t,h)\in B_n^\circ} w_h(\|\psi_{t,h}\|_2^{-1}|\int \psi_{t,h}(s)dW_s| - \sqrt{2\log(\nu/h)})$ has the same distribution as $w_{K^{-1}}\max(|\xi_1|,\ldots,|\xi_K|) - w_{K^{-1}}\sqrt{2\log(\nu K)}$. It is easy to show that

$$\sqrt{\log \nu K} = \sqrt{\log K} + \frac{\log\nu}{2\sqrt{\log K}} + O\Big(\frac{1}{\log^{3/2} K}\Big)$$

and

$$\Big|\frac{1}{w_{K^{-1}}} - \frac{\log\log K}{\sqrt{\frac{1}{2}\log K}}\Big| = O\left(\frac{\log\log K}{\log^{3/2} K}\right).$$

Assume that $\eta_n \to 0$ and $\eta_n \log\log K \to \infty$. Then for sufficiently large n,

$$\begin{aligned}
&\mathbb{P}\big(T_n^\circ > -\tfrac{1}{4} + \eta_n\big) \\
&= \mathbb{P}\Big(\max(|\xi_1|, \ldots, |\xi_K|) > \big(-\tfrac{1}{4} + \eta_n\big)/w_{K^{-1}} + \sqrt{2\log \nu K}\Big) \\
&= \mathbb{P}\Big(\max(|\xi_1|, \ldots, |\xi_K|) > \\
&\qquad \big(-1 + 4\eta_n\big)\frac{\log\log K}{\sqrt{8\log K}} + \sqrt{2\log K} + \frac{\log \nu}{\sqrt{2\log K}} + O\big(\frac{\log\log K}{\log^{3/2} K}\big)\Big) \\
&\leq \mathbb{P}\Big(\max(|\xi_1|, \ldots, |\xi_K|) > a_K + b_K 2\eta_n \log\log K\Big) \to 0.
\end{aligned}$$

Similarly,

$$\mathbb{P}\big(T_n^\circ \leq -\tfrac{1}{4} - \eta_n\big) \leq \mathbb{P}\Big(\max(|\xi_1|, \ldots, |\xi_K|) \leq a_K - b_K \eta_n \log\log K\Big) \to 0.$$

$\square$

In order to illustrate the general multiscale statistic discussed in Section 2, let us show in the subsequent example that it is also possible to choose B_n in order to construct (level-dependent) values for simultaneous wavelet thresholding.

EXAMPLE C.1. *Observe that* $\widehat{d}_{j,k} = T_{k2^{-j},2^{-j}}$ *and* $d_{j,k} = \mathbb{E}T_{k2^{-j},2^{-j}} = \int \psi_{k2^{-j},2^{-j}}(s)g(s)ds = \int \psi(2^j s - k)g(s)ds$ *are the (estimated) wavelet coefficients and if* j_{0n} *and* j_{1n} *are integers satisfying* $2^{-j_{1n}} n \log^{-3} n \to \infty$ *and* $j_{0n} \to \infty$*, then for* $\alpha \in (0,1)$ *and*

$$B_n = \big\{ (k2^{-j}, 2^{-j}) \big|\ k = 0, 1, \ldots, 2^j - 1,\ j_{0n} \leq j \leq j_{1n},\ j \in \mathbb{N}\ \big\},$$

Theorem 1 yields in a natural way level-dependent thresholds $q_{j,k}(\alpha)$*, such that*

$$\lim_{n\to\infty} \mathbb{P}\Big(\big|\widehat{d}_{j,k} - d_{j,k}\big| \leq q_{j,k}(\alpha),\ \textit{for all } j,k,\ \textit{with } (k2^{-j}, 2^{-j}) \in B_n\Big) = 1 - \alpha.$$

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