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Motivic multiple zeta values and superstring amplitudes

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Abstract

The structure of tree-level open and closed superstring amplitudes is analyzed. For the open superstring amplitude we find a striking and elegant form, which allows one to disentangle its α'-expansion into several contributions accounting for different classes of multiple zeta values. This form is bolstered by the decomposition of motivic multiple zeta values, i.e. the latter encapsulate the α'-expansion of the superstring amplitude. Moreover, a morphism induced by the coproduct maps the α'-expansion onto a non-commutative Hopf algebra. This map represents a generalization of the symbol of a transcendental function. In terms of elements of this Hopf algebra the α'-expansion assumes a very simple and symmetric form, which carries all the relevant information. Equipped with these results we can also cast the closed superstring amplitude into a very elegant form.

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1. Introduction

One important question in quantum field theory is how to find a simple principle to easily compute physical quantities such as Feynman integrals describing higher-order quantum corrections. Analytic results for Feynman integrals are encoded by transcendental functions such as multiple polylogarithms or elliptic functions [1]. These functions, which depend on the kinematic invariants, have a rich algebraic structure and obey a variety of different classes of relations among each other. Although these equations may allow one to obtain a short and simple answer, in practice it is not straightforward how to concretely apply and disentangle these relations to arrive at this simple answer. Hence, a guiding principle to get a grip on these relations is important.

A recent step towards an implicit application of these relations, which also leads to quite remarkable simplifications [2], is the concept of the symbol of a transcendental function, which maps the combinatorics of relations among different multiple polylogarithms to the combinatorics of a tensor algebra [3]. All the functional identities between the polylogarithms are mapped to simple algebraic relations in the tensor algebra over the group of rational functions. A generalization of the symbol approach is the coproduct structure of multiple polylogarithms [4, 5]. The advantage of the coproduct structure is that it also keeps track of multiple zeta values (MZVs) in contrast to the symbol S, for which we have $S(\pi), S(\zeta) = 0$. Recently, in [6] the coproduct structure has been applied for a concrete physical amplitude.

The properties of scattering amplitudes in both gauge and gravity theories suggest a deeper understanding from string theory; see [7] for a recent review. Many field theory objects and relations such as Bern–Carrasco–Johansson (BCJ) [8] or Kawai–Lewellen–Tye (KLT) [9] relations can be easily derived from and understood in string theory by tracing these identities back to the monodromy properties of the string world-sheet [10, 11]. In this context we would also like to mention the question of transcendentality of a Feynman integral [12], which can be related to superstring tree-level amplitudes given by generalized Euler integrals [13]. Moreover, the concept of symbols and the coproduct structure for Feynman integrals might have a natural appearance in string theory. In fact, in this work we shall demonstrate that the aforementioned coproduct structure allows one to cast the α'-expansion of the tree-level open and closed superstring amplitude into a short and symmetric form.

Generically, the string amplitudes are given by integrals over vertex operator positions on the Riemann surface describing the interacting string world-sheet. At higher loops there is also an integral over the moduli space of this manifold. At tree-level such integrals over positions boil down to generalized Euler integrals [14]. Expanding the latter w.r.t. to powers in the string tension α' yields higher-order string corrections to Yang–Mills (YM) theory. Their expansion coefficients are given by MZVs multiplying some polynomials in the kinematic invariants: at each order in α' only a set of MZVs of a fixed transcendentality degree (transcendentality level [12]) appears. In practice, extracting these orders from the integrals [14, 15], which boils down to computing generalized Euler–Zagier sums, is both cumbersome and provides quite complicated expressions: the appearance of various MZVs of different depths seems to lack any sorted structure. Furthermore, there is no selection principle to choose the right basis of MZVs in the α'-expansion. As for computing amplitudes in field theory, a lot of their simplicity and symmetry structure is lost by not using the most appropriate approach. In other words, though the final result may have a simple structure, the actual computation might not be able to reproduce this simplicity and yield a difficult answer.

In fact, by passing from the MZVs to their motivic versions [4, 5] and then mapping the latter to elements of a Hopf algebra endows the superstring amplitude with its motivic structure. More precisely, the isomorphism ϕ, which is induced by the coproduct, maps the α'-expansion of the open superstring amplitude $\mathcal{A}$ into the very short and intriguing form in terms of elements f_i of a non-commutative Hopf algebra:

$$\mathcal{A} \xrightarrow{\phi} \left(\sum_{k=0}^{\infty} f_2^k\, P_{2k}\right) \left\{\sum_{p=0}^{\infty} \sum_{\substack{i_1,\ldots,i_p\\ \in 2\mathbb{N}^+ +1}} f_{i_1} f_{i_2} \ldots f_{i_p}\, M_{i_p} \ldots M_{i_2} M_{i_1}\right\} A. \tag{1.1}$$

In equation (1.1) the vector A encompasses a basis of YM subamplitudes, the matrices P_{2k} and M_{2n+1} encode polynomials of degree $2k$ and $2n+1$, respectively in α' and the kinematic invariants. As the vector A the string amplitude $\mathcal{A}$ represents a vector of the same dimension; see section 3 for further notational details. All the relevant information of the α'-expansion of the open superstring amplitude is encapsulated in (1.1) without further specifying the latter explicitly in terms of MZVs. This way all relations between MZVs are automatically built in as simple algebraic relations following from the coalgebra structure. Furthermore, the result is independent on any particular selection of a basis of MZVs. Finally, in contrast to the symbol the map ϕ, which is invertible, does not lose any information on the amplitude.

The present work is organized as follows. In section 2 we review those aspects of MZVs, which will be needed in the sequel. In section 3 we present our findings for the α'-expansion of the N-point open superstring amplitude. After some short exhibition on the work of Brown [5] on motivic MZVs in section 4 we compute the decompositions of motivic MZVs from weight 11 to weight 16 and compare the results with the structure of the open superstring amplitude. Equipped with these results, in section 5 we investigate the motivic structure of the open superstring amplitude and derive (1.1). In section 6 we use our open superstring results to also cast the closed string amplitude into a compact form. In the appendix we present some more results on the decomposition of motivic MZVs.

2. Aspects of multiple zeta values

One prime object in both quantum field theory and string theory are MZVs:

$$\zeta_{n_1,\ldots,n_r} := \zeta(n_1,\ldots,n_r) = \sum_{0<k_1<\cdots<k_r} \prod_{l=1}^{r} k_l^{-n_l}, \quad n_l \in \mathbf{N}^+, \quad n_r \geqslant 2. \tag{2.1}$$

In this section we review some of their aspects. They can be written as special cases [16]

$$\zeta_{n_1,\ldots,n_r} = (-1)^r\, G(\underbrace{0,\ldots,0}_{n_r-1},1\ldots,\underbrace{0,\ldots,0}_{n_1-1},1;1) \tag{2.2}$$

of multiple polylogarithms [16, 17]

$$G(a_1,\ldots,a_n;z) = \int_0^z \frac{\mathrm{d}t}{t-a_1}\, G(a_2,\ldots,a_n;t), \tag{2.3}$$

with $G(z) = 1$ and $a_i, z \in \mathbf{C}$. In (2.1) the sum $w = \sum_{l=1}^{r} n_l$ is called the transcendentality degree or weight of (2.1) and r its depth. The integral representation (2.2) is useful for establishing various properties and relations of (2.1). The set of integral linear combinations of MZVs (2.1) is a ring, since the product of any two values can be expressed by a (positive) integer linear combination of the other MZVs [18], e.g.

$$\zeta_m\, \zeta_n = \zeta_{m,n} + \zeta_{n,m} + \zeta_{m+n}. \tag{2.4}$$

This relation is known as the quasi-shuffle or stuffle relation. There are many relations over $\mathbf{Q}$ among MZVs, e.g. $\zeta_{1,4} = 2\zeta_5 - \zeta_2\zeta_3$. We define the (commutative) $\mathbf{Q}$-algebra $\mathcal{Z}$ spanned by all MZVs over $\mathbf{Q}$. The latter is the (conjecturally direct) sum over the $\mathbf{Q}$-vector spaces $\mathcal{Z}_N$ spanned by the set of MZVs (2.1) of total weight $w = N$, with $n_r \geqslant 2$, i.e. $\mathcal{Z} = \bigoplus_{k\geqslant 0} \mathcal{Z}_k$. For a given weight $w \in \mathbf{N}$ the dimension $\dim_{\mathbf{Q}}(\mathcal{Z}_N)$ of the space $\mathcal{Z}_N$ is conjecturally given by $\dim_{\mathbf{Q}}(\mathcal{Z}_N) = d_N$, with $d_N = d_{N-2} + d_{N-3}$, $N \geqslant 3$ and $d_0 = 1$, $d_1 = 0$, $d_2 = 1$ [18]. Starting at weight $w = 8$ MZVs of depth greater than 1 $r > 1$ appear in the basis. By applying stuffle, shuffle, doubling, generalized doubling relations and duality it is possible to reduce the MZVs of a given weight to a minimal set. Strictly speaking this is explicitly proven only up to weight 26 [19]. For $D_{w,r}$ being the number of independent MZVs at weight $w > 2$ and depth r, which cannot be reduced to primitive MZVs of smaller depth and their products, it is believed that $D_{8,2} = 1, D_{10,2} = 1$, $D_{11,3} = 1$, $D_{12,2} = 1$ and $D_{12,4} = 1$ [20]. For $Z = \frac{\mathcal{Z}_{>0}}{\mathcal{Z}_{>0}\mathcal{Z}_{>0}}$ the graded space of irreducible MZVs we have: $dim(Z_w) \equiv \sum_r D_{w,r} = 1, 0, 1, 0, 1, 1, 1, 1, 2, 2, 3, 3, 4, 5$ for $w = 3, \ldots, 16$, respectively [19, 20].

The selection of a basis of MZVs can be performed by following some principles. For instance, a minimal depth representation may be preferable. In addition, one may write as many elements of the basis as possible with positive odd indices n_l only. However, it is not possible to achieve this for the whole basis, i.e. a number of basis elements needs at least

Table 1. Basis elements for $\mathcal{Z}_w$, with $2 \leqslant w \leqslant 12$.

w	2	3	4	5	6	7	8	9	10	11		12		
$\mathcal{Z}_w$	ζ_2	ζ_3	ζ_2^2	ζ_5 $\zeta_2\,\zeta_3$	ζ_3^2 ζ_2^3	ζ_7 $\zeta_2\,\zeta_5$ $\zeta_2^2\,\zeta_3$	$\zeta_{3,5}$ $\zeta_3\,\zeta_5$ $\zeta_2\,\zeta_3^2$ ζ_2^4	ζ_9 ζ_3^3 $\zeta_2\,\zeta_7$ $\zeta_2^2\,\zeta_5$ $\zeta_2^3\,\zeta_3$	$\zeta_{3,7}$ $\zeta_3\,\zeta_7$ ζ_5^2 $\zeta_2\,\zeta_{3,5}$ $\zeta_2\,\zeta_3\,\zeta_5$ $\zeta_2^2\,\zeta_3^2$ ζ_2^5	$\zeta_{3,3,5}$ $\zeta_{3,5}\,\zeta_3$ ζ_{11} $\zeta_3^2\,\zeta_5$	$\zeta_2\,\zeta_3^3$ $\zeta_2\,\zeta_9$ $\zeta_2^2\,\zeta_7$ $\zeta_2^3\,\zeta_5$ $\zeta_2^4\,\zeta_3$	$\zeta_{1,1,4,6}$ $\zeta_{3,9}$ $\zeta_3\,\zeta_9$ $\zeta_5\,\zeta_7$ ζ_3^4	$\zeta_2\,\zeta_{3,7}$ $\zeta_2^2\,\zeta_{3,5}$ $\zeta_2\,\zeta_5^2$ $\zeta_2\,\zeta_3\,\zeta_7$ $\zeta_2^2\,\zeta_3\,\zeta_5$ $\zeta_2^3\,\zeta_3^2$ ζ_2^6
d_w	1	1	1	2	2	3	4	5	7	9		12		

Table 2. Basis elements for $\mathcal{Z}_w$, with $13 \leqslant w \leqslant 15$.

w	13		14		15		
$\mathcal{Z}_w$	$\zeta_{3,3,7}$ $\zeta_{3,5,5}$ ζ_{13} $\zeta_{3,7}\,\zeta_3$ $\zeta_{3,5}\,\zeta_5$ $\zeta_3^2\,\zeta_7$ $\zeta_3\,\zeta_5^2$	$\zeta_2\,\zeta_{3,3,5}$ $\zeta_2\,\zeta_3\,\zeta_{3,5}$ $\zeta_2\,\zeta_{11}$ $\zeta_2\,\zeta_3^2\,\zeta_5$ $\zeta_2^2\,\zeta_3^3$ $\zeta_2^2\,\zeta_9$ $\zeta_2^3\,\zeta_7$ $\zeta_2^4\,\zeta_5$ $\zeta_2^5\,\zeta_3$	$\zeta_{3,3,3,5}$ $\zeta_{3,11}$ $\zeta_{5,9}$ $\zeta_{3,3,5}\,\zeta_3$ $\zeta_{3,5}\,\zeta_3^2$ $\zeta_3\,\zeta_{11}$ $\zeta_3^3\,\zeta_5$ $\zeta_5\,\zeta_9$ ζ_7^2	$\zeta_2\,\zeta_{1,1,4,6}$ $\zeta_2\,\zeta_{3,9}$ $\zeta_2\,\zeta_3\,\zeta_9$ $\zeta_2\,\zeta_5\,\zeta_7$ $\zeta_2\,\zeta_3^4$ $\zeta_2^2\,\zeta_{3,7}$ $\zeta_2^3\,\zeta_{3,5}$ $\zeta_2^2\,\zeta_5^2$ $\zeta_2^2\,\zeta_3\,\zeta_7$ $\zeta_2^3\,\zeta_3\,\zeta_5$ $\zeta_2^4\,\zeta_3^2$ ζ_2^7	$\zeta_{1,1,3,4,6}$ $\zeta_{3,3,9}$ $\zeta_{5,3,7}$ ζ_{15} $\zeta_{1,1,4,6}\,\zeta_3$ $\zeta_{3,9}\,\zeta_3$ $\zeta_9\,\zeta_3^2$ $\zeta_3\,\zeta_5\,\zeta_7$ ζ_3^5 $\zeta_{3,7}\,\zeta_5$ ζ_5^3 $\zeta_{3,5}\,\zeta_7$	$\zeta_2\,\zeta_{3,3,7}$ $\zeta_2\,\zeta_{3,5,5}$ $\zeta_2\,\zeta_{13}$ $\zeta_2\,\zeta_3\,\zeta_{3,7}$ $\zeta_2\,\zeta_5\,\zeta_{3,5}$ $\zeta_2\,\zeta_3^2\,\zeta_7$ $\zeta_2\,\zeta_3\,\zeta_5^2$	$\zeta_2^2\,\zeta_{3,3,5}$ $\zeta_2^2\,\zeta_3\,\zeta_{3,5}$ $\zeta_2^2\,\zeta_{11}$ $\zeta_2^2\,\zeta_3^2\,\zeta_5$ $\zeta_2^3\,\zeta_3^3$ $\zeta_2^3\,\zeta_9$ $\zeta_2^4\,\zeta_7$ $\zeta_2^5\,\zeta_5$ $\zeta_2^6\,\zeta_3$
d_w	16		21		28		

two even entries [19]. Up to weight $w = 16$, one can choose the following basis elements, displayed in the following three tables; see tables 1–3.

A slight generalization of (2.3) represents the iterated integral I_γ over a product of closed 1-forms [16]

$$I_\gamma(a_0; a_1, \ldots, a_n; a_{n+1}) = \int_{\Delta_{n,\gamma}} \frac{\mathrm{d}z_1}{z_1 - a_1} \wedge \cdots \wedge \frac{\mathrm{d}z_n}{z_n - a_n}, \tag{2.5}$$

with γ a path in $M = \mathbf{C}/\{a_1, \ldots, a_n\}$ with endpoints $\gamma(0) = a_0 \in M,\ \gamma(1) = a_{n+1} \in M$ and $\Delta_{n,\gamma}$ a simplex consisting of all ordered n-tuples of points $(z_1, \ldots, z_n)$ on γ. For the map

$$\rho(n_1, \ldots, n_r) = 10^{n_1-1} \cdots 10^{n_r-1}, \tag{2.6}$$

with $n_r \geqslant 2$ Kontsevich observed that:

$$\zeta_{n_1,\ldots,n_r} = (-1)^r\, I_\gamma(0; \rho(n_1, \ldots, n_r); 1). \tag{2.7}$$

This defines an element in the category $MT(\mathbf{Z})$ of mixed Tate motives over $\mathbf{Z}$. It is an Abelian tensor category, whose simple objects are the Tate motives $\mathbf{Q}(n)$. The periods of $MT(\mathbf{Z})$ are $\mathbf{Q}\left[\frac{1}{2\pi i}\right]$-linear combinations of $\zeta_{n_1,\ldots,n_r}$ [21].

3. Open superstring amplitude

The string S-matrix, which describes string scattering processes involving on-shell string states as external states, comprises a perturbative expansion in the string tension α' and the string coupling constant g_{string}. From this expansion one may extract for a given order in α' and g_{string} the relevant interaction terms of the low-energy effective action.

Table 3. Basis elements for $\mathcal{Z}_{16}$.

w	16			
$\mathcal{Z}_w$	$\zeta_{1,1,6,8}$	$\zeta_2\,\zeta_3\,\zeta_{3,3,5}$	$\zeta_2\,\zeta_{3,3,3,5}$	$\zeta_2^2\,\zeta_{1,1,4,6}$
	$\zeta_{3,3,3,7}$	$\zeta_2\,\zeta_3^2\,\zeta_{3,5}$	$\zeta_2\,\zeta_{3,11}$	$\zeta_2^2\,\zeta_{3,9}$
	$\zeta_{3,3,5,5}$	$\zeta_2\,\zeta_3\,\zeta_{11}$	$\zeta_2\,\zeta_{5,9}$	$\zeta_2^2\,\zeta_5\,\zeta_7$
	$\zeta_{3,13}$	$\zeta_2\,\zeta_3^3\,\zeta_5$	$\zeta_2\,\zeta_5\,\zeta_9$	$\zeta_2^3\,\zeta_{3,7}$
	$\zeta_{5,11}$	$\zeta_2^2\,\zeta_3^4$	$\zeta_2\,\zeta_7^2$	$\zeta_2^4\,\zeta_{3,5}$
	$\zeta_3\,\zeta_{3,3,7}$	$\zeta_2^2\,\zeta_3\,\zeta_9$		$\zeta_2^3\,\zeta_5^2$
	$\zeta_3\,\zeta_{3,5,5}$	$\zeta_2^3\,\zeta_3\,\zeta_7$		ζ_2^8
	$\zeta_3\,\zeta_{13}$	$\zeta_2^4\,\zeta_3\,\zeta_5$		
	$\zeta_{3,7}\,\zeta_3^2$	$\zeta_2^5\,\zeta_3^2$		
	$\zeta_{3,5}\,\zeta_3\,\zeta_5$			
	$\zeta_3^3\,\zeta_7$			
	$\zeta_3^2\,\zeta_5^2$			
	$\zeta_7\,\zeta_9$			
	$\zeta_{3,5}^2$			
	$\zeta_5\,\zeta_{11}$			
	$\zeta_{3,3,5}\,\zeta_5$			
d_w	37			

Open superstring theory contains a massless vector identified as a gauge boson. Its interactions are studied by gluon scattering amplitudes. Geometrically, at tree-level (i.e. at leading order in g_{string}) the latter are described by a disc with (integrated) insertions of gluon vertex operators. Due to the extended nature of strings the amplitudes generically represent non-trivial functions of the string tension α'. In the effective field theory description this α'-dependence gives rise to a series of infinitely many higher order gauge operators governed by positive integer powers in α'. The classical YM term is reproduced in the zero-slope limit $\alpha' \to 0$, while its modification can be derived by studying the higher orders in α' of the tree-level gluon scattering amplitudes.

At string tree-level the complete open string N-point superstring amplitude has been computed in [22, 23]. The main result is written in a strikingly compact form[3]

$$\mathcal{A}(1,\ldots,N) = \sum_{\sigma\in S_{N-3}} A_{\text{YM}}(1,2_\sigma,\ldots,(N-2)_\sigma,N-1,N)\,F^\sigma_{(1,\ldots,N)}(\alpha'), \tag{3.1}$$

where A_{YM} represent $(N-3)!$ color ordered YM subamplitudes, $F^\sigma(\alpha')$ are generalized Euler integrals encoding the full α'-dependence of the string amplitude and $i_\sigma = \sigma(i)$. The labels $(1,\ldots,N)$ in $F^\sigma_{(1,\ldots,N)}$ are related to the integration region of the integrals: choosing an ordering of the vertex operator positions z_i along the boundary of the disc determines the color-ordering of the superstring subamplitude. The system of $(N-3)!$ multiple hypergeometric functions F^σ appearing in (3.1) are given as generalized Euler integrals (with $z_1 = 0$, $z_{N-1} = 1$ and $z_N = \infty$)

$$\begin{aligned} F^{(23,\ldots,N-2)}_{(1,\ldots,N)}(s_{ij}) &= (-1)^{N-3}\int_{z_i<z_{i+1}} \prod_{j=2}^{N-2} dz_j \left(\prod_{i<l}|z_{il}|^{s_{il}}\right)\left\{\prod_{k=2}^{N-2}\sum_{m=1}^{k-1}\frac{s_{mk}}{z_{mk}}\right\},\\ &= (-1)^{N-3}\int_{z_i<z_{i+1}} \prod_{j=2}^{N-2} dz_j \left(\prod_{i<l}|z_{il}|^{s_{il}}\right)\\ &\quad\times\left\{\left(\prod_{k=2}^{[N/2]}\sum_{m=1}^{k-1}\frac{s_{mk}}{z_{mk}}\right)\left(\prod_{k=[N/2]+1}^{N-2}\sum_{n=k+1}^{N-1}\frac{s_{kn}}{z_{kn}}\right)\right\}, \end{aligned} \tag{3.2}$$

[3] A very compact expression for $D = 4$ maximal helicity violating N-gluon amplitudes has been derived in [13].

with permutations $\sigma \in S_{N-3}$ acting on all indices within the curly brace. Above, $[\cdots]$ denotes the Gauss bracket $[x] = \max_{n\in\mathbf{Z},n\leqslant x} n$, which picks out the nearest integer smaller than or equal to its argument. The α'-dependence of (3.2) is encoded in the kinematic invariants

$$s_{ij} = \alpha' \, (k_i + k_j)^2, \tag{3.3}$$

with the external gluon momenta k_i satisfying the on-shell condition $k_i^2 = 0$. For further details we refer the reader to [22, 23].

The result (3.1) is valid in any space–time dimension D, for any compactification and any amount of supersymmetry. Furthermore, the expression (3.1) does not make any reference to any kinematical or space–time helicity choices. Hence, the same is true for our results throughout this paper. The integrals (3.2) boil down to linear combinations of the following generalized Euler or Selberg integrals [23]

$$B_N\,[n] = \left(\prod_{i=1}^{N-3}\int_0^1 \mathrm{d}x_i\right)\prod_{j=1}^{N-3} x_j^{s_{12\ldots j+1}+n_j}\prod_{l=j}^{N-3}\left(1-\prod_{k=j}^{l}x_k\right)^{s_{j+1,l+2}+n_{jl}}, \tag{3.4}$$

with the set of $\frac{1}{2}N(N-3)$ integers $n_j, n_{jl} \in \mathbf{Z}$ as well as $s_{i\ldots l} = \alpha'(k_i+\cdots+k_l)^2$ and $s_{i,j} \equiv s_{ij}$. The integrals B_N share a very interesting mathematical structure [14, 23]. For a given N the functions (3.2) represent integrals on the moduli space of Riemann spheres with N marked points $\mathcal{M}_{0,N}$ [24, 25]. These spaces have an N-fold symmetry following from N-fold cyclic transformations on the disc; see [23] for more details. The lowest terms of the α'-expansion of the functions F^σ assume the form [23]

$$\begin{aligned} F^\sigma &= 1 + p_2^\sigma\,\zeta_2 + p_3^\sigma\,\zeta_3 + \cdots, \quad \sigma = (23,\ldots,N-2),\\ F^\sigma &= p_2^\sigma\,\zeta_2 + p_3^\sigma\,\zeta_3 + \cdots, \quad \sigma \neq (23,\ldots,N-2), \end{aligned} \tag{3.5}$$

with some polynomials p_n^σ of degree n in the kinematic invariants s_{ij} and $s_{i,\ldots,l}$. Note that starting at $N \geqslant 7$ subsets of F^σ start at even higher order in α', i.e. $p_2^\sigma,\ldots,p_\nu^\sigma = 0$ for some $\nu \geqslant 2$. In [24, 25] it is proven that at lowest order in α' these integrals always lead to linear $\mathbf{Q}$ combinations of MZVs of weight $w \leqslant N-3$. Consequently, to all orders in α' only combinations of MZVs show up.

In the following let us discuss the cases $N = 4$ and $N = 5$ in more detail before moving to the general case afterwards.

3.1. $N = 4$

For $N = 4$ equation (3.1) becomes:

$$\mathcal{A}(1,2,3,4) = A_{\mathrm{YM}}(1,2,3,4)\,F, \tag{3.6}$$

with the function

$$F := F^{(2)}_{(1,2,3,4)} = s\int_0^1 \mathrm{d}x\, x^{s-1}\,(1-x)^u = \frac{\Gamma(1+s)\,\Gamma(1+u)}{\Gamma(1+s+u)}, \tag{3.7}$$

and the two kinematic invariants $s = \alpha'(k_1+k_2)^2$ and $u = \alpha'(k_1+k_4)^2$. With the identities

$$\begin{aligned} \frac{\Gamma(1+x)}{\Gamma(1-x)} &= \exp\left\{-2\sum_{n=1}^{\infty}\frac{x^{2n+1}}{2n+1}\,\zeta_{2n+1}\right\}\,\exp\{-2\gamma_E x\},\\ \pi\,\frac{s\,u}{s+u}\,\frac{\sin[\pi\,(s+u)]}{\sin(\pi s)\,\sin(\pi u)} &= \exp\left\{2\sum_{n=1}^{\infty}\frac{\zeta_{2n}}{2n}\,[s^{2n}+u^{2n}-(s+u)^{2n}]\right\}, \end{aligned}$$

we may bring (3.6) into the following form

$$\mathcal{A}(1,2,3,4) = P\ \exp\left\{\sum_{n\geqslant 1} \zeta_{2n+1}\ M_{2n+1}\right\}\ A_{\rm YM}(1,2,3,4), \tag{3.8}$$

with:

$$P = \exp\left\{\sum_{n=1}^{\infty} \frac{\zeta_{2n}}{2n}\ [s^{2n} + u^{2n} - (s+u)^{2n}]\right\},$$
$$M_{2n+1} = -\frac{1}{2n+1}\ \left[s^{2n+1} + u^{2n+1} - (s+u)^{2n+1}\ \right]. \tag{3.9}$$

In (3.8) we observe a disentanglement of Riemann zeta functions of even and odd arguments. Furthermore, no MZVs of depth greater than one $r > 1$ appear.

3.2. $N = 5$

For $N = 5$ we have a basis of two color ordered superstring amplitudes $\mathcal{A}(1,2,3,4,5)$ and $\mathcal{A}(1,3,2,4,5)$. According to (3.1) they take the form:

$$\begin{aligned}\mathcal{A}(1,2,3,4,5) &= A_{\rm YM}(1,2,3,4,5)\ F_1 + A_{\rm YM}(1,3,2,4,5)\ F_2,\\ \mathcal{A}(1,3,2,4,5) &= A_{\rm YM}(1,3,2,4,5)\ \widetilde{F}_1 + A_{\rm YM}(1,2,3,4,5)\ \widetilde{F}_2,\end{aligned} \tag{3.10}$$

with the hypergeometric functions (3.2):

$$\begin{aligned}F_1 := F^{(23)}_{(1,2,3,4,5)} &= s_{12}\ s_{34}\ \int_0^1 {\rm d}x \int_0^1 {\rm d}y\ x^{s_{45}}\ y^{s_{12}-1}\ (1-x)^{s_{34}-1}\ (1-y)^{s_{23}}\ (1-xy)^{s_{24}}\\ &= \frac{\Gamma(1+s_{12})\ \Gamma(1+s_{23})\ \Gamma(1+s_{34})\ \Gamma(1+s_{45})}{\Gamma(1+s_{12}+s_{23})\ \Gamma(1+s_{34}+s_{45})}\\ &\quad\times\ {}_3F_2\left[\begin{matrix} s_{12},\ 1+s_{45},\ -s_{24}\\ 1+s_{12}+s_{23},\ 1+s_{34}+s_{45}\end{matrix};1\right],\\ F_2 := F^{(32)}_{(1,2,3,4,5)} &= s_{13}\ s_{24}\ \int_0^1 {\rm d}x \int_0^1 {\rm d}y\ x^{s_{45}}\ y^{s_{12}}\ (1-x)^{s_{34}}\ (1-y)^{s_{23}}\ (1-xy)^{s_{24}-1}\\ &= s_{13}\ s_{24}\ \frac{\Gamma(1+s_{12})\ \Gamma(1+s_{23})\ \Gamma(1+s_{34})\ \Gamma(1+s_{45})}{\Gamma(2+s_{12}+s_{23})\ \Gamma(2+s_{34}+s_{45})}\\ &\quad\times\ {}_3F_2\left[\begin{matrix} 1+s_{12},\ 1+s_{45},\ 1-s_{24}\\ 2+s_{12}+s_{23},\ 2+s_{34}+s_{45}\end{matrix};1\right].\end{aligned} \tag{3.11}$$

Furthermore, we have:

$$\widetilde{F}_1 = F_1|_{2\leftrightarrow 3}\,,\quad \widetilde{F}_2 = F_2|_{2\leftrightarrow 3}\,. \tag{3.12}$$

When investigating the α'-expansions[4] of (3.10) one makes the following intriguing observation[5]:

$$\mathcal{A} = P\ Q\ :\exp\left\{\sum_{n\geqslant 1} \zeta_{2n+1}\ M_{2n+1}\right\}:\ A, \tag{3.13}$$

[4] Expanding Gaussian hypergeometric functions ${}_pF_q\left[\begin{smallmatrix} a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{smallmatrix}\right]$ w.r.t. small parameters a_i, b_j can conveniently be performed up to weight eight with the help of the programs HypExp [26] or XSummer [27]. However, for higher weights we used a FORM code to Taylor expand the hypergeometric function, manipulate the resulting harmonic sums and express the latter in terms of MZVs. Eventually, these MZVs are expanded w.r.t. to the MZV basis of [19] by using the tables thereof.

[5] We have tested this formula up to weight 16. Work beyond this order is in progress [28].

with the vectors

$$A = \begin{pmatrix} A_{\mathrm{YM}}(1,2,3,4,5) \\ A_{\mathrm{YM}}(1,3,2,4,5) \end{pmatrix}, \quad \mathcal{A} = \begin{pmatrix} \mathcal{A}(1,2,3,4,5) \\ \mathcal{A}(1,3,2,4,5) \end{pmatrix}, \tag{3.14}$$

and the matrices

$$M_{2n+1} = \begin{pmatrix} F_1 & F_2 \\ \widetilde{F}_2 & \widetilde{F}_1 \end{pmatrix}\Bigg|_{\zeta_{2n+1}},$$

$$P = \begin{pmatrix} \sum\limits_{n\geqslant 0} p_{2n}\,\zeta_2^n & \sum\limits_{n\geqslant 0} q_{2n}\,\zeta_2^n \\ \sum\limits_{n\geqslant 0} \widetilde{q}_{2n}\,\zeta_2^n & \sum\limits_{n\geqslant 0} \widetilde{p}_{2n}\,\zeta_2^n \end{pmatrix} = 1 + \sum_{n\geqslant 1} \zeta_2^n\, P_{2n}, \tag{3.15}$$

where $\widetilde{p}_{2n} = p_{2n}|_{2\leftrightarrow 3}$, $\widetilde{q}_{2n} = q_{2n}|_{2\leftrightarrow 3}$. Furthermore, we have the matrix:

$$Q = 1 + \sum_{n\geqslant 8} Q_n, \tag{3.16}$$

with:

$$\begin{aligned}
Q_8 &= \frac{1}{5}\,\zeta_{3,5}\,[M_5, M_3], \quad Q_9 = 0,\\
Q_{10} &= \left\{\tfrac{3}{14}\,\zeta_5^2 + \tfrac{1}{14}\,\zeta_{3,7}\right\}\,[M_7, M_3],\\
Q_{11} &= \left\{9\,\zeta_2\,\zeta_9 + \tfrac{6}{25}\,\zeta_2^2\,\zeta_7 - \tfrac{4}{35}\,\zeta_2^3\,\zeta_5 + \tfrac{1}{5}\,\zeta_{3,3,5}\right\}\,[M_3,[M_5, M_3]],\\
Q_{12} &= \left\{\tfrac{2}{9}\,\zeta_5\,\zeta_7 + \tfrac{1}{27}\,\zeta_{3,9}\right\}\,[M_9, M_3]\\
&\quad + \tfrac{48}{691}\,\left\{\tfrac{18}{35}\,\zeta_2^3\,\zeta_3^2 + \tfrac{1}{5}\,\zeta_2^2\,\zeta_3\,\zeta_5 - 10\,\zeta_2\,\zeta_3\,\zeta_7 - \tfrac{7}{2}\,\zeta_2\,\zeta_5^2 - \tfrac{3}{5}\,\zeta_2^2\,\zeta_{3,5} - 3\,\zeta_2\,\zeta_{3,7}\right.\\
&\quad \left.-\tfrac{1}{12}\,\zeta_3^4 - \tfrac{467}{108}\,\zeta_5\,\zeta_7 + \tfrac{799}{72}\,\zeta_3\,\zeta_9 + \tfrac{2665}{648}\,\zeta_{3,9} + \zeta_{1,1,4,6}\right\}\,\{[M_9, M_3] - 3\,[M_7, M_5]\}\,,\\
Q_{13} &= \left\{\tfrac{11}{4}\,\zeta_2\,\zeta_{11} - \tfrac{2}{35}\,\zeta_2^2\,\zeta_9 - \tfrac{16}{245}\,\zeta_2^3\,\zeta_7 - \tfrac{3}{35}\,\zeta_{3,5,5} + \tfrac{1}{14}\,\zeta_{3,3,7}\right\}\,[M_3,[M_7, M_3]]\\
&\quad + \left\{\tfrac{11}{2}\,\zeta_2\,\zeta_{11} + \tfrac{2}{5}\,\zeta_2^2\,\zeta_9 + \tfrac{1}{5}\,\zeta_5\,\zeta_{3,5} + \tfrac{1}{25}\,\zeta_{3,5,5}\right\}\,[M_5,[M_5, M_3]],\\
Q_{14} &= \left\{4\,\zeta_2\,\zeta_5\,\zeta_7 + \tfrac{4}{175}\,\zeta_2^3\,\zeta_{3,5} - \tfrac{647\,287}{11\,880}\,\zeta_7^2 - \tfrac{12\,775}{198}\,\zeta_5\,\zeta_9 + \tfrac{232}{81}\,\zeta_{5,9}\right.\\
&\quad \left.+\tfrac{2}{3}\,\zeta_2\,\zeta_{3,9} - \tfrac{12\,841}{1188}\,\zeta_{3,11} + \tfrac{1}{5}\,\zeta_{3,3,3,5}\right\}\,[M_3,[M_3,[M_5, M_3]]]\\
&\quad + \left\{-\tfrac{235}{396}\,\zeta_7^2 - \tfrac{23}{33}\,\zeta_5\,\zeta_9 + \tfrac{1}{27}\,\zeta_{5,9} - \tfrac{23}{198}\,\zeta_{3,11}\right\}\,[M_{11}, M_3]\\
&\quad + \left\{\tfrac{55}{36}\,\zeta_7^2 + \tfrac{5}{3}\,\zeta_5\,\zeta_9 + \tfrac{5}{18}\,\zeta_{3,11} - \tfrac{2}{27}\,\zeta_{5,9}\right\}\,[M_9, M_5],\\
Q_{15} &= \left\{\tfrac{1339}{30}\,\zeta_2\,\zeta_{13} + \tfrac{128}{45}\,\zeta_2^2\,\zeta_{11} - \tfrac{236}{4725}\,\zeta_2^3\,\zeta_9 - \tfrac{184}{2625}\,\zeta_2^4\,\zeta_7 - \tfrac{64}{5775}\,\zeta_2^5\,\zeta_5\right.\\
&\quad \left.-\tfrac{2}{45}\,\zeta_5^3 - \tfrac{1}{15}\,\zeta_7\,\zeta_{3,5} - \tfrac{2}{45}\,\zeta_5\,\zeta_{3,7} + \tfrac{1}{27}\,\zeta_{3,3,9}\right\}\,[M_3,[M_9, M_3]]\\
&\quad + \left\{-\tfrac{143}{20}\,\zeta_2\,\zeta_{13} - \tfrac{11}{35}\,\zeta_2^2\,\zeta_{11} + \tfrac{68}{1225}\,\zeta_2^3\,\zeta_9 + \tfrac{11}{70}\,\zeta_5^3 + \tfrac{24}{875}\,\zeta_2^4\,\zeta_7 + \tfrac{48}{13\,475}\,\zeta_2^5\,\zeta_5\right.\\
&\quad \left.+\tfrac{1}{5}\,\zeta_7\,\zeta_{3,5} + \tfrac{3}{35}\,\zeta_5\,\zeta_{3,7} - \tfrac{1}{70}\,\zeta_{5,3,7}\right\}\,[M_5,[M_7, M_3]] + \tfrac{2}{15}\,\zeta_{5,3,7}\,[M_3,[M_7, M_5]]\\
&\quad + \tfrac{48}{7601}\,\left\{-8\,\zeta_2\,\zeta_3\,\zeta_5^2 + \tfrac{21}{2}\,\zeta_2\,\zeta_5\,\zeta_{3,5} - \tfrac{14}{5}\,\zeta_2\,\zeta_{3,5,5} + 2\,\zeta_2\,\zeta_{3,3,7} - 26\,\zeta_2\,\zeta_3^2\,\zeta_7\right.\\
&\quad - \tfrac{6417\,649}{2880}\,\zeta_2\,\zeta_{13} - 6\,\zeta_2\,\zeta_3\,\zeta_{3,7} - \tfrac{8495\,287}{15\,120}\,\zeta_2^2\,\zeta_{11} - \tfrac{23}{10}\,\zeta_2^2\,\zeta_3^2\,\zeta_5 - \tfrac{8}{5}\,\zeta_2^2\,\zeta_3\,\zeta_{3,5}\\
&\quad + 4\,\zeta_2^2\,\zeta_{3,3,5} + \tfrac{12}{35}\,\zeta_2^3\,\zeta_3^3 + \tfrac{54\,263\,011}{396\,900}\,\zeta_2^3\,\zeta_9 + \tfrac{57\,847}{15\,750}\,\zeta_2^4\,\zeta_7 - \tfrac{1714\,624}{121\,275}\,\zeta_2^5\,\zeta_5\\
&\quad + \tfrac{1451\,972}{716\,625}\,\zeta_2^6\,\zeta_3 + \tfrac{1185\,701}{30\,240}\,\zeta_5^3 - \tfrac{74}{3}\,\zeta_3\,\zeta_5\,\zeta_7 - \tfrac{1}{15}\,\zeta_3^5 + \tfrac{6775}{144}\,\zeta_3^2\,\zeta_9 + \tfrac{2188}{945}\,\zeta_5\,\zeta_{3,7}\\
&\quad \left.-\tfrac{12\,199}{720}\,\zeta_7\,\zeta_{3,5} + \tfrac{29}{9}\,\zeta_3\,\zeta_{3,9} + \zeta_3\,\zeta_{1,1,4,6} + \tfrac{17\,203}{3360}\,\zeta_{5,3,7} - \tfrac{853}{648}\,\zeta_{3,3,9} + \zeta_{1,1,3,4,6}\right\}\\
&\quad \times \Big\{\,[M_3,[M_9, M_3]] - 3\,[M_3,[M_7, M_5]]\,\Big\},\\
Q_{16} &= \tfrac{1}{50}\,\zeta_{3,5}^2\,([M_5, M_3])^2 + \left\{\tfrac{210}{121}\,\zeta_9\,\zeta_7 + \tfrac{9}{11}\,\zeta_{11}\,\zeta_5 - \tfrac{5}{242}\,\zeta_{5,11} + \tfrac{3}{22}\,\zeta_{3,13}\right\}\,[M_{11}, M_5]\\
&\quad + \left\{-\tfrac{1275}{1573}\,\zeta_9\,\zeta_7 - \tfrac{57}{143}\,\zeta_{11}\,\zeta_5 + \tfrac{3}{242}\,\zeta_{5,11} - \tfrac{19}{286}\,\zeta_{3,13}\right\}\,[M_{13}, M_3]\\
&\quad + \left\{\tfrac{24}{35}\,\zeta_7\,\zeta_5\,\zeta_2^2 + \tfrac{6}{245}\,\zeta_5^2\,\zeta_2^3 + \tfrac{2}{245}\,\zeta_{3,7}\,\zeta_2^3 + \tfrac{4}{35}\,\zeta_{3,9}\,\zeta_2^2 + \tfrac{967}{56}\,\zeta_7^2\,\zeta_2\right.
\end{aligned}$$

$$+ \tfrac{363}{14}\,\zeta_9\,\zeta_5\,\zeta_2 - \tfrac{47}{42}\,\zeta_{5,9}\,\zeta_2 + \tfrac{121}{28}\,\zeta_{3,11}\,\zeta_2 - \tfrac{2272\,973}{330\,330}\,\zeta_9\,\zeta_7 - \tfrac{601\,677}{40\,040}\,\zeta_{11}\,\zeta_5$$
$$+ \tfrac{23\,181}{67\,760}\,\zeta_{5,11} - \tfrac{200\,559}{80\,080}\,\zeta_{3,13} - \tfrac{3}{35}\,\zeta_{3,3,5,5} + \tfrac{1}{14}\,\zeta_{3,3,3,7}\Big\}\ [M_3,[M_3,[M_7,M_3]]]$$
$$+ \Big\{-\tfrac{8}{25}\,\zeta_7\,\zeta_5\,\zeta_2^2 - \tfrac{2}{35}\,\zeta_5^2\,\zeta_2^3 - \tfrac{4}{75}\,\zeta_{3,9}\,\zeta_2^2 - \tfrac{333}{20}\,\zeta_7^2\,\zeta_2$$
$$- 21\,\zeta_9\,\zeta_5\,\zeta_2 + \zeta_{5,9}\,\zeta_2 - \tfrac{7}{2}\,\zeta_{3,11}\,\zeta_2 - \tfrac{299\,373}{7150}\,\zeta_9\,\zeta_7 - \tfrac{21\,033}{1300}\,\zeta_{11}\,\zeta_5$$
$$+ \tfrac{909}{2200}\,\zeta_{5,11} - \tfrac{7011}{2600}\,\zeta_{3,13} + \tfrac{1}{5}\,\zeta_5\,\zeta_{3,3,5} + \tfrac{1}{25}\,\zeta_{3,3,5,5}\Big\}\ [M_3,[M_5,[M_5,M_3]]]$$
$$+ \tfrac{720}{3617}\,\Big\{-\tfrac{21\,331}{525}\,\zeta_5\,\zeta_7\,\zeta_2^2 - \tfrac{284}{245}\,\zeta_5^2\,\zeta_2^3 + \tfrac{108}{875}\,\zeta_{3,5}\,\zeta_2^4 - \tfrac{62}{245}\,\zeta_{3,7}\,\zeta_2^3$$
$$- \tfrac{8954}{1575}\,\zeta_{3,9}\,\zeta_2^2 - \tfrac{78\,201}{140}\,\zeta_7^2\,\zeta_2 - \tfrac{12\,443}{14}\,\zeta_5\,\zeta_9\,\zeta_2 + \tfrac{697}{21}\,\zeta_{5,9}\,\zeta_2 - \tfrac{1991}{14}\,\zeta_{3,11}\,\zeta_2$$
$$- 137\,\zeta_{11}\,\zeta_3\,\zeta_2 - \tfrac{11}{7}\,\zeta_9\,\zeta_3\,\zeta_2^2 + \tfrac{848}{245}\,\zeta_7\,\zeta_3\,\zeta_2^3 + \tfrac{48}{35}\,\zeta_5\,\zeta_3\,\zeta_2^4 + \tfrac{408}{2695}\,\zeta_3^2\,\zeta_2^5$$
$$- \tfrac{4}{7}\,\zeta_3^2\,\zeta_5^2 - \tfrac{1}{3}\,\zeta_3^3\,\zeta_7 + \tfrac{4850\,713}{6600}\,\zeta_7\,\zeta_9 + \tfrac{455\,534}{525}\,\zeta_5\,\zeta_{11} + \tfrac{8497}{42}\,\zeta_3\,\zeta_{13} + \tfrac{1}{7}\,\zeta_3^2\,\zeta_{3,7}$$
$$- \tfrac{114\,307}{7392}\,\zeta_{5,11} + \tfrac{2217\,053}{16\,800}\,\zeta_{3,13} - \tfrac{2}{5}\,\zeta_5\,\zeta_{3,3,5} - \tfrac{6}{7}\,\zeta_3\,\zeta_{3,5,5} + \tfrac{5}{7}\,\zeta_{3,3,7}\,\zeta_3$$
$$+ \tfrac{542}{175}\,\zeta_{3,3,5,5} - \tfrac{19}{7}\,\zeta_{3,3,3,7} + \zeta_{1,1,6,8}\Big\}\ \Big\{\tfrac{7}{11}\,[M_{11},M_5] - \tfrac{2}{11}\,[M_{13},M_3] - [M_9,M_7]$$
$$+ \tfrac{6493}{9240}\,[M_3,[M_3,[M_7,M_3]]] - \tfrac{751}{100}\,[M_3,[M_5,[M_5,M_3]]]\Big\}\,. \tag{3.17}$$

Finally, in (3.13) the ordering colons $:\cdots:$ are defined such that matrices with larger subscripts multiply matrices with smaller subscripts from the left,

$$:M_i\,M_j: := \begin{cases} M_i\,M_j, & i \geqslant j, \\ M_j\,M_i, & i < j. \end{cases} \tag{3.18}$$

The generalization to iterated matrix products $:M_{i_1}M_{i_2}\cdots M_{i_p}:$ is straightforward.

To illustrate the structure of the matrices P and M, given in (3.15), let us display P_2 and M_3:

$$P_2 = \begin{pmatrix} -s_3s_4 + s_1\,(s_3 - s_5) & s_{13}\,s_{24} \\ s_1\,s_3 & (s_1+s_2)\,(s_2+s_3) - s_4s_5 \end{pmatrix}, \qquad M_3 = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}, \tag{3.19}$$

with

$$m_{11} = s_3\,[-s_1\,(s_1+2s_2+s_3) + s_3s_4 + s_4^2] + s_1s_5\,(s_1+s_5),$$
$$m_{12} = -s_{13}\,s_{24}\,(s_1+s_2+s_3+s_4+s_5), \quad m_{21} = s_1\,s_3\,[s_1+s_2+s_3-2\,(s_4+s_5)],$$
$$m_{22} = (s_2+s_3)\,[(s_1+s_2)(s_1+s_3) - 2\,s_1s_4] - [2s_1s_3 - s_4^2 + 2s_2\,(s_3+s_4)]s_5 + s_4s_5^2, \tag{3.20}$$

and

$$s_i \equiv \alpha'(k_i+k_{i+1})^2, \quad i = 1,\ldots,5, \tag{3.21}$$

subject to cyclic identification $k_{i+N} \equiv k_i$. The expression (3.13) allows one to conveniently extract any order in α' of the superstring amplitude by simple matrix manipulations. For example, at weight $w = 8$ from (3.13) we obtain the expressions

$$\mathcal{A}\,|_{\zeta_3\zeta_5} = M_5\,M_3\,A,$$
$$\mathcal{A}\,|_{\zeta_{3,5}} = \tfrac{1}{5}\,[M_5,M_3]\,A,$$
$$\mathcal{A}\,|_{\zeta_2\zeta_3^2} = \tfrac{1}{2}\,P_2\,M_3\,M_3\,A,$$
$$\mathcal{A}\,|_{\zeta_2^4} = P_8\,A, \tag{3.22}$$

while for weight $w = 10$ we get:

$$\mathcal{A}\,|_{\zeta_3\zeta_7} = M_7\,M_3\,A,$$
$$\mathcal{A}\,|_{\zeta_{3,7}} = \tfrac{1}{14}\,[M_7,M_3]\,A,$$
$$\mathcal{A}\,|_{\zeta_5^2} = \big(\tfrac{1}{2}\,M_5\,M_5 + \tfrac{3}{14}\,[M_7,M_3]\big)\,A,$$
$$\mathcal{A}\,|_{\zeta_2\zeta_3\zeta_5} = P_2\,M_5\,M_3\,A,$$

$$\begin{aligned}\mathcal{A}\,|_{\zeta_2\zeta_{3,5}} &= \tfrac{1}{5}\,P_2\ [M_5, M_3]\ A,\\ \mathcal{A}\,|_{\zeta_2^2\zeta_3^2} &= \tfrac{1}{2}\,P_4\,M_3\,M_3\,A,\\ \mathcal{A}\,|_{\zeta_2^5} &= P_{10}\,A.\end{aligned} \tag{3.23}$$

The terms M_5M_3A in (3.22) and M_7M_3A, $P_2M_5M_3A$ in (3.23) use the ordering prescription (3.18) introduced in (3.13) for the matrices M_i stemming from the exponential.

3.3. General N

For generic N in (3.1) we have a basis of $(N-3)!$ color ordered superstring amplitudes $\mathcal{A}(1, 2_\sigma, \ldots, (N-2)_\sigma, N-1, N)$. Putting these $(N-3)!$ amplitudes into an $(N-3)!$-dimensional vector $\mathcal{A}$ according to (3.1) the latter can be expressed by an $(N-3)! \times (N-3)!$-matrix F acting on the vector A encoding an $(N-3)!$-dimensional YM-basis as:

$$\mathcal{A} = F\,A. \tag{3.24}$$

The matrix F encodes the full α'-dependence of the superstring amplitude (3.24). We conjecture that the α'-dependence of the latter assumes the same form (3.13) as for the case $N = 5$

$$F = P\,Q\ :\exp\left\{\sum_{n\geqslant 1} \zeta_{2n+1}\,M_{2n+1}\right\}:, \tag{3.25}$$

with the matrices P, M and Q now being $(N-3)! \times (N-3)!$ matrices, following from

$$\begin{aligned}M_{2n+1} &= F\,|_{\zeta_{2n+1}}\,,\\ P &= 1 + \sum_{n\geqslant 1}\zeta_2^n\,P_{2n} := 1 + \sum_{n\geqslant 1}\zeta_2^n\ \ F\,|_{\zeta_2^n}\,,\end{aligned} \tag{3.26}$$

with $P_{2n} = P|_{\zeta_2^n}$ and Q given in (3.16). The polynomial structure of the matrices M, P and Q is further exhibited[6] in [28].

What makes the form (3.25) appealing is the disentanglement of the full α'-expansion into several contributions accounting for different classes of MZVs: P comprising powers of ζ_2, M accounting for ζ_{2n+1} and powers thereof and Q encapsulating the MZVs $\zeta_{n_1,\ldots,n_r}$ of depth $r > 1$ greater than 1. As we shall see in section 4 the specific form (3.25) is bolstered by the decomposition of motivic MZVs. It is interesting to note that in (3.16) MZVs of depth greater than 1 $r > 1$ appear with commutators as:

$$\zeta_{n_1,\ldots,n_r}\ \ [M_{n_2}, [M_{n_3}, \ldots, [M_{n_r}, M_{n_1}]] \ldots]. \tag{3.27}$$

This property turns out to have a crucial impact on the closed string amplitude; see section 6.

At weight 16 in (3.16) the term $\frac{1}{50}\,\zeta_{3,5}^2\ ([M_5, M_3])^2$ gives rise to the speculation that all terms in Q follow from expanding an exponential:

$$Q = \exp\left\{\tfrac{1}{5}\,\zeta_{3,5}\,[M_5, M_3] + \left(\tfrac{3}{14}\,\zeta_5^2 + \tfrac{1}{14}\,\zeta_{3,7}\right)\ [M_7, M_3] + \cdots\right\}. \tag{3.28}$$

In fact, at weight 18 we find the following terms[7]

$$\begin{aligned}\mathcal{A}|_{\zeta_{3,5}\zeta_{3,7}} = \tfrac{1}{5}\,\tfrac{1}{14}\,[M_7, M_3]\,[M_5, M_3] + \tfrac{208\,926}{894\,845}\,[M_3, [M_3, [M_7, M_5]]]\\ -\tfrac{69\,642}{894\,845}\,[M_3, [M_3, [M_9, M_3]]],\end{aligned}$$

[6] While the world-sheet integrals (3.2) for $N = 5$ can still be reduced to a set of single (variable) Gaussian hypergeometric functions ${}_3F_2$, their six- and higher-point versions comprise multiple Gaussian hypergeometric functions [14] of the type (3.4), whose expansions in α' are much more involved. Though computing some of these expansions has been accomplished at the six- [14, 15] and seven-point level [23, 29], a systematic approach is still lacking and will be presented in [28].

[7] Note the commutator relations: $[M_7, M_3][M_5, M_3] = [M_5, M_3][M_7, M_3]$ and $[M_3, [M_5, [M_7, M_3]]] = [M_5, [M_3, [M_7, M_3]]]$.

$$\begin{aligned}\mathcal{A}|_{\zeta_{3,5}\zeta_5^2} = \tfrac{1}{2}\,\tfrac{1}{5}\,[M_5, M_3]M_5^2 + \tfrac{1}{5}\,\tfrac{3}{14}\,[M_7, M_3]\,[M_5, M_3] + \tfrac{1}{5}\,[M_5, [M_5, M_3]]M_5 \\ + \tfrac{1800}{43\,867}\,[M_{11}, M_7] - \tfrac{22\,500}{570\,271}\,[M_{13}, M_5] + \tfrac{7200}{570\,271}\,[M_{15}, M_3] \\ - \tfrac{7044\,111\,243\,797}{6415\,252\,209\,080}\,[M_3, [M_3, [M_7, M_5]]] + \tfrac{2792\,059}{5702\,710}\,[M_5, [M_5, [M_5, M_3]]] \\ - \tfrac{2432\,943}{7983\,794}\,[M_5, [M_3, [M_7, M_3]]] - \tfrac{2818\,807\,834\,641}{6415\,252\,209\,080}\,[M_3, [M_3, [M_9, M_3]]],\end{aligned} \tag{3.29}$$

in agreement with the Ansatz (3.28).

Obviously, for $N = 4$ in (3.25) we have $Q = 1$ as all commutators vanish for the scalars M_{2n+1} given in (3.9). With this information (3.25) boils down to (3.8). So far, for $N = 6$ we have verified (3.25) up to α'^8. Further tests are in progress [28] and confirm (3.25).

3.4. Minimal depth representation with Euler sums

The choice of basis elements may follow some minimal intrinsic representation guided by the minimal depth representation and the choice of positive odd indices only. For MZVs this is achieved by also allowing for Euler sums as basis elements:

$$\zeta(\epsilon_1 n_1, \ldots, \epsilon_r n_r) = \sum_{0<k_1<\ldots<k_r} \prod_{l=1}^{r} \epsilon_l^{k_l}\, k_l^{-n_l}, \quad n_l \in \mathbf{N}^+,\ n_r \geqslant 2 \tag{3.30}$$

with signs $\epsilon_l = \pm 1$. For $M_{w,r}$ being the number of basis elements for MZVs when expressed in terms of Euler sums in a minimal depth representation at weight $w > 2$ and depth r we have $M_{12,2} = 2, M_{12,4} = 0,\ M_{15,3} = 3,\ M_{15,5} = 0, M_{16,2} = 3$ and $M_{16,4} = 2$ [20]. At weight 12 one may get rid of the basis element $\zeta_{1,1,4,6}$ with even entries at the cost of introducing the Euler sum $\zeta_{\bar{5},\bar{7}} := \zeta(-5, -7)$ [19]:

$$\begin{aligned}\zeta_{1,1,4,6} = -\tfrac{5045}{648}\,\zeta_{3,9} + 3\,\zeta_2\,\zeta_{3,7} + \tfrac{3}{5}\,\zeta_2^2\,\zeta_{3,5} - \tfrac{799}{72}\,\zeta_3\,\zeta_9 - \tfrac{5747}{432}\,\zeta_5\,\zeta_7 + 10\,\zeta_2\,\zeta_3\,\zeta_7 \\ + \tfrac{7}{2}\,\zeta_2\,\zeta_5^2 - \tfrac{1}{5}\,\zeta_2^2\,\zeta_3\,\zeta_5 + \tfrac{1}{12}\,\zeta_3^4 - \tfrac{18}{35}\,\zeta_2^3\,\zeta_3^2 + \tfrac{694\,891}{2837\,835}\,\zeta_2^6 - \tfrac{64}{27}\,\zeta_{\bar{5},\bar{7}}.\end{aligned} \tag{3.31}$$

Similarly, we may use the Euler sum $\zeta_{\bar{3},\bar{9}} := \zeta(-3, -9)$ to arrive at [30]:

$$\begin{aligned}\zeta_{1,1,4,6} = \tfrac{371}{144}\,\zeta_{3,9} + 3\,\zeta_2\,\zeta_{3,7} + \tfrac{3}{5}\,\zeta_2^2\,\zeta_{3,5} - \tfrac{3131}{144}\,\zeta_3\,\zeta_9 + \tfrac{107}{24}\,\zeta_5\,\zeta_7 + 10\,\zeta_2\,\zeta_3\,\zeta_7 \\ + \tfrac{7}{2}\,\zeta_2\,\zeta_5^2 - \tfrac{1}{5}\,\zeta_2^2\,\zeta_3\,\zeta_5 + \tfrac{1}{12}\,\zeta_3^4 - \tfrac{18}{35}\,\zeta_2^3\,\zeta_3^2 - \tfrac{117\,713}{2627\,625}\,\zeta_2^6 + \tfrac{64}{9}\,\zeta_{\bar{3},\bar{9}}.\end{aligned} \tag{3.32}$$

In [19] the object $A_{5,7}$

$$A_{5,7} = \zeta_{\bar{5},\bar{7}} + \zeta_{5,7} \tag{3.33}$$

has been argued to play a special status within the Euler sums, since it is quite similar to the MZVs. With this (3.31) can be written:

$$\begin{aligned}\zeta_{1,1,4,6} = -\tfrac{7967}{1944}\,\zeta_{3,9} + 3\,\zeta_2\,\zeta_{3,7} + \tfrac{3}{5}\,\zeta_2^2\,\zeta_{3,5} - \tfrac{799}{72}\,\zeta_3\,\zeta_9 + \tfrac{11\,431}{1296}\,\zeta_5\,\zeta_7 + 10\,\zeta_2\,\zeta_3\,\zeta_7 \\ + \tfrac{7}{2}\,\zeta_2\,\zeta_5^2 - \tfrac{1}{5}\,\zeta_2^2\,\zeta_3\,\zeta_5 + \tfrac{1}{12}\,\zeta_3^4 - \tfrac{18}{35}\,\zeta_2^3\,\zeta_3^2 - \tfrac{5607\,853}{6081\,075}\,\zeta_2^6 - \tfrac{64}{27}\,A_{5,7}.\end{aligned} \tag{3.34}$$

Clearly, the above three equations (3.31), (3.32) and (3.34) are related by the identities:

$$\begin{aligned}\zeta_{5,7} &= \tfrac{14}{9}\,\zeta_{3,9} + \tfrac{28}{3}\,\zeta_5\,\zeta_7 - \tfrac{776\,224}{1576\,575}\,\zeta_2^6, \\ \zeta_{\bar{3},\bar{9}} &= -\tfrac{1}{3}\,\zeta_{\bar{5},\bar{7}} - \tfrac{13\,429}{9216}\,\zeta_{3,9} + \tfrac{1533}{1024}\,\zeta_3\,\zeta_9 - \tfrac{7673}{3072}\,\zeta_5\,\zeta_7 + \tfrac{10\,275\,263}{252252000}\,\zeta_2^6.\end{aligned} \tag{3.35}$$

We can write the weight 12 part Q_{12} of (3.16) in terms of Euler sums in a minimal depth representation and positive odd indices only in the following three ways corresponding to (3.31), (3.32) and (3.34), respectively;

$$\begin{aligned}Q_{12} = \left\{\tfrac{2}{9}\,\zeta_5\,\zeta_7 + \tfrac{1}{27}\,\zeta_{3,9}\right\}\,[M_9, M_3] + \tfrac{48}{691}\,\{[M_9, M_3] - 3\,[M_7, M_5]\} \\ \times \left\{\tfrac{694\,891}{2837\,835}\,\zeta_2^6 - \tfrac{7615}{432}\,\zeta_5\,\zeta_7 - \tfrac{595}{162}\,\zeta_{3,9} - \tfrac{64}{27}\,\zeta_{\bar{5},\bar{7}}\right\}\end{aligned}$$

$$= \left\{\tfrac{2}{9}\,\zeta_5\,\zeta_7 + \tfrac{1}{27}\,\zeta_{3,9}\right\}\,[M_9,M_3] + \tfrac{48}{691}\,\{[M_9,M_3] - 3\,[M_7,M_5]\}$$
$$\times\left\{-\tfrac{117\,713}{2627\,625}\,\zeta_2^6 + \tfrac{29}{216}\,\zeta_5\,\zeta_7 - \tfrac{511}{48}\,\zeta_3\,\zeta_9 + \tfrac{8669}{1296}\,\zeta_{3,9} + \tfrac{64}{9}\,\zeta_{\overline{3},\overline{9}}\right\}$$
$$= \left\{\tfrac{2}{9}\,\zeta_5\,\zeta_7 + \tfrac{1}{27}\,\zeta_{3,9}\right\}\,[M_9,M_3] + \tfrac{48}{691}\,\{[M_9,M_3] - 3\,[M_7,M_5]\}$$
$$\times\left\{-\tfrac{5607\,853}{6081\,075}\,\zeta_2^6 + \tfrac{5827}{1296}\,\zeta_5\,\zeta_7 + \tfrac{7}{486}\,\zeta_{3,9} - \tfrac{64}{27}\,A_{5,7}\right\}. \tag{3.36}$$

At weight 15 in (3.16) one may get rid of the basis element $\zeta_{1,1,3,4,6}$ with even entries at the cost of introducing the Euler sum $\zeta_{\overline{3},\overline{5},\overline{7}} := \zeta(-3,-5,-7)$ [19]:

$$\zeta_{1,1,3,4,6} = \tfrac{16\,663}{11\,664}\,\zeta_{3,3,9} + \tfrac{150\,481}{68\,040}\,\zeta_{5,3,7} - \tfrac{20651486329}{4082\,400}\,\zeta_{15} + \tfrac{1903}{120}\,\zeta_7\,\zeta_{3,5} - \tfrac{101\,437}{38\,880}\,\zeta_5\,\zeta_{3,7}$$
$$- \tfrac{1520\,827}{38\,880}\,\zeta_5^3 + 10\,\zeta_3\,\zeta_{1,1,4,6} + \tfrac{162\,823}{3888}\,\zeta_3\,\zeta_{3,9} - \tfrac{93\,619}{1296}\,\zeta_3\,\zeta_5\,\zeta_7 + \tfrac{3601}{48}\,\zeta_3^2\,\zeta_9$$
$$- \tfrac{17}{20}\,\zeta_3^5 + \tfrac{14}{5}\,\zeta_2\,\zeta_{3,5,5} - 2\,\zeta_2\,\zeta_{3,3,7} + \tfrac{31\,753\,363}{12\,960}\,\zeta_2\zeta_{13} - \tfrac{21}{2}\,\zeta_2\,\zeta_5\,\zeta_{3,5}$$
$$- 27\,\zeta_2\,\zeta_3\,\zeta_{3,7} - \tfrac{61}{2}\,\zeta_2\,\zeta_3\,\zeta_5^2 - 84\,\zeta_2\,\zeta_3^2\,\zeta_7 - 4\,\zeta_2^2\,\zeta_{3,3,5} + \tfrac{979\,621}{1701}\,\zeta_2^2\,\zeta_{11}$$
$$- 5\,\zeta_2^2\,\zeta_3\,\zeta_{3,5} + \tfrac{9}{2}\,\zeta_2^2\,\zeta_3^2\,\zeta_5 - \tfrac{490670609}{3572\,100}\,\zeta_2^3\,\zeta_9 + \tfrac{186}{35}\,\zeta_2^3\,\zeta_3^3 - \tfrac{1455\,253}{283\,500}\,\zeta_2^4\,\zeta_7$$
$$+ \tfrac{4049\,341}{311\,850}\,\zeta_2^5\,\zeta_5 + \tfrac{12\,073\,102}{1488\,375}\,\zeta_2^6\,\zeta_3 + \tfrac{1408}{81}\,A_{3,5,7}. \tag{3.37}$$

More precisely, with the relations (3.37) and (3.34) the combination $\zeta_3\zeta_{1,1,4,6} + \zeta_{1,1,3,4,6}$ can be eliminated to cast the weight 15 part Q_{15} in terms of Euler sums in a minimal depth representation and positive odd indices only:

$$Q_{15} = \left\{\tfrac{1339}{30}\,\zeta_2\,\zeta_{13} + \tfrac{128}{45}\,\zeta_2^2\,\zeta_{11} - \tfrac{236}{4725}\,\zeta_2^3\,\zeta_9 - \tfrac{184}{2625}\,\zeta_4^2\,\zeta_7 - \tfrac{64}{5775}\,\zeta_2^5\,\zeta_5\right.$$
$$\left.-\tfrac{2}{45}\,\zeta_5^3 - \tfrac{1}{15}\,\zeta_7\,\zeta_{3,5} - \tfrac{2}{45}\,\zeta_5\,\zeta_{3,7} + \tfrac{1}{27}\,\zeta_{3,3,9}\right\}\,[M_3,[M_9,M_3]]$$
$$+\left\{-\tfrac{143}{20}\,\zeta_2\,\zeta_{13} - \tfrac{11}{35}\,\zeta_2^2\,\zeta_{11} + \tfrac{68}{1225}\,\zeta_2^3\,\zeta_9 + \tfrac{11}{70}\,\zeta_5^3 + \tfrac{24}{875}\,\zeta_2^4\,\zeta_7 + \tfrac{48}{13\,475}\,\zeta_2^5\,\zeta_5\right.$$
$$\left.+\tfrac{1}{5}\,\zeta_7\,\zeta_{3,5} + \tfrac{3}{35}\,\zeta_5\,\zeta_{3,7} - \tfrac{1}{70}\,\zeta_{5,3,7}\right\}\,[M_5,[M_7,M_3]] + \tfrac{2}{15}\,\zeta_{5,3,7}\,[M_3,[M_7,M_5]]$$
$$+\tfrac{48}{7601}\,\left\{\tfrac{1408}{81}\,A_{3,5,7} - \tfrac{704}{27}\,A_{5,7}\,\zeta_3 - \tfrac{20\,651\,486\,329}{4082\,400}\,\zeta_{15} + \tfrac{1149\,577}{5184}\,\zeta_2\,\zeta_{13}\right.$$
$$+\tfrac{1912\,097}{136\,080}\,\zeta_2^2\,\zeta_{11} - \tfrac{230\,351}{357\,210}\,\zeta_2^3\,\zeta_9 - \tfrac{414\,007}{283\,500}\,\zeta_2^4\,\zeta_7 - \tfrac{45\,779}{39\,690}\,\zeta_2^5\,\zeta_5 - \tfrac{24\,257}{3869\,775}\,\zeta_2^6\,\zeta_3$$
$$+\tfrac{77}{648}\,\zeta_3\,\zeta_5\,\zeta_7 + \tfrac{77}{3888}\,\zeta_{3,9}\,\zeta_3 + \tfrac{319}{3402}\,\zeta_5^3 - \tfrac{15\,983}{54\,432}\,\zeta_{3,7}\,\zeta_5 - \tfrac{781}{720}\,\zeta_{3,5}\,\zeta_7$$
$$\left.+\tfrac{1995\,367}{272\,160}\,\zeta_{5,3,7} + \tfrac{1309}{11\,664}\,\zeta_{3,3,9}\right\}\{[M_3,[M_9,M_3]] - 3\,[M_3,[M_7,M_5]]\}. \tag{3.38}$$

4. Motivic multiple zeta values

In this section we want to compare our findings (3.25) with the excellent work of Brown on the decomposition of motivic MZVs [5]. For this purpose, after reviewing some aspects of motivic MZVs, we determine the decomposition of motivic MZVs for weights 11 to 16.

4.1. Motivic aspects of multiple zeta values

An important question is how to explicitly describe the structure of the algebra $\mathcal{Z}$, which eventually allows one to get a grip on all algebraic MZV identities over **Q**. For this purpose the actual MZVs (2.1) are replaced by symbols (or motivic MZVs), which are elements of a certain algebra.

In this section we review some aspects of motivic MZVs [5]. The task is to lift the ordinary iterated integrals I_γ given in (2.5) to motivic versions I^m such that the standard relations are fulfilled. With an embedding $\sigma : F \hookrightarrow \mathbf{C}$ the iterated integrals I_γ can be upgraded to a framed mixed Tate motive over F (motivic iterated integral)

$$I^m(a_0;a_1,\ldots,a_n;a_{n+1}) \in \mathcal{H}(F), \quad a_0,\ldots,a_{n+1} \in F, \tag{4.1}$$

with $p_\sigma(I^m(a_0; a_1, \ldots, a_n; a_{n+1})) = I(\sigma(a_0); \sigma(a_1), \ldots, \sigma(a_n); \sigma(a_{n+1}))$ [4] and some number field F. The latter is a finite degree field extension of the field of rational numbers $\mathbf{Q}$. The symbols (4.1) are elements of a commutative graded Hopf algebra $\mathcal{H}(F)$:

$$\mathcal{H} = \bigoplus_{n\geqslant 0} \mathcal{H}_n. \tag{4.2}$$

The Hopf algebra[8] $\mathcal{H}$ implies a product given by the shuffle product

$$I^m(x; a_1, \ldots, a_r; y) \cdot I^m(x; a_{r+1}, \ldots, a_{r+s}; y) = \sum_{\sigma\in\Sigma(r,s)} I^m(x; a_{\sigma(1)}, \ldots, a_{\sigma(r+s)}; y), \tag{4.3}$$

with $\Sigma(r, s) = \{\sigma \in \Sigma(r + s) \mid \sigma^{-1}(1) < \cdots < \sigma^{-1}(r) \cap \sigma^{-1}(r + 1) < \cdots < \sigma^{-1}(r + s)\}$ and $a_i, x, y \in \{0, 1\}$ and the coproduct Δ acting on the elements I^m as [4]

$$\Delta\, I^m(a_0; a_1, \ldots, a_n; a_{n+1}) = \sum_{0=i_0<i_1<\cdots<i_k<i_{k+1}=n+1} I^m(a_0; a_{i_1}, \ldots, a_{i_k}; a_{n+1})$$
$$\otimes \prod_{p=0}^{k} I^m(a_{i_p}; a_{i_p+1}, \ldots, a_{i_{p+1}-1}; a_{i_{p+1}}), \tag{4.4}$$

with $0 \leqslant k \leqslant n$ and $a_i \in F$. As in (2.7) by (4.1) with $a_i \in \{0, 1\}$ we may define the motivic versions $\zeta^m_{n_1,\ldots,n_r}$ of the MZVs $\zeta_{n_1,\ldots,n_r}$, i.e. by (4.1) the motivic MZVs are defined as

$$\zeta^m_{n_1,\ldots,n_r} = (-1)^r\, I^m(0; \rho(n_1, \ldots, n_r); 1) \in \mathcal{H}_w(\mathbf{Z}), \tag{4.5}$$

with the weight $w = \sum_{l=1}^r n_l$ and ρ given in (2.6). Any symbol $I^m(a_0; a_1, \ldots, a_n; a_{n+1})$, with $a_i \in \{0, 1\}$, can be reduced to a linear combination of elements of the form (4.5), with $n_i \geqslant 1$, $n_r \geqslant 2$ and $w = N$. The dimension of the space of motivic MZVs of weight k is equal to d_k, i.e. $\dim_{\mathbf{Q}}(\mathcal{H}_k) = d_k$. The map $\mathcal{H}_k \to \mathcal{Z}_k$ is surjective, i.e. $\dim_{\mathbf{Q}}(\mathcal{Z}_k) \leqslant \dim_{\mathbf{Q}}(\mathcal{H}_k) = d_k$ [21, 31]. By this certain identities between MZVs can be lifted to their motivic versions [5].

There is a non-canonical isomorphism[9]

$$\mathcal{H} \simeq \mathcal{A} \otimes_{\mathbf{Q}} \mathbf{Q}[\zeta^m_2], \quad \mathcal{A} = \mathcal{H}/\zeta^m_2\mathcal{H}, \tag{4.6}$$

with the first factor graded by the weight, i.e. $\mathcal{A} = \bigoplus_{n\geqslant 0} \mathcal{A}_n$.

To explicitly describe the structure of $\mathcal{H}$ one introduces the (trivial) algebra-comodule:

$$\mathcal{U} = \mathbf{Q}\langle f_3, f_5, \ldots\rangle \otimes_{\mathbf{Q}} \mathbf{Q}[f_2]. \tag{4.7}$$

The first factor $\mathcal{U}' = \mathcal{U}/f_2\mathcal{U}$ is a cofree Hopf-algebra on the cogenerators f_{2r+1} in degree $2r+1 \geqslant 3$, whose basis consists of all non-commutative words in the f_{2i+1}. The multiplication on $\mathcal{U}'$ is given by the shuffle product ш

$$f_{i_1}\ldots f_{i_r} ш f_{i_{r+1}}\ldots f_{i_{r+s}} = \sum_{\sigma\in\Sigma(r,s)} f_{i_{\sigma(1)}}\ldots f_{i_{\sigma(r+s)}}, \tag{4.8}$$

with $\Sigma(r, s)$ given after equation (4.3). The Hopf-algebra $\mathcal{U}'$ is isomorphic to the space of non-commutative polynomials in f_{2i+1}. The element f_2 commutes with all f_{2r+1}. Again, there is a grading $\mathcal{U}_k$ on $\mathcal{U}$, with $\dim(\mathcal{U}_k) = d_k$. Then, there exists a morphism ϕ of graded algebra-comodules

$$\phi : \mathcal{H} \longrightarrow \mathcal{U}, \tag{4.9}$$

[8] Coalgebras and comodules are dualizations of algebras and modules. A coalgebra over a field K (K-coalgebra) is a K-module V over K endowed with the coproduct $\Delta : V \to V \otimes V$ and the counit $\epsilon : V \to K$. Moreover, a Hopf algebra is an algebra $\mathcal{A}$ with multiplication $\mu : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$, i.e. $\mu(x_1 \otimes x_2) = x_1 \cdot x_2$ and associativity. At the same time it is also a coalgebra with coproduct $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ and coassociativity such that the product and coproduct are compatible: $\Delta(x_1 \cdot x_2) = \Delta(x_1) \cdot \Delta(x_2)$, with $x_1, x_2 \in \mathcal{A}$.

[9] Note that in contrast to [4], in this setup ζ^m_2 is non-zero.

normalized[10] by:

$$\phi\big(\zeta_n^m\big) = f_n, \quad n \geqslant 2. \tag{4.10}$$

The map (4.9) sends every motivic MZV to a non-commutative polynomial in the f_i. Furthermore, (4.9) respects the shuffle multiplication rule (4.8):

$$\phi(x_1x_2) = \phi(x_1) ш \phi(x_2), \quad x_1, x_2 \in \mathcal{H}. \tag{4.11}$$

It is believed that the isomorphism $\mathcal{Z}_k \simeq \mathcal{U}_k$ of graded algebras over $\mathbf{Q}$ holds.

The motivic MZVs have a hidden structure, which is revealed by the action of motivic derivations. The latter are derived from the coaction $\Delta : \mathcal{H} \to \mathcal{A} \otimes_{\mathbf{Q}} \mathcal{H}$ [5, 21]

$$\Delta\, I^m(a_0; a_1, \ldots, a_n; a_{n+1}) = \sum_{\substack{0=i_0<i_1<\cdots<\\ <i_k<i_{k+1}=n+1}} \Pi\left(\prod_{p=0}^{k} I^m(a_{i_p}; a_{i_p+1}, \ldots, a_{i_{p+1}-1}; a_{i_{p+1}})\right) \otimes I^m(a_0; a_{i_1}, \ldots, a_{i_k}; a_{n+1}), \tag{4.12}$$

which represents a modification of the coproduct (4.4). Here, Π is the projector $\Pi : \mathcal{H} \to \mathcal{A}$ acting on ζ_2^m as $\zeta_2^m \stackrel{\Pi}{\longrightarrow} 0$. The derivations $D_r : \mathcal{H}_n \to \mathcal{A}_r \otimes_{\mathbf{Q}} \mathcal{H}_{n-r} \stackrel{\pi\otimes id}{\to} \mathcal{L}_r \otimes_{\mathbf{Q}} \mathcal{H}_{n-r}$ on $\mathcal{H}$ are defined as the infinitesimal version of the coaction (4.12) [5]

$$D_r\, I^m(a_0; a_1, \ldots, a_n; a_{n+1}) = \sum_{p=0}^{n-r} \pi\left(I^a(a_p; a_{p+1}, \ldots, a_{p+r}; a_{p+r+1})\right) \otimes I^m(a_0; a_1, \ldots, a_p, a_{p+r+1}, \ldots, a_n; a_{n+1}), \tag{4.13}$$

with the projection $\pi : \mathcal{A} \to \mathcal{L}$ onto the Lie coalgebra $\mathcal{L} = \frac{\mathcal{A}_{>0}}{\mathcal{A}_{>0}\mathcal{A}_{>0}}$ describing all indecomposable (irreducible) elements of $\mathcal{A}$. By this we have $D_{2r}I^m \equiv 0$. Above, the symbols I^a denote elements of the quotient $\mathcal{A} = \mathcal{H}/\zeta_2\mathcal{H}$.

4.2. On the decomposition of motivic multi zeta values

The coalgebra structure (4.7) underlying the motivic MZVs can be used to decompose any MZV into a basis. Let us now describe the decomposition of motivic MZVs up to some weight $M \geqslant 2$ [5].

We are looking for decompositions in the $\mathbf{Q}$-vector space $\mathcal{H}_N,\ 2 \leqslant N \leqslant M$ spanned by the symbols (4.5), with $w = N$ and $n_i \geqslant 1,\ n_r \geqslant 2$. To check that a (conjectural) polynomial basis B of motivic MZVs $\bigoplus_{2\leqslant n\leqslant M} \mathcal{H}_n$ up to weight M indeed represents a polynomial basis of motivic MZVs up to weight M for $n \leqslant N$ for each set B_n of elements of B of weight n, one constructs the map (4.9):

$$\phi : B_n \longrightarrow \mathcal{U}_n, \quad n \leqslant N. \tag{4.14}$$

This map assigns to every element of our basis B (of weight at most N) a $\mathbf{Q}$-linear combination of monomials

$$f_{2i_1+1} \cdots f_{2i_r+1}\, f_2^k, \quad r, k \geqslant 0, i_1, \ldots, i_r \geqslant 1,\ 2\,(i_1 + \cdots + i_r) + r + 2k = n, \tag{4.15}$$

which are basis elements of the $\mathbf{Q}$-vector space $\mathcal{U}_n$ supplemented by the multiplication rule $ш : \mathcal{U}_m \times \mathcal{U}_n \to \mathcal{U}_{n+m}$ given in (4.8). Actually, ϕ can be extended to the vector space $\mathcal{H}_n$:

$$\phi : \mathcal{H}_n \longrightarrow \mathcal{U}_n, \quad n \leqslant N. \tag{4.16}$$

For the basis B we must have: $\dim_{\mathbf{Q}}(\langle B\rangle_N) = d_N,\ 2 \leqslant N \leqslant M$, with $\langle B\rangle_N$ the $\mathbf{Q}$-vector space spanned by monomials in the elements of B of total additive weight N. Furthermore, we have

$$B \supset B^0 = \{\zeta_2^m\} \cup \{\zeta_3^m, \ldots, \zeta_{2r+1}^m\}, \tag{4.17}$$

[10] Note that there is no canonical choice of ϕ and the latter depends on the choice of motivic generators of $\mathcal{H}$.

with $r = \lfloor (M-1)/2 \rfloor$. For the elements of B^0 the map ϕ is given by (4.10). For the remaining elements of B the explicit construction of ϕ is performed inductively, i.e. from (4.14) the case $n = N+1$ is determined. To find $\phi(\xi)$ for a general $\xi \in B_{N+1}$, with $\xi = I^m(a_0; a_1, \ldots, a_{N+1}; a_{N+2})$ according to (4.5), we need to compute the coefficients

$$\xi_{2r+1} = \sum_{p=0}^{N-2r} c^{\phi}_{2r+1}\left(I^m(a_p; a_{p+1}, \ldots, a_{p+2r+1}; a_{p+2r+2})\right) \\ \times \phi\left(I^m(a_0; a_1, \ldots, a_p, a_{p+2r+2}, \ldots, a_{N+1}; a_{N+2})\right) \in \mathcal{U}_{N-2r}, \quad 3 \leqslant 2r+1 \leqslant N \tag{4.18}$$

in the expansion:

$$\phi(\xi) = \sum_{3\leqslant 2r+1 \leqslant N} f_{2r+1}\, \xi_{2r+1} \in \mathcal{U}_{N+1}. \tag{4.19}$$

Above, the operator $c^{\phi}_{2r+1}(\xi)$ with $\xi \in \mathcal{H}_{2r+1}$ determines the rational coefficient of f_{2r+1} in the monomial $\phi(\xi) \in \mathcal{U}_{2r+1}$. Note that the right hand side of (4.18) only involves elements I^m from $\mathcal{H}_{\leqslant N}$ for which ϕ has already been determined.

The above construction allows one to assign a **Q**-linear combination of monomials to every element $\zeta^m_{n_1,\ldots,n_r}$. The map[11] ϕ sends every motivic MZV of weight less than or equal to N to a non-commutative polynomial in the f_is. Inverting this map gives the decomposition of $\zeta^m_{n_1,\ldots,n_r}$ w.r.t. the basis B_n, with $n = \sum_{l=1}^{r} n_l$. In other words, the derivations (4.20) are used to detect elements in $\mathcal{U}$ and to decompose any motivic MZV ξ into a candidate basis B.

In [5] the map (4.14) and the decomposition are explicitly worked out up to weight 10. For example, one finds

$$\phi(\zeta^m_{3,5}) = -5\, f_5 f_3, \quad \phi(\zeta^m_{3,7}) = -14\, f_7 f_3 - 6\, f_5 f_5, \tag{4.21}$$

and at weight 10 one has for $\xi_{10} \in \mathcal{H}_{10}$ the following decomposition

$$\xi_{10} = a_0\left(\zeta^m_2\right)^5 + a_1\left(\zeta^m_2\right)^2\left(\zeta^m_3\right)^2 + a_2\, \zeta^m_2\, \zeta^m_3\, \zeta^m_5 \\ + a_3\left(\zeta^m_5\right)^2 + a_4\, \zeta^m_2\, \zeta^m_{3,5} + a_5\, \zeta^m_3\, \zeta^m_7 + a_6\, \zeta^m_{3,7}, \tag{4.22}$$

with the operators:

$$a_1 = \tfrac{1}{2}\, c_2^2\, \partial_3^2, \;\; a_2 = c_2\, \partial_5 \partial_3, \;\; a_3 = \tfrac{1}{2}\, \partial_5^2 + \tfrac{3}{14}\, [\partial_7, \partial_3], \\ a_4 = \tfrac{1}{5}\, c_2\, [\partial_5, \partial_3], \;\; a_5 = \partial_7 \partial_3, \;\; a_6 = \tfrac{1}{14}\, [\partial_7, \partial_3] \tag{4.23}$$

acting on $\phi(\xi_{10})$. The derivation operators $\partial_{2n+1} : \mathcal{U} \to \mathcal{U}$ are defined as [5]:

$$\partial_{2n+1}(f_{i_1}, \ldots, f_{i_r}) = \begin{cases} f_{i_2}, \ldots, f_{i_r}, & i_1 = 2n+1, \\ 0, & \text{otherwise}, \end{cases} \tag{4.24}$$

with $\partial_{2n+1} f_2 = 0$. Furthermore, we have the product rule for the shuffle product:

$$\partial_{2n+1}(a \,⧢\, b) = \partial_{2n+1} a \,⧢\, b + a \,⧢\, \partial_{2n+1} b, \quad a, b \in \mathcal{U}'. \tag{4.25}$$

Finally, c_2^n takes the coefficient of f_2^n.

It seems very amusing that the coefficients (4.23) and the commutator structure agree exactly with (3.23). Therefore, MZVs encapsulate the α'-expansion of the open superstring amplitude.

[11] The choice of ϕ describes for each weight $2r+1$ the motivic derivation operators ∂^{ϕ}_{2r+1} acting on the space of motivic MZVs $\partial^{\phi}_{2r+1} : \mathcal{H} \to \mathcal{H}$ [5]

$$\partial^{\phi}_{2r+1} = (c^{\phi}_{2r+1} \otimes id) \circ D_{2r+1}, \tag{4.20}$$

with D_{2n+1} given in (4.13) and the coefficient function c^{ϕ}_{2r+1}, introduced above.

4.3. Decomposition of motivic multi zeta values for weights 11 *through* 16

In order to bolster this connection, in the following subsections we determine the decompositions ξ_w of any motivic MZV for the weights $11 \leqslant w \leqslant 16$.

For a given weight w we proceed as described in [5]: in the lines of tables 1–3 at weight w we first detect the new elements B_w to be added to constitute the conjectural basis B up to weight w. For these new elements B_w we then compute their coefficients (4.18) or motivic derivations ∂^{ϕ}_{2r+1} by applying the relations $(R0)-(R4)$ given in section 5.1 of [5]. Equipped with these results we then determine the map (4.19) by using the findings from the lower weights. After having derived the map (4.19) for all d_w basis elements of $\langle B\rangle_w$ we can construct the basis for $\mathcal{U}_w$ and eventually the operator ξ_w.

For the depth two case $\zeta^m_{n_1,n_2}$ there exists a closed formula, which computes the map $\phi(\zeta^m_{n_1,n_2})$ directly [32]. Our results for $\phi(\zeta^m_{3,9}), \phi(\zeta^m_{3,11}), \phi(\zeta^m_{5,9}), \phi(\zeta^m_{3,13})$ and $\phi(\zeta^m_{5,11})$ agree with what this formula gives. However, as will become clear in the following, beyond depth two the computations involve new aspects and become rather involved.

4.3.1. Decomposition at weight 11. At weight 11 we take the following set of motivic MZVs

$$B = \{\, \zeta^m_2,\ \zeta^m_3,\ \zeta^m_5,\ \zeta^m_7,\ \zeta^m_{3,5},\ \zeta^m_9,\ \zeta^m_{3,7},\ \zeta^m_{11},\ \zeta^m_{3,3,5}\, \} \tag{4.26}$$

as independent algebra generators up to weight 11. In [5] up to weight $n \leqslant 10$ to each element of B an element of $\mathcal{U}$ is associated by the map ϕ given in (4.14). Hence, we only need to compute $\phi(\zeta^m_{3,3,5})$, which according to (4.18) requires the following derivatives:

$$\begin{aligned} &\partial^{\phi}_3 \zeta^m_{3,3,5} = 0, && \partial^{\phi}_5 \zeta^m_{3,3,5} = -5\,\zeta^m_{3,3} = -\tfrac{5}{2}\,(\zeta^m_3)^2 + \tfrac{4}{7}\,(\zeta^m_2)^3,\\ &\partial^{\phi}_7 \zeta^m_{3,3,5} = -\tfrac{6}{5}\,(\zeta^m_2)^2, && \partial^{\phi}_9 \zeta^m_{3,3,5} = -45\,\zeta^m_2. \end{aligned} \tag{4.27}$$

From these results the expression (4.19) gives rise to:

$$\phi(\zeta^m_{3,3,5}) = -\tfrac{5}{2}\, f_5(f_3 \,ш\, f_3) + \tfrac{4}{7}\, f_5 f_2^3 - \tfrac{6}{5}\, f_7 f_2^2 - 45\, f_9 f_2. \tag{4.28}$$

Gathering the information about the lower weight basis $\mathcal{U}_{k\leqslant 10}$ with (4.28) we can construct the following basis for $\mathcal{U}_{11}$:

$$\begin{aligned} &-\tfrac{5}{2}\, f_5(f_3 \,ш\, f_3) + \tfrac{4}{7}\, f_5 f_2^3 - \tfrac{6}{5}\, f_7 f_2^2 - 45\, f_9 f_2,\\ &\qquad -5\,(f_5 f_3)\,ш\, f_3,\ f_{11},\ f_3 \,ш\, f_3 \,ш\, f_5,\ f_3 \,ш\, f_3 \,ш\, f_3 f_2,\\ &\qquad f_9 f_2,\ f_7 f_2^2,\ f_5 f_2^3,\ f_3 f_2^4. \end{aligned} \tag{4.29}$$

This basis gives rise to the following decomposition of any motivic MZV ξ_{11} of weight 11

$$\begin{aligned} \xi_{11} = {}& a_1\, \zeta^m_{3,3,5} + a_2\, \zeta^m_{3,5}\, \zeta^m_3 + a_3\, \zeta^m_{11} + a_4\, (\zeta^m_3)^2\, \zeta^m_5 + a_5\, \zeta^m_2\, (\zeta^m_3)^3\\ &+ a_6\, \zeta^m_2\, \zeta^m_9 + a_7\, (\zeta^m_2)^2\, \zeta^m_7 + a_8\, (\zeta^m_2)^3\, \zeta^m_5 + a_9\, (\zeta^m_2)^4\, \zeta^m_3 \end{aligned} \tag{4.30}$$

with[12] the following operators

$$\begin{aligned} &a_1 = \tfrac{1}{5}\,[\partial_3,[\partial_5,\partial_3]],\ a_2 = \tfrac{1}{5}\,[\partial_5,\partial_3]\partial_3,\\ &a_3 = \partial_{11},\ a_4 = \tfrac{1}{2}\,\partial_5\partial_3^2,\ a_5 = \tfrac{1}{6}\,c_2\,\partial_3^3,\\ &a_6 = c_2\,\partial_9 + 9\,[\partial_3,[\partial_5,\partial_3]],\ a_7 = c_2^2\,\partial_7 + \tfrac{6}{25}\,[\partial_3,[\partial_5,\partial_3]],\\ &a_8 = c_2^3\,\partial_5 - \tfrac{4}{35}\,[\partial_3,[\partial_5,\partial_3]],\ a_9 = c_2^4\,\partial_3 \end{aligned} \tag{4.31}$$

acting on $\phi(\xi_{11})$.

[12] The following relations $[\partial_3,[\partial_5,\partial_3]] f_3 \,ш\, f_3 \,ш\, f_5 = 0$ and $[\partial_3,[\partial_5,\partial_3]] f_5 f_3 \,ш\, f_3 = 0$ are useful. More generally, we have: $[\partial_a,[\partial_b,\partial_c]] f_a \,ш\, f_b \,ш\, f_c = 0$ and $[\partial_a,[\partial_b,\partial_a]] f_b f_a \,ш\, f_a = 0$.

4.3.2. Decomposition at weight 12. Next, at weight 12 we take the set of motivic MZVs

$$B = \{ \zeta_2^m,\ \zeta_3^m,\ \zeta_5^m,\ \zeta_7^m,\ \zeta_{3,5}^m,\ \zeta_9^m,\ \zeta_{3,7}^m,\ \zeta_{11}^m,\ \zeta_{3,3,5}^m,\ \zeta_{3,9}^m,\ \zeta_{1,1,4,6}^m \} \tag{4.32}$$

as independent algebra generators up to weight 12. We need to compute $\phi(\zeta_{3,9}^m)$ and $\phi(\zeta_{1,1,4,6}^m)$, which require the following derivatives

$$\begin{aligned} &\partial_3^\phi \zeta_{3,9}^m = 0, && \partial_7^\phi \zeta_{3,9}^m = -15\ \zeta_5^m, \\ &\partial_5^\phi \zeta_{3,9}^m = -6\ \zeta_7^m, && \partial_9^\phi \zeta_{3,9}^m = -27\ \zeta_3^m, \end{aligned} \tag{4.33}$$

and

$$\begin{aligned} \partial_3^\phi \zeta_{1,1,4,6}^m &= \tfrac{1}{3}\ (\zeta_3^m)^3 - \tfrac{799}{72}\ \zeta_9^m + 10\ \zeta_7^m \zeta_2^m - \tfrac{1}{5}\ \zeta_5^m \big(\zeta_2^m\big)^2 - \tfrac{36}{35}\ \zeta_3^m \big(\zeta_2^m\big)^3, \\ \partial_5^\phi \zeta_{1,1,4,6}^m &= 29\ \zeta_7^m - 11\ \zeta_5^m \zeta_2^m - \tfrac{16}{5}\ \zeta_3^m \big(\zeta_2^m\big)^2, \\ \partial_7^\phi \zeta_{1,1,4,6}^m &= \tfrac{1133}{16}\ \zeta_5^m - 32\ \zeta_3^m \zeta_2^m, \\ \partial_9^\phi \zeta_{1,1,4,6}^m &= \tfrac{1799}{18}\ \zeta_3^m, \end{aligned} \tag{4.34}$$

respectively. With the derivatives (4.33) and (4.34) we determine the following maps:

$$\begin{aligned} \phi(\zeta_{3,9}^m) &= -6\ f_5 f_7 - 15\ f_7 f_5 - 27\ f_9 f_3, \\ \phi(\zeta_{1,1,4,6}^m) &= \tfrac{1799}{18}\ f_9 f_3 - 32\ f_7 f_3 f_2 + \tfrac{1133}{16}\ f_7 f_5 + 29\ f_5 f_7 - 11\ f_5^2 f_2 - \tfrac{16}{5}\ f_5 f_3 f_2^2 \\ &\quad + \tfrac{1}{3}\ f_3 (f_3 ш f_3 ш f_3) - \tfrac{799}{72}\ f_3 f_9 + 10\ f_3 f_7 f_2 - \tfrac{1}{5}\ f_3 f_5 f_2^2 - \tfrac{36}{35}\ f_3^2 f_2^3. \end{aligned} \tag{4.35}$$

Inspecting the lower weight basis $\mathcal{U}_{k\leqslant 12}$ with (4.35) we have the following basis for $\mathcal{U}_{12}$:

$$\begin{aligned} &\tfrac{1799}{18}\ f_9 f_3 - 32\ f_7 f_3 f_2 + \tfrac{1133}{16}\ f_7 f_5 + 29\ f_5 f_7 - 11\ f_5^2 f_2 - \tfrac{16}{5}\ f_5 f_3 f_2^2 \\ &\quad + \tfrac{1}{3}\ f_3 (f_3 ш f_3 ш f_3) - \tfrac{799}{72}\ f_3 f_9 + 10\ f_3 f_7 f_2 - \tfrac{1}{5}\ f_3 f_5 f_2^2 - \tfrac{36}{35}\ f_3^2 f_2^3, \\ &\quad -6\ f_5 f_7 - 15\ f_7 f_5 - 27\ f_9 f_3,\ f_3 ш f_9,\ f_5 ш f_7,\ f_3 ш f_3 ш f_3 ш f_3, \\ &\quad (-14 f_7 f_3 - 6 f_5^2) f_2,\ -5\ f_5 f_3 f_2^2,\ f_5 ш f_5 f_2,\ f_3 ш f_7 f_2, \\ &\quad f_3 ш f_5 f_2^2,\ f_3 ш f_3 f_2^3,\ f_2^6. \end{aligned} \tag{4.36}$$

Therefore, the decomposition of any motivic MZV ξ_{12} of weight 12 assumes the form

$$\begin{aligned} \xi_{12} &= a_1\ \zeta_{1,1,4,6}^m + a_2\ \zeta_{3,9}^m + a_3\ \zeta_9^m\ \zeta_3^m + a_4\ \zeta_7^m\ \zeta_5^m + a_5\ \big(\zeta_3^m\big)^4 + a_6\ \zeta_{3,7}^m\ \zeta_2^m \\ &\quad + a_7\ \zeta_{3,5}^m\ \big(\zeta_2^m\big)^2 + a_8\ \big(\zeta_5^m\big)^2\ \zeta_2^m + a_9\ \zeta_7^m\ \zeta_3^m\ \zeta_2^m + a_{10}\ \zeta_5^m\ \zeta_3^m\ \big(\zeta_2^m\big)^2 \\ &\quad + a_{11}\ \big(\zeta_3^m\big)^2\ \big(\zeta_2^m\big)^3 + a_{12}\ \big(\zeta_2^m\big)^6, \end{aligned} \tag{4.37}$$

with the following operators

$$\begin{aligned} &a_1 = \tfrac{48}{691}\ ([\partial_9, \partial_3] - 3\ [\partial_7, \partial_5]),\ a_2 = \tfrac{1}{27}\ [\partial_9, \partial_3] + \tfrac{2665}{648}\ a_1, \\ &a_3 = \partial_9 \partial_3 + \tfrac{799}{72}\ a_1,\ a_4 = \partial_7 \partial_5 + \tfrac{2}{9}\ [\partial_9, \partial_3] - \tfrac{467}{108}\ a_1,\ a_5 = \tfrac{1}{24}\ \partial_3^4 - \tfrac{1}{12}\ a_1, \\ &a_6 = \tfrac{1}{14}\ c_2\ [\partial_7, \partial_3] - 3\ a_1,\ a_7 = \tfrac{1}{5}\ c_2^2\ [\partial_5, \partial_3] - \tfrac{3}{5}\ a_1, \\ &a_8 = c_2\ \big(\tfrac{1}{2}\ \partial_5^2 + \tfrac{3}{14}\ [\partial_7, \partial_3]\big) - \tfrac{7}{2}\ a_1,\ a_9 = c_2\ \partial_7 \partial_3 - 10\ a_1, \\ &a_{10} = c_2^2\ \partial_5 \partial_3 + \tfrac{1}{5}\ a_1,\ a_{11} = \tfrac{1}{2}\ c_2^3\ \partial_3^2 + \tfrac{18}{35}\ a_1,\ a_{12} = c_2^6 \end{aligned} \tag{4.38}$$

acting on $\phi(\xi_{12})$.

4.3.3. Decomposition at weight 13. At weight 13 the following set of motivic MZVs

$$B = \{ \zeta_2^m,\ \zeta_3^m,\ \zeta_5^m,\ \zeta_7^m,\ \zeta_{3,5}^m,\ \zeta_9^m,\ \zeta_{3,7}^m,\ \zeta_{11}^m,\ \zeta_{3,3,5}^m,\ \zeta_{3,9}^m,\ \zeta_{1,1,4,6}^m,\ \zeta_{3,3,7}^m,\ \zeta_{3,5,5}^m \} \tag{4.39}$$

represents independent algebra generators up to weight 13. We need to compute $\phi(\zeta_{3,3,7}^m)$ and $\phi(\zeta_{3,5,5}^m)$, which require the following derivatives

$$\begin{aligned} &\partial_3^\phi \zeta_{3,3,7}^m = 0, && \partial_9^\phi \zeta_{3,3,7}^m = -\tfrac{56}{5}\ \big(\zeta_2^m\big)^2, \\ &\partial_5^\phi \zeta_{3,3,7}^m = -6\ \zeta_{3,5}^m, && \partial_{11}^\phi \zeta_{3,3,7}^m = -\tfrac{407}{2}\ \zeta_2^m, \\ &\partial_7^\phi \zeta_{3,3,7}^m = -7\ \big(\zeta_3^m\big)^2 + \tfrac{32}{35}\ \big(\zeta_2^m\big)^3, \end{aligned} \tag{4.40}$$

and

$$\begin{aligned} &\partial_3^\phi \zeta^m_{3,5,5} = 0, && \partial_9^\phi \zeta^m_{3,5,5} = -10\,(\zeta^m_2)^2,\\ &\partial_5^\phi \zeta^m_{3,5,5} = -5\,\zeta^m_{3,5}, && \partial_{11}^\phi \zeta^m_{3,5,5} = -\tfrac{275}{2}\,\zeta^m_2,\\ &\partial_7^\phi \zeta^m_{3,5,5} = 0, \end{aligned} \tag{4.41}$$

respectively. The derivatives (4.40) and (4.41) give rise to the maps:

$$\begin{aligned} \phi(\zeta^m_{3,3,7}) &= 30\,f_5^2 f_3 - 7\,f_7(f_3 ш f_3) + \tfrac{32}{35}\,f_7 f_2^3 - \tfrac{56}{5}\,f_9 f_2^2 - \tfrac{407}{2}\,f_{11} f_2,\\ \phi(\zeta^m_{3,5,5}) &= 25\,f_5^2 f_3 - 10\,f_9 f_2^2 - \tfrac{275}{2}\,f_{11} f_2. \end{aligned} \tag{4.42}$$

Collecting the information about the lower weight basis $\mathcal{U}_{k\leqslant 13}$ with (4.42) we have the following basis for $\mathcal{U}_{13}$:

$$\begin{aligned} &30\,f_5^2 f_3 - 7\,f_7(f_3 ш f_3) + \tfrac{32}{35}\,f_7 f_2^3 - \tfrac{56}{5}\,f_9 f_2^2 - \tfrac{407}{2}\,f_{11} f_2,\\ &25\,f_5^2 f_3 - 10\,f_9 f_2^2 - \tfrac{275}{2}\,f_{11} f_2,\ f_{13},\ (-14 f_7 f_3 - 6 f_5^2)\,ш\, f_3,\\ &\qquad -5\,(f_5 f_3)\,ш\, f_5,\ f_7\,ш\, f_3\,ш\, f_3,\ f_5\,ш\, f_5\,ш\, f_5,\\ &\qquad -\tfrac{5}{2}\,f_5(f_3\,ш\, f_3) f_2 + \tfrac{4}{7}\,f_5 f_2^4 - \tfrac{6}{5}\,f_7 f_2^3 - 45\,f_9 f_2^2,\ -5(f_5 f_3)\,ш\, f_3 f_2,\\ &\qquad f_{11} f_2,\ f_5\,ш\, f_3\,ш\, f_3 f_2,\ f_3\,ш\, f_3\,ш\, f_3 f_2^2,\ f_9 f_2^2,\ f_7 f_2^3,\ f_5 f_2^4,\ f_3 f_2^5. \end{aligned} \tag{4.43}$$

Therefore, we have the following decomposition of any motivic MZV ξ_{13} of weight 13:

$$\begin{aligned} \xi_{13} &= a_1\,\zeta^m_{3,3,7} + a_2\,\zeta^m_{3,5,5} + a_3\,\zeta^m_{13} + a_4\,\zeta^m_{3,7}\,\zeta^m_3 + a_5\,\zeta^m_{3,5}\,\zeta^m_5 + a_6\,\zeta^m_7(\zeta^m_3)^2 + a_7\,(\zeta^m_5)^2\zeta^m_3\\ &\quad + a_8\,\zeta^m_{3,3,5}\,\zeta^m_2 + a_9\,\zeta^m_{3,5}\,\zeta^m_3\,\zeta^m_2 + a_{10}\,\zeta^m_{11}\,\zeta^m_2 + a_{11}\,\zeta^m_5\,(\zeta^m_3)^2\,\zeta^m_2\\ &\quad + a_{12}\,(\zeta^m_3)^3\,(\zeta^m_2)^2 + a_{13}\,\zeta^m_9\,(\zeta^m_2)^2 + a_{14}\,\zeta^m_7\,(\zeta^m_2)^3 + a_{15}\,\zeta^m_5\,(\zeta^m_2)^4 + a_{16}\,\zeta^m_3\,(\zeta^m_2)^5, \end{aligned} \tag{4.44}$$

with the following operators

$$\begin{aligned} &a_1 = \tfrac{1}{14}\,[\partial_3,[\partial_7,\partial_3]],\ a_2 = \tfrac{1}{25}\,[\partial_5,[\partial_5,\partial_3]] - \tfrac{3}{35}\,[\partial_3,[\partial_7,\partial_3]],\ a_3 = \partial_{13},\\ &a_4 = \tfrac{1}{14}\,[\partial_7,\partial_3]\partial_3,\ a_5 = \tfrac{1}{5}\,\partial_5[\partial_5,\partial_3],\ a_6 = \tfrac{1}{2}\partial_7\partial_3^2,\ a_7 = \tfrac{3}{14}\,[\partial_7,\partial_3]\partial_3 + \tfrac{1}{2}\,\partial_5^2\partial_3,\\ &a_8 = \tfrac{1}{5}\,c_2\,[\partial_3,[\partial_5,\partial_3]],\ a_9 = \tfrac{1}{5}\,c_2[\partial_5,\partial_3]\partial_3,\\ &a_{10} = c_2\partial_{11} + \tfrac{11}{2}\,[\partial_5,[\partial_5,\partial_3]] + \tfrac{11}{4}\,[\partial_3,[\partial_7,\partial_3]],\ a_{11} = \tfrac{1}{2}\,c_2\partial_5\partial_3^2,\ a_{12} = \tfrac{1}{6}\,c_2^2\partial_3^3,\\ &a_{13} = c_2^2\partial_9 + 9\,c_2[\partial_3,[\partial_5,\partial_3]] + \tfrac{2}{5}\,[\partial_5,[\partial_5,\partial_3]] - \tfrac{2}{35}\,[\partial_3,[\partial_7,\partial_3]],\\ &a_{14} = c_2^3\partial_7 + \tfrac{6}{25}\,c_2[\partial_3,[\partial_5,\partial_3]] - \tfrac{16}{245}\,[\partial_3,[\partial_7,\partial_3]],\\ &a_{15} = c_2^4\partial_5 - \tfrac{4}{35}\,c_2[\partial_3,[\partial_5,\partial_3]],\ a_{16} = c_2^5\partial_3 \end{aligned} \tag{4.45}$$

acting on $\phi(\xi_{13})$.

4.3.4. *Decomposition at weight* 14.

At weight 14 we take the following set of motivic MZVs

$$\begin{aligned} B = \{&\zeta^m_2,\ \zeta^m_3,\ \zeta^m_5,\ \zeta^m_7,\ \zeta^m_{3,5},\ \zeta^m_9,\ \zeta^m_{3,7},\ \zeta^m_{11},\ \zeta^m_{3,3,5},\ \zeta^m_{3,9},\ \zeta^m_{1,1,4,6},\ \zeta^m_{3,3,7},\ \zeta^m_{3,5,5},\\ &\zeta^m_{3,3,3,5},\ \zeta^m_{3,11},\ \zeta^m_{5,9}\,\} \end{aligned} \tag{4.46}$$

as independent algebra generators up to weight 14. Hence, we only need to compute the maps $\phi(\zeta^m_{3,11}),\ \phi(\zeta^m_{5,9})$ and $\phi(\zeta^m_{3,3,3,5})$, which require the following derivatives

$$\begin{aligned} &\partial_3^\phi \zeta^m_{3,11} = 0, && \partial_9^\phi \zeta^m_{3,11} = -28\,\zeta^m_5,\\ &\partial_5^\phi \zeta^m_{3,11} = -6\,\zeta^m_9, && \partial_{11}^\phi \zeta^m_{3,11} = -44\,\zeta^m_3,\\ &\partial_7^\phi \zeta^m_{3,11} = -15\,\zeta^m_7, \end{aligned} \tag{4.47}$$

and

$$\begin{aligned}&\partial_3^\phi \zeta_{5,9}^m = 0, && \partial_9^\phi \zeta_{5,9}^m = -69\ \zeta_5^m,\\ &\partial_5^\phi \zeta_{5,9}^m = 0, && \partial_{11}^\phi \zeta_{3,5,5}^m = -165\ \zeta_3^m,\\ &\partial_7^\phi \zeta_{5,9}^m = -15\ \zeta_7^m, &&\end{aligned} \tag{4.48}$$

and

$$\begin{aligned}&\partial_3^\phi \zeta_{3,3,3,5}^m = 0, && \partial_9^\phi \zeta_{3,3,3,5}^m = -\tfrac{405}{2}\ \zeta_5^m + 90\ \zeta_3^m\ \zeta_2^m,\\ &\partial_5^\phi \zeta_{3,3,3,5}^m = -\tfrac{5}{6}\left(\zeta_3^m\right)^3 - \tfrac{5}{3}\ \zeta_9^m + \tfrac{4}{7}\ \zeta_3^m\left(\zeta_2^m\right)^3, && \partial_{11}^\phi \zeta_{3,3,3,5}^m = -15\ \zeta_3^m,\\ &\partial_7^\phi \zeta_{3,3,3,5}^m = -51\ \zeta_7^m + 30\ \zeta_5^m\ \zeta_2^m, &&\end{aligned} \tag{4.49}$$

respectively. These derivatives give rise to:

$$\begin{aligned}\phi(\zeta_{3,11}^m) &= -6\ f_5f_9 - 15\ f_7^2 - 28\ f_9f_5 - 44\ f_{11}\ f_3,\\ \phi(\zeta_{5,9}^m) &= -15\ f_7^2 - 69\ f_9f_5 - 165\ f_{11}\ f_3,\\ \phi(\zeta_{3,3,3,5}^m) &= -\tfrac{5}{6}\ f_5\ (f_3 ш f_3 ш f_3) - \tfrac{5}{3}\ f_5f_9 + \tfrac{4}{7}\ f_5f_3f_2^3 - 51f_7^2\\ &\quad +30\ f_7f_5f_2 - \tfrac{405}{2}\ f_9f_5 + 90\ f_9f_3f_2 - 15\ f_{11}f_3,\end{aligned} \tag{4.50}$$

respectively. Gathering the information about the lower weight basis $\mathcal{U}_{k\leqslant 13}$ with (4.50) we can construct the basis for $\mathcal{U}_{14}$ displayed in (A.1). This basis (A.1) gives rise to the following decomposition of any motivic MZV ξ_{14} of weight 14

$$\begin{aligned}\xi_{14} = a_1\ \zeta_{3,3,3,5}^m &+ a_2\ \zeta_{3,11}^m + a_3\ \zeta_{5,9}^m + a_4\ \zeta_{3,3,5}^m\ \zeta_3^m + a_5\ \zeta_{3,5}^m\left(\zeta_3^m\right)^2 + a_6\ \zeta_3^m\ \zeta_{11}^m\\ &+a_7\left(\zeta_3^m\right)^3\ \zeta_5^m + a_8\ \zeta_5^m\ \zeta_9^m + a_9\left(\zeta_7^m\right)^2 + a_{10}\ \zeta_{1,1,4,6}^m\ \zeta_2^m + a_{11}\ \zeta_{3,9}^m\ \zeta_2^m\\ &+a_{12}\ \zeta_3^m\ \zeta_9^m\ \zeta_2^m + a_{13}\ \zeta_5^m\ \zeta_7^m\ \zeta_2^m + a_{14}\left(\zeta_3^m\right)^4\ \zeta_2^m + a_{15}\ \zeta_{3,7}^m\left(\zeta_2^m\right)^2\\ &+a_{16}\ \zeta_{3,5}^m\left(\zeta_2^m\right)^3 + a_{17}\left(\zeta_5^m\right)^2\left(\zeta_2^m\right)^2 + a_{18}\ \zeta_7^m\ \zeta_3^m\left(\zeta_2^m\right)^2 + a_{19}\ \zeta_5^m\ \zeta_3^m\left(\zeta_2^m\right)^3\\ &+a_{20}\left(\zeta_3^m\right)^2\left(\zeta_2^m\right)^4 + a_{21}\left(\zeta_2^m\right)^7\end{aligned} \tag{4.51}$$

with the operators a_i acting on $\phi(\xi_{14})$ and given in (A.2).

4.3.5. Decomposition at weight 15. At weight 15 we have the following set of motivic MZVs

$$\begin{aligned}B = \{\ &\zeta_2^m,\ \zeta_3^m,\ \zeta_5^m,\ \zeta_7^m,\ \zeta_{3,5}^m,\ \zeta_9^m,\ \zeta_{3,7}^m,\ \zeta_{11}^m,\ \zeta_{3,3,5}^m,\ \zeta_{3,9}^m,\ \zeta_{1,1,4,6}^m,\ \zeta_{3,3,7}^m,\ \zeta_{3,5,5}^m,\\ &\zeta_{3,3,3,5}^m,\ \zeta_{3,11}^m,\ \zeta_{5,9}^m,\ \zeta_{5,3,7}^m,\ \zeta_{3,3,9}^m,\ \zeta_{1,1,3,4,6}^m\}\end{aligned} \tag{4.52}$$

as independent algebra generators up to weight 15. Hence, we only need to compute the maps $\phi(\zeta_{5,3,7}^m)$, $\phi(\zeta_{3,3,9}^m)$ and $\phi(\zeta_{1,1,3,4,6}^m)$, which require the following derivatives

$$\begin{aligned}&\partial_3^\phi \zeta_{5,3,7}^m = 0, && \partial_9^\phi \zeta_{5,3,7}^m = \tfrac{136}{35}\left(\zeta_2^m\right)^3,\\ &\partial_5^\phi \zeta_{5,3,7}^m = -3\left(\zeta_5^m\right)^2 + \tfrac{96}{385}\left(\zeta_2^m\right)^5 + 6\ \zeta_{3,7}^m, && \partial_{11}^\phi \zeta_{5,3,7}^m = -22\left(\zeta_2^m\right)^2,\\ &\partial_7^\phi \zeta_{5,3,7}^m = -14\ \zeta_{5,3}^m = -14\ \zeta_3^m\ \zeta_5^m + 14\ \zeta_{3,5}^m + \tfrac{48}{25}\left(\zeta_2^m\right)^4, && \partial_{13}^\phi \zeta_{5,3,7}^m = -\tfrac{1001}{2}\ \zeta_2^m\end{aligned} \tag{4.53}$$

and

$$\begin{aligned}&\partial_3^\phi \zeta_{3,3,9}^m = 0, && \partial_9^\phi \zeta_{3,3,9}^m = -\tfrac{27}{2}\left(\zeta_3^m\right)^2 - \tfrac{116}{35}\left(\zeta_2^m\right)^3,\\ &\partial_5^\phi \zeta_{3,3,9}^m = -6\ \zeta_{3,7}^m, && \partial_{11}^\phi \zeta_{3,3,9}^m = -\tfrac{252}{5}\left(\zeta_2^m\right)^2,\\ &\partial_7^\phi \zeta_{3,3,9}^m = -\tfrac{72}{175}\left(\zeta_2^m\right)^4 - 15\ \zeta_{3,5}^m, && \partial_{13}^\phi \zeta_{3,3,9}^m = -\tfrac{1209}{2}\ \zeta_2^m,\end{aligned} \tag{4.54}$$

and

$$\begin{aligned}
\partial_3^\phi \zeta^m_{1,1,3,4,6} &= \tfrac{74}{3}\,\zeta^m_5\zeta^m_7 - 83\,\zeta^m_3\zeta^m_9 - \tfrac{29}{9}\,\zeta^m_{3,9} - \zeta^m_{1,1,4,6} + 6\,\zeta^m_{3,7}\,\zeta^m_2 + \tfrac{8}{5}\,\zeta^m_{3,5}\,(\zeta^m_2)^2\\
&\quad + 8\,(\zeta^m_5)^2\,\zeta^m_2 + 42\,\zeta^m_3\zeta^m_7\zeta^m_2 + \tfrac{24}{5}\,\zeta^m_3\zeta^m_5(\zeta^m_2)^2 - \tfrac{1451\,972}{716\,625}\,(\zeta^m_2)^6\\
\partial_5^\phi \zeta^m_{1,1,3,4,6} &= -\tfrac{12\,263}{112}\,(\zeta^m_5)^2 - \tfrac{245}{2}\,\zeta^m_3\zeta^m_7 + \tfrac{145}{112}\,\zeta^m_{3,7} - \tfrac{25}{2}\,\zeta^m_{3,5}\zeta^m_2 + \tfrac{87}{2}\,\zeta^m_3\zeta^m_5\zeta^m_2\\
&\quad + \tfrac{15}{2}\,(\zeta^m_3)^2(\zeta^m_2)^2 + \tfrac{19\,939}{1617}\,(\zeta^m_2)^5,\\
\partial_7^\phi \zeta^m_{1,1,3,4,6} &= \tfrac{31}{4}\,\zeta^m_3\zeta^m_5 + \tfrac{481}{20}\,\zeta^m_{3,5} - 12\,(\zeta^m_3)^2\zeta^m_2 + \tfrac{6404}{2625}\,(\zeta^m_2)^4,\\
\partial_9^\phi \zeta^m_{1,1,3,4,6} &= -\tfrac{5599}{72}\,(\zeta^m_3)^2 + \tfrac{25\,687}{630}\,(\zeta^m_2)^3,\\
\partial_{11}^\phi \zeta^m_{1,1,3,4,6} &= \tfrac{28\,519}{60}\,(\zeta^m_2)^2,\\
\partial_{13}^\phi \zeta^m_{1,1,3,4,6} &= \tfrac{56\,717}{120}\,\zeta^m_2,
\end{aligned} \tag{4.55}$$

respectively. These derivatives give rise to:

$$\begin{aligned}
\phi(\zeta^m_{5,3,7}) &= -3\,f_5\,(f_5 ш f_5) + \tfrac{96}{385}\,f_5f_2^5 - 6\,f_5\,(14f_7f_3 + 6f_5^2) - 14\,f_7(f_3 ш f_5)\\
&\quad -70\,f_7f_5f_3 + \tfrac{48}{25}\,f_7f_2^4 + \tfrac{136}{35}\,f_9f_2^3 - 22\,f_{11}f_2^2 - \tfrac{1001}{2}\,f_{13}f_2,\\
\phi(\zeta^m_{3,3,9}) &= 6\,f_5\,(14f_7f_3 + 6f_5^2) - \tfrac{72}{175}\,f_7f_2^4 + 75\,f_7f_5f_3 - \tfrac{27}{2}\,f_9\,(f_3 ш f_3)\\
&\quad -\tfrac{116}{35}\,f_9f_2^3 - \tfrac{252}{5}\,f_{11}f_2^2 - \tfrac{1209}{2}\,f_{13}f_2,\\
\phi(\zeta^m_{1,1,3,4,6}) &= -\tfrac{29}{9}\,f_3\,\phi(\zeta^m_{3,9}) - f_3\,\phi(\zeta^m_{1,1,4,6}) + \tfrac{74}{3}\,f_3(f_5 ш f_7) - 83\,f_3(f_3 ш f_9)\\
&\quad -6\,f_3(14f_7f_3 + 6f_5^2)f_2 - 8\,f_3f_5f_3f_2^2 + 8\,f_3(f_5 ш f_5)f_2\\
&\quad +42\,f_3(f_3 ш f_7)f_2 + \tfrac{24}{5}\,f_3(f_3 ш f_5)f_2^2 - \tfrac{1451\,972}{716\,625}\,f_3f_2^6\\
&\quad -\tfrac{12\,263}{112}\,f_5(f_5 ш f_5) - \tfrac{245}{2}\,f_5(f_3 ш f_7) - \tfrac{145}{112}\,f_5(14f_7f_3 + 6f_5^2)\\
&\quad +\tfrac{125}{2}\,f_5^2f_3f_2 + \tfrac{87}{2}\,f_5(f_3 ш f_5)f_2 + \tfrac{15}{2}\,f_5(f_3 ш f_3)f_2^2 + \tfrac{19\,939}{1617}\,f_5f_2^5,\\
&\quad +\tfrac{31}{4}\,f_7(f_3 ш f_5) - \tfrac{481}{4}\,f_7f_5f_3 - 12\,f_7(f_3 ш f_3)f_2 + \tfrac{6404}{2625}\,f_7f_2^4,\\
&\quad -\tfrac{5599}{72}\,f_9(f_3 ш f_3) + \tfrac{25\,687}{630}\,f_9f_2^3 + \tfrac{28\,519}{60}\,f_{11}\,f_2^2 + \tfrac{56\,717}{120}\,f_{13}f_2,
\end{aligned} \tag{4.56}$$

respectively. The maps $\phi(\zeta^m_{3,9})$ and $\phi(\zeta^m_{1,1,4,6})$ are given in (4.35). With the information about the lower weight basis $\mathcal{U}_{k\leqslant 14}$ with (4.56) we can construct the basis for $\mathcal{U}_{15}$ shown in (A.6). This basis (A.6) gives rise to the following decomposition of any motivic MZV ξ_{15} of weight 15

$$\begin{aligned}
\xi_{15} &= a_1\,\zeta^m_{1,1,3,4,6} + a_2\,\zeta^m_{3,3,9} + a_3\,\zeta^m_{5,3,7} + a_4\,\zeta^m_{15} + a_5\,\zeta^m_{1,1,4,6}\,\zeta^m_3 + a_6\,\zeta^m_{3,9}\,\zeta^m_3\\
&\quad + a_7\,\zeta^m_9\,(\zeta^m_3)^2 + a_8\,\zeta^m_3\,\zeta^m_5\,\zeta^m_7 + a_9\,(\zeta^m_3)^5 + a_{10}\,\zeta^m_{3,7}\,\zeta^m_5 + a_{11}\,(\zeta^m_5)^3 + a_{12}\,\zeta^m_{3,5}\,\zeta^m_7\\
&\quad + a_{13}\,\zeta^m_2\,\zeta^m_{3,3,7} + a_{14}\,\zeta^m_2\,\zeta^m_{3,5,5} + a_{15}\,\zeta^m_2\,\zeta^m_{13} + a_{16}\,\zeta^m_2\,\zeta^m_3\,\zeta^m_{3,7} + a_{17}\,\zeta^m_2\,\zeta^m_5\,\zeta^m_{3,5}\\
&\quad + a_{18}\,\zeta^m_2\,(\zeta^m_3)^2\,\zeta^m_7 + a_{19}\,\zeta^m_2\,\zeta^m_3\,(\zeta^m_5)^2 + a_{20}\,(\zeta^m_2)^2\,\zeta^m_{3,3,5} + a_{21}\,(\zeta^m_2)^2\,\zeta^m_3\,\zeta^m_{3,5}\\
&\quad + a_{22}\,(\zeta^m_2)^2\,\zeta^m_{11} + a_{23}\,\zeta^m_5\,(\zeta^m_2)^2\,(\zeta^m_3)^2 + a_{24}\,(\zeta^m_2)^3\,(\zeta^m_3)^3 + a_{25}\,(\zeta^m_2)^3\,\zeta^m_9\\
&\quad + a_{26}\,(\zeta^m_2)^4\,\zeta^m_7 + a_{27}\,(\zeta^m_2)^5\,\zeta^m_5 + a_{28}\,(\zeta^m_2)^6\,\zeta^m_3,
\end{aligned} \tag{4.57}$$

with the operators a_i acting on $\phi(\xi_{15})$ and collected in (A.7).

4.3.6. Decomposition at weight 16. Finally, at weight 16 we have the following set of motivic MZVs

$$\begin{aligned}
B = \{&\zeta^m_2,\ \zeta^m_3,\ \zeta^m_5,\ \zeta^m_7,\ \zeta^m_{3,5},\ \zeta^m_9,\ \zeta^m_{3,7},\ \zeta^m_{11},\ \zeta^m_{3,3,5},\ \zeta^m_{3,9},\ \zeta^m_{1,1,4,6},\ \zeta^m_{3,3,7},\ \zeta^m_{3,5,5},\ \zeta^m_{3,3,3,5},\\
&\zeta^m_{3,11},\ \zeta^m_{5,9},\ \zeta^m_{5,3,7},\ \zeta^m_{3,3,9},\ \zeta^m_{1,1,3,4,6},\ \zeta^m_{3,3,3,7},\ \zeta^m_{3,3,5,5},\ \zeta^m_{3,13},\ \zeta^m_{5,11},\ \zeta^m_{1,1,6,8}\}
\end{aligned} \tag{4.58}$$

as independent algebra generators up to weight 16. Hence, we only need to compute the maps $\phi(\zeta^m_{3,3,3,7})$, $\phi(\zeta^m_{3,3,5,5})$, $\phi(\zeta^m_{3,13})$, $\phi(\zeta^m_{5,11})$ and $\phi(\zeta^m_{1,1,6,8})$, which require the following derivatives

$$\begin{aligned}
&\partial^\phi_3 \zeta^m_{3,3,3,7} = 0, \quad \partial^\phi_5 \zeta^m_{3,3,3,7} = -6\, \zeta^m_{3,3,5},\\
&\partial^\phi_7 \zeta^m_{3,3,3,7} = -\tfrac{775}{6}\, \zeta^m_9 - \tfrac{7}{3}\, \big(\zeta^m_3\big)^3 + 63\, \zeta^m_7 \zeta^m_2 + \tfrac{36}{5}\, \zeta^m_5 \big(\zeta^m_2\big)^2 + \tfrac{8}{5}\, \zeta^m_3 \big(\zeta^m_2\big)^3,\\
&\partial^\phi_9 \zeta^m_{3,3,3,7} = -476\, \zeta^m_7 + 280\, \zeta^m_5 \zeta^m_2,\\
&\partial^\phi_{11} \zeta^m_{3,3,3,7} = -\tfrac{3723}{4}\, \zeta^m_5 + 407\, \zeta^m_3 \zeta^m_2, \quad \partial^\phi_{13} \zeta^m_{3,3,3,7} = -165\, \zeta^m_3,
\end{aligned} \tag{4.59}$$

$$\begin{aligned}
&\partial^\phi_3 \zeta^m_{3,3,5,5} = 0, \quad \partial^\phi_5 \zeta^m_{3,3,5,5} = -5\, \zeta^m_{3,3,5},\\
&\partial^\phi_7 \zeta^m_{3,3,5,5} = \tfrac{381}{2}\, \zeta^m_9 - 105\, \zeta^m_7 \zeta^m_2 - 6\, \zeta^m_5 \big(\zeta^m_2\big)^2,\\
&\partial^\phi_9 \zeta^m_{3,3,5,5} = 70\, \zeta^m_7 + 25\, \zeta^m_5 \zeta^m_2 - 36\, \zeta^m_3 \big(\zeta^m_2\big)^2,\\
&\partial^\phi_{11} \zeta^m_{3,3,5,5} = -\tfrac{1881}{4}\, \zeta^m_5 + 275\, \zeta^m_3 \zeta^m_2, \quad \partial^\phi_{13} \zeta^m_{3,3,5,5} = \tfrac{99}{2}\, \zeta^m_3,
\end{aligned} \tag{4.60}$$

$$\begin{aligned}
&\partial^\phi_3 \zeta^m_{3,13} = 0, && \partial^\phi_9 \zeta^m_{3,13} = -28\, \zeta^m_7,\\
&\partial^\phi_5 \zeta^m_{3,13} = -6\, \zeta^m_{11}, && \partial^\phi_{11} \zeta^m_{3,13} = -45\, \zeta^m_5,\\
&\partial^\phi_7 \zeta^m_{3,13} = -15\, \zeta^m_9, && \partial^\phi_{13} \zeta^m_{3,13} = -65\, \zeta^m_3,
\end{aligned} \tag{4.61}$$

$$\begin{aligned}
&\partial^\phi_3 \zeta^m_{5,11} = 0, && \partial^\phi_9 \zeta^m_{5,11} = -70\, \zeta^m_7,\\
&\partial^\phi_5 \zeta^m_{5,11} = 0, && \partial^\phi_{11} \zeta^m_{5,11} = -209\, \zeta^m_5,\\
&\partial^\phi_7 \zeta^m_{5,11} = -15\, \zeta^m_9, && \partial^\phi_{13} \zeta^m_{5,11} = -429\, \zeta^m_3,
\end{aligned} \tag{4.62}$$

and

$$\begin{aligned}
\partial^\phi_3 \zeta^m_{1,1,6,8} &= -\tfrac{5}{7}\, \zeta^m_{3,3,7} + \tfrac{6}{7}\, \zeta^m_{3,5,5} - \tfrac{2}{7}\, \zeta^m_3 \zeta^m_{3,7} - \tfrac{8497}{42}\, \zeta^m_{13} + \tfrac{8}{7}\, \zeta^m_3 \big(\zeta^m_5\big)^2 + \big(\zeta^m_3\big)^2 \zeta^m_7\\
&\quad + 137\, \zeta^m_{11} \zeta^m_2 + \tfrac{11}{7}\, \zeta^m_9 \big(\zeta^m_2\big)^2 - \tfrac{848}{245}\, \zeta^m_7 \big(\zeta^m_2\big)^3 - \tfrac{48}{35}\, \zeta^m_5 \big(\zeta^m_2\big)^4 - \tfrac{816}{2695}\, \zeta^m_3 \big(\zeta^m_2\big)^5,\\
\partial^\phi_5 \zeta^m_{1,1,6,8} &= -\tfrac{2}{5}\, \zeta^m_{3,3,5} - \tfrac{18\,211}{240}\, \zeta^m_{11} + \big(\zeta^m_3\big)^2 \zeta^m_5 + \tfrac{71}{2}\, \zeta^m_9 \zeta^m_2 + \tfrac{163}{25}\, \zeta^m_7 \big(\zeta^m_2\big)^2\\
&\quad + \tfrac{36}{35}\, \zeta^m_5 \big(\zeta^m_2\big)^3 - \tfrac{132}{175}\, \zeta^m_3 \big(\zeta^m_2\big)^4,\\
\partial^\phi_7 \zeta^m_{1,1,6,8} &= 72\, \zeta^m_9 + \big(\zeta^m_3\big)^3 - 22\, \zeta^m_7 \zeta^m_2 - 7\, \zeta^m_5 \big(\zeta^m_2\big)^2 - \tfrac{116}{35}\, \zeta^m_3 \big(\zeta^m_2\big)^3,\\
\partial^\phi_9 \zeta^m_{1,1,6,8} &= \tfrac{26\,921}{72}\, \zeta^m_7 - \tfrac{277}{2}\, \zeta^m_5 \zeta^m_2 - 41\, \zeta^m_3 \big(\zeta^m_2\big)^2,\\
\partial^\phi_{11} \zeta^m_{1,1,6,8} &= \tfrac{11\,536}{15}\, \zeta^m_5 - \tfrac{727}{2}\, \zeta^m_3 \zeta^m_2, \quad \partial^\phi_{13} \zeta^m_{1,1,6,8} = \tfrac{28\,513}{25}\, \zeta^m_3,
\end{aligned} \tag{4.63}$$

respectively. These derivatives give rise to:

$$\begin{aligned}
\phi(\zeta^m_{3,3,3,7}) &= -6\, f_5\, \phi\big(\zeta^m_{3,3,5}\big) - \tfrac{775}{6}\, f_7 f_9 - \tfrac{7}{3}\, f_7 (f_3 ш f_3 ш f_3) + 63\, f_7^2 f_2 + \tfrac{36}{5}\, f_7 f_5 f_2^2\\
&\quad + \tfrac{8}{5}\, f_7 f_3 f_2^3 - 476\, f_9 f_7 + 280\, f_9 f_5 f_2 - \tfrac{3723}{4}\, f_{11} f_5 + 407\, f_{11} f_3 f_2 - 165\, f_{13} f_3,\\
\phi(\zeta^m_{3,3,5,5}) &= -5\, f_5\, \phi\big(\zeta^m_{3,3,5}\big) + \tfrac{381}{2}\, f_7 f_9 - 105\, f_7^2 f_2 - 6\, f_7 f_5 f_2^2 + 70\, f_9 f_7 + 25\, f_9 f_5 f_2\\
&\quad - 36\, f_9 f_3 f_2^2 - \tfrac{1881}{4}\, f_{11} f_5 + 275\, f_{11} f_3 f_2 + \tfrac{99}{2}\, f_{13} f_3,\\
\phi(\zeta^m_{3,13}) &= -6\, f_5 f_{11} - 15\, f_7 f_9 - 28\, f_9 f_7 - 45\, f_{11} f_5 - 65\, f_{13} f_3,\\
\phi(\zeta^m_{5,11}) &= -15\, f_7 f_9 - 70\, f_9 f_7 - 209\, f_{11} f_5 - 429\, f_{13} f_3,\\
\phi(\zeta^m_{1,1,6,8}) &= -\tfrac{5}{7}\, f_3\, \phi(\zeta^m_{3,3,7}) + \tfrac{6}{7}\, f_3\, \phi(\zeta^m_{3,5,5}) + \tfrac{2}{7}\, f_3 [f_3 ш (14 f_7 f_3 + 6 f_5^2)] - \tfrac{8497}{42}\, f_3 f_{13}\\
&\quad + \tfrac{8}{7}\, f_3 (f_3 ш f_5 ш f_5) + f_3 (f_3 ш f_3 ш f_7) + 137\, f_3 f_{11} f_2 + \tfrac{11}{7}\, f_3 f_9 f_2^2\\
&\quad - \tfrac{848}{245}\, f_3 f_7 f_2^3 - \tfrac{48}{35}\, f_3 f_5 f_2^4 - \tfrac{816}{2695}\, f_3 f_3 f_2^5 - \tfrac{2}{5}\, f_5\, \phi\big(\zeta^m_{3,3,5}\big) - \tfrac{18\,211}{240}\, f_5 f_{11}\\
&\quad + f_5 (f_3 ш f_3 ш f_5) + \tfrac{71}{2}\, f_5 f_9 f_2 + \tfrac{163}{25}\, f_5 f_7 f_2^2 + \tfrac{36}{35}\, f_5 f_5 f_2^3 - \tfrac{132}{175}\, f_5 f_3 f_2^4,\\
&\quad + 72\, f_7 f_9 + f_7 (f_3 ш f_3 ш f_3) - 22\, f_7 f_7 f_2 - 7\, f_7 f_5 f_2^2 - \tfrac{116}{35}\, f_7 f_3 f_2^3 + \tfrac{26\,921}{72}\, f_9 f_7\\
&\quad - \tfrac{277}{2}\, f_9 f_5 f_2 - 41\, f_9 f_3 f_2^2 + \tfrac{11\,536}{15}\, f_{11} f_5 - \tfrac{727}{2}\, f_{11} f_3 f_2 + \tfrac{28\,513}{25}\, f_{13} f_3,
\end{aligned} \tag{4.64}$$

respectively. The maps $\phi\big(\zeta^m_{3,3,5}\big), \phi(\zeta^m_{3,3,7})$ and $\phi(\zeta^m_{3,5,5})$ are given in (4.28) and (4.42), respectively. Gathering the information about the lower weight basis $\mathcal{U}_{k\leqslant 15}$ with (4.64) we can construct the basis for $\mathcal{U}_{16}$ shown in (A.8). This basis (A.8) gives rise to the following decomposition of any motivic MZV ξ_{16} of weight 16

$$\begin{aligned}\xi_{16} = {} & a_1\, \zeta^m_{1,1,6,8} + a_2\, \zeta^m_{3,3,3,7} + a_3\, \zeta^m_{3,3,5,5} + a_4\, \zeta^m_{3,13} + a_5\, \zeta^m_{5,11} + a_6\, \zeta^m_3\, \zeta^m_{3,3,7} \\ & + a_7\, \zeta^m_3\, \zeta^m_{3,5,5} + a_8\, \zeta^m_{13}\, \zeta^m_3 + a_9\, \zeta^m_{3,7}\, \big(\zeta^m_3\big)^2 + a_{10}\, \zeta^m_{3,5}\, \zeta^m_3\, \zeta^m_5 + a_{11}\, \zeta^m_7 \big(\zeta^m_3\big)^3 \\ & + a_{12}\, \big(\zeta^m_5\big)^2\, \big(\zeta^m_3\big)^2 + a_{13}\, \zeta^m_9\, \zeta^m_7 + a_{14}\, (\zeta^m_{3,5})^2 + a_{15}\, \zeta^m_{11}\, \zeta^m_5 + a_{16}\, \zeta^m_{3,3,5}\, \zeta^m_5 \\ & + a_{17}\, \zeta^m_2\, \zeta^m_3\, \zeta^m_{3,3,5} + a_{18}\, \zeta^m_{3,5}\, \big(\zeta^m_3\big)^2\, \zeta^m_2 + a_{19}\, \zeta^m_{11}\, \zeta^m_3\, \zeta^m_2 + a_{20}\, \zeta^m_5\, \big(\zeta^m_3\big)^3\, \zeta^m_2 \\ & + a_{21}\, \big(\zeta^m_3\big)^4\, \big(\zeta^m_2\big)^2 + a_{22}\, \zeta^m_9\, \zeta^m_3\, \big(\zeta^m_2\big)^2 + a_{23}\, \zeta^m_7\, \zeta^m_3\, \big(\zeta^m_2\big)^3 + a_{24}\, \zeta^m_5\, \zeta^m_3\, \big(\zeta^m_2\big)^4 \\ & + a_{25}\, \big(\zeta^m_3\big)^2\, \big(\zeta^m_2\big)^5 + a_{26}\, \zeta^m_{3,3,3,5}\, \zeta^m_2 + a_{27}\, \zeta^m_{3,11}\, \zeta^m_2 + a_{28}\, \zeta^m_{5,9}\, \zeta^m_2 + a_{29}\, \zeta^m_9\, \zeta^m_5\, \zeta^m_2 \\ & + a_{30}\, \big(\zeta^m_7\big)^2\, \zeta^m_2 + a_{31}\, \zeta^m_{1,1,4,6}\, \big(\zeta^m_2\big)^2 + a_{32}\, \zeta^m_{3,9}\, \big(\zeta^m_2\big)^2 + a_{33}\, \zeta^m_7\, \zeta^m_5\, \big(\zeta^m_2\big)^2 \\ & + a_{34}\, \zeta^m_{3,7}\, \big(\zeta^m_2\big)^3 + a_{35}\, \zeta^m_{3,5}\, \big(\zeta^m_2\big)^4 + a_{36}\, \big(\zeta^m_5\big)^2\, \big(\zeta^m_2\big)^3 + a_{37}\, \big(\zeta^m_2\big)^8, \end{aligned} \tag{4.65}$$

with the operators a_i acting on $\phi(\xi_{16})$ and listed in (A.9).

4.3.7. Comments on regularizing the coproduct and the map ϕ. Some terms in the sum of the coproduct (4.4) may imply divergences [4, 6, 16]. Divergences of multiple polylogarithms are end-point divergences, i.e. the poles in the integrand (2.5) coincide with the endpoints of the path γ. A canonical regularization has been introduced in [16] by shifting the endpoints by a small parameter ϵ:

$$I^m(0; a_1, \dots, a_n; 1) \to I^m(\epsilon; a_1, \dots, a_n; 1-\epsilon). \tag{4.66}$$

Expanding the latter w.r.t. small ϵ gives a polynomial in $\ln \epsilon$. Its constant term defines the regularized value $\hat{I}^m(0; a_1, \dots, a_n; 1)$. The coproduct in the non-generic case is defined by replacing in the sum of (4.4) every multiple polylogarithm $I^m(0; a_1, \dots, a_n; 1)$ by its regularized value $\hat{I}^m(0; a_1, \dots, a_n; 1)$ [4, 16].

Also the coaction (4.12) and therefore (4.13) and (4.18) may be plagued by divergences. We have regularized the terms in the sum (4.18) in the same way as described above for the coproduct (4.4). The problem, which only affects the first factor $c^\phi_{2r+1}(\cdots)$ of the terms in (4.18), occurs only in the computation of the maps $\phi(\zeta^m_{1,1,4,6}), \phi(\zeta^m_{1,1,3,4,6})$ and $\phi(\zeta^m_{1,1,6,8})$. In addition, in the above three cases $c^\phi_{2r+1}(\cdots)$ computes the coefficient of ζ^m_{2r+1}, which does not depend on the regularization, i.e. it is independent on ϵ.

Let us demonstrate the regularization at the computation of $\partial^\phi_3(\zeta^m_{1,1,6,8})$, whose result is given in (4.63). With $\zeta^m_{1,1,6,8} = I^m(0; 1, 1, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0; 1)$ computing (4.18) for $r = 1$ yields:

$$\begin{aligned}\xi_3 = {} & c^\phi_3[I^m(0; 1, 1, 0; 1) + I^m(0; 1, 0, 1; 1)]\, I^m(0; 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0; 1) \\ & -c^\phi_3\, \big[I^m(0; 0, 0, 1; 1)\big]\, I^m(0; 1, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0; 1). \end{aligned} \tag{4.67}$$

Above, the integral $I^m(0; 1, 0, 1; 1)$ has to be replaced by its regularized value $\hat{I}^m(0; 1, 0, 1; 1)$. The latter is computed from expanding

$$\begin{aligned} I^m(\epsilon; 1, 0, 1; 1-\epsilon) &\simeq \int_\epsilon^{1-\epsilon} \frac{dt_3}{1-t_3} \int_\epsilon^{t_3} \frac{dt_2}{t_2} \int_\epsilon^{t_2} \frac{dt_1}{1-t_1} \\ &= -\zeta^m_2\, \ln\epsilon - 2\, \zeta^m_3 + [2 + \zeta^m_2 - (\ln\epsilon)^2]\, \epsilon + \mathcal{O}(\epsilon^2) \end{aligned} \tag{4.68}$$

w.r.t. small ϵ. Hence, we have[13]:

$$\hat{I}^m(0; 1, 0, 1; 1) = -2\, \zeta_3^m. \tag{4.69}$$

Note, that this agrees with what one would obtain by applying the shuffle rule (4.3)

$$I^m(0; 1, 0; 1)\, I^m(0; 1; 1) = I^m(0; 1, 0, 1; 1) + 2\, I^m(0; 1, 1, 0; 1), \tag{4.70}$$

from which we obtain:

$$I^m(0; 1, 0, 1; 1) = I^m(0; 1, 0; 1)\, I^m(0; 1; 1) - 2\, I^m(0; 1, 1, 0; 1). \tag{4.71}$$

With $I^m(0; 1, 1, 0; 1) = \zeta_{1,2}^m = \zeta_3^m$ the two expressions (4.68) and (4.71) give the same finite piece. This is a consequence of the fact that the shuffle relation also holds for the canonical regularization of multiple polylogarithms [16]. Another way to arrive at the conclusion (4.69) follows from simply identifying $I^m(a_0; a_1; a_2) \simeq 0$ for $a_i \in \{0, 1\}$ in the shuffle relation (4.70); see [5].

4.4. Motivic decomposition operators and α'-expansion

By comparing the decomposition operators ξ_l given for $l = 10, \ldots, 16$ in (4.23), (4.31), (4.38), (4.45), (A.2), (A.7) and (A.9), respectively, with the corresponding order α'^l in the expansion of (3.25) (with the operators (3.26) and (3.17)) we see an exact match in the coefficient and commutator structure by identifying the motivic derivation operators (4.20) and the matrix operators (3.26)

$$\partial_{2n+1} \simeq M_{2n+1}, \tag{4.72}$$

and the coefficient operator c_2 with the matrix operators (3.26):

$$c_2^k \simeq P_{2k}, \quad k \geqslant 1. \tag{4.73}$$

We can further strengthen this connection. Let $\mathcal{L}' = \mathbf{Q}\langle e_3, e_5, \ldots\rangle$ be the free graded Lie algebra (some vector space over $\mathbf{Q}$) freely generated by the generators e_{2r+1} of degree $-(2r+1)$ with the Lie-bracket $(e_i, e_j) \mapsto [e_i, e_j]$ and the Jacobi relations:

$$[e_i, [e_j, e_k]] + [e_j, [e_k, e_i]] + [e_k, [e_i, e_j]] = 0. \tag{4.74}$$

With $\mathcal{L} = \mathbf{Q}[e_2] \oplus \mathcal{L}'$ the underlying graded vector space over $\mathbf{Q}$ is generated by the following elements [33]:

$$e_2,\ e_3,\ e_5,\ e_7,\ [e_3, e_5],\ e_9,\ [e_3, e_7],\ e_{11},\ [e_3, [e_5, e_3]],\ [e_3, e_9],\ [e_5, e_7],\ \ldots, \tag{4.75}$$

e.g. at weight 11 the elements e_{11} and $[e_3, [e_3, e_5]]$ generate $\mathcal{L}'_{11}$. For $f_3, f_5, \ldots$ being the functionals on the vector space generated by the vectors $e_3, e_5, \ldots$ such that $\langle f_i, e_j\rangle = \delta_{ij}$ the dual to the universal enveloping algebra $U(\mathcal{L})$ is isomorphic to the space $\mathcal{U}$ of non-commutative polynomials in f_{2n+1} with the shuffle product [16, 34].

In fact, the Lie algebra $\mathcal{L}$ generators e_i can be identified with the matrices M_{2n+1} and P_2 introduced in (3.26), i.e.

$$\begin{aligned} e_{2n+1} &\simeq M_{2n+1}, \\ e_2 &\simeq P_2, \end{aligned} \tag{4.76}$$

and of course the matrices M_{2n+1} fulfil the Jacobi identity (4.74):

$$[M_i, [M_j, M_k]] + [M_j, [M_k, M_i]] + [M_k, [M_i, M_j]] = 0. \tag{4.77}$$

To conclude, motivic MZVs encapsulate the α'-expansion of the open superstring amplitude.

[13] With this result equation (4.67) becomes: $\xi_3 = c_3^\phi\left(\zeta_{1,2}^m - 2\zeta_3^m\right)\ \phi(\zeta_{5,8}^m) + c_3^\phi\left(-\zeta_3^m\right)\ \phi(\zeta_{1,4,8}^m) = -\phi(\zeta_{5,8}^m) - \phi(\zeta_{1,4,8}^m)$.

5. Motivic structure of the open superstring amplitude

The symbol of a transcendental function represents a motivic road map encoding all the relevant information about the function without further specifying the latter explicitly in terms of multiple polylogarithms [2, 3, 35]. In particular, the various relations among different multiple polylogarithms become simple algebraic identities in the corresponding tensor algebra. In this section we show that the isomorphism ϕ, which is induced by the coaction (4.12), encapsulates all the relevant information of the α'-expansion of the open superstring amplitude without further specifying the latter explicitly in terms of MZVs. By passing from the MZVs $\zeta \in \mathcal{Z}$ to their motivic versions $\zeta^m \in \mathcal{H}$ and then mapping the latter to elements $\phi(\zeta^m)$ of the Hopf algebra $\mathcal{U}$ the map ϕ endows the superstring amplitude with its motivic structure: it maps the α'-expansion into a very short and intriguing form (1.1) in terms of the non-commutative Hopf algebra $\mathcal{U}$. In particular, the various relations among different MZVs become simple algebraic identities in the Hopf algebra $\mathcal{U}$. Moreover, in this writing the final result (1.1) for the superstring expansion does not depend on the choice of a specific set[14] of MZVs as basis elements.

In this section we apply the isomorphism ϕ to the motivic version $\mathcal{A}^m$ of the open superstring amplitude expression (3.24)

$$\phi(\mathcal{A}^m) = \phi(F^m)\, A, \tag{5.1}$$

with

$$\begin{aligned} F^m &= P^m\, Q^m \ : \exp\left\{ \sum_{n\geqslant 1} \zeta^m_{2n+1}\, M_{2n+1} \right\} :, \\ P^m &= P|_{\zeta_2 \to \zeta_2^m}\,, \quad Q^m = Q|_{\zeta_{n_1,\ldots,n_r} \to \zeta^m_{n_1,\ldots,n_r}}\,, \end{aligned} \tag{5.2}$$

and the matrices P, M and Q defined in (3.26) and (3.17), respectively. The action (4.9) of ϕ on the motivic MZVs is explained in the previous section.

5.1. Motivic structure up to weight 16

The first hint of a simplification under ϕ occurs in (3.22) at weight $w = 8$, where the commutator term $[M_5, M_3]$ together with the prefactor $\frac{1}{5}\zeta^m_{3,5}$ conspires[15] into:

$$\phi\left(\zeta_3^m \zeta_5^m\, M_5 M_3 + Q_8^m \right) = f_3 f_5\, M_5 M_3 \ + \ f_5 f_3\, M_3 M_5. \tag{5.3}$$

The right-hand side obviously treats the objects f_3, M_3 and f_5, M_5 in a democratic way. The effect of the map ϕ becomes even more drastic[16] at weight $w = 11$ at the permutations of $M_3 M_3 M_5$:

$$\begin{aligned} \phi\left(F^m|_{w=11}\right) &= \phi\Big(Q^m_{11} + Q^m_8\, \zeta^m_3\, M_3 + \tfrac{1}{2}\, (\zeta_3^m)^2 \zeta_5^m\, M_5 M_3^2 + \zeta^m_{11}\, M_{11} + \tfrac{1}{6}\, (\zeta_3^m)^3 \zeta_2^m\, P_2 M_3^3 \\ &\quad + \zeta_9^m \zeta_2^m\, P_2 M_9 + \zeta_7^m (\zeta_2^m)^2\, P_4 M_7 + \zeta_5^m (\zeta_2^m)^3\, P_6 M_5 + \zeta_3^m (\zeta_2^m)^4\, P_8 M_3 \Big) \\ &= f_{11}\, M_{11} + f_3^2 f_5\, M_5 M_3^2 + f_3 f_5 f_3\, M_3 M_5 M_3 + f_5 f_3^2\, M_3^2 M_5 \\ &\quad + P_2 f_2 \left(f_9\, M_9 \ + \ f_3^3\, M_3^3 \right) + P_4 f_2^2 f_7\, M_7 + P_6 f_2^3 f_5\, M_5 + P_8 f_2^4 f_3\, M_3. \end{aligned} \tag{5.4}$$

From (5.3) and (5.4) we observe that in the Hopf algebra $\mathcal{U}$, every non-commutative word of odd letters f_{2k+1} multiplies the associated reverse product of matrices M_{2k+1}. Powers f_2^k of

[14] For instance, instead of taking a basis containing the depth one elements ζ^m_{2n+1} one also could choose the set of Lyndon words in the Hoffman elements $\zeta^m_{n_1,\ldots,n_r}$, with $n_i = 2, 3$ [19, 21].

[15] Note the useful relation $\phi(Q_8^m) = f_5 f_3\, [M_3, M_5]$ for $Q_8^m = \frac{1}{5}\zeta^m_{3,5}\, [M_5, M_3]$.

[16] We use the identity: $\phi(Q^m_{11}) = f_5 f_3^2\, [M_3, [M_3, M_5]]$.

the commuting generator f_2 are accompanied by P_{2k}, which multiplies all the operators M_{2k+1} from the left. Most notably, in contrast to the representation in terms of motivic MZVs, the numerical factors become unity, i.e. all the rational numbers in (3.16) drop out. Our explicit results confirm that the beautiful structure with the combination of operators $M_{i_p}, \ldots, M_{i_2}M_{i_1}$ accompanying the word $f_{i_1}f_{i_2}, \ldots, f_{i_p}$, continues to hold through weight $w = 16$:

$$\begin{aligned}
\phi\,(F^m) \;=\; & \big(1 + f_2P_2 + f_2^2P_4 + f_2^3P_6 + f_2^4P_8 + f_2^5P_{10} + f_2^6P_{12} + f_2^7P_{14} + f_2^8P_{16} + \cdots\big) \\
& \times \big(1 + f_3\, M_3 + f_5\, M_5 + f_3^2\, M_3^2 + f_7\, M_7 + f_3f_5\, M_5M_3 + f_5f_3\, M_3M_5 \\
& + f_9\, M_9 + f_3^3\, M_3^3 + f_5^2\, M_5^2 + f_3f_7\, M_7M_3 + f_7f_3\, M_3M_7 + f_{11}\, M_{11} \\
& + f_3^2f_5\, M_5M_3^2 + f_3f_5f_3\, M_3M_5M_3 + f_5f_3^2\, M_3^2M_5 + f_3^4\, M_3^4 + f_3f_9\, M_9M_3 \\
& + f_9f_3\, M_3M_9 + f_5f_7\, M_7M_5 + f_7f_5\, M_5M_7 + f_{13}\, M_{13} + f_3^2f_7\, M_7M_3^2 \\
& + f_3f_7f_3\, M_3M_7M_3 + f_7f_3^2\, M_3^2M_7 + f_3f_5^2\, M_5^2M_3 + f_5f_3f_5\, M_5M_3M_5 \\
& + f_5^2f_3\, M_3M_5^2 + f_7^2\, M_7^2 + f_3f_{11}\, M_{11}M_3 + f_{11}f_3\, M_3M_{11} + f_5f_9\, M_9M_5 \\
& + f_9f_5\, M_5M_9 + f_3^3f_5\, M_5M_3^3 + f_3^2f_5f_3\, M_3M_5M_3^2 + f_3f_5f_3^2\, M_3^2M_5M_3 \\
& + f_5f_3^3\, M_3^3M_5 + f_{15}\, M_{15} + f_5^3\, M_5^3 + f_3^5\, M_3^5 + f_3^2f_9\, M_9M_3^2 + f_3f_9f_3\, M_3M_9M_3 \\
& + f_9f_3^2\, M_3^2M_9 + f_3f_5f_7\, M_7M_5M_3 + f_3f_7f_5\, M_5M_7M_3 + f_5f_3f_7\, M_7M_3M_5 \\
& + f_5f_7f_3\, M_3M_7M_5 + f_7f_3f_5\, M_5M_3M_7 + f_7f_5f_3\, M_3M_5M_7 + f_7f_9\, M_9M_7 \\
& + f_9f_7\, M_7M_9 + f_{11}f_5\, M_5M_{11} + f_5f_{11}\, M_{11}M_5 + f_3f_{13}\, M_{13}M_3 + f_{13}f_3\, M_3M_{13} \\
& + f_3^2f_5^2\, M_5^2M_3^2 + f_5^2f_3^2\, M_3^2M_5^2 + f_3f_5^2f_3\, M_3M_5^2M_3 + f_5f_3^2f_5\, M_5M_3^2M_5 \\
& + f_3f_5f_3f_5\, M_5M_3M_5M_3 + f_5f_3f_5f_3\, M_3M_5M_3M_5 + f_3^3f_7\, M_7M_3^3 \\
& + f_3^2f_7f_3\, M_3M_7M_3^2 + f_3f_7f_3^2\, M_3^2M_7M_3 + f_7f_3^3\, M_3^3M_7 + \cdots\big)\,.
\end{aligned} \tag{5.5}$$

Writing the amplitude (5.1) in terms of elements of the algebra comodule $\mathcal{U}$, with $\phi(F^m)$ given above encodes all the information contained in (3.17).

5.2. Motivic structure at general weight

Motivated by the observation that every non-commutative word constructed from odd generators f_{2k+1} shows up in (5.8) we write down the following formula

$$\phi(F^m) = \left(\sum_{k=0}^{\infty} f_2^k\, P_{2k}\right)\left(\sum_{p=0}^{\infty}\ \sum_{\substack{i_1,\ldots,i_p \\ \in 2\mathbb{N}^+ +1}} f_{i_1}f_{i_2}\ldots f_{i_p}\, M_{i_p}\ldots M_{i_2}M_{i_1}\right), \tag{5.6}$$

for the image[17] $\phi(F^m)$ valid for any weight. In (5.6) the sum over the combinations $f_{i_1}f_{i_2}\ldots f_{i_p}M_{i_p}\ldots M_{i_2}M_{i_1}$ includes *all* possible non-commutative words $f_{i_1}f_{i_2}\ldots f_{i_p}$ with coefficients $M_{i_p}\ldots M_{i_2}M_{i_1}$ graded by their length p. Matrices P_{2k} associated with the powers f_2^k always act by left multiplication. The commutative nature of f_2 w.r.t. the odd generators f_{2k+1} ties in with the fact that in the matrix ordering the matrices P_{2k} have the well-defined place left of all matrices M_{2k+1}. With (5.8) one easily checks that (5.6) is compatible through weights less than or equal to 16. Combining equations (5.1) and (5.6) gives the final result[18] (1.1):

[17] Note that a different normalization (4.10) or choice of ϕ (see footnote 8) can be compensated by an appropriate modification of the definition of the matrices M_{2n+1} such that the form of (5.6) stays unchanged.

[18] The combinations $f_{i_1}f_{i_2}\ldots f_{i_p}M_{i_p}\ldots M_{i_2}M_{i_1}$ in (1.1) reflect the agreement of the coefficients in the α' expansion $\mathcal{A}|_w$ of the superstring amplitude with those in the motivic decomposition operators ξ_w observed in subsection 4.4.

$$\phi(\mathcal{A}^m) = \left(\sum_{k=0}^{\infty} f_2^k \, P_{2k}\right) \left\{\sum_{p=0}^{\infty} \sum_{\substack{i_1,\ldots,i_p \\ \in 2\mathbb{N}^+ +1}} f_{i_1} f_{i_2} \ldots f_{i_p} \, M_{i_p} \ldots M_{i_2} M_{i_1}\right\} A. \tag{5.7}$$

In the following we shall give further evidence that the validity extends to higher weights. In subsection 4.4 we have already pointed out that the decomposition formula ξ_w for MZVs of weight w exactly matches the corresponding α'^w-part of the superstring amplitude subject to the replacements (4.72) and (4.73). If this mapping holds to arbitrary weight, then the simplicity of our final result (5.6) reflects the role of $\phi(\xi_w)$ being the unit operator projected to weight w, e.g.

$$\begin{aligned}\phi(\xi_{10}) = \; & f_2^5 c_2^5 \; + \; f_2^2 f_3^2 c_2^2 \partial_3^2 \; + \; f_2\big(f_3 f_5 \partial_5 \partial_3 \; + \; f_5 f_3 \partial_3 \partial_5\big) c_2 \\ & + f_5^2 \partial_5^2 \; + \; f_7 f_3 \partial_3 \partial_7 \; + \; f_3 f_7 \partial_7 \partial_3 = \mathrm{id}\big|_{w=10}\end{aligned} \tag{5.8}$$

maps any non-commutative weight ten polynomial in f_2, f_3, f_5, f_7, f_9 to itself. More generally, the differential operator $c_2^k \partial_{i_p} \ldots \partial_{i_2} \partial_{i_1}$ annihilates all $\mathcal{U}$ elements except for $f_2^k f_{i_1} f_{i_2} \ldots f_{i_p}$. Hence, the weight w identity operator is given by

$$\begin{aligned}\phi(\xi_w) = \sum_{k=0}^{\infty} f_2^k c_2^k \sum_{p=0}^{\infty} \sum_{\substack{i_1,\ldots,i_p \\ \in 2\mathbb{N}^+ +1}} & f_{i_1} f_{i_2} \ldots f_{i_p} \partial_{i_p} \ldots \partial_{i_2} \partial_{i_1} \\ & \times \delta\big(i_1 + i_2 + \cdots + i_p + 2k - w\big) = \; \mathrm{id}\big|_w,\end{aligned} \tag{5.9}$$

where the $\delta(\cdots)$ function makes sure that the correct weight is picked up. Clearly, (5.9) maps to the weight w contributions of (5.6) under the replacements (4.72) and (4.73). In this sense, the image under ϕ of the disc amplitude at weight w is closely related to the identity operator in the algebra comodule $\mathcal{U}$, restricted to weight w.

6. Closed superstring amplitude

The string world-sheet describing the tree-level string S-matrix of N gravitons has the topology of a complex sphere with N (integrated) insertions of graviton vertex operators. One of the key properties of graviton amplitudes in string theory is that at tree-level they can be expressed as sum over squares of (color ordered) gauge amplitudes in the left- and right-moving sectors. This map, known as KLT relations [9], gives a relation between a closed string tree-level amplitude $\mathcal{M}$ involving N closed strings and a sum of squares of (partial ordered) open string tree-level amplitudes. We may write these relations in matrix notation as follows

$$\mathcal{M}(1,\ldots,N) = \mathcal{A}^t \, S \, \mathcal{A}, \tag{6.1}$$

with the vector $\mathcal{A}$ encoding the $(N-3)!$ independent color ordered open string subamplitudes and some $(N-3)! \times (N-3)!$ matrix S. The latter encodes the sin-factors from the KLT relations [9] and the contributions from the monodromy relations [10, 11] to express both left- and right-movers in terms of the same open string basis $\mathcal{A}$. Hence, in superstring theory the tree-level computation of graviton amplitudes boils down to considering squares of tree-level gauge amplitudes $\mathcal{A}$ given in (3.1). For this sector the explicit expressions (3.24) and (3.25) and subsequent results from the previous sections can be used. The KLT relations are insensitive to the compactification details or the amount of supersymmetries of the superstring background. Hence, the following discussions and results are completely general.

In the sequel we shall discuss the implication of (3.25) to the closed string amplitude (6.1). Especially, we shall be interested in the structure of its α'-expansion. The latter has been already investigated up to the order α'^8 for the cases $N = 4, 5$ and $N = 6$ with the remarkable observation that the graviton amplitudes do not allow for powers of ζ_2 in their α'-expansions up to the order α'^8 [36]. With the explicit expression (3.25) for the open superstring amplitude we are now able to reveal the pattern and more general framework behind these findings.

6.1. $N = 4$

For $N = 4$ the KLT relation (6.1) can be written as:

$$\mathcal{M}(1, 2, 3, 4) = \mathcal{A}^t \, S \, \mathcal{A}, \tag{6.2}$$

with the basis $\mathcal{A} = \mathcal{A}(1, 2, 3, 4)$ of open string amplitudes (3.8) and the scalar:

$$S = \sin(\pi s) \, \frac{\sin(\pi u)}{\sin(\pi t)}. \tag{6.3}$$

With (3.8) and

$$P = \left\{ \pi \, \frac{s \, u}{s + u} \, \frac{\sin[\pi \, (s + u)]}{\sin(\pi s) \, \sin(\pi u)} \right\}^{1/2}, \tag{6.4}$$

equation (6.2) yields:

$$\mathcal{M}(1, 2, 3, 4) = \pi \, \frac{su}{s + u} \, \exp\left\{ 2 \sum_{n \geqslant 1} \zeta_{2n+1} \, M_{2n+1} \right\} \, |A|^2, \tag{6.5}$$

with the YM subamplitude $A = A_{\rm YM}(1, 2, 3, 4)$ and M_{2n+1} given in (3.9). Obviously, in the 4-graviton amplitude (6.5), not any Riemann zeta function with even entries shows up.

The field-theory contribution from (6.2) arises from $P = 1$ and $\mathcal{A} = A$, i.e.

$$\mathcal{M}(1, 2, 3, 4)|_{\rm FT} = A^t \, S_0 \, A, \tag{6.6}$$

with

$$S_0 \equiv S|_{\rm FT} = \pi \, s \, \frac{u}{t}. \tag{6.7}$$

We observe that:

$$P^t SP = S_0. \tag{6.8}$$

This equation guarantees the absence of powers of ζ_2 in (6.5). Stated differently, the absence of powers of ζ_2 in (6.2) allows one to determine the scalar $P = P^t$ from the equation (6.8) as:

$$P = S_0^{1/2} \, (S^{-1})^{1/2}. \tag{6.9}$$

6.2. $N = 5$

For $N = 5$ the closed string amplitude (6.1) can be cast into

$$\mathcal{M}(1, 2, 3, 4, 5) = \mathcal{A}^t \, S \, \mathcal{A}, \tag{6.10}$$

with the basis $\mathcal{A}$ of open string amplitudes given in (3.14) and the symmetric matrix S encoding the diagonal matrix $\text{diag}\{\sin(\pi s_{12}) \sin(\pi s_{34}), \ \sin(\pi s_{13}) \sin(\pi s_{24})\}$ from the KLT relation [9] and further sin-factors from the monodromy relations [10, 11] expressing the string amplitudes $\mathcal{A}(2, 1, 4, 3, 5)$ and $\mathcal{A}(3, 1, 4, 2, 5)$ in terms of the basis elements $\mathcal{A}(1, 2, 3, 4, 5)$ and $\mathcal{A}(1, 3, 2, 4, 5)$. More precisely, we have

$$S = [\sin(\pi s_{35}) \, \sin(\pi s_{25}) \, \sin(\pi s_{14})]^{-1} \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12} & \Sigma_{22} \end{pmatrix}, \tag{6.11}$$

with

$$\Sigma_{11} = \tfrac{1}{4}\ \sin(\pi s_1)\ \sin(\pi s_3)\ [\sin\pi(s_1 - s_2 - s_3) - \sin\pi(s_1 + s_2 - s_3) + \sin\pi(s_1 + s_2 + s_3) + \sin\pi(s_1 + s_2 - s_3 - 2s_4) + \sin\pi(-s_1 + s_2 + s_3 - 2s_5) - \sin\pi(s_1 + s_2 + s_3 - 2s_4 - 2s_5)\ ],$$

$$\Sigma_{12} = -\sin(\pi s_1)\ \sin(\pi s_3)\ \sin(\pi s_{13})\ \sin(\pi s_{24})\ \sin\pi(s_4 + s_5),$$

$$\Sigma_{22} = \tfrac{1}{4}\sin(\pi s_{13})\ \sin(\pi s_{24}) \times [\sin\pi(s_1 + s_2 - s_3 - s_4 - s_5) - \sin\pi(s_1 + s_2 - s_3 - s_4 + s_5) - \sin\pi(s_1 + s_2 + s_3 - s_4 - s_5) + \sin\pi(s_1 - s_2 - s_3 - s_4 + s_5) - \sin\pi(s_1 - s_2 - s_3 + s_4 + s_5) + \sin\pi(s_1 + s_2 + s_3 + s_4 + s_5)\ ], \qquad (6.12)$$

and the kinematic invariants defined in (3.3) and (3.21).

The field-theory contribution from (6.10) arises from $P = \left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)$ and $\mathcal{A} = A$, with the YM basis vector A given in (3.14), i.e.

$$\mathcal{M}(1,2,3,4,5)|_{\rm FT} = A^t\ S_0\ A, \qquad (6.13)$$

with

$$S_0 \equiv S|_{\rm FT} = \pi^2\ (s_{25}\ s_{35}\ s_{14})^{-1} \begin{pmatrix}\sigma_{11} & \sigma_{12}\\ \sigma_{12} & \sigma_{22}\end{pmatrix}, \qquad (6.14)$$

and:

$$\sigma_{11} = s_1 s_3\ [s_4(s_3 - s_5)(-s_2 + s_4 + s_5) + s_1(-s_3(s_4 + s_5) + s_5(-s_2 + s_4 + s_5))],$$

$$\sigma_{12} = -s_1 s_3\ s_{13}\ s_{24}\ (s_4 + s_5),$$

$$\sigma_{22} = -s_{13}\ s_{24}\ [s_1 s_4(s_2 + s_3) + s_1 s_3 s_5 + s_2 s_5(s_3 + s_4)]. \qquad (6.15)$$

By considering the closed superstring amplitude (6.10) and analyzing its α'-expansion [36] we find that the following matrix equation holds:

$$P^t\ SP = S_0. \qquad (6.16)$$

We have checked the validity of (6.26) up to the order α'^{18}. As a consequence of the relation (6.26), the contribution of the matrix P stemming from the open superstring amplitudes (3.13) and accounting for powers of ζ_2 drops out of the α'-expansion of (6.10). In addition, we find the relation

$$M_l^t\ S_0 = S_0\ M_l, \qquad (6.17)$$

which we have verified up to weight $l = 19$. For commutators $\mathcal{Q}_{(r)}$ of M_l, equation (6.17) implies

$$\begin{aligned} S_0\ \mathcal{Q}_{(2)} + \mathcal{Q}^t_{(2)}\ S_0 = 0, &\quad \mathcal{Q}_{(2)} = [M_l, M_m],\\ S_0\ \mathcal{Q}_{(3)} - \mathcal{Q}^t_{(3)}\ S_0 = 0, &\quad \mathcal{Q}_{(3)} = [M_l, [M_m, M_n]],\\ S_0\ \mathcal{Q}_{(4)} + \mathcal{Q}^t_{(4)}\ S_0 = 0, &\quad \mathcal{Q}_{(4)} = [M_k, [M_l, [M_m, M_n]]] \end{aligned} \qquad (6.18)$$

generalizing to

$$S_0\ \mathcal{Q}_{(r)} + (-1)^r\ \mathcal{Q}^t_{(r)} S_0 = 0, \quad \mathcal{Q}_{(r)} = [M_{n_2}, [M_{n_3}, \ldots, [M_{n_r}, M_{n_1}]]\ldots] \qquad (6.19)$$

for nested commutators $\mathcal{Q}_{(r)}$ of generic depth r. In contrast to (6.19) in the closed string amplitude the nested commutators $\mathcal{Q}_{(r)}$ show up in the combination $S_0\mathcal{Q}_{(r)} + \mathcal{Q}^t_{(r)}S_0$, which only vanishes for commutators of even depth $r \in 2\mathbf{N}$. Assuming for Q the exponential form (3.28) the relation

$$S_0\ {\rm e}^{\mathcal{Q}_{(r)}} = {\rm e}^{(-1)^{r+1}\mathcal{Q}^t_{(r)}}\ S_0 \qquad (6.20)$$

following from (6.19) guarantees the decoupling of any power of nested commutators $\mathcal{Q}_{(r)}$ of even depth $r \in 2\mathbf{N}$ in (6.10).

On the basis of (6.26) and (6.17), we obtain the final form

$$\begin{aligned}\mathcal{M}(1,2,3,4,5) &= A^t \left(:\exp\left\{ \sum_{r\in 2\mathbf{N}^+ +1} \zeta_r\, M_r \right\} : \right)^t Q^t S_0 Q\ :\exp\left\{ \sum_{s\in 2\mathbf{N}^+ +1} \zeta_s\, M_s \right\} :\ A \\ &= A^t\, S_0 \left(:\exp\left\{ \sum_{r\in 2\mathbf{N}^+ +1} \zeta_r\, M_r^t \right\} : \right)^t \widetilde{Q}\, Q\ :\exp\left\{ \sum_{s\in 2\mathbf{N}^+ +1} \zeta_s\, M_s \right\} :\ A, \end{aligned} \tag{6.21}$$

where the ordering colons enclosing the exponentials[19] are defined in (3.18) and the matrix $\widetilde{Q}$ is obtained from Q by replacing commutators $\mathcal{Q}_{(r)}$ as follows:

$$\widetilde{Q} = Q\,|_{\mathcal{Q}_{(r)}\to(-1)^{r+1}\mathcal{Q}_{(r)}}\,. \tag{6.22}$$

As a consequence, terms with commutator factors $\mathcal{Q}_{(2n)}$ of even depth do not show up in the product[20]:

$$\widetilde{Q}\,Q = 1 + 2\,Q_{11} + 2\,Q_{13} + 2\,Q_{15} + \cdots. \tag{6.23}$$

Hence, we observe that in (6.21) MZVs of even weight or depth $\geqslant 2$ only enter through the product (6.23) starting at weight $w = 11$. This result is in agreement with the observation made in [36]. We now have verified this observation through weight 18. Let us display the expansion of (6.21) through the order α'^{14}

$$\begin{aligned}\mathcal{M}(1,2,3,4,5) = A^t\, S_0\, \big(1 &+ 2\,\zeta_3 M_3 + 2\,\zeta_5 M_5 + 2\,\zeta_3^2 M_3^2 + 2\,\zeta_7 M_7 + 2\,\zeta_3\zeta_5\{M_3, M_5\} \\ &+\ 2\,\zeta_9 M_9 + \tfrac{4}{3}\,\zeta_3^3 M_3^3 + 2\,\zeta_5^2 M_5^2 + 2\,\zeta_3\zeta_7\{M_3, M_7\} + 2\,Q_{11} + 2\,\zeta_{11} M_{11} \\ &+\ \zeta_3^2\zeta_5\{M_3, \{M_3, M_5\}\} + \tfrac{2}{3}\,\zeta_3^4 M_3^4 + 2\,\zeta_3\zeta_9\{M_3, M_9\} + 2\,\zeta_5\zeta_7\{M_5, M_7\} \\ &+\ 2\,Q_{13} + 2\,\zeta_{13} M_{13} + \zeta_3^2\zeta_7\{M_3, \{M_3, M_7\}\} + 2\,\zeta_3\zeta_5^2\{M_3, M_5^2\} \\ &+\ 2\,\zeta_3\{M_3, Q_{11}\} + 2\,\zeta_7^2 M_7^2 + 2\,\zeta_3\zeta_{11}\{M_3, M_{11}\} + 2\,\zeta_5\zeta_9\{M_5, M_9\} \\ &+\ \tfrac{1}{3}\,\zeta_3^3\zeta_5\{M_3, \{M_3, \{M_3, M_5\}\}\} + \cdots \big)\, A, \end{aligned} \tag{6.24}$$

with the anticommutator $\{A, B\} = AB + BA$. Up to the order shown, MZVs of depth $r \geqslant 2$ enter through the objects Q_{11}, Q_{13} and $\{M_3, Q_{11}\}$. In the single zeta sector, the coefficient of the general power $(\zeta_{2k+1} M_{2k+1})^p$ is given by $2^p/p!$.

6.3. General N

Let us now phrase the observation from above for general multiplicities N. The general form of the N-point closed string amplitude is given in (6.1),

$$\mathcal{M}(1,\ldots,N) = \mathcal{A}^t\, S\, \mathcal{A}, \tag{6.25}$$

with the $(N-3)! \times (N-3)!$ matrix S specified above and the vector $\mathcal{A}$ encoding the $(N-3)!$ open string subamplitudes (3.24). Just as in the five-point case, the relations

$$P^t\, SP = S_0, \tag{6.26}$$

$$M_l^t\, S_0 = S_0\, M_l, \tag{6.27}$$

$$S_0\, \mathcal{Q}_{(r)} + (-1)^r\, \mathcal{Q}_{(r)}^t S_0 = 0, \tag{6.28}$$

with $S_0 \equiv S|_{\text{FT}}$ and $\mathcal{Q}_{(r)} = [M_{n_2}, [M_{n_3}, \ldots, [M_{n_r}, M_{n_1}]] \ldots]$ imply that from (3.25) both the matrix P and the part of Q with admixtures of even depth commutators $\mathcal{Q}_{(2n)}$ are canceled in

[19] Note that the transpositions involved in the expression $\left(:\exp\left\{\sum_r \zeta_r\, M_r^t\right\}:\right)^t$ lead to a reversal of the matrix multiplication order compared to the ordered product $:\exp\left\{\sum_s \zeta_s\, M_s\right\}:$ without transposition, i.e.: $\left(:\exp\left\{\sum_{r\in 2\mathbf{N}^+ +1} \zeta_r\, M_r^t\right\}:\right)^t = 1 + \zeta_3 M_3 + \zeta_5 M_5 + \frac{1}{2}\zeta_3^2 M_3^2 + \zeta_7 M_7 + \zeta_3\zeta_5 M_3 M_5 + \frac{1}{6}\zeta_3^3 M_3^3 + \zeta_9 M_9 + \frac{1}{2}\zeta_5^2 M_5^2 + \zeta_3\zeta_7 M_3 M_7 + \frac{1}{2}\zeta_3^2\zeta_5 M_3^2 M_5 + \zeta_{11} M_{11} + \cdots$.

[20] The exponential form (3.28) leads us to expect even weight contributions to $\widetilde{Q}Q$ starting at weight 22, e.g. $Q_{22} = \frac{1}{2}Q_{11}^2 + \cdots$ such that $\widetilde{Q}\,Q|_{w=22} = 2\,Q_{11}^2$.

the N-point closed string amplitude (6.25). With the information from (6.26) and (6.27) the closed superstring (6.25) amplitude for any number N of external states takes the generic form

$$\mathcal{M}(1,\ldots,N) = A^t\, S_0\, \left(: \exp\left\{ \sum_{r\in 2\mathbb{N}^+ +1} \zeta_r\, M_r^t \right\} : \right)^t \widetilde{Q}\, Q : \exp\left\{ \sum_{s\in 2\mathbb{N}^+ +1} \zeta_s\, M_s \right\} :\, A, \tag{6.29}$$

with the $(N-3)!$-dimensional vector A specifying a YM basis $A \equiv A_{\text{YM}}$, the $(N-3)! \times (N-3)!$ matrix S_0 introduced above and the $(N-3)! \times (N-3)!$ matrices M_{2n+1} defined in (3.26). The ordering colons enclosing the exponentials are defined in (3.18) and the matrix $\widetilde{Q}$ is obtained from Q according to (6.22). Due to (6.28) and the exponential form (3.28), the product

$$\widetilde{Q}\, Q = 1 + 2\, Q_{11} + 2\, Q_{13} + 2\, Q_{15} + \cdots \tag{6.30}$$

in (6.29) is free of even depth commutators $\mathcal{Q}_{(2n)}$. Finally, the α'-expansion of the N-point amplitude (6.29) assumes the same form as (6.24), with the matrices M_{2n+1} given in (3.26).

We would like to mention two final remarks: as for the $N = 4$ case (6.9) one can constrain P from the matrix equation (6.26); moreover, equation (6.27) provides restrictive relations between entries of the matrices M_{2k+1}. Their information content on the polynomial structure of P and M_{2k+1} is further investigated and exhibited in more detail in [28]. Of course, with the explicit expression for P and M the relations (6.26) and (6.27) and hence (6.29) can be verified to all orders.

6.4. Motivic structure of the closed superstring amplitude

Experiencing the simplicity in the open string sector suggests that one should also investigate the image under ϕ of the gravity amplitude (6.25). We insert the result (5.6) for $\phi(\mathcal{A}^m)$ into (6.25). The multiplication rule (4.11) of the isomorphism ϕ yields:

$$\phi(\mathcal{M}^m) = A^t \left\{ \sum_{p=0}^{\infty} \sum_{\substack{i_1,\ldots,i_p \\ \in 2\mathbb{N}^+ +1}} f_{i_1} f_{i_2} \ldots f_{i_p}\, M_{i_p} \ldots M_{i_2} M_{i_1} \right\}^t \\ ш\; S_0 \left\{ \sum_{q=0}^{\infty} \sum_{\substack{j_1,\ldots,j_q \\ \in 2\mathbb{N}^+ +1}} f_{j_1} f_{j_2} \ldots f_{j_q}\, M_{j_q} \ldots M_{j_2} M_{j_1} \right\} A. \tag{6.31}$$

The sum over $f_2^k P_{2k}$ in the open string amplitudes $\mathcal{A}^t$, $\mathcal{A}$ has already been dropped, taking into account the motivic version of the relation (6.26). As a result the commutative Hopf algebra element f_2 is absent in $\phi(\mathcal{M}^m)$.

In order to simplify (6.31) we can make use of (6.27) to convert all the M_i^t from the left moving open string amplitude to M_i factors multiplying S_0 from the right:

$$\phi(\mathcal{M}^m) = A^t\, S_0 \left\{ \sum_{p=0}^{\infty} \sum_{\substack{i_1,\ldots,i_p \\ \in 2\mathbb{N}^+ +1}} f_{i_1} f_{i_2} \ldots f_{i_p}\, M_{i_1} M_{i_2} \ldots M_{i_p} \right. \\ \left. ш \sum_{q=0}^{\infty} \sum_{\substack{j_1,\ldots,j_q \\ \in 2\mathbb{N}^+ +1}} f_{j_1} f_{j_2} \ldots f_{j_q}\, M_{j_q} \ldots M_{j_2} M_{j_1} \right\} A \\ = A^t\, S_0 \left\{ \sum_{p=0}^{\infty} \sum_{\substack{i_1,\ldots,i_p \\ \in 2\mathbb{N}^+ +1}} M_{i_1} M_{i_2} \ldots M_{i_p} \sum_{k=0}^{p} f_{i_1} f_{i_2} \ldots f_{i_k}\, ш\, f_{i_p} f_{i_{p-1}} \ldots f_{i_{k+1}} \right\} A. \tag{6.32}$$

On the way to the last line of (6.32), the double sum over non-commutative words in f_i has been rearranged to identify the overall coefficient of $A^t S_0 M_{i_1} M_{i_2} \ldots M_{i_p} A$. Symmetry of the shuffle product implies that each string of matrices $M_{i_1} M_{i_2} \ldots M_{i_p}$ multiplies the same f_i polynomials as its reverse $M_{i_p} \ldots M_{i_2} M_{i_1}$. In particular, this assigns the symmetric coefficient $2 f_i ш f_j = \phi(2\zeta_i^m \zeta_j^m)$ to the matrix products $M_i M_j$ of length 2, reflecting the absence of the first double zetas along with $\mathcal{Q}_{(2)}$.

Let us present the momentum expansion of (6.32) up to weight 14:

$$\begin{aligned}\phi(\mathcal{M}^m) = A^t\, S_0\, \big(&1 + 2\, f_3 M_3 + 2\, f_5 M_5 + 4\, f_3^2 M_3^2 + 2\, f_7 M_7 + 2\, f_3 ш f_5 \{M_3, M_5\}\\ &+ 2\, f_9 M_9 + 8 f_3^3 M_3^3 + 4\, f_5^2 M_5^2 + 2 f_3 ш f_7 \{M_3, M_7\} + 2 f_{11} M_{11}\\ &+ f_3 ш f_3 ш f_5 \{M_3, \{M_3, M_5\}\} + 2\, f_5 f_3^2\, [M_3, [M_3, M_5]] + 16 f_3^4 M_3^4\\ &+ 2\, f_3 ш f_9 \{M_3, M_9\} + 2 f_5 ш f_7 \{M_5, M_7\} + 2 f_{13} M_{13}\\ &+ f_3 ш f_3 ш f_7 \{M_3, \{M_3, M_7\}\} + 2 f_7 f_3^2\, [M_3, [M_3, M_7]]\\ &+ f_5 ш f_5 ш f_3 \{M_5, \{M_5, M_3\}\} + 2 f_3 f_5^2\, [M_5, [M_5, M_3]]\\ &+ 4\, f_7^2 M_7^2 + 2 f_{11} ш f_3 \{M_3, M_{11}\} + 2 f_9 ш f_5 \{M_5, M_9\}\\ &+ \tfrac{1}{3}\, f_3 ш f_3 ш f_3 ш f_5 \{M_3, \{M_3, \{M_3, M_5\}\}\}\\ &+ 2\, f_3 ш (f_5 f_3^2) \{M_3, [M_3, [M_3, M_5]]\} + \cdots \big)\, A. \end{aligned} \tag{6.33}$$

Starting from weight 11, some of the f_i polynomials cannot be represented as a pure shuffle product $f_{i_1} ш f_{i_2} ш \cdots ш f_{i_p}$ reflecting the presence of depth $\geqslant 2$ MZVs in (6.29) due to $\widetilde{Q}Q$. As expected from $Q_{11}, Q_{13}, \ldots$ given in (3.17), they multiply nested M_i commutators $\mathcal{Q}_{(3)}, \mathcal{Q}_{(5)}, \ldots$ of odd depth, see e.g. $\cdots + 2 f_5 f_3^2 [M_3, [M_3, M_5]] + \cdots$ or the last line of (6.33).

To summarize, we have shown that the closed string tree amplitude also has an α' expansion whose beautiful motivic structure is revealed through the ϕ isomorphism.

7. Conclusion

In this work we have investigated the structure of the α'-expansion of the open and closed superstring amplitude at tree-level with particular emphasis on their transcendentality properties. The strict matching of powers α'^w with their associated MZV prefactors of weight w constituting a well-confirmed pattern has been considerably refined.

The main point is to replace the **C** valued MZVs ζ by more abstract versions thereof, the so-called motivic MZVs ζ^m, which are endowed by a Hopf algebra structure. Furthermore, through the isomorphism ϕ the motivic MZVs are mapped into an algebra comodule generated by the non-commutative words in generators $f_3, f_5, f_7, \ldots$ and an additional commutative element f_2. In the same way as the symbol conveniently captures patterns of field theory amplitudes the isomorphism ϕ yields a strikingly simple and compact expression (5.7) for the open superstring disc amplitude: the systematics of the α'-dependence is written in closed and short form to all weights. In contrast to the symbol, the map ϕ does not lose any information and can be inverted to recover the tree amplitude in terms of motivic MZVs.

In the closed superstring sector the properties of the matrix P encoded in (6.26) and the commutation relations (6.27) between matrices M_{2r+1} and S_0 result in the compact form (6.29), where MZVs of even weight or depth $\geqslant 2$ only enter through (6.30) starting at weight $w = 11$. On the other hand, after applying the map ϕ this result turns into (6.31), in which the element f_2 is absent and all matrices M_{2r+1} and Hopf-algebra generators f_{2s+1} are treated democratically without the necessity for the ordering prescription (3.18) in (6.29).

Note that, based on the observation (3.13) for $N = 5$, which we have proven through weight $w = 16$, we have conjectured (3.25) to hold for generic N of the open superstring amplitude. The polynomial structure of the matrices M and P and various other aspects of α'-expansions are further elaborated in [28]. Moreover, in this reference further evidence for the generic form (3.25) is given for $N = 5$ through weight $w = 22$, for $N = 6$ through weight $w = 9$ and $N = 7$ through weight $w = 7$. On the same grounds the form (6.24) for the $N = 5$ closed superstring amplitude in (6.29) is conjectured to hold for any N. At any rate, in [37] a mathematical proof for the generic form (3.25) for any N will be presented.

The structure underlying the motivic open and closed superstring amplitudes in terms of a Hopf algebra is not only a tool to conveniently express these amplitudes but seems rather to be an intrinsic feature, which might allow one to compute the latter by first principles.

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Appendix. Decomposition of motivic multi zeta values

A.1. Decomposition at weight 14

Gathering the information about the lower weight basis $\mathcal{U}_{k\leqslant 13}$ with (4.50) we can construct the following basis for $\mathcal{U}_{14}$:

$$\begin{aligned}
&-\tfrac{5}{6}\, f_5\,(f_3 \,ш\, f_3 \,ш\, f_3) - \tfrac{5}{3}\, f_5 f_9 + \tfrac{4}{7}\, f_5 f_3 f_2^3 - 51 f_7^2 + 30\, f_7 f_5 f_2 - \tfrac{405}{2}\, f_9 f_5\\
&+90\, f_9 f_3 f_2 - 15\, f_{11} f_3,\\
&-6\, f_5 f_9 - 15\, f_7^2 - 28\, f_9 f_5 - 44\, f_{11}\, f_3,\ -15\, f_7^2 - 69\, f_9 f_5 - 165\, f_{11}\, f_3,\\
&\left(-\tfrac{5}{2}\, f_5(f_3 \,ш\, f_3) + \tfrac{4}{7}\, f_5 f_2^3 - \tfrac{6}{5}\, f_7 f_2^2 - 45\, f_9 f_2\right) ш\, f_3,\ -5\,(f_5 f_3)\,ш\, f_3 \,ш\, f_3,\\
&f_{11} \,ш\, f_3,\ f_3 \,ш\, f_3 \,ш\, f_5 \,ш\, f_3,\ f_9 \,ш\, f_5,\ f_7 \,ш\, f_7,\\
&\left(\tfrac{1799}{18}\, f_9 f_3 - 32\, f_7 f_3 f_2 + \tfrac{1133}{16}\, f_7 f_5 + 29\, f_5 f_7 - 11\, f_5^2 f_2 - \tfrac{16}{5}\, f_5 f_3 f_2^2\right.\\
&\left.+\tfrac{1}{3}\, f_3(f_3 \,ш\, f_3 \,ш\, f_3) - \tfrac{799}{72}\, f_3 f_9 + 10\, f_3 f_7 f_2 - \tfrac{1}{5}\, f_3 f_5 f_2^2 - \tfrac{36}{35}\, f_3^2 f_2^3\right) f_2,\\
&(-6\, f_5 f_7 - 15\, f_7 f_5 - 27\, f_9 f_3)\, f_2,\ f_9 \,ш\, f_3 f_2,\ f_7 \,ш\, f_5 f_2,\ f_3 \,ш\, f_3 \,ш\, f_3 \,ш\, f_3 f_2,\\
&(-14 f_7 f_3 - 6 f_5^2) f_2^2,\ -5\, f_5 f_3 f_2^3,\ f_5 \,ш\, f_5 f_2^2,\ f_3 \,ш\, f_7 f_2^2,\ f_3 \,ш\, f_5 f_2^3,\ f_3 \,ш\, f_3 f_2^4,\ f_2^7. \qquad \text{(A.1)}
\end{aligned}$$

The operators a_i of the decomposition (4.51) are:

$$\begin{aligned}
&a_1 = \tfrac{1}{5}\,[\partial_3,[\partial_3,[\partial_5,\partial_3]]],\ a_2 = -\tfrac{23}{198}\,[\partial_{11},\partial_3] + \tfrac{5}{18}\,[\partial_9,\partial_5] - \tfrac{12\,841}{1188}\,[\partial_3,[\partial_3,[\partial_5,\partial_3]]],\\
&a_3 = -\tfrac{2}{27}\,[\partial_9,\partial_5] + \tfrac{1}{27}\,[\partial_{11},\partial_3] + \tfrac{232}{81}\,[\partial_3,[\partial_3,[\partial_5,\partial_3]]],
\end{aligned}$$

$$a_4 = \tfrac{1}{5}\,[\partial_3,[\partial_5,\partial_3]]\partial_3,\ a_5 = \tfrac{1}{10}\,[\partial_5,\partial_3]\partial_3^2,\ a_6 = \partial_{11}\partial_3,\ a_7 = \tfrac{1}{6}\,\partial_5\partial_3^3,$$
$$a_8 = \partial_9\partial_5 - \tfrac{23}{33}\,[\partial_{11},\partial_3] + \tfrac{5}{3}\,[\partial_9,\partial_5] - \tfrac{12\,775}{198}\,[\partial_3,[\partial_3,[\partial_5,\partial_3]]],$$
$$a_9 = \tfrac{1}{2}\,\partial_7^2 - \tfrac{235}{396}\,[\partial_{11},\partial_3] + \tfrac{55}{36}\,[\partial_9,\partial_5] - \tfrac{647\,287}{11\,880}\,[\partial_3,[\partial_3,[\partial_5,\partial_3]]]$$
$$a_{10} = c_2a_0,\ a_{11} = c_2\left(\tfrac{1}{27}\,[\partial_9,\partial_3] + \tfrac{2665}{648}\,a_0\right) + \tfrac{2}{3}\,[\partial_3,[\partial_3,[\partial_5,\partial_3]]],$$
$$a_{12} = c_2\left(\partial_9\partial_3 + \tfrac{799}{72}\,a_0\right) + 9\,[\partial_3,[\partial_5,\partial_3]]\partial_3,$$
$$a_{13} = c_2\left(\partial_7\partial_5 + \tfrac{2}{9}\,[\partial_9,\partial_3] - \tfrac{467}{108}\,a_0\right) + 4\,[\partial_3,[\partial_3,[\partial_5,\partial_3]]],$$
$$a_{14} = c_2\ \left(\tfrac{1}{24}\partial_3^4 - \tfrac{1}{12}\,a_0\right),\ a_{15} = \tfrac{1}{14}\,c_2^2\,[\partial_7,\partial_3] - 3\,c_2a_0,$$
$$a_{16} = \tfrac{1}{5}c_2^3\,[\partial_5,\partial_3] - \tfrac{3}{5}\,c_2\,a_0 + \tfrac{4}{175}\,[\partial_3,[\partial_3,[\partial_5,\partial_3]]],$$
$$a_{17} = \tfrac{1}{2}\,c_2^2\left(\partial_5^2 + \tfrac{3}{7}\,[\partial_7,\partial_3]\right) - \tfrac{7}{2}\,c_2a_0,$$
$$a_{18} = c_2^2\,\partial_7\partial_3 - 10\,c_2a_0 + \tfrac{6}{25}\,[\partial_3,[\partial_5,\partial_3]]\partial_3,$$
$$a_{19} = c_2^3\,\partial_5\partial_3 + \tfrac{1}{5}\,c_2a_0 - \tfrac{4}{35}\,[\partial_3,[\partial_5,\partial_3]]\partial_3,$$
$$a_{20} = \tfrac{1}{2}\,c_2^4\partial_3^2 + \tfrac{18}{35}\,c_2a_0,\ a_{21} = c_2^7 \tag{A.2}$$

acting on $\phi(\xi_{14})$. Above we have introduced the operator:

$$a_0 = \tfrac{48}{691}\ \left(\,[\partial_9,\partial_3] - 3\,[\partial_7,\partial_5]\,\right). \tag{A.3}$$

Furthermore, we have used some useful formulae exhibited in the following. Nested commutators involving the derivatives ∂_3 and ∂_5 acting on various products of f_3 and f_5 have a 'diagonal' structure:

$$\begin{aligned}&[\partial_3,[\partial_3,[\partial_3,\partial_5]]]f_5f_3f_3f_3 = 1,\\ &[\partial_3,[\partial_3,\partial_5]]\partial_3(f_5f_3f_3) \amalg f_3 = 1,\\ &[\partial_3,\partial_5]\partial_3^2(f_5f_3) \amalg f_3 \amalg f_3 = 2,\\ &\partial_5\partial_3^3f_5 \amalg f_3 \amalg f_3 \amalg f_3 = 6.\end{aligned} \tag{A.4}$$

On the other hand, all the other combinations of differential operators

$$[\partial_3,[\partial_3,[\partial_3,\partial_5]]],\ [\partial_3,[\partial_3,\partial_5]]\partial_3,\ [\partial_3,\partial_5]\partial_3^2,\ \partial_5\partial_3^3$$

acting on the products $\{f_5f_3^3,\ (f_5f_3f_3) \amalg f_3,\ (f_5f_3) \amalg f_3 \amalg f_3,\ f_5 \amalg f_3 \amalg f_3 \amalg f_3\}$ vanish, e.g. $[\partial_3,[\partial_3,[\partial_3,\partial_5]]]$ annihilates all of $(f_5f_3f_3) \amalg f_3,\ (f_5f_3) \amalg f_3 \amalg f_3,\ f_5 \amalg f_3 \amalg f_3 \amalg f_3$. More generally, we have:

$$\underbrace{[\partial_3,[\partial_3,[\ldots,[\partial_3,\partial_5]\ldots]]]}_{(k-p)\text{-fold commutator}}\ \partial_3^p(f_5f_3^{k-q})(\,\amalg f_3)^q = p!\ \delta_{p,q}. \tag{A.5}$$

A.2. Decomposition at weight 15

At weight 15 we collect the information about the lower weight basis $\mathcal{U}_{k\leqslant 14}$ and with (4.56) we can construct the following basis for $\mathcal{U}_{15}$:

$$\begin{aligned}&\phi(\zeta^m_{1,1,3,4,6}),\ \phi(\zeta^m_{3,3,9}),\ \phi(\zeta^m_{5,3,7}),\ f_{15},\ f_3 \amalg \phi(\zeta^m_{1,1,4,6}),\ f_3 \amalg \phi(\zeta^m_{3,9}),\\ &f_9 \amalg f_3 \amalg f_3,\ f_3 \amalg f_5 \amalg f_7,\ f_3 \amalg f_3 \amalg f_3 \amalg f_3 \amalg f_3,\\ &(-14f_7f_3 - 6f_5^2) \amalg f_5,\ f_5 \amalg f_5 \amalg f_5,\ (-5f_5f_3) \amalg f_7,\\ &\phi(\zeta^m_{3,3,7})f_2,\ \phi(\zeta^m_{3,5,5})f_2,\ f_{13}f_2,\ (-14f_7f_3 - 6f_5^2) \amalg f_3f_2,\ (-5f_5f_3) \amalg f_5f_2,\\ &f_7 \amalg f_3 \amalg f_3f_2,\ f_5 \amalg f_5 \amalg f_3f_2,\\ &\phi\big(\zeta^m_{3,3,5}\big)f_2^2,\ (-5f_5f_3) \amalg f_3f_2^2,\ f_{11}f_2^2,\ f_5 \amalg f_3 \amalg f_3f_2^2,\ f_3 \amalg f_3 \amalg f_3f_2^3,\\ &f_9f_2^3,\ f_7f_2^4,\ f_5f_2^5,\ f_3f_2^6,\end{aligned} \tag{A.6}$$

with $\phi(\zeta^m_{3,3,5})$, $\phi(\zeta^m_{3,9})$, $\phi(\zeta^m_{1,1,4,6})$, $\phi(\zeta^m_{3,3,7})$, $\phi(\zeta^m_{3,5,5})$, $\phi(\zeta^m_{1,1,3,4,6})$, $\phi(\zeta^m_{3,3,9})$ and $\phi(\zeta^m_{5,3,7})$ given in (4.28), (4.35), (4.42) and (4.56), respectively. The operators a_i of the decomposition (4.57) are:

$$\begin{aligned}
&a_1 = \tfrac{48}{7601}\ ([\partial_3,[\partial_9,\partial_3]] - 3\,[\partial_3,[\partial_7,\partial_5]])\,,\ a_2 = \tfrac{1}{27}\,[\partial_3,[\partial_9,\partial_3]] - \tfrac{853}{648}\,a_1,\\
&a_3 = \tfrac{2}{15}\,[\partial_3,[\partial_7,\partial_5]] - \tfrac{1}{70}\,[\partial_5,[\partial_7,\partial_3]] + \tfrac{17\,203}{3360}\,a_1,\ a_4 = \partial_{15},\\
&a_5 = a_1 + a_0\,\partial_3,\ a_6 = \tfrac{1}{27}\,[\partial_9,\partial_3]\partial_3 + \tfrac{2665}{648}\,a_0\partial_3 + \tfrac{29}{9}\,a_1,\\
&a_7 = \tfrac{1}{2}\,\partial_9\partial_3^2 + \tfrac{799}{72}\,a_0\partial_3 + \tfrac{6775}{144}\,a_1,\\
&a_8 = \partial_7\partial_5\partial_3 + \tfrac{2}{9}\,[\partial_9,\partial_3]\partial_3 - \tfrac{467}{108}\,a_0\partial_3 - \tfrac{74}{3}\,a_1,\\
&a_9 = \tfrac{1}{5!}\,\partial_3^5 - \tfrac{1}{12}\,a_0\partial_3 - \tfrac{1}{15}\,a_1,\\
&a_{10} = \tfrac{1}{14}\,[\partial_7,\partial_3]\partial_5 + \tfrac{2188}{945}\,a_1 + \tfrac{3}{35}\,[\partial_5,[\partial_7,\partial_3]] - \tfrac{2}{45}\,[\partial_3,[\partial_9,\partial_3]],\\
&a_{11} = \tfrac{1}{6}\,\partial_5^3 + \tfrac{3}{14}\,[\partial_7,\partial_3]\partial_5 + \tfrac{1185\,701}{30\,240}\,a_1 + \tfrac{11}{70}\,[\partial_5,[\partial_7,\partial_3]] - \tfrac{2}{45}\,[\partial_3,[\partial_9,\partial_3]]\\
&a_{12} = \tfrac{1}{5}\,[\partial_5,\partial_3]\partial_7 - \tfrac{12\,199}{720}\,a_1 + \tfrac{1}{5}\,[\partial_5,[\partial_7,\partial_3]] - \tfrac{1}{15}\,[\partial_3,[\partial_9,\partial_3]]\\
&a_{13} = \tfrac{1}{14}\,c_2\,[\partial_3,[\partial_7,\partial_3]] + 2\,a_1,\\
&a_{14} = -\tfrac{3}{35}\,c_2\,[\partial_3,[\partial_7,\partial_3]] + \tfrac{1}{25}\,c_2\,[\partial_5,[\partial_5,\partial_3]] - \tfrac{14}{5}\,a_1,\\
&a_{15} = c_2\,\partial_{13} - \tfrac{6417\,649}{2880}\,a_1 - \tfrac{143}{20}\,[\partial_5,[\partial_7,\partial_3]] + \tfrac{1339}{30}\,[\partial_3,[\partial_9,\partial_3]],\\
&a_{16} = \tfrac{1}{14}\,c_2\,[\partial_7,\partial_3]\partial_3 - 3\,a_0\partial_3 - 6\,a_1,\\
&a_{17} = \tfrac{1}{5}\,c_2\,[\partial_5,\partial_3]\partial_5 + \tfrac{1}{5}\,c_2\,[\partial_5,[\partial_5,\partial_3]] + \tfrac{21}{2}\,a_1,\\
&a_{18} = \tfrac{1}{2}\,c_2\,\partial_7\partial_3^2 - 10\,a_0\partial_3 - 26\,a_1,\\
&a_{19} = \tfrac{1}{2}\,c_2\ \partial_5^2\partial_3 + \tfrac{3}{14}\,c_2\,[\partial_7,\partial_3]\partial_3 - \tfrac{7}{2}\,a_0\partial_3 - 8\,a_1,\\
&a_{20} = \tfrac{1}{5}\,c_2^2\,[\partial_3,[\partial_5,\partial_3]] + 4\,a_1,\ a_{21} = \tfrac{1}{5}\,c_2^2\,[\partial_5,\partial_3]\partial_3 - \tfrac{3}{5}\,a_0\partial_3 - \tfrac{8}{5}\,a_1,\\
&a_{22} = c_2^2\,\partial_{11} + \tfrac{11}{4}\,c_2\,[\partial_3,[\partial_7,\partial_3]] + \tfrac{11}{2}\,c_2\,[\partial_5,[\partial_5,\partial_3]] - \tfrac{8495\,287}{15\,120}\,a_1\\
&\qquad - \tfrac{11}{35}\,[\partial_5,[\partial_7,\partial_3]] + \tfrac{128}{45}\,[\partial_3,[\partial_9,\partial_3]],\ a_{23} = \tfrac{1}{2}\,c_2^2\,\partial_5\partial_3^2 + \tfrac{1}{5}\,a_0\partial_3 - \tfrac{23}{10}\,a_1,\\
&a_{24} = \tfrac{1}{6}\,c_2^3\,\partial_3^3 + \tfrac{18}{35}\,a_0\partial_3 + \tfrac{12}{35}\,a_1,\\
&a_{25} = c_2^3\,\partial_9 + 9\,c_2^2\,[\partial_3,[\partial_5,\partial_3]] - \tfrac{2}{35}\,c_2\,[\partial_3,[\partial_7,\partial_3]] + \tfrac{2}{5}\,c_2\,[\partial_5,[\partial_5,\partial_3]]\\
&\qquad + \tfrac{54\,263\,011}{396\,900}\,a_1 + \tfrac{68}{1225}\,[\partial_5,[\partial_7,\partial_3]] - \tfrac{236}{4725}\,[\partial_3,[\partial_9,\partial_3]],\\
&a_{26} = c_2^4\,\partial_7 + \tfrac{6}{25}\,c_2^2\,[\partial_3,[\partial_5,\partial_3]] - \tfrac{16}{245}\,c_2\,[\partial_3,[\partial_7,\partial_3]] + \tfrac{57\,847}{15\,750}\,a_1\\
&\qquad + \tfrac{24}{875}\,[\partial_5,[\partial_7,\partial_3]] - \tfrac{184}{2625}\,[\partial_3,[\partial_9,\partial_3]],\\
&a_{27} = c_2^5\,\partial_5 - \tfrac{4}{35}\,c_2^2\,[\partial_3,[\partial_5,\partial_3]] - \tfrac{1714\,624}{121\,275}\,a_1 + \tfrac{48}{13\,475}\,[\partial_5,[\partial_7,\partial_3]],\\
&\qquad - \tfrac{64}{5775}\,[\partial_3,[\partial_9,\partial_3]],\ a_{28} = c_2^6\,\partial_3 + \tfrac{1451\,972}{716\,625}\,a_1
\end{aligned} \tag{A.7}$$

acting on $\phi(\xi_{15})$. Above we have used the operator a_0 defined in (A.3).

A.3. Decomposition at weight 16

Gathering the information about the lower weight basis $\mathcal{U}_{k\leqslant 15}$ with (4.56) we can construct the following basis for $\mathcal{U}_{16}$:

$$\begin{aligned}
&\phi(\zeta^m_{1,1,6,8}),\ \phi(\zeta^m_{3,3,3,7}),\ \phi(\zeta^m_{3,3,5,5}),\ \phi(\zeta^m_{3,13}),\ \phi(\zeta^m_{5,11})\\
&f_3 ш \phi(\zeta^m_{3,3,7}),\ f_3 ш \phi(\zeta^m_{3,5,5}),\ f_3 ш f_{13},\ (-14 f_7 f_3 - 6 f_5^2) ш f_3 ш f_3,\\
&(-5 f_5 f_3) ш f_3 ш f_5,\ f_3 ш f_3 ш f_3 ш f_7,\ f_3 ш f_3 ш f_5 ш f_5,\ f_7 ш f_9,\\
&25\,(f_5 f_3) ш (f_5 f_3),\ f_5 ш f_{11},\ f_5 ш \phi(\zeta^m_{3,3,5}),\\
&f_3 ш \phi(\zeta^m_{3,3,5}) f_2,\ f_3 ш f_3 ш (-5 f_5 f_3) f_2,\ f_3 ш f_{11} f_2,\ f_3 ш f_3 ш f_3 ш f_5 f_2\\
&f_3 ш f_3 ш f_3 ш f_3 f_2^2,\ f_3 ш f_9 f_2^2,\ f_3 ш f_7 f_2^3,\ f_3 ш f_5 f_2^4,\ f_3 ш f_3 f_2^5,
\end{aligned}$$

$$\phi(\zeta^m_{3,3,3,5})f_2,\ \phi(\zeta^m_{3,11})f_2,\ \phi(\zeta^m_{5,9})f_2,\ f_5 ш f_9 f_2,\ f_7 ш f_7 f_2,$$
$$\phi(\zeta^m_{1,1,4,6})f_2^2,\ \phi(\zeta^m_{3,9})f_2^2,\ f_5 ш f_7 f_2^2,\ (-14 f_7 f_3 - 6 f_5^2)f_2^3, -5 f_5 f_3 f_2^4,$$
$$f_5 ш f_5 f_2^3,\ f_2^8, \tag{A.8}$$

with the maps $\phi(\zeta^m_{3,3,5}),\ \phi(\zeta^m_{3,9}), \phi(\zeta^m_{1,1,4,6}),\ \phi(\zeta^m_{3,3,7}), \phi(\zeta^m_{3,5,5}),\ \phi(\zeta^m_{3,3,3,5}), \phi(\zeta^m_{3,11}), \phi(\zeta^m_{5,9})$ $\phi(\zeta^m_{3,3,3,7}), \phi(\zeta^m_{3,3,5,5}), \phi(\zeta^m_{3,13}), \phi(\zeta^m_{5,11})$ and $\phi(\zeta^m_{1,1,6,8})$ given in (4.28), (4.35), (4.42), (4.50) and (4.64), respectively. The operators a_i of the decomposition (4.65) are:

$$a_1 = \tfrac{720}{3617}\left\{\tfrac{7}{11}\,[\partial_{11},\partial_5] - \tfrac{2}{11}\,[\partial_{13},\partial_3] - [\partial_9,\partial_7] + \tfrac{6493}{9240}\,[\partial_3,[\partial_3,[\partial_7,\partial_3]]] \right.$$
$$\left. -\tfrac{751}{100}\,[\partial_3,[\partial_5,[\partial_5,\partial_3]]]\right\},$$
$$a_2 = -\tfrac{19}{7}\,a_1 + \tfrac{1}{14}\,[\partial_3,[\partial_3,[\partial_7,\partial_3]]],$$
$$a_3 = \tfrac{542}{175}\,a_1 - \tfrac{3}{35}\,[\partial_3,[\partial_3,[\partial_7,\partial_3]]] + \tfrac{1}{25}\,[\partial_3,[\partial_5,[\partial_5,\partial_3]]],$$
$$a_4 = -\tfrac{19}{286}\,[\partial_{13},\partial_3] + \tfrac{3}{22}\,[\partial_{11},\partial_5] + \tfrac{2\,217\,053}{16\,800}\,a_1 - \tfrac{200\,559}{80\,080}\,[\partial_3,[\partial_3,[\partial_7,\partial_3]]]$$
$$-\tfrac{7011}{2600}\,[\partial_3,[\partial_5,[\partial_5,\partial_3]]]$$
$$a_5 = \tfrac{3}{242}\,[\partial_{13},\partial_3] - \tfrac{5}{242}\,[\partial_{11},\partial_5] - \tfrac{114\,307}{7392}\,a_1 + \tfrac{23\,181}{67\,760}\,[\partial_3,[\partial_3,[\partial_7,\partial_3]]]$$
$$+\tfrac{909}{2200}\,[\partial_3,[\partial_5,[\partial_5,\partial_3]]],\ a_6 = -\tfrac{1}{14}\,[\partial_3,[\partial_3,\partial_7]]\partial_3 + \tfrac{5}{7}\,a_1,$$
$$a_7 = -\tfrac{1}{25}\,[\partial_5,[\partial_3,\partial_5]]\partial_3 + \tfrac{3}{35}\,[\partial_3,[\partial_3,\partial_7]]\partial_3 - \tfrac{6}{7}\,a_1,\ a_8 = \partial_{13}\partial_3 + \tfrac{8497}{42}\,a_1,$$
$$a_9 = \tfrac{1}{28}\,[\partial_7,\partial_3]\partial_3^2 + \tfrac{1}{7}\,a_1,\ a_{10} = \tfrac{1}{5}\,[\partial_5,\partial_3]\partial_5\partial_3 + \tfrac{1}{5}\,[\partial_5,[\partial_5,\partial_3]]\partial_3,$$
$$a_{11} = \tfrac{1}{3!}\,\partial_7\partial_3^3 - \tfrac{1}{3}\,a_1,\ a_{12} = \tfrac{1}{2}\,\left(\tfrac{1}{2}\,\partial_5^2 + \tfrac{3}{14}\,[\partial_7,\partial_3]\right)\partial_3^2 - \tfrac{4}{7}\,a_1,$$
$$a_{13} = \partial_9\partial_7 + \tfrac{4\,850\,713}{6600}\,a_1 - \tfrac{2\,272\,973}{330\,330}\,[\partial_3,[\partial_3,[\partial_7,\partial_3]]] - \tfrac{299\,373}{7150}\,[\partial_3,[\partial_5,[\partial_5,\partial_3]]]$$
$$-\tfrac{1275}{1573}\,[\partial_{13},\partial_3] + \tfrac{210}{121}\,[\partial_{11},\partial_5],\ a_{14} = \tfrac{1}{50}\,[\partial_5,\partial_3]^2,$$
$$a_{15} = \partial_{11}\partial_5 + \tfrac{455\,534}{525}\,a_1 - \tfrac{601\,677}{40\,040}\,[\partial_3,[\partial_3,[\partial_7,\partial_3]]] - \tfrac{21\,033}{1300}\,[\partial_3,[\partial_5,[\partial_5,\partial_3]]]$$
$$-\tfrac{57}{143}\,[\partial_{13},\partial_3] + \tfrac{9}{11}\,[\partial_{11},\partial_5],$$
$$a_{16} = \tfrac{1}{5}\,[\partial_3,[\partial_5,\partial_3]]\partial_5 + \tfrac{1}{5}\,[\partial_3,[\partial_5,[\partial_5,\partial_3]]] - \tfrac{2}{5}\,a_1,$$
$$a_{17} = \tfrac{1}{5}\,c_2\,[\partial_3,[\partial_5,\partial_3]]\partial_3,\ a_{18} = \tfrac{1}{10}\,c_2\,[\partial_5,\partial_3]\partial_3^2,$$
$$a_{19} = c_2\,\partial_{11}\partial_3 - \tfrac{11}{4}\,[\partial_3,[\partial_3,\partial_7]]\partial_3 - \tfrac{11}{2}\,[\partial_5,[\partial_3,\partial_5]]\partial_3 - 137\,a_1,$$
$$a_{20} = \frac{1}{3!}\,c_2\,\partial_5\partial_3^3,\ a_{21} = \frac{1}{4!}\,c_2^2\,\partial_3^4 - \tfrac{1}{12}\,c_2^2\,a_0,$$
$$a_{22} = c_2^2\,\partial_9\partial_3 + 9\,c_2\,[\partial_3,[\partial_5,\partial_3]]\partial_3 + \tfrac{799}{72}\,c_2^2\,a_0 - \tfrac{2}{35}\,[\partial_3,[\partial_7,\partial_3]]\partial_3$$
$$+\tfrac{2}{5}\,[\partial_5,[\partial_5,\partial_3]]\partial_3 - \tfrac{11}{7}\,a_1,$$
$$a_{23} = c_2^3\,\partial_7\partial_3 + \tfrac{6}{25}\,c_2\,[\partial_3,[\partial_5,\partial_3]]\partial_3 - 10\,c_2^2\,a_0 - \tfrac{16}{245}\,[\partial_3,[\partial_7,\partial_3]]\partial_3 + \tfrac{848}{245}\,a_1,$$
$$a_{24} = c_2^4\,\partial_5\partial_3 - \tfrac{4}{35}\,c_2\,[\partial_3,[\partial_5,\partial_3]]\partial_3 + \tfrac{1}{5}\,c_2^2\,a_0 + \tfrac{48}{35}\,a_1,$$
$$a_{25} = \tfrac{1}{2}\,c_2^5\,\partial_3^2 + \tfrac{18}{35}\,c_2^2\,a_0 + \tfrac{408}{2695}\,a_1,\ a_{26} = \tfrac{1}{5}\,c_2\,[\partial_3,[\partial_3,[\partial_5,\partial_3]]],$$
$$a_{27} = -\tfrac{23}{198}\,c_2\,[\partial_{11},\partial_3] + \tfrac{5}{18}\,c_2\,[\partial_9,\partial_5] - \tfrac{12\,841}{1188}\,c_2\,[\partial_3,[\partial_3,[\partial_5,\partial_3]]] - \tfrac{1991}{14}\,a_1$$
$$+\tfrac{121}{28}\,[\partial_3,[\partial_3,[\partial_7,\partial_3]]] - \tfrac{7}{2}\,[\partial_3,[\partial_5,[\partial_5,\partial_3]]],$$
$$a_{28} = \tfrac{1}{27}\,c_2\,[\partial_{11},\partial_3] - \tfrac{2}{27}\,c_2\,[\partial_9,\partial_5] + \tfrac{232}{81}\,c_2\,[\partial_3,[\partial_3,[\partial_5,\partial_3]]] + \tfrac{697}{21}\,a_1$$
$$-\tfrac{47}{42}\,[\partial_3,[\partial_3,[\partial_7,\partial_3]]] + [\partial_3,[\partial_5,[\partial_5,\partial_3]]],$$
$$a_{29} = c_2\,\partial_9\partial_5 + 9\,[\partial_3,[\partial_5,\partial_3]]\partial_5 - \tfrac{23}{33}\,c_2\,[\partial_{11},\partial_3] + \tfrac{5}{3}\,c_2\,[\partial_9,\partial_5] - \tfrac{12\,443}{14}\,a_1$$
$$-\tfrac{12\,775}{198}\,c_2\,[\partial_3,[\partial_3,[\partial_5,\partial_3]]] + \tfrac{363}{14}\,[\partial_3,[\partial_3,[\partial_7,\partial_3]]] - 21\,[\partial_3,[\partial_5,[\partial_5,\partial_3]]],$$
$$a_{30} = \tfrac{1}{2}\,c_2\,\partial_7^2 - \tfrac{235}{396}\,c_2\,[\partial_{11},\partial_3] + \tfrac{55}{36}\,c_2\,[\partial_9,\partial_5] - \tfrac{647\,287}{11\,880}\,c_2\,[\partial_3,[\partial_3,[\partial_5,\partial_3]]]$$
$$-\tfrac{78\,201}{140}\,a_1 + \tfrac{967}{56}\,[\partial_3,[\partial_3,[\partial_7,\partial_3]]] - \tfrac{333}{20}\,[\partial_3,[\partial_5,[\partial_5,\partial_3]]],\ a_{31} = c_2^2\,a_0,$$

$$a_{32} = \tfrac{1}{27}\, c_2^2\, [\partial_9, \partial_3] + \tfrac{2665}{648}\, c_2^2\, a_0 + \tfrac{2}{3}\, c_2\, [\partial_3, [\partial_3, [\partial_5, \partial_3]]] - \tfrac{8954}{1575}\, a_1 + \tfrac{4}{35}\, [\partial_3, [\partial_3, [\partial_7, \partial_3]]] - \tfrac{4}{75}\, [\partial_3, [\partial_5, [\partial_5, \partial_3]]],$$

$$a_{33} = c_2^2\, \partial_7 \partial_5 + \tfrac{2}{9}\, c_2^2\, [\partial_9, \partial_3] + \tfrac{6}{25}\, [\partial_3, [\partial_5, \partial_3]]\partial_5 - \tfrac{467}{108}\, c_2^2\, a_0 + 4\, c_2\, [\partial_3, [\partial_3, [\partial_5, \partial_3]]] - \tfrac{21\,331}{525}\, a_1 + \tfrac{24}{35}\, [\partial_3, [\partial_3, [\partial_7, \partial_3]]] - \tfrac{8}{25}\, [\partial_3, [\partial_5, [\partial_5, \partial_3]]],$$

$$a_{34} = \tfrac{1}{14}\, c_2^3\, [\partial_7, \partial_3] - 3\, c_2^2\, a_0 - \tfrac{62}{245}\, a_1 + \tfrac{2}{245}\, [\partial_3, [\partial_3, [\partial_7, \partial_3]]],$$

$$a_{35} = \tfrac{1}{5}\, c_2^4\, [\partial_5, \partial_3] - \tfrac{3}{5}\, c_2^2\, a_0 + \tfrac{4}{175}\, c_2\, [\partial_3, [\partial_3, [\partial_5, \partial_3]]] + \tfrac{108}{875}\, a_1,$$

$$a_{36} = c_2^3\, \left(\tfrac{1}{2}\, \partial_5^2 + \tfrac{3}{14}\, [\partial_7, \partial_3]\right) - \tfrac{4}{35}\, [\partial_3, [\partial_5, \partial_3]]\partial_5 - \tfrac{7}{2}\, c_2^2\, a_0 - \tfrac{284}{245}\, a_1 + \tfrac{6}{245}\, [\partial_3, [\partial_3, [\partial_7, \partial_3]]] - \tfrac{2}{35}\, [\partial_3, [\partial_5, [\partial_5, \partial_3]]], \quad a_{37} = c_2^8, \tag{A.9}$$

acting on $\phi(\xi_{16})$. Again, we have used the operator a_0 defined in (A.3).

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