Non-commutative dual representation for quantum systems on Lie groups

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Abstract. We provide a short overview of some recent results on the application of group Fourier transform to quantum mechanics and quantum gravity. We close by pointing out some future research directions.

1. Introduction

The group Fourier transform is an integral transform from functions on a Lie group to functions on a non-commutative dual space. It was first formulated for functions on SO(3) [1, 2], later generalized to SU(2) [2–4] and other Lie groups [5, 6], and is based on the quantum group structure of Drinfel'd double of the group [7, 8]. The transform provides a unitarily equivalent representation of quantum systems with a Lie group configuration space in terms of non-commutative algebra of functions on the classical dual space, the dual of the Lie algebra. As such, it has proven useful in several different ways. Most importantly, it provides a clear connection between the quantum system and the corresponding classical one, allowing for a better physical insight into the system. In the case of quantum gravity models this has turned out to be particularly helpful in unraveling the geometrical content of the models, because the dual variables have an intuitive interpretation as classical geometrical quantities.

We will first review the general formulation of the group Fourier transform, and then mention its recent applications to quantum mechanics on $SO(3)$, and to Loop Quantum Gravity and spin foam models. We will close by pointing out some future reseach directions.

2. Group Fourier transform

Let $\mathcal{G}$ be a (finite dimensional) Lie group, and $\mathfrak{g}^*$the dual of the Lie algebra $\mathfrak{g}$ of $\mathcal{G}$. The group Fourier transform is an isometry between $L^2(\mathcal{G}, \mathrm{d}g)$ and a non-commutative function space $L^2_\star(\mathfrak{g}^*, \mathrm{d}X)$ defined in the following way [6]: We choose a function $E : \mathcal{T}^*\mathcal{G} = \mathcal{G} \times \mathfrak{g}^* \to U(1)$, $(g, X) \mapsto E_g(X)$, the non-commutative plane wave, such that we may decompose the delta distribution on $\mathcal{G}$ in terms of the Fourier modes in $\mathfrak{g}^*$ as $\frac{1}{\kappa^d}\delta_e(g) = \int_{\mathfrak{g}^*} \frac{\mathrm{d}X}{(2\pi\hbar)^d} E_g(X)$. Here $\mathrm{d}X$ the Lebesgue measure on $\mathfrak{g}^*$, $d = \dim \mathcal{G}$, $\kappa \in \mathbb{R}_+$ is a deformation parameter, which determines the physical dimensions on $\mathcal{G}$, and $\hbar$ is the Planck constant. We then define a non-commutative $\star$-product via the relation $E_g(X) \star E_h(X) \equiv E_{gh}(X)$. The $\star$-product is extended to functions on $\mathfrak{g}^*$ through linearity by defining the group Fourier transform of $\phi \in L^2(\mathcal{G}, \mathrm{d}g)$ as $\tilde{\phi}(X) = \int_{\mathcal{G}} \kappa^d \mathrm{d}g\ E_g(X)\phi(g)$, where $\mathrm{d}g$ is the (left invariant) Haar measure on $\mathcal{G}$. Due to above

Journal of Physics: Conference Series **360** (2012) 012052 doi:10.1088/1742-6596/360/1/012052

properties, the inverse transform is obtained as $\phi(g) = \int_{\mathfrak{g}^*} \frac{\mathrm{d}X}{(2\pi\hbar)^d} E_g(X) \star \tilde{\phi}(X)$. We denote the image of $L^2(\mathcal{G}, \mathrm{d}g)$ under the transform as $L^2_\star(\mathfrak{g}^*, \mathrm{d}X)$, since by requiring $E_{g^{-1}}(X) = \overline{E_g(X)}$ the transform is an isometry from $L^2(\mathcal{G}, \mathrm{d}g)$ to $L^2_\star(\mathfrak{g}^*, \mathrm{d}X)$, i.e.,

$$\int_{\mathcal{G}} \kappa^d \mathrm{d}g\, \overline{\phi(g)} \phi'(g) = \int_{\mathfrak{g}^*} \frac{\mathrm{d}X}{(2\pi\hbar)^d} \overline{\tilde{\phi}(X)} \star \tilde{\phi}'(X) \ . \tag{1}$$

Accordingly, we obtain a unitarily equivalent non-commutative dual representation of fields living on $\mathcal{G}$ via the group Fourier transform.

The choice for the explicit form of the non-commutative plane waves $E_g(X)$ is not obvious. Typically the plane waves are of the form $E_g(X) = \exp(i \sum_i Z_i(g) \cdot X^i)$, where $Z_i(g)$ are some coordinates on the group manifold. In particular, [1, 2, 9, 10] use the coordinates $Z^i(g) = -\frac{i}{2} \operatorname{tr}(g\sigma^i)$ for $SO(3)$, where the trace is taken in the fundamental spin-$\frac{1}{2}$-representation and σ^i are the Pauli matrices. For $SU(2)$ the construction must be modified, as for example in [2–4], since the above coordinates are two-to-one in this case, $Z^i(g^{-1}) = Z^i(-g)$. In [2,3] an extra parameter was introduced to make the transform one-to-one for $SU(2)$. For an exponential Lie group the canonical choice for coordinate functions Z^i on $\mathcal{G}$ can be obtained from the inverse of the exponential map $\exp : \mathfrak{g} \simeq \mathbb{R}^d \to \mathcal{G}$, $Z \mapsto \exp Z$, for a portion of $\mathfrak{g}$ containing the origin, where the exponential map is one-to-one. With this choice we have the natural inner product $\cdot : \mathfrak{g} \times \mathfrak{g}^* \to \mathbb{R}$ to use for the plane waves $E_g(X) = \exp(iZ(g) \cdot X)$.[1]

The non-commutative variables $X \in \mathfrak{g}^*$ can be seen to correspond to the classical dual cotangent space variables in the way of deformation quantization: The canonical Lie derivative operators $\hat{X}_i = -i\frac{\hbar}{\kappa}\mathcal{L}_i$, where $\mathcal{L}_i$ are Lie derivatives with respect to a basis in the Lie algebra, satisfy the commutation relations $[\hat{X}_i, \hat{X}_j] = -i\frac{\hbar}{\kappa} c_{ij}{}^k \hat{X}_k$, where $c_{ij}{}^k$ are the structure constants of $\mathfrak{g}$. These commutators correctly reflect the classical Poisson algebra of the phase space $\mathcal{T}^*\mathcal{G}$ arising from the symplectic structure of the cotangent bundle. If as usual we have $\mathcal{L}_i E_e(X) = i\frac{\kappa}{\hbar} X_i$, then the non-commutative plane wave is the generating function for $\star$-polynomials of coordinates of the non-commutative dual space, $\hat{X}_{i_1} \cdots \hat{X}_{i_n} E_g(X) = E_g(X) \star X_{i_1} \star \cdots \star X_{i_n}$, and we recover the Lie algebraic commutation relations for the non-commutative coordinates $[X_i, X_j]_\star = X_i \star X_j - X_j \star X_i = -i\frac{\hbar}{\kappa} c_{ij}{}^k X_k$. Here we see that the quantity $(\hbar/\kappa)$ controls the non-commutativity, but it is κ^{-1}, which is the deformation parameter associated to the $\star$-product in this case, since $\hbar$ plays a distinct role [10]. In particular, κ determines the physical dimensions of the group, present even in the classical theory, whereas $\hbar$ is the parameter controlling the quantum fluctuations. Therefore there are two distinct limits of a quantum theory with the phase space $\mathcal{T}^*\mathcal{G}$ to be considered, the classical limit $\hbar \to 0$ and the commutative limit $\kappa^{-1} \to 0$. In the commutative limit the Lie algebra commutators vanish, and the group coincides essentially with the tangent space $\mathfrak{g} \simeq \mathbb{R}^d$ at the unit element.

3. Phase space path integral for quantum mechanics on $SO(3)$

Taking advantage of the group Fourier transform, we obtain a dual non-commutative representation of quantum mechanics on $\mathcal{G}$ in terms of a set of states $\{|X\rangle \mid X \in \mathfrak{g}^*\}$ by setting $\langle g|X\rangle := E_g(X)$, where Dirac notation is used. These states constitute a basis with respect to the non-commutative $\star$-product [6, 10]. We may then write down the first order phase space path integral. As usual, we have a Hamiltonian operator $\hat{H}$ generating the time-evolution of the system. Then, the propagator for the quantum system in $\mathcal{G}$-basis reads $\langle g', t'|g, t\rangle := \langle g'|e^{-i(t'-t)\hat{H}/\hbar}|g\rangle$. We then introduce time-splitting $(t'-t) \equiv \epsilon N$ and insert

[1] In fact, since the $\star$-product may be defined only under integration, we only need $\mathcal{G}$ to be weakly exponential, i.e., the image of the exponential map to be dense in $\mathcal{G}$. (See [11] and references therein for details on exponential Lie groups.) If this is not the case, it may be possible to generalize to several coordinate patches as in [3].

resolutions of identity $\hat{\mathbb{1}} \equiv \int_{\mathcal{G}} \kappa^d \mathrm{d}g\ |g\rangle\langle g|$ and $\hat{\mathbb{1}} \equiv \int_{\mathfrak{g}^*} \frac{\mathrm{d}X}{(2\pi\hbar)^d}\ |X\rangle \star \langle X|$ for each timestep. To obtain the path integral, we take the limits $\epsilon \to 0$, $N \to \infty$, while $\epsilon N = (t' - t)$. For the path integral to satisfy the Schrödinger equation, only the linear order in ϵ must be taken into account [12], and we may approximate $\langle X_k|e^{-i\epsilon \hat{H}/\hbar}|g_k\rangle \approx e^{-i\epsilon H_\star(g_k,X_k)/\hbar} \star \langle X_k|g_k\rangle$, where $\langle g|\hat{H}|X\rangle =: \langle g|X\rangle \star H_\star(g,X)$. Further, using the properties of the plane waves, we arrive at the discrete timestep expression for the path integral

$$\langle g', t'|g, t\rangle = \lim_{\substack{\epsilon\to 0\\ N\to\infty}} \left[\prod_{k=1}^{N-1} \int_{\mathcal{G}} \kappa^d \mathrm{d}g_k\right] \left[\prod_{k=0}^{N-1} \int_{\mathfrak{g}^*} \frac{\mathrm{d}X_k}{(2\pi\hbar)^d}\right] \exp\left\{\frac{i}{\hbar}\sum_{k=0}^{N-1} \epsilon\left(\frac{1}{\epsilon} V_k \cdot X_k - H_{q,k}\right)\right\}\ , \quad (2)$$

where $V_k := Z(g_k^{-1} g_{k+1})$, $H_{q,k} := H_q(g_k, X_k)$, $Z^i(g)$ are the coordinates on $\mathcal{G}$ used in the plane waves, and $H_q(g,X) := [\omega(-i\partial_X^i)]^{-1} H_\star(g,X)$ is *the quantum corrected Hamiltonian* for the system [6]. Here $\omega(Z^i(g))\mathrm{d}^d Z \equiv \mathrm{d}g$, where $\mathrm{d}^d Z$ is the Lebesgue measure on the coordinates Z^i. These quantum corrections to the classical Hamiltonian, which are due to the non-commutative structure of the phase space, are crucial for the propagator to satisfy the Schrödinger equation [10].

For the special case of $\mathcal{G} = \mathbb{R}^d$ with the coordinates $Z^i(g)$ obtained via the inverse exponential map (2) agrees with the usual expressions for the first order path integral. In [10] the stationary phase approximation was studied for $SO(3)$ with coordinates $Z^i(g) = -(i/2)\,\mathrm{tr}(g\sigma^i)$ and found to yield the correct classical equations of motion in the limit $\hbar \to 0$. This further confirms the identification of the non-commutative dual variables as corresponding to the classical canonically conjugate variables. Also, it was shown that for the case of a free particle on $SO(3)$ the quantum corrections agree with the well-established results [12–14]. In the limit $\kappa \to \infty$ the path integral coincides with the one for $\mathcal{G} = \mathbb{R}^d$ as the diameter of the group becomes infinite. Thus the first order path integral facilitates access to semi-classical as well as 'semi-commutative' approximations for quantum systems on Lie groups.

4. Non-commutative metric variables for quantum gravity

The group Fourier transform has recently provided a better insight into the geometrical content of quantum gravity models. The configuration space of Loop Quantum Gravity (LQG), for example, which corresponds to a graph Γ representing the quantum geometry of a spatial slice of spacetime, is $SO(3)^E/SO(3)^V$, where E and V are the numbers of edges and vertices in Γ [15]. Here the $SO(3)^E$ part corresponds to parallel transports along the edges of the embedded graph, and $SO(3)^V$ corresponds to local gauge transformations at the vertices[2]. From the canonical analysis of LQG it can be seen that the dual variables to the parallel transports along edges of Γ taking values in $SO(3)$ are area bivectors valued in $\mathfrak{so}(3)$ associated to faces dual to the edges of Γ. In [16] the group Fourier transform for $SO(3)$ was used to formulate the metric representation for the LQG state space $L^2(SO(3)^E/SO(3)^V)$. In this case the non-commutative variables of the dual space $L^2_\star((\mathfrak{so}(3)^*)^E)$ to $L^2(SO(3)^E)$ correspond to these area bivectors. Integrating out the gauge degrees of freedom imposes geometrical closure constraints on the bivectors, so that the faces bound polyhedra dual to the vertices of Γ. Therefore the non-commutative metric representation obtained via the group Fourier transform provides a precise connection between LQG and simplicial geometry.

Group field theories are quantum field theoretical models with field arguments taking values in Lie groups, which give a covariant path integral formulation of quantum gravity as a sum over

[2] Usually the group $SU(2)$, the double cover of $SO(3)$, is used for the group variables associated to the edges of the graph. However, we will restrict to $SO(3)$, since the construction of the non-commutative metric representation for LQG has been carried out only in this case. The generalization to $SU(2)$ should be straightforward by using one of the already existing transforms for $SU(2)$.

spin foam geometries. (See e.g. [17, 18] for further details.) Using the group Fourier transform, a metric representation for 3d group field theory, the Boulatov model, which corresponds to a path integral quantization of discrete 3d BF theory, was formulated in [5, 9, 19]. (For spin foam models the corresponding non-commutative methods were considered in [20], and the relation between group field theory models and deformed κ-Poincaré symmetry via group Fourier transform was also considered in [21].) Importantly, the non-commutative metric representation led to the realization of diffeomorphism symmetry in the colored 3d model [22], which however due to the deformed coproduct of the symmetry algebra still needs to be properly implemented on the quantum level in the framework of braided quantum field theory [23] as pointed out in [9, 19]. Another important feature of the metric representation is that the simplicity constraints on the bivector variables, taking the 4d BF theory to general relativity, can be imposed directly and in a geometrically clear way on the classical phase space variables [24, 25].

5. Future directions

The group Fourier transform was recently generalized [6] to other Lie groups besides copies of $SO(3)$ and $SU(2)$, which opens up the possibility for formulating the non-commutative metric representation of 4d Lorenzian spin foam models. This should allow for a host of developments in the 4d case, as it has done in the 3d case. In particular, the metric representation should allow for a convenient imposition of the simplicity constraints at the level of the classical phase space variables, and a better understanding of the geometrical symmetries of the models. Also, the study of the semi-classical limit of the models is expected to be greatly facilitated by the non-commutative methods, since no excursion to the elaborate representation theory and special functions is necessary. In addition to these interesting possibilities, the commutative limit can also be used to develop convenient approximations to the full model. A pursuit in this direction of research is currently under way.

References

[1] Freidel L and Livine E 2006 *Class. Quant. Grav.* **23** 2021 (*Preprint* arXiv:hep-th/0502106)
Freidel L and Livine E 2006 *Phys. Rev. Lett.* **96** 221301 (*Preprint* arXiv:hep-th/0512113)
[2] Freidel L and Majid S 2008 *Class. Quant. Grav.* **25** 045006 (*Preprint* arXiv:hep-th/0601004)
[3] Joung E, Mourad J and Noui K 2009 *J. Math. Phys.* **50** 052503 (*Preprint* arXiv:0806.4121)
[4] Dupuis M, Girelli F and Livine E 2011 *Preprint* arXiv:1107.5693
[5] Livine E 2009 *Class. Quant. Grav.* **26** 195014 (*Preprint* arXiv:0811.1462)
[6] Raasakka M 2011 *Preprint* arXiv:1111.6481
[7] Majid S 2000 *Foundations of Quantum Group Theory* (Cambridge: Cambridge University Press)
[8] Batista E and Majid S 2003 *J. Math. Phys.* **44** 107 (*Preprint* arXiv:hep-th/0205128)
[9] Baratin A and Oriti D 2010 *Phys. Rev. Lett.* **105** 221302 (*Preprint* arXiv:1002.4723)
[10] Oriti D and Raasakka M 2011 *Phys. Rev.* D **84** 025003 (*Preprint* arXiv:1103.2098)
[11] Moskowitz M and Sacksteder R 2008 *Math. Res. Lett.* **15** 1197
[12] DeWitt B 1957 *Rev. Mod. Phys.* **29** 377
[13] Chaichian M and Demichev A 2001 *Path Integrals in Physics, Volume I* (London: IOP Publishing)
[14] Marinov M S and Terentyev M V 1979 *Fortsch. Phys.* **27** 511
[15] Thiemann T 2007 *Modern Canonical Quantum General Relativity* (Cambridge: Cambridge University Press)
[16] Baratin A, Dittrich B, Oriti D and Tambornino J 2011 *Class. Quant. Grav.* **28** 175011 (*Preprint* arXiv:1004.3450)
[17] Freidel L 2005 *Int. J. Theor. Phys.* **44** 1769 (*Preprint* arXiv:hep-th/0505016)
[18] Oriti D 2011 *Foundations of Space and Time*, ed J Murugan *et al* (Cambridge: Cambridge University Press)
[19] Girelli F and Livine E 2010 *Class. Quant. Grav.* **27** 245018 (*Preprint* arXiv:1001.2919)
[20] Bonzom V and Livine E 2009 *Phys. Rev.* D **79** 064034 (*Preprint* arXiv:0812.3456)
[21] Girelli F, Livine E and Oriti D 2010 *Phys. Rev.* D **81** 024015 (*Preprint* arXiv:0903.3475)
[22] Baratin A, Girelli F and Oriti D 2011 *Phys. Rev.* D **83** 104051 (*Preprint* arXiv:1101.0590)
[23] Oeckl R 2001 *Comm. Math. Phys.* **217** 451 (*Preprint* arXiv:hep-th/9906225)
[24] Baratin A and Oriti D 2011 *Preprint* arXiv:1108.1178
[25] Baratin A and Oriti D 2011 *Preprint* arXiv:1111.5842