Massive motion in area metric spacetimes

Raffaele Punzi,[1] Frederic P. Schuller,[2,*] and Mattias N. R. Wohlfarth[1,†]

[1]*Zentrum für Mathematische Physik und II. Institut für Theoretische Physik, Universität Hamburg, Luruper Chaussee 149, 22761 Hamburg, Germany*

[2]*Max Planck Institut für Gravitationsphysik, Albert Einstein Institut, Am Mühlenberg 1, 14467 Potsdam, Germany*

(Received 5 February 2009; published 16 June 2009)

The motion of massive point matter in area metric spacetimes, which arise as refinements of the Lorentzian spacetime structure in various situations of physical interest, is shown to be governed by a Finslerian geometry. This finding is based on the study of interacting string fluids on area metric manifolds, and extends previous findings on the motion of light rays.

DOI: 10.1103/PhysRevD.79.124025 PACS numbers: 04.50.Kd

I. INTRODUCTION

What we know about the geometric structure of spacetime, we infer from the properties of matter inhabiting it. Remarkably, in some situations one is led to the conclusion that the geometric structure of spacetime is best described not by the familiar metric geometry, but by an area metric manifold [1,2]. This is simply a four-dimensional smooth manifold M equipped with a covariant tensor field G of fourth rank, the area metric, which is symmetric under exchange of the first and second pair of entries, and antisymmetric under exchange of the individual entries within any one pair,

$$G_{abcd} = G_{cdab} \quad \text{and} \quad G_{bacd} = -G_{abcd}, \tag{1}$$

and for which there exists an inverse, i.e., a contravariant tensor field of the fourth rank, such that

$$G^{abmn}G_{mncd} = 4\delta^{[a}_{[c}\delta^{b]}_{d]}, \tag{2}$$

where small Latin indices run through $0, \ldots, 3$. While any metric manifold (M, g) immediately gives rise to an area metric manifold by virtue of letting $(G_g)_{abcd} = g_{ac}g_{bd} - g_{ad}g_{bc}$, the converse is only true in three dimensions [3]. In particular, a generic four-dimensional area metric is not induced from some metric, but rather presents a refinement of metric geometry. A scalar density $|\det G|^{1/6}$ of weight one, and thus a volume form $\omega_{G abcd} = |\det G|^{1/6}\epsilon_{abcd}$, are induced by virtue of the fact that the components of an area metric may be arranged as a symmetric 6×6 matrix, say by considering the six antisymmetric index pairs in the order [01], [02], [03], [23], [31], [12], and defining $\det G$ precisely as the determinant over this 6×6 matrix [4].

Area metrics emerge, for instance, as the effective spacetime geometry seen by photons on curved Lorentzian spacetime backgrounds [5], if the first order quantum corrections are absorbed into the geometry [6]. In this context it is the effective area metric, not the metric, which governs the ticking of light clocks that operate with photons rather than classical light rays. Another instance concerns quantum strings. A string moving in a coherent background of massless string excitations feels an area metric geometry [1]. This result extends to Dirichlet branes, whose world volume is that of an area metric manifold. As a third instance we mention that combining the scalar and tensor data of Brans-Dicke gravity [7] into an area metric, one does not only discover that of the entire one-parameter family of Brans-Dicke theories only one presents an area metric gravity theory, but remarkably also that this one member agrees with general relativity to first post-Newtonian order if [8,9] and only if [10] matter couples also to the scalar, as in this case geometrically dictated by the area metric spacetime structure [11]. More indirect, but no less intriguing, motivations for area metric manifolds come from various approaches to quantum gravity [12–17], where areas and two-dimensionality are seen to emerge on a deeper level of the spacetime structure. Quite generally, when studying the implications of an area metric spacetime structure for the motion of matter, one may consider area metric manifolds independently of the various different mechanisms giving rise to it in the first place. Operationally, this amounts to the hypothesis that physical spacetime is an area metric manifold, and this is the view we take in this paper.

It is then the purpose of this article to derive the implications of this assumption for the motion of massive point matter. For light rays, i.e., massless point matter, this study has been completed in [18], building on earlier results obtained in the context of premetric electrodynamics [19]. In the present paper, we arrive at the conclusion that the worldlines x of free massive point particles in an area metric spacetime stationarize the *quartic* functional

$$S[x] = \int d\lambda [\mathcal{G}_{abcd}\dot{x}^a\dot{x}^b\dot{x}^c\dot{x}^d]^{1/4}, \tag{3}$$

where $\dot{x}$ denotes the worldline tangent vector taken with respect to the worldline parameter λ, and the totally symmetric tensor $\mathcal{G}$ is the following algebraic expression in terms of the area metric G and the volume form ω_{G^C}, where $G^C_{abcd} = G_{abcd} - G_{[abcd]}$ is the cyclic part of the

*fps@aei.mpg.de
†mattias.wohlfarth@desy.de

area metric:

$$\mathcal{G}_{abcd} = -\tfrac{1}{24}\omega^{ijkl}_{G^C}\omega^{mnpq}_{G^C}G_{ijm(a}G_{b|kn|c}G_{d)lpq}. \qquad (4)$$

Despite its apparent complexity, this totally symmetric tensor emerges naturally within area metric geometry, and was in fact first discovered by Rubilar [20] in the context of general linear backgrounds in electrodynamics. In the case of a purely metric-induced area metric, Eq. (4) reduces to the bi-quadratic form $\mathcal{G}_{ijkl} = g_{(ij}g_{kl)}$ and thus expression (3) to an action functional whose stationary points are the familiar metric geodesics. In the case of a generic area metric, however, the tensor field $\mathcal{G}$ cannot be generated from a metric, but defines a truly quartic (pseudo-) norm. Thus, we find that the area metric geometry impresses itself on the motion of point particles as a Finslerian geometry, and the action (3) gives rise to Finslerian geodesics.

Of course, one could simply postulate that the dynamics of a point particle be provided by (3). While perfectly acceptable as a phenomenological model of particle motion, such an approach raises the question of compatibility with other predictions of the theory. Indeed, studying the geometric-optical limit of Maxwell theory on an area metric spacetime, we showed in [18] from first principles that light rays are described by stationary curves of precisely the same functional, but with the constraint that $\mathcal{G}_{abcd}\dot{x}^a\dot{x}^b\dot{x}^c\dot{x}^d = 0$. Similarly, in the present paper, we derive the motion of massive point particles from a thorough investigation of the motion of string fluids. The central role of the functional (3) is underlined by the fact that it arises with the same inevitability in both contexts, the massless and massive one.

Technically, we obtain the class of distinguished curves that present free point particle motion as follows. We start in Sec. II from general observations concerning the description of matter in area metric spacetimes, with a special focus on perfect fluids. The latter are described by three macroscopic parameters (rather than the usual two in metric spacetime), and may be viewed as being constituted by classical strings, rather than point particles. In Sec. III, we will then be prepared to perform the two crucial steps toward the identification of the worldlines of free point particles. The first step consists in the isotropization of the string dust, i.e., a geometrically well-defined spatial averaging over the orientations of the strings that constitute the fluid. The insight that this first steps leads to an effective fluid of interacting point particles then prompts the second step, namely, the identification of the appropriate local string interactions that are needed to make the string fluid lump together such as to give rise to a noninteracting particle fluid. The trajectories of this noninteracting particle fluid are recognized to follow Finslerian geodesics according to the action (3), which are fully determined by the area metric. This result underlines the importance of Finsler geometry in testing generalized background geometries through the motion of light and matter. In Sec. IV, we demonstrate the consistency of our results with any diffeomorphism invariant gravity action for area metric spacetimes. We conclude with a discussion in Sec. V.

II. STRING FLUIDS ON AREA METRIC BACKGROUNDS

In this section we make some general observations concerning the coupling of matter to area metric backgrounds, most importantly the conservation equation for matter sources. Then we explain why perfect fluids on area metric backgrounds feature a refined structure, which can be viewed as arising from strings rather than point particles being their constitutive matter. We briefly review and elaborate on some known results on string fluids in order to prepare our discussion of massive motion in the following section.

A. Matter on area metric backgrounds

It is not difficult to derive some generic properties of matter sources coupling to an area metric background. Indeed, consider an action

$$S[G, \psi] = \int_M \omega_G L[G, \psi] \qquad (5)$$

for a generic matter field ψ, which couples to the area metric G via a scalar Lagrangian $L[G, \psi]$. Variation of this action with respect to the area metric will always yield a fourth-rank source tensor $T^{ijkl} = |\det G|^{1/6}\delta S/\delta G_{ijkl}$, see [2]. Important for the present paper is the observation that independent of any assumed gravitational dynamics, diffeomorphism invariance of the matter action implies a conservation equation for this source tensor. To see this note that invariance of the matter action under diffeomorphisms requires

$$0 = \delta S = \int_M \delta G_{ijkl}\frac{\delta S}{\delta G_{ijkl}} + \delta\psi\frac{\delta S}{\delta\psi}, \qquad (6)$$

where the second term vanishes due to the equations of motion of the matter field ψ. The variation of the area metric under a diffeomorphism generated by a vector field ξ can be written as the Lie derivative $\delta G_{ijkl} = (\mathcal{L}_\xi G)_{ijkl}$. Straightforward substitution of this expression and partial integration then lead to the source conservation equation

$$-|\det G|^{1/6}T^{ijkl}\partial_p G_{ijkl} + 4\partial_a(|\det G|^{1/6}T^{ijka}G_{ijkp}) = 0. \qquad (7)$$

When considering matter given directly in terms of a source tensor T^{abcd}, rather than in terms of an action, one needs to impose the conservation equation by hand. This is, *mutatis mutandis*, the case when considering perfect fluids in metric spacetime, and so it is for the string fluids studied in the following section. Indeed, the latter present the perfect fluids on area metric backgrounds.

B. String fluids

We now consider a particular form of matter on area metric backgrounds, namely, string fluids [2,21,22]. As on metric backgrounds [23–26], these can be thought of as collections of strings. Geometrically, their description features a field of local tangent areas $\Omega = \dot{x} \wedge x'$, i.e., $\Omega^{ij} = \dot{x}^i x'^j - \dot{x}^j x'^i$, to the two-dimensional string world sheets $x:\Sigma \to M$. This is analogous to perfect fluids in general relativity, which can be understood as a collection of point particles, and whose description involves the velocity field tangent to the particle worldlines. Even though string fluids are not derived from an action, their source tensor must satisfy the conservation equation above in order to ensure a consistent coupling to any theory of area metric gravity, which is derived from an action by variation with respect to the area metric G.

The simplest string fluid is noninteracting string dust with source tensor

$$T^{ijkl} = \tilde{\rho}\Omega^{ij}\Omega^{kl}. \tag{8}$$

That this indeed describes noninteracting strings will now be shown by proving that the source conservation equation is equivalent to the equation of motion of the free classical string, i.e., the minimal surface equation, and the string continuity equation.

To see this, consider the string worldsheet tangent areas to be normalized as $G(\Omega, \Omega) = -1$ for $\Omega = u \wedge v$, where $u = \dot{x}$, $v = x'$. Substituting the source tensor (8) into the conservation Eq. (7), one obtains

$$\begin{aligned} 0 = |\det G|^{1/6}\tilde{\rho}\Big\{ & v^q \partial_q (G_{apcd} u^a u^c v^d) \\ & + u^q \partial_q (G_{pbcd} v^b u^c v^d) - \tfrac{1}{2}\partial_p G_{abcd} u^a v^b u^c v^d \Big\} \\ & + G_{ijkp} u^i v^j [\partial_l (|\det G|^{1/6}\tilde{\rho}\Omega^{kl})]. \end{aligned} \tag{9}$$

The minimal surface equation for strings on area metric backgrounds is derived as the stationarity condition of the integrated worldsheet area [1], and requires the vanishing of the curly brackets in the expression above. The continuity equation [2] on the other hand requires the vanishing of the square brackets. Hence, both these conditions together imply source conservation; this direction of the argument was already given in [2].

Now to show also the converse, observe that the term in curly brackets in Eq (9) vanishes if contracted with Ω^{pq}; hence, the other term does. With the notation $\Omega_{pq} = G_{pqrs}\Omega^{rs}$ one thus concludes

$$\Sigma_{mk}\partial_l(|\det G|^{1/6}\tilde{\rho}\Omega^{kl}) = 0 \tag{10}$$

for $\Sigma_{mk} = \Omega_{mq}\Omega^{qp}\Omega_{pk}$. The next step is to show that Σ_{mk} in this equation can be replaced simply by Ω_{mk}. This follows from the easily checked identity $\Omega^{ab}\Omega_{bc}\Omega^{cd} = \Omega^{ad}$, whence $\Sigma_{mk}\Omega^{kl} = \Omega_{mk}\Omega^{kl}$. By linear independence of the vectors u and v (otherwise the tangent area $\Omega = 0$ would be degenerate), it follows that $\Sigma_{mk}u^k = \Omega_{mk}u^k$ and $\Sigma_{mk}v^k = \Omega_{mk}v^k$, i.e., Σ_{mk} can be replaced by Ω_{mk} in contractions with u and v. This is precisely what we need for the replacement of Σ by Ω in (10), and so we see from (9) that source conservation implies the minimal surface condition for string dust. Finally, we may rewrite (10) as

$$\begin{aligned} & G(u, v, u, \cdot)\partial_l(|\det G|^{1/6}\tilde{\rho}v^l) \\ & \quad - G(u, v, v, \cdot)\partial_l(|\det G|^{1/6}\tilde{\rho}u^l) = 0. \end{aligned} \tag{11}$$

Evaluating this one-form on u and v, respectively, shows that both divergence terms must vanish separately. Hence, also the continuity equation, in the form of vanishing square brackets in (9), holds.

With this new converse result, it is now rigorously proven that noninteracting string dust on area metric backgrounds is described by the source tensor given in (8). Any modification of this source tensor by other terms depending on the background geometry G or the worldsheet tangent areas Ω hence describes an interacting string fluid. In the cosmological context, for instance, the most general interacting string fluid is given by a source tensor

$$T^{ijkl} = (\tilde{\rho} + \tilde{p})\Omega^{ij}\Omega^{kl} + \tilde{p}G^{ijkl} + (\tilde{\rho} + \tilde{q})G^{[ijkl]}. \tag{12}$$

String dust is thus recovered by imposing the equations of state $\tilde{p} = 0$ and $\tilde{q} = -\tilde{\rho}$. The effect of switching on a generalized pressure term $\tilde{p}$ is that of an according modification of the string continuity equation, and that of generating a nonzero mean curvature (given by the projection of the gradient $d\tilde{p}$ to the string tangent areas Ω) of the strings constituting the fluid. In other words, switching on a nonconstant generalized pressure $\tilde{p}$ causes the strings in the fluid to be nonstationary, which reveals that they must be interacting [2]. Finally, one can show that string fluids modelling radiation have $\tilde{q} = 0$ [22]. In this paper, we will not make use of these results in detail. But in the next section, we will identify interaction terms that cause a string fluid to lump together such as to move as an effective particle fluid, and it may be helpful to already have the intuition concerning interactions we sketched here.

Finally, consider the familiar case of a purely metric spacetime (M, g) where areas are simply measured by the induced area metric $G_{abcd} = 2g_{a[c}g_{d]b}$. Then the source conservation Eq. (7) for string dust with source tensor (8) reduces to $\nabla_a T_{\text{eff}}{}^a{}_b = 0$ for $T_{\text{eff}}{}^a{}_b = \tilde{\rho}\Omega^{ap}\Omega_{pb}$. Writing $\Omega = u \wedge v$ as we did before, and choosing the basis $g(u, v) = 0$, $g(u, u) = -1$, and $g(v, v) = 1$, one obtains

$$T_{\text{eff}}{}^a{}_b = \tilde{\rho}(u^a u_b - v^a v_b) \tag{13}$$

for string dust. This special case of our construction is known from the literature, and has been used to describe string energy momentum coupled to standard metric theories of gravity [23,24].

III. EFFECTIVE FINSLER GEODESICS

We are now prepared to derive the key result of this paper. In this section we will demonstrate the existence of a class of interacting string fluids that behave precisely the same way as noninteracting particle dust. We prove that these string fluids effectively propagate along non-null Finsler geodesics with respect to a special Finsler norm determined by the area metric.

Consider again the case of noninteracting strings on a metric background with effective energy momentum (13). It is clear that such energy momentum cannot be interpreted as that of a point particle fluid: the string worldsheet singles out a preferred spatial direction v, which destroys isotropy around the particle trajectory u. The term $\tilde{\rho} v^a v_b$ represents anisotropic pressure. From this observation, which is not new for metric backgrounds, we learn the following lesson for area metric geometry. Two issues must be taken into account to derive string fluids that effectively behave like noninteracting point particle fluids: we must

(i) isotropically superpose string fluids by implementing an average over the spatial directions of the respective world sheets; and

(ii) adjust the string interaction terms to achieve effective point particle motion.

This is the procedure we will now implement. In Sec. III A, we will define the isotropic average of string dust matter; in Sec. III B, we then determine the necessary string interaction terms responsible for noninteracting particle motion. The fact that interaction terms have to be added to an isotropic average of noninteracting strings agrees with the physical intuition that strings, in order to effectively behave like point particle dust, must clump together by some form of interaction.

A. Isotropization of string dust

As discussed above, the first step in finding string fluids that effectively move as noninteracting particle fluids is the definition of an isotropic average over the spatial worldsheet directions. We define this average with respect to a vector field u, which later emerges as the velocity field of the resulting particle fluid. The construction will be independent of coordinates.

Note first that the local tangent spaces can be decomposed as $TM = \langle u\rangle \oplus V$ (the following construction will not be affected by the nonuniqueness of the complement V), which in turn induces a decomposition of the antisymmetric tensor bundle as $\Lambda^2 TM = \Lambda^2_u TM \oplus \Lambda^2_V TM$ for

$$\Lambda^2_u TM = \{\Omega \in \Lambda^2 TM \mid \Omega \wedge u = 0\}. \tag{14}$$

Thus, any element of the space $\Lambda^2_u TM$ can be written in the form $\Omega = u \wedge v$ for $v \in V$, but $\Lambda^2_u TM$ is independent of the choice of complement V, since $u \wedge v = u \wedge (v + \lambda u)$ for any scalar λ. Moreover, $\Lambda^2_u TM$ is a three-dimensional linear subspace of $\Lambda^2 TM$ to which the area metric hence can be sensibly restricted. The restriction then defines a unique metric $\tilde{g}$: $\Lambda^2_u TM \times \Lambda^2_u TM \to \mathbb{R}$ by

$$\tilde{g} = -G|_{\Lambda^2_u TM} \tag{15}$$

on all areas in the set $u \wedge V$ independent of the choice of V. We assume that $\tilde{g}$ is positive definite; note that this is not a restriction on the background geometry, but distinguishes particular vector fields u that can play the role of velocity field in our final particle fluid.

The sought-for isotropic average over the spatial worldsheet directions will now essentially be the integration over the linear subspace $\Lambda^2_u TM$ with metric measure $\tilde{g}$. But since the volume of this space is noncompact, we must restrict the integration to the two-dimensional unit sphere S^2_u consisting of areas with $\tilde{g}(\Omega, \Omega) = 1$. Let ϕ^* denote the pullback from $\Lambda^2_u TM$ to S^2_u and use coordinates θ^1, θ^2. Then vol $S^2_u = \int d^2\theta \sqrt{\det \phi^* \tilde{g}} = 4\pi$, which is most easily seen by choosing the Cartesian frame $\{e_{\hat{0}}, e_{\hat{\alpha}}\}$ with $e_{\hat{0}} = u$ and $\langle e_{\hat{\alpha}}\rangle = V$ such that $\tilde{g}_{\hat{\alpha}\hat{\beta}} = \tilde{g}(e_{\hat{0}} \wedge e_{\hat{\alpha}}, e_{\hat{0}} \wedge e_{\hat{\beta}}) = \delta_{\hat{\alpha}\hat{\beta}}$. We now calculate the average of the string dust source tensor (8) over S^2_u,

$$\langle T^{abcd}\rangle = \frac{\tilde{\rho}}{\mathrm{vol} S^2_u} \int_{S^2_u} d^2\theta \sqrt{\det \phi^* \tilde{g}}\, \Omega(\theta)^{ab} \Omega(\theta)^{cd}. \tag{16}$$

The integral is performed using the same Cartesian frame as above; for elements of S^2_u we then have $\Omega = \Omega^{\hat{\alpha}} e_{\hat{0}} \wedge e_{\hat{\alpha}} / \sqrt{\tilde{g}(\Omega, \Omega)}$ and hence the coordinates $\Omega^{\hat{\alpha}}/||\Omega||$. As a result we find

$$\langle T^{\hat{0}\hat{\alpha}\hat{0}\hat{\beta}}\rangle = \tfrac{1}{3}\tilde{\rho}\,\delta^{\hat{\alpha}\hat{\beta}}. \tag{17}$$

Note that in this frame $\delta^{\hat{\alpha}\hat{\beta}} = \tilde{g}^{\hat{\alpha}\hat{\beta}}$, which is defined as the inverse of $\tilde{g}_{\hat{\alpha}\hat{\beta}}$, is regarded as a 3×3 matrix

$$\tilde{g}^{\,\hat{\alpha}\hat{\beta}} = \frac{1}{2 \det \tilde{g}} \epsilon^{\hat{0}\hat{\alpha}\hat{\mu}\hat{\nu}} \epsilon^{\hat{0}\hat{\beta}\hat{\rho}\hat{\sigma}} \tilde{g}_{\hat{\mu}\hat{\rho}} \tilde{g}_{\hat{\nu}\hat{\sigma}}. \tag{18}$$

Since the areas over which we averaged are elements of $\Lambda^2_u TM$, the result of the average, which is a linear operation, must be a tensor $\Lambda^{2*}_u TM \times \Lambda^{2*}_u TM \to \mathbb{R}$. Hence, there is an extension $\tilde{g}^{-1}$: $\Lambda^{2*}_u TM \times \Lambda^{2*}_u TM \to \mathbb{R}$ with components $\tilde{g}^{-1abcd}$ that in the frame chosen above reduces to $\tilde{g}^{-1\hat{0}\hat{\alpha}\hat{0}\hat{\beta}} = \tilde{g}^{\hat{\alpha}\hat{\beta}}$. In other words, this allows us to write the result of the average in the fully covariant form

$$\langle T^{abcd}\rangle = \tfrac{1}{3}\tilde{\rho}\,\tilde{g}^{-1abcd} = \tfrac{1}{3}\tilde{\rho}\tfrac{1}{2} u^{[a} u^{[c} h^{d]b]}, \tag{19}$$

with

$$h^{ab} = \mathcal{G}(u,u,u,u)^{-1} \omega^{armn}_{G^C} \omega^{bspq}_{G^C} G^C_{rmtp} G^C_{vnsq} u^t u^v, \tag{20}$$

$$\mathcal{G}_{abcd} = -\tfrac{1}{24} \omega^{ijkl}_{G^C} \omega^{mnpq}_{G^C} G^C_{ijm(a} G^C_{b|kn|c} G^C_{d)lpq}. \tag{21}$$

The antisymmetrizations in $\tilde{g}^{-1}$ act only on the index pairs,

G^C denotes the cyclic part $G_{abcd} - G_{[abcd]}$ of the area metric, and $\omega_{G^C}^{abcd} = |\det G^C|^{-1/6}\epsilon^{abcd}$.

We remark that the equations that could now be obtained from the conservation of the isotropic averaged source tensor (19) only involve the vector field u and the background geometry determined by the area metric G, so they are already equations for a particle fluid, albeit an interacting one.

B. Matter trajectories

We now come to the second part of the program for this section, and determine the necessary string interaction terms that have to be added to the isotropic averaged source tensor (19) so that the resulting string fluid moves as a noninteracting particle fluid.

It will turn out to be sufficient to consider interaction terms $\Sigma(G)^{abcd}$ that only depend locally on the background geometry. Our ansatz for the particle string fluid source tensor therefore is

$$T^{abcd} = \tfrac{1}{3}\tilde{\rho}\tilde{g}^{-1abcd} + \tfrac{4}{3}\tilde{\rho}\Sigma^{abcd}. \tag{22}$$

We will now determine the term Σ so that the source conservation Eq. (7) implies the standard continuity equation for point particles

$$\partial_l(|\det G|^{1/6}\tilde{\rho}\tilde{A}u^l) = 0 \tag{23}$$

for effective energy density $\tilde{\rho}\tilde{A}$, in which also $\tilde{A}(G)$ depends only locally on the background. In a second step we will then be able to derive the equation of motion for the point particle fluid; this will turn out to be the equation for non-null Finsler geodesics.

We substitute the ansatz (22) for the particle string fluid source tensor into the source conservation equation. The result can be rewritten in the form

$$\begin{aligned} 0 = {} & |\det G|^{1/6}\det\tilde{g}^{-1}\tilde{\rho}\,\partial_p\tilde{\mathcal{G}}_{ijkl}u^iu^ju^ku^l \\ & - 4\partial_l\left(|\det G|^{1/6}\tilde{\rho}\,\frac{\mathcal{G}_{pijk}u^iu^ju^k}{\mathcal{G}(u,u,u,u)}u^l\right) \\ & - |\det G|^{1/6}\tilde{\rho}\Sigma^{ijkl}\partial_pG_{ijkl} \\ & + 4\partial_l(|\det G|^{1/6}\tilde{\rho}\Sigma^{lijk}G_{pijk}) - 2\partial_p(|\det G|^{1/6}\tilde{\rho}). \end{aligned} \tag{24}$$

Here, $\tilde{\mathcal{G}}_{abcd} = |\det G^C|^{1/3}\mathcal{G}_{abcd}$. The derivation of this result requires the following identities, whose proof is rather technical, but can be performed with relative ease in the frame $\{e_{\hat{0}}, e_{\hat{\alpha}}\}$, with $e_{\hat{0}} = u$

$$\tilde{g}^{-1aijk}G_{pijk} = -2\delta_p^a - 4\frac{\mathcal{G}_{pijk}u^iu^ju^k}{\mathcal{G}(u,u,u,u)}u^a, \tag{25}$$

$$\frac{\delta\tilde{\mathcal{G}}(u,u,u,u)}{\delta G^C_{abcd}} = -\frac{1}{4}\det\tilde{g}\,\tilde{g}^{-1abcd}. \tag{26}$$

A first restriction on the term Σ can now be obtained by using the fact that it depends only locally on the background geometry: hence, any condition on Σ that is derived for constant area metric components G_{abcd} must also hold for general backgrounds. We therefore set all partial derivatives of G in (24) to zero, and contract with u^p. This is the only possible scalar contraction, and so must imply the continuity Eq. (23). This requires

$$\Sigma^{lijk}G_{pijk} = \tfrac{1}{2}\delta_p^l. \tag{27}$$

Substituting this into the full Eq. (24), the last line is precisely cancelled. We again contract with u^p to obtain a scalar equation, now for generic backgrounds G. To conveniently simplify the calculation we use the normalization $\mathcal{G}(u,u,u,u) = 1 = |\det G^C|^{-1/3}\det\tilde{g}$ in terms of the totally symmetric tensor $\mathcal{G}$ defined in (21). It is then straightforward to show that the continuity Eq. (23) can be obtained for interaction terms Σ that also satisfy the condition

$$\Sigma^{ijkl}\partial_pG_{ijkl} = \partial_p\ln B \tag{28}$$

for some scalar density $B(G)$. The function $\tilde{A}$ is then determined by $\tilde{A} = \tilde{A}_0|\det G^C|^{-1/12}B^{1/4}$.

We now employ the two conditions (27) and (28) for Σ in the source conservation Eq. (24), which yields the simplified equivalent expression

$$\begin{aligned} 0 = {} & \partial_\tau(\mathcal{G}_{pijk}u^iu^ju^k) - \tfrac{1}{4}\partial_p\mathcal{G}_{ijkl}u^iu^ju^ku^l \\ & - \mathcal{G}_{pijk}u^iu^ju^k\partial_\tau\ln\tilde{A} + \partial_p\ln\tilde{A}, \end{aligned} \tag{29}$$

where $\partial_\tau = u^p\partial_p$. Note that all dependence on the string fluid's energy density $\tilde{\rho}$ has cancelled. It is not hard to prove now that this equation can be equivalently derived as the stationarity equation from the point particle action

$$\int d\tau\tilde{A}^{-1}\mathcal{G}(\dot{x},\dot{x},\dot{x},\dot{x})^{1/4} \tag{30}$$

together with the normalization constraint $\mathcal{G}(\dot{x},\dot{x},\dot{x},\dot{x}) = 1$.

Thus, we have shown that all string fluids with the source tensor (22) and string interaction terms Σ solving both conditions (27) and (28) behave as noninteracting particle fluids. The source conservation equation for these string fluids not only implies the particle fluid conservation Eq. (23), but also an equation of motion for the fluid worldlines: the equation for non-null Finsler geodesics [27] with respect to the Finsler norm $\tilde{A}^{-1}\mathcal{G}(\dot{x},\dot{x},\dot{x},\dot{x})^{1/4}$. This Finsler norm is fully determined by the area metric G through its associated totally symmetric dual Fresnel tensor $\mathcal{G}$, see definition (21).

We conclude this section with some comments. First, note from the continuity Eq. (23) that the appearance of the function $\tilde{A}$ originates from a redefinition of the resulting particle fluid's energy density $\rho = \tilde{\rho}\tilde{A}$ in terms of the string fluid variable $\tilde{\rho}$. Mathematically, $\tilde{A}$ simply presents a conformal rescaling of the Finsler norm. This function can be fixed as $\tilde{A} = 1$ by identifying the tension per area $\tilde{\rho}$ of the strings with the energy density ρ of the effective point particles [26].

Second, it is worthwhile to comment on possible interactions that may modify the effective particle dust constructed here. In our discussion of string fluids in Sec. II we learned that such interactions simply arise from adding further terms to the averaged string fluid source tensor (22). As an example we now discuss the standard isotropic pressure term on metric-induced backgrounds $G_{abcd} = 2g_{a[c}g_{d]b}$. To do so we define the second-rank effective energy momentum tensor ${T_{\text{eff}}}^a{}_b = 4T^{aijk}G_{bijk}$ from the source tensor. If the area metric is metric induced, the source conservation equation simplifies to give standard covariant conservation $\nabla_a {T_{\text{eff}}}^a{}_b = 0$ of this effective energy momentum tensor. Also, it can be verified that the noninteracting particle fluid with source tensor (22) produces ${T_{\text{eff}}}^a{}_b \sim \tilde{\rho} u^a u_b$, which is the standard expression for particle dust. It is now obvious that including a term $\tilde{p}G^{abcd}$ into (22) will yield a contribution proportional to $\tilde{p}\delta^a_b$ in ${T_{\text{eff}}}^a{}_b$, which is precisely the isotropic pressure term.

IV. CONSISTENCY

Recall that a diffeomorphism invariant coupling of point particles to any gravity theory for a metric spacetime already determines the motion of these particles along Riemannian geodesics. This is because diffeomorphism invariance will result in some Bianchi identity for the gravitational curvature tensor and energy momentum conservation of the point particles. The latter requires the worldlines to follow geodesics. So given the same initial conditions, point particles will follow the same worldlines, independent of their masses. We will now explicitly demonstrate an analogous result for area metric spacetimes. Also, here the motion of point particles along Finsler geodesics can be understood as a consequence of diffeomorphism invariance; thus, these point particles can be consistently coupled to any area metric gravitational field equations.

In the previous section we found that the point particle limit of a string fluid leads to the action (30) with $\tilde{A} = 1$. The corresponding source tensor is then

$$T_{ijkl}(y) = -m \int d\tau \frac{\delta(y - x(\tau))}{|\det G|^{1/6}} \frac{\delta \mathcal{G}_{abcd}(y)}{\delta G^{ijkl}} \dot{x}^a \dot{x}^b \dot{x}^c \dot{x}^d, \tag{31}$$

where m denotes either the energy of a photon, or the mass of a massive particle. Since this source tensor is obtained from an action, it satisfies the conservation Eq. (7). We will now show that this equation can be rewritten as

$$-m \int d\tau \delta(y - x(\tau)) \Big[\partial_p \mathcal{G}_{abcd} \dot{x}^a \dot{x}^b \dot{x}^c \dot{x}^d + 4 \frac{d}{d\tau} (\mathcal{G}_{pabc} \dot{x}^a \dot{x}^b \dot{x}^c) \Big] = 0. \tag{32}$$

Since this equation must hold for any point y, the expression in square brackets must vanish, which is precisely the equation of a geodesic in a Finsler geometry determined by the Fresnel tensor $\mathcal{G}$.

Indeed, the variation of the Fresnel tensor with respect to the inverse area metric, as it appears in the source tensor (31), can be expressed as a variation with respect to the cyclic part G^C of the area metric G as

$$\frac{\delta \mathcal{G}_{abcd}}{\delta G^{ijkl}} = \frac{\delta \mathcal{G}_{abcd}}{\delta G^C_{\alpha\beta\gamma\delta}} \frac{\delta(G_{\alpha\beta\gamma\delta} - G_{[\alpha\beta\gamma\delta]})}{\delta G^{ijkl}} = -\frac{1}{4} \frac{\delta \mathcal{G}_{abcd}}{\delta G^C_{\alpha\beta\gamma\delta}} (G_{\alpha\beta ij} G_{\gamma\delta kl} - G_{ij[\alpha\beta} G_{\gamma\delta]kl}). \tag{33}$$

The first term of the source conservation equation can now be evaluated as

$$\text{first term} = m \int d\tau \delta(y - x(\tau)) \frac{\delta \mathcal{G}_{abcd}}{\delta G^C_{\alpha\beta\gamma\delta}} (\delta^{rstu}_{\alpha\beta\gamma\delta} - \delta^{rstu}_{[\alpha\beta\gamma\delta]}) \partial_p G_{rstu} \dot{x}^a \dot{x}^b \dot{x}^c \dot{x}^d, \tag{34}$$

which via the chain rule already yields the first term of the Finsler geodesic equation, in the square brackets of (32). The simplification of the second term of the source conservation equation is slightly more involved. First, observe that

$$\frac{\delta \mathcal{G}_{abcd}}{\delta G^C_{\alpha\beta\gamma\delta}} = -\frac{1}{12} (G^C)^{-1\alpha\beta\gamma\delta} \mathcal{G}_{abcd} + |\det G|^{-1/3} \frac{\delta \tilde{\mathcal{G}}_{abcd}}{\delta G^C_{\alpha\beta\gamma\delta}}, \tag{35}$$

where we defined the Fresnel tensor density $\tilde{\mathcal{G}}_{abcd} = |\det G^C|^{1/3} \mathcal{G}_{abcd}$. Using this the second term becomes

$$-2m \int d\tau \frac{\partial}{\partial y^p} \delta(y - x(\tau)) \mathcal{G}_{abcd} \dot{x}^a \dot{x}^b \dot{x}^c \dot{x}^d + 4m \int d\tau \frac{\partial}{\partial y^\beta} \delta(y - x(\tau)) |\det G|^{-1/3} A^\beta{}_p, \tag{36}$$

where we have introduced a shorthand for the quantity

$$A^\beta{}_p = \frac{\delta \tilde{\mathcal{G}}_{abcd} \dot{x}^a \dot{x}^b \dot{x}^c \dot{x}^d}{\delta G^C_{\alpha\beta\gamma\delta}} G^C_{\gamma\delta\alpha p}. \tag{37}$$

This quantity is most efficiently calculated using a nonholonomic frame $\{e_{\hat{k}}\}$ with $e_{\hat{0}} = u$, and a dual frame $e^{\hat{k}}$. After considerable algebra one finally arrives at

$$A^{\beta}{}_{p} = |\det G|^{1/3}(\mathcal{G}_{pabc}\dot{x}^a\dot{x}^b\dot{x}^c\dot{x}^\beta + \tfrac{1}{2}\delta^\beta_p \mathcal{G}_{abcd}\dot{x}^a\dot{x}^b\dot{x}^c\dot{x}^d). \quad (38)$$

Insertion of this result into (36) then provides the second term of Eq. (32)

$$\text{second term} = 4m\int d\tau \dot{x}^p \frac{\partial}{\partial y^p}\delta(y - x(\tau))\mathcal{G}_{abcp}\dot{x}^a\dot{x}^b\dot{x}^c. \quad (39)$$

Hence, in any diffeomorphism invariant theory of area metric gravity, the field equations consistently couple to effective point particles propagating along Finsler geodesics. This coupling is universal for all point particles, irrespective of their mass (or energy for light). This in fact shows consistency with the experimentally supported weak equivalence principle.

V. CONCLUSIONS

In this article, we have calculated the paths of massive pointlike matter on general area metric manifolds. Even though pointlike particles do not arise as fundamental mechanical objects on area metric backgrounds, their effective description is of phenomenological relevance.

Intriguingly, we find that an area metric background impresses itself as a Finsler geometry on the motion of all pointlike matter. Free motion is described by Finsler geodesics, and the relevant Finsler norm is determined by the area metric. To obtain this result, we have constructed the class of classical string fluids that admit a particle fluid limit through a geometrically well-defined averaging process. It turned out that the massive case considered here is governed by precisely the same Finsler geometry as the propagation of light [18]. Taking into account recent studies of Finsler geometries in connection to the quantization of deformed general relativity [28,29] and to quantum generalizations of the Poincaré algebra [30], it is interesting to note that the area metric structure of spacetime attaches a prominent role to a particular Finsler geometry when it comes to the description of the effective motion of light and matter.

Of course, the Finsler geodesics found here simply reduce to the standard metric geodesics in case the area metric is induced by a metric. However, already $\omega = 0$ Brans-Dicke theory determines a true area metric background with an additional degree of freedom, namely, the scalar field [11]. In this case, the Finsler geodesics are the geodesics of a particular conformally rescaled metric, with the result that the theory is rendered consistent with solar system physics.

The general result that light and massive matter propagate along Finsler geodesics becomes inevitable if the area metric structure is taken seriously as the geometry of spacetime. Indeed, spacetime backgrounds described by a more general structure than metric geometry arise in various approaches to quantum gravity, and in string theory where additional massless background fields appear. Especially important for the physical interpretation of generalized geometries are the classical tests of gravity in the solar system that require a model of planetary motion via distinguished curves. Similarly in cosmology, the trajectories of galaxies in an area metric spacetime must be understood as arising from a fluid model of the cosmological matter distribution. Our result identifies this mechanism.

Thus, the findings of this article are essential in order to test the viability of the hypothesis of those generalized geometric backgrounds that can be cast in area metric form.

ACKNOWLEDGMENTS

Our co-author, colleague, and above all friend Raffaele Punzi died in a tragic accident in his beloved Italian mountains before the final completion of this paper. He is sorely missed. RP and MNRW acknowledge full financial support from the German Research Foundation DFG, through Grant No. WO 1447/1-1.

[1] F. P. Schuller and M. N. R. Wohlfarth, J. High Energy Phys. 02 (2006) 059.
[2] R. Punzi, F. P. Schuller, and M. N. R. Wohlfarth, J. High Energy Phys. 02 (2007) 030.
[3] É. Cartan, *Les Espaces Métriques Fondés sur la Notion D'aire* (Hermann, Paris, 1933).
[4] The proof that det$G$ transforms as a scalar density of weight six under transformations of the four-dimensional tangent space basis is not entirely obvious and requires considering the determinant of a tensor product.
[5] I. T. Drummond and S. J. Hathrell, Phys. Rev. D **22**, 343 (1980).
[6] R. Punzi, F. P. Schuller, and M. N. R. Wohlfarth, arXiv:0902.1811.
[7] C. Brans and R. H. Dicke, Phys. Rev. **124**, 925 (1961).
[8] G. Magnano and L. M. Sokolowski, Phys. Rev. D **50**, 5039 (1994).
[9] N. Deruelle, M. Sasaki, and Y. Sendouda, Phys. Rev. D **77**, 124024 (2008).
[10] C. M. Will, Living Rev. Relativity **9**, 3 (2005).
[11] R. Punzi, F. P. Schuller, and M. N. R. Wohlfarth, Phys. Lett. B **670**, 161 (2008).
[12] C. Rovelli and L. Smolin, Nucl. Phys. **B442**, 593 (1995); **B456**, 753(E) (1995).

[13] J. Ambjorn, J. Jurkiewicz, and R. Loll, Phys. Rev. Lett. **95**, 171301 (2005).
[14] J. Ambjorn, J. Jurkiewicz, and R. Loll, Phys. Rev. D **72**, 064014 (2005).
[15] O. Lauscher and M. Reuter, Lect. Notes Phys. **721**, 265 (2007).
[16] O. Lauscher and M. Reuter, J. High Energy Phys. 10 (2005) 050.
[17] K. Krasnov, Mod. Phys. Lett. A **22**, 3013 (2007).
[18] R. Punzi, M. N. R. Wohlfarth, and F. P. Schuller, Classical Quantum Gravity **26**, 035024 (2009).
[19] F. W. Hehl and Y. N. Obukhov, *Foundations of Classical Electrodynamics* (Birkhäuser, Basel, 2003).
[20] G. F. Rubilar, Ann. Phys. (N.Y.) **11**, 717 (2002).
[21] R. Punzi, F. P. Schuller, and M. N. R. Wohlfarth, Phys. Rev. D **76**, 101501 (2007).
[22] F. P. Schuller and M. N. R. Wohlfarth, J. Cosmol. Astropart. Phys. 12 (2007) 013.
[23] P. S. Letelier, Phys. Rev. D **20**, 1294 (1979).
[24] J. Stachel, Phys. Rev. D **21**, 2171 (1980).
[25] J. Stachel, Phys. Rev. D **21**, 2182 (1980).
[26] P. S. Letelier, Phys. Rev. D **28**, 2414 (1983).
[27] Z. Shen, *Lectures on Finsler Geometry* (World Scientific Singapore, 2001).
[28] F. Girelli, S. Liberati, and L. Sindoni, Phys. Rev. D **75**, 064015 (2007).
[29] S. I. Vacaru, arXiv:0707.1526.
[30] G. W. Gibbons, J. Gomis, and C. N. Pope, Phys. Rev. D **76**, 081701 (2007).