

Generalized conformal realizations of Kac–Moody algebras

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(Received 13 November 2007; accepted 8 December 2008;
published online 20 January 2009)

We present a construction which associates an infinite sequence of Kac–Moody algebras, labeled by a positive integer n , to one single Jordan algebra. For $n=1$, this reduces to the well known Kantor–Koecher–Tits construction. Our generalization utilizes a new relation between different generalized Jordan triple systems, together with their known connections to Jordan and Lie algebras. Applied to the Jordan algebra of Hermitian 3×3 matrices over the division algebras \mathbb{R} , \mathbb{C} , \mathbb{H} , \mathbb{O} , the construction gives the exceptional Lie algebras \mathfrak{f}_4 , \mathfrak{e}_6 , \mathfrak{e}_7 , \mathfrak{e}_8 for $n=2$. Moreover, we obtain their infinite-dimensional extensions for $n \geq 3$. In the case of 2×2 matrices, the resulting Lie algebras are of the form $\mathfrak{so}(p+n, q+n)$ and the concomitant nonlinear realization generalizes the conformal transformations in a spacetime of signature (p, q) . © 2009 American Institute of Physics. [DOI: 10.1063/1.3063628]

I. INTRODUCTION

Jordan algebras are commutative but nonassociative algebras, which were originally studied in order to understand the foundations of quantum mechanics.^{1,2} Through their connection to Lie algebras, Jordan algebras play an important role in fundamental physics and can be used to define generalized spacetimes.^{3,4} The origin of this connection lies in the observation that the triple product

$$(x, y, z) \mapsto (xyz) \equiv [[x, \tau(y)], z] \quad (1.1)$$

in the subspace \mathfrak{g}_{-1} of a three-graded Lie algebra $\mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$, where τ is an involution $\mathfrak{g}_{-1} \rightarrow \mathfrak{g}_1$, has the same general properties as the triple product

$$(x, y, z) \mapsto (xyz) \equiv (xy)z + x(yz) - y(xz) \quad (1.2)$$

formed from the multiplication in a Jordan algebra. In the Kantor–Koecher–Tits construction,^{5–7} any Jordan algebra gives rise to a 3-graded Lie algebra, such that the two triple products coincide. In the present paper, we generalize this construction for a certain kind of Jordan algebras. We will show that any such Jordan algebra gives not only one Lie algebra but an infinite sequence of Lie algebras, labeled by a positive integer n . For $n=1$, we get back the original Kantor–Koecher–Tits construction.

The Kantor–Koecher–Tits construction has already been generalized by Kantor from Jordan algebras to Jordan triple systems and further to generalized Jordan triple systems.⁸ The generalization of Jordan algebras to Jordan triple systems is needed for the inverse of the Kantor–Koecher–Tits construction—any 3-graded Lie algebra with an involution τ gives rise to a Jordan triple system, but not all of them can be obtained from a Jordan algebra by (1.2). Generalized Jordan triple systems correspond to graded Lie algebras in general, not necessarily 3-graded.

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These well known connections between Jordan algebras, (generalized) Jordan triple systems, and graded Lie algebras are tools that we will use to derive the results in this paper.

Our construction is based on a new generalization of a single generalized Jordan triple system to an infinite sequence of such triple systems. We study the case when the Lie algebra associated with the first one (the original generalized Jordan triple system) is a finite Kac–Moody algebra, which means that it can be characterized by a Dynkin diagram. We find that continuing the sequence then corresponds to adding more nodes to the Dynkin diagram. Each node will be connected to the previous one by a single line, starting from an arbitrary node in the original Dynkin diagram. For the classical Lie algebras \mathfrak{b}_r ($r \geq 3$) and \mathfrak{d}_r ($r \geq 4$) and for the exceptional Lie algebras \mathfrak{f}_4 , \mathfrak{e}_6 , \mathfrak{e}_7 , \mathfrak{e}_8 , there is a unique node in the Dynkin diagram such that we get the affine extension if we connect an additional node to it by a single line. In this case, our construction only gives the current algebra extension, which means that the central element and the derivation must be added by hand. In all other cases we get the full Kac–Moody algebra, whether it is finite dimensional or not.

This work is motivated by the “magic square” constructions,^{9–12} which associate a Lie algebra $M(\mathbb{K}, \mathbb{K}')$ with any pair $(\mathbb{K}, \mathbb{K}')$ of division algebras $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$. These constructions involve the simple Jordan algebras $H_3(\mathbb{K})$ of Hermitian 3×3 matrices over the division algebras $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$, where the product is the symmetrized matrix product. Our construction gives a Lie algebra for each simple Jordan algebra and each positive integer value of a parameter n in the following way. The Jordan algebra first leads to a Jordan triple system by (1.2) which in turn generalizes to infinitely many generalized Jordan triple systems. Each of them has an associated Lie algebra. We will show that when we apply the construction to the Jordan algebras $H_3(\mathbb{K})$, we obtain the third row in the magic square for $n=1$ (since this is the ordinary Kantor–Koecher– Tits construction) and the fourth row for $n=2$. Moreover, we get the current algebra extension of the algebras in the fourth row for $n=3$ (since the node that we add is the “affine” one) and their hyperbolic extensions for $n=4$. Thus, our construction not only unifies the third and the fourth row but also extends the magic square with infinitely many new rows. In particular, for $\mathbb{K} = \mathbb{O}$, we get a unified construction of \mathfrak{e}_8 , \mathfrak{e}_9 , \mathfrak{e}_{10} and further extensions. When we instead apply our construction to the Jordan algebras $H_2(\mathbb{K})$ of Hermitian 2×2 matrices over the division algebras $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$, then the associated Lie algebras will always be finite dimensional, and we consider in this case not only the complex Lie algebras but also their real forms. In particular, for $\mathbb{K} = \mathbb{O}$, we get the pseudo-orthogonal algebras $\mathfrak{so}(1+n, 9+n)$, with the conformal algebra in a ten-dimensional Minkowski spacetime as the well known case $n=1$.

Our method is useful for studying nonlinear realizations of Lie algebras.^{13,14} Any graded Lie algebra can be realized nonlinearly on its subspaces of negative (or positive) degree.¹⁵ This nonlinear realization can be expressed in terms of the corresponding generalized Jordan triple system. When this in turn is obtained from an original one for some n in the way that we will describe, then the nonlinear realization can be expressed in terms of the original generalized Jordan triple system. We will illustrate this for $\mathfrak{so}(p+n, q+n)$.

The paper is organized as follows. In Sec. II, we show that any generalized Jordan triple system corresponding to a finite Kac–Moody algebra generalizes to an infinite sequence of such triple systems, labeled by a positive integer n , and that this corresponds to adding nodes to the original associated Dynkin diagram. In Sec. III, we review the relation between Jordan algebras and the magic square of Lie algebras. Then we show that the associated Lie algebras in the $H_3(\mathbb{K})$ case are the exceptional Lie algebras and their extensions. The nonlinear realization of $\mathfrak{so}(p+n, q+n)$, with the linearly realized subalgebra $\mathfrak{so}(p, q)$, is given in Sec. IV. In the Appendix we review in detail how any generalized Jordan triple system gives rise to a graded Lie algebra and how the graded Lie algebra can be nonlinearly realized.

II. KAC–MOODY ALGEBRAS

In this section, we will prove our main result (Theorem 2.1) about Kac–Moody algebras and generalized Jordan triple systems. First, we will briefly recall how a complex Kac–Moody algebra can be constructed from its (generalized) Cartan matrix or, equivalently, from its Dynkin diagram

(for details, see Ref. 16) and then how it can be given a grading. We will assume that the determinant of the Cartan matrix is nonzero, which, in particular, means that we leave the affine case for now. The Kac–Moody algebra will then be finite dimensional (or simply *finite*) if and only if the Cartan matrix is positive definite.

The Cartan matrix is of type $r \times r$, where r is the *rank* of the Lie algebra. Its entries are integers satisfying $A^{ii}=2$ (no summation) and

$$i \neq j \Rightarrow A^{ij} \leq 0, \quad A^{ij} = 0 \Leftrightarrow A^{ji} = 0 \quad (2.1)$$

for $i, j=1, 2, \dots, r$. The Dynkin diagram consists of r nodes, and two nodes i, j are connected by a line if $A^{ij}=A^{ji}=-1$ but disconnected if $A^{ij}=A^{ji}=0$ (these are the only two cases that we will consider).

In the construction of a Lie algebra from its Cartan matrix, one starts with $3r$ generators e_i, f_i, h_i satisfying the *Chevalley relations* (no summation)

$$\begin{aligned} [e_i, f_j] &= \delta_{ij} h_j, & [h_i, h_j] &= 0, \\ [h_i, e_j] &= A_{ij} e_j, & [h_i, f_j] &= -A_{ij} f_j. \end{aligned} \quad (2.2)$$

The elements h_i span the Abelian *Cartan subalgebra* \mathfrak{g}_0 . Further basis elements of \mathfrak{g} will then be multiple commutators of either e_i or f_i generated by these elements modulo the *Serre relations* (no summation)

$$(\text{ad } e_i)^{1-A^{ji}} e_j = 0, \quad (\text{ad } f_i)^{1-A^{ji}} f_j = 0. \quad (2.3)$$

It follows from (2.2) that these multiple commutators (as well as the elements e_i and f_i themselves) are eigenvectors of $\text{ad } h$ for any $h \in \mathfrak{g}_0$, and thus each of them defines an element α in the dual space of \mathfrak{g}_0 , such that $\alpha(h)$ is the corresponding eigenvalue. These elements α are the *roots* of \mathfrak{g} and the eigenvectors are called *root vectors*. In particular, e_i are root vectors of the *simple roots* α_i , which form a basis of the dual space of \mathfrak{g}_0 . In this basis, an arbitrary root $\mu = \mu^i \alpha_i$ has integer components μ^i , either all non-negative (if α is a *positive root*) or all nonpositive (if α is a *negative root*).

For finite Kac–Moody algebras, the space of root vectors corresponding to any root is one dimensional. Furthermore, if α is a root, then $-\alpha$ is a root as well but no other multiples of α . For any positive root α of a finite Kac–Moody algebra \mathfrak{g} , we let e_α and f_α be root vectors corresponding to α and $-\alpha$, respectively, such that they are multiple commutators of e_i or f_i . (This requirement fixes the normalization up to a sign.) Thus, a basis of \mathfrak{g} is formed by these root vectors e_α, f_α for all positive roots α and by the Cartan elements h_i for all $i=1, 2, \dots, r$.

A. Graded Lie algebras

A Lie algebra \mathfrak{g} is *graded*, or has a *grading*, if it is the direct sum of subspaces $\mathfrak{g}_k \subset \mathfrak{g}$ for all integers k , such that $[\mathfrak{g}_m, \mathfrak{g}_n] \subset \mathfrak{g}_{m+n}$ for all integers m, n . If there is a positive integer m such that $\mathfrak{g}_{\pm m} \neq 0$ but $\mathfrak{g}_{\pm k} = 0$ for all $k > m$, then the Lie algebra \mathfrak{g} is $(2m+1)$ -*graded*. We will occasionally use the notation $\mathfrak{g}_\pm = \mathfrak{g}_{\pm 1} + \mathfrak{g}_{\pm 2} + \dots$.

Any simple root α_i of a Kac–Moody algebra \mathfrak{g} *generates* a grading of \mathfrak{g} , such that \mathfrak{g}_k is spanned by all root vectors e_μ or f_μ with the component $\mu^i = -k$ (the minus sign is a convention) corresponding to α_i in the basis of simple roots and, if $k=0$, by the Cartan elements h_j .

A *graded involution* τ on the Lie algebra \mathfrak{g} is an automorphism such that $\tau^2(x) = x$ for any $x \in \mathfrak{g}$ and $\tau(\mathfrak{g}_k) = \mathfrak{g}_{-k}$ for any integer k . The simplest example of a graded involution in a graded Kac–Moody algebra is given by $e_\alpha \leftrightarrow \pm f_\alpha$ and $h_i \leftrightarrow -h_i$. (With the minus sign, this is the *Chevalley involution*.)

On the subspace \mathfrak{g}_{-1} of a graded Lie algebra \mathfrak{g} with a graded involution τ , we can define a triple product, that is, a trilinear map $(\mathfrak{g}_{-1})^3 \rightarrow \mathfrak{g}_{-1}$, given by

$$(x, y, z) \mapsto (xyz) = [[x, \tau(y)], z]. \tag{2.4}$$

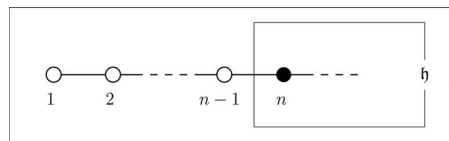
Then, due to the Jacobi identity and the fact that τ is an involution, this triple product will satisfy the identity

$$(uv(xyz)) - (xy(uvz)) = ((uvx)yz) - (x(vuy)z), \tag{2.5}$$

which means that \mathfrak{g}_{-1} is a *generalized Jordan triple system*. As this name suggests and as we have already mentioned, this kind of triple systems is related to Jordan algebras. We will explain the relation in more detail in Sec. IV A.

B. Extensions of graded Lie algebras

We now consider the situation when a finite Kac–Moody algebra \mathfrak{h} is extended to another one \mathfrak{g} in the following way for an arbitrary integer $n \geq 2$:



The black node, which \mathfrak{g} and \mathfrak{h} have in common, generates a grading of \mathfrak{g} as well as of \mathfrak{h} . (Here and below, this meaning of a black node in a Dynkin diagram should not be confused with any different meaning used elsewhere.) We want to investigate how the triple systems \mathfrak{g}_{-1} and \mathfrak{h}_{-1} , corresponding to these two gradings, are related to each other. It is clear that $\dim \mathfrak{g}_{-1} = n \dim \mathfrak{h}_{-1}$, which means that \mathfrak{g}_{-1} as a vector space is isomorphic to the direct sum $(\mathfrak{h}_{-1})^n$ of n vector spaces, each isomorphic to \mathfrak{h}_{-1} . The question is if we can define a triple product on $(\mathfrak{h}_{-1})^n$ such that \mathfrak{g}_{-1} and $(\mathfrak{h}_{-1})^n$ are isomorphic also as triple systems. To answer this question, we write a general element in $(\mathfrak{h}_{-1})^n$ as $(x_1)^1 + (x_2)^2 + \dots + (x_n)^n$, where x_1, x_2, \dots are elements in \mathfrak{h}_{-1} . Furthermore, we define for any graded involution τ on \mathfrak{h} a bilinear form on \mathfrak{h}_{-1} associated with τ by $(e_\mu, \tau(f_\nu)) = \delta_{\mu\nu}$ for root vectors $e_\mu \in \mathfrak{h}_{-1}$ and $f_\nu \in \mathfrak{h}_1$. The answer is then given by the following theorem, which is the main result of this paper.

Theorem 2.1: *The vector space $(\mathfrak{h}_{-1})^n$, together with the triple product given by*

$$(x^a y^b z^c) = \delta^{ab} [[x, \tau(y)], z]^c - \delta^{ab} (x, y) z^c + \delta^{bc} (x, y) z^a \tag{2.6}$$

for $a, b, \dots = 1, 2, \dots, n$ and $x, y, z \in \mathfrak{h}_{-1}$, is a triple system isomorphic to the triple system \mathfrak{g}_{-1} with the triple product $(uvw) = [[u, \tau(v)], w]$, where the involution τ is extended from \mathfrak{h} to \mathfrak{g} by $\tau(e_i) = -f_i$ for the simple root vectors.

Proof: A basis of \mathfrak{h}_{-1} consists of all root vectors e_μ such that the component of μ corresponding to α_n in the basis of simple roots is equal to 1. A basis of \mathfrak{g}_{-1} consists of all such basis elements e_μ of \mathfrak{h}_{-1} together with all commutators $[e^i, e_\mu]$, where e^i , for $i = 1, 2, \dots, n-1$, is the root vector

$$e^i = [\dots [[e_i, e_{i+1}], e_{i+2}], \dots, e_{n-1}] \tag{2.7}$$

of the \mathfrak{a}_{n-1} subalgebra of \mathfrak{g} . We also define the root vector

$$f^i = (-1)^{n-1-i} [\dots [[f_i, f_{i+1}], f_{i+2}], \dots, f_{n-1}] \tag{2.8}$$

for the corresponding negative root, and the element

$$h^i = h_i + h_{i+1} + h_{i+2} + \dots + h_{n-1} \tag{2.9}$$

in the Cartan subalgebra of \mathfrak{g} , such that

$$[e^i, f^i] = h^i, \quad [h^i, e^i] = 2e^i, \quad [h^i, f^i] = -2f^i \tag{2.10}$$

(no summation). If $i \neq j$, then

$$[h^i, e^j] = e^j, \quad [h^i, f^j] = -f^j, \quad (2.11)$$

while $[e^i, e^j]$ is either zero or a root vector of \mathfrak{g} that does not belong to \mathfrak{g}_{-1} . (We stress the difference between having the indices i, j, \dots on e, f, h upstairs and downstairs. The root vectors e_i correspond to the simple roots of the \mathfrak{a}_{n-1} subalgebra, while the root vectors e^i correspond to roots of the \mathfrak{a}_{n-1} subalgebra for which the component corresponding to the simple root α_{n-1} is equal to 1, and these roots are not simple, except for α_{n-1} itself.) Using the relations (2.10) and (2.11), we get

$$\begin{aligned} [[e^i, f^j], e_\mu] &= -\delta^{ij} e_\mu, & [[e^i, f^j], f_\nu] &= \delta^{ij} f_\nu, \\ [[e_\mu, f_\nu], e^i] &= -\delta_{\mu\nu} e^i, & [[e_\mu, f_\nu], f^j] &= \delta_{\mu\nu} f^j, \end{aligned} \quad (2.12)$$

and then

$$\begin{aligned} [[e^i, e_\mu], [f^j, f_\nu]] &= -\delta^{ij} [e_\mu, f_\nu] - \delta_{\mu\nu} [e^i, f^j], \\ [[e^i, e_\mu], f_\nu] &= \delta_{\mu\nu} e^i, & [e_\mu, [f^j, f_\nu]] &= -\delta_{\mu\nu} f^j. \end{aligned} \quad (2.13)$$

Finally we have

$$[[e^i, f^j], e^k] = \delta^{jk} e^i + \delta^{ij} e^k. \quad (2.14)$$

We introduce the bilinear form on \mathfrak{h}_{-1} associated with τ , defined by $(e_\mu, \tau(f_\nu)) = \delta_{\mu\nu}$. Then, from (2.12) and (2.13) we get

$$\begin{aligned} [[e_\mu, \tau(e_\nu)], e^i] &= -(e_\mu, e_\nu) e^i, & [[e_\mu, \tau(e_\nu)], f^j] &= (e_\mu, e_\nu) f^j, \\ [[e^i, e_\mu], [f^j, \tau(e_\nu)]] &= -\delta^{ij} [e_\mu, \tau(e_\nu)] - (e_\mu, e_\nu) [e^i, f^j], \\ [[e^i, e_\mu], \tau(e_\nu)] &= (e_\mu, e_\nu) e^i, & [e_\mu, [f^j, \tau(e_\nu)]] &= -(e_\mu, e_\nu) f^j. \end{aligned} \quad (2.15)$$

Consider now the direct sum $(\mathfrak{h}_{-1})^n$ of n vector spaces, each isomorphic to \mathfrak{h}_{-1} , and write a general element in $(\mathfrak{h}_{-1})^n$ as $(x_1)^1 + (x_2)^2 + \dots + (x_{n+1})^n$, where x_1, x_2, \dots are elements in \mathfrak{h}_{-1} . It is easy to see that the map ψ , defined by $e_\mu^1 \mapsto e_\mu$ and $(e_\mu)^{i+1} \mapsto [e^i, e_\mu]$ for $i=1, 2, \dots, n-1$ is one to one. Using the relations above, it is straightforward to show, case by case, that also $\psi((uvw)) = (\psi(u)\psi(v)\psi(w))$ for all $u, v, w \in (\mathfrak{h}_{-1})^n$. \square

III. JORDAN ALGEBRAS AND MAGIC SQUARES

In the Kantor–Koecher–Tits construction, any Jordan algebra J is associated with a 3-graded Lie algebra $\mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$ spanned by the operators

$$\begin{aligned} u &\in \mathfrak{g}_{-1}: x \mapsto u \quad (\text{constant}), \\ [u, \tau(v)] &\in \mathfrak{g}_0: x \mapsto (uvx) \quad (\text{linear}), \\ \tau(u) &\in \mathfrak{g}_1: x \mapsto -\frac{1}{2}(xux) \quad (\text{quadratic}) \end{aligned} \quad (3.1)$$

acting on the Jordan algebra, where

$$(xyz) = (xy)z + x(yz) - y(xz). \quad (3.2)$$

The associated Lie algebra $\mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$ is the *conformal algebra* $\text{con } J$, and \mathfrak{g}_0 is the *structure algebra* $\text{str } J$. If J has an identity element, then all scalar multiplications form a one dimensional ideal of $\text{str } J$. Factoring out this ideal, we obtain the *reduced structure algebra* $\text{str}' J$, which in

turn contains the Lie algebra $\mathfrak{der} J$ of all *derivations* of J . Thus, $\mathfrak{g}_0 = \mathfrak{str} J \supset \mathfrak{str}' J \supset \mathfrak{der} J$. To see why the resulting Lie algebra is called “conformal” we consider the Jordan algebras $H_2(\mathbb{K})$ of Hermitian 2×2 matrices over the division algebras $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$. We have

$$\mathfrak{der} H_2(\mathbb{K}) = \mathfrak{so}(d-1),$$

$$\mathfrak{str}' H_2(\mathbb{K}) = \mathfrak{so}(1, d-1),$$

$$\mathfrak{con} H_2(\mathbb{K}) = \mathfrak{so}(2, d) \tag{3.3}$$

for $d=3, 4, 6, 10$, respectively.¹⁷ It is well known that $\mathfrak{con} H_2(\mathbb{K})$ is the algebra that generates conformal transformations in a d -dimensional Minkowski spacetime. Furthermore, $\mathfrak{str}' H_2(\mathbb{K})$ is the Lorentz algebra and $\mathfrak{der} H_2(\mathbb{K})$ its spatial part.

The Kantor–Koecher–Tits construction can be applied also to the Jordan algebras $H_3(\mathbb{K})$ of Hermitian 3×3 matrices over \mathbb{K} . Then we obtain the first three rows in a magic square of Lie algebras.^{9–12,17} The magic square construction associates a Lie algebra $M(\mathbb{K}, \mathbb{K}')$ with any pair $(\mathbb{K}, \mathbb{K}')$ of division algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ in a natural way that leads to the following symmetric 4×4 array:

| | | | | |
|-----------------------------------|------------------|--|------------------|------------------|
| $\mathbb{K} \setminus \mathbb{K}$ | \mathbb{R} | \mathbb{C} | \mathbb{H} | \mathbb{O} |
| \mathbb{R} | \mathfrak{a}_1 | \mathfrak{a}_2 | \mathfrak{c}_3 | \mathfrak{f}_4 |
| \mathbb{C} | \mathfrak{a}_2 | $\mathfrak{a}_2 \oplus \mathfrak{a}_2$ | \mathfrak{a}_5 | \mathfrak{e}_6 |
| \mathbb{H} | \mathfrak{c}_3 | \mathfrak{a}_5 | \mathfrak{d}_6 | \mathfrak{e}_7 |
| \mathbb{O} | \mathfrak{f}_4 | \mathfrak{e}_6 | \mathfrak{e}_7 | \mathfrak{e}_8 |

For simplicity, we only specify the complex Lie algebras here. In this magic square, the real Lie algebras would actually be the compact forms of the complex Lie algebras that we have specified, but we also get other magic squares of real Lie algebras if we replace \mathbb{K} or \mathbb{K}' by the corresponding “split” algebras $\mathbb{C}^s, \mathbb{H}^s, \mathbb{O}^s$.¹⁸ When \mathbb{K}' is split and \mathbb{K} nonsplit, we get the derivation, reduced structure, and conformal algebras of $H_3(\mathbb{K})$ as the first three rows. When \mathbb{K} and \mathbb{K}' are both split, we get the split real forms of the complex Lie algebras above. We focus on the 3×3 subsquare in the lower right corner, consisting of simply laced algebras, with the following Dynkin diagrams:

| | | | |
|-----------------------------------|--------------|--------------|--------------|
| $\mathbb{K} \setminus \mathbb{K}$ | \mathbb{C} | \mathbb{H} | \mathbb{O} |
| \mathbb{C} | | | |
| \mathbb{H} | | | |
| \mathbb{O} | | | |

In the middle row of the 3×3 subsquare above, the black node generates the 3-grading of the conformal algebra. The outermost node next to it in the last row generates the unique 5-grading where the subspaces $\mathfrak{g}_{\pm 2}$ are one dimensional. With this 5-grading, the algebras in the last row are called “quasiconformal,” associated with Freudenthal triple systems.^{13,19,20} This is usually the way \mathfrak{e}_8 is included in the context of Jordan algebras and octonions. Here, we are more interested in the grading generated by the black node itself. Then we have the same situation in the last row as in Sec. II B for $n=2$. With the notation used in Sec. II B we thus have \mathfrak{g} in the last row and \mathfrak{h} in the second last row. Theorem 2.1 now implies that the algebras in the last row follow after those in the second last row in the sequence of Lie algebras that is obtained from $H_3(\mathbb{K})$ via the (generalized) Jordan triple systems. (This holds even for $\mathbb{K} = \mathbb{R}$, although we have not included it in the illustration above.) However, this sequence does not end with $n=2$ but can be continued to infinity, with one Kac–Moody algebra for each positive integer value of n . Since the node that we add to each diagram in the last row is the affine one, we will get the corresponding current algebra for $n=3$ and the hyperbolic extension for $n=4$.

If we apply Theorem 2.1 to the Jordan algebras $H_2(\mathbb{K})$ instead of $H_3(\mathbb{K})$, then we get \mathfrak{d}_{r+n} where $r=2,3,5$ for $\mathbb{K}=\mathbb{C},\mathbb{H},\mathbb{O}$, respectively. We will show this in detail in the next section.

IV. APPLICATION TO PSEUDO-ORTHOGONAL ALGEBRAS

In this section, we first give the nonlinear realization of $\mathfrak{so}(p+n, q+n)$ with a linearly realized subalgebra $\mathfrak{so}(p, q)$. Then we show that the special case $n=1$, which corresponds to conformal transformations in a spacetime of signature (p, q) , is related to the general case in the way it should according to Theorem 2.1. Finally, we relate the generalized conformal realization to the Jordan algebras $H_2(\mathbb{K})$ for Minkowski spacetimes in three, four, six, and ten dimensions.

We start with some basic facts. Let V be a real vector space with an inner product. The real Lie group $SO(V)$ consists of all endomorphisms F of V which preserve the inner product,

$$(F(u), F(v)) = (u, v), \quad (4.1)$$

for all $u, v \in V$. The corresponding real Lie algebra $\mathfrak{so}(V)$ consists of all endomorphisms f of V which are antisymmetric with respect to the inner product,

$$(f(u), v) + (u, f(v)) = 0, \quad (4.2)$$

for all $u, v \in V$. If V is nondegenerate and finite dimensional with signature (p, q) , then we can identify $SO(V)$ with the real Lie group $SO(p, q)$ consisting of all real $(p+q) \times (p+q)$ matrices X such that

$$X^t \eta X = \eta, \quad \det X = 1, \quad (4.3)$$

where η is the diagonal matrix associated with the inner product. Correspondingly, we can identify $\mathfrak{so}(V)$ with the real Lie algebra $\mathfrak{so}(p, q)$ consisting of all $(p+q) \times (p+q)$ matrices x such that

$$x^t \eta + \eta x = 0. \quad (4.4)$$

In other words, $\mathfrak{so}(p, q)$ consists of all real matrices of the form

$$x = \begin{pmatrix} a & c^t \\ c & b \end{pmatrix}, \quad (4.5)$$

where a and b are orthogonal $p \times p$ and $q \times q$ matrices, respectively. These groups and algebras are said to be *pseudo-orthogonal* or, if $p=0$, *orthogonal*, written simply as $SO(q)$ and $\mathfrak{so}(q)$.

We consider now the pseudo-orthogonal algebra $\mathfrak{so}(p+n, q+n)$, with the inner product given by

$$\eta = \text{diag}(\underbrace{-1, \dots, -1}_p, \underbrace{+1, \dots, +1}_q, \underbrace{-1, \dots, -1}_n, \underbrace{+1, \dots, +1}_n) \quad (4.6)$$

for some arbitrary positive integers n, p, q . It is spanned by all matrices G^I_J , where the entry in row L , column K is given by

$$(G^I_J)^K_L = \delta^I_L \delta^K_J - \eta^{IK} \eta_{JL}, \quad (4.7)$$

and $I, J, \dots = 0, 1, \dots, p+q+2n-1$. It follows that $(G^I_J)^t = G^J_I$. If $I \neq J$, then the entry of G^I_J in row I , column J is 1 while the entry in row J , column I is ± 1 and all the others are zero. If $I=J$, then $G^I_J=0$. These matrices satisfy the commutation relations

$$[G^I_J, G^K_L] = \delta^I_L G^K_J - \delta^K_J G^I_L + \eta^{IK} \eta_{JM} G^M_L - \eta_{JL} \eta^{IM} G^K_M, \quad (4.8)$$

and all those G^I_J with $I < J$ (say) form a basis of $\mathfrak{so}(p+n, q+n)$.

For $\mu, \nu, \dots = 0, 1, \dots, p+q-1$ and $a, b, \dots = 1, 2, \dots, n$, with $\mu < \nu$ and $a < b$, we take the linear combinations

$$\begin{aligned}
 K_{ab} &= \frac{1}{2}(-G^{a+m+n}_{b+m+n} + G^{a+m}_{b+m+n} - G^{a+m+n}_{b+m} + G^{a+m}_{b+m}), \\
 K^{\mu}_a &= -G^{\mu}_{a+m+n} - G^{\mu}_{a+m}, \\
 D^a_b &= \frac{1}{2}(G^{a+m+n}_{b+m+n} + G^{a+m}_{b+m+n} + G^{a+m+n}_{b+m} + G^{a+m}_{b+m}), \\
 P^a_{\mu} &= -G^{a+m+n}_{\mu} - G^{a+m}_{\mu}, \\
 P^{ab} &= \frac{1}{2}(-G^{a+m+n}_{b+m+n} - G^{a+m}_{b+m+n} + G^{a+m+n}_{b+m} + G^{a+m}_{b+m})
 \end{aligned}
 \tag{4.9}$$

as a new basis, where we have set $m=p+q-1$ for convenience. We note that K_{ab} and P^{ab} vanish when $n=1$, since they are antisymmetric in the indices a, b . The basis elements (4.9) satisfy the commutation relations

$$\begin{aligned}
 [G^{\mu}_\nu, D^a_b] &= [G^{\mu}_\nu, P^{ab}] = [G^{\mu}_\nu, K_{ab}] = 0, \\
 [G^{\mu}_\nu, G^{\rho}_\sigma] &= \delta^{\mu}_{\sigma} G^{\rho}_{\nu} - \delta^{\rho}_{\nu} G^{\mu}_{\sigma} + \eta^{\mu\rho} \eta_{\nu\lambda} G^{\lambda}_{\sigma} - \eta_{\nu\sigma} \eta^{\mu\lambda} G^{\rho}_{\lambda}, \\
 [D^a_b, D^c_d] &= \delta^a_d D^c_b - \delta^c_b D^a_d, \\
 [P^a_{\mu}, K^{\nu}_b] &= 2(\delta^a_b G^{\nu}_{\mu} - \delta^{\nu}_{\mu} D^a_b), \\
 [G^{\mu}_\nu, P^a_{\rho}] &= \delta^{\mu}_{\rho} P^a_{\nu} - \eta_{\nu\rho} \eta^{\mu\lambda} P^a_{\lambda}, \quad [G^{\mu}_\nu, K^{\rho}_a] = -\delta^{\rho}_{\nu} K^{\mu}_a + \eta^{\mu\rho} \eta_{\nu\lambda} K^{\lambda}_a, \\
 [D^a_b, P^c_{\mu}] &= -\delta^c_b P^a_{\mu}, \quad [D^a_b, K^{\mu}_c] = \delta^a_c K^{\mu}_b, \\
 [P^a_{\mu}, P^b_{\nu}] &= -2\eta_{\mu\nu} P^{ab}, \quad [K^{\mu}_a, K^{\nu}_b] = -2\eta^{\mu\nu} K_{ab}, \\
 [P^{ab}, P^c_{\mu}] &= 0, \quad [K_{ab}, K^{\mu}_c] = 0, \\
 [D^a_b, P^{cd}] &= \delta^d_b P^{ac} - \delta^c_b P^{ad}, \quad [D^a_b, K_{cd}] = \delta^a_c K_{bd} - \delta^a_d K_{bc}, \\
 [K^{\mu}_a, P^{bc}] &= \delta^c_a \eta^{\mu\lambda} P^b_{\lambda} - \delta^b_a \eta^{\mu\lambda} P^c_{\lambda}, \quad [P^a_{\mu}, K_{bc}] = \delta^a_c \eta_{\mu\lambda} K^{\lambda}_b - \delta^a_b \eta_{\mu\lambda} K^{\lambda}_c, \\
 [P^{ab}, P^{cd}] &= 0, \quad [K_{ab}, K_{cd}] = 0, \\
 [P^{ab}, K_{cd}] &= \delta^a_c D^b_d - \delta^b_c D^a_d - \delta^a_d D^b_c + \delta^b_d D^a_c.
 \end{aligned}
 \tag{4.10}$$

We see that $\mathfrak{so}(p+n, q+n)$ has the following 5-grading, which reduces to a 3-grading when $n=1$:

| | | | | | |
|----------|---------------------|---------------------|------------------------|------------------|------------------|
| Subspace | \mathfrak{g}_{-2} | \mathfrak{g}_{-1} | \mathfrak{g}_0 | \mathfrak{g}_1 | \mathfrak{g}_2 |
| Basis | P^{ab} | P^a_{μ} | G^{μ}_{ν}, D^a_b | K^{μ}_a | K_{ab} |

Furthermore, we see that D^a_b satisfy the commutation relations for $\mathfrak{gl}(n, \mathbb{R})$. Since they also commute with G^{μ}_{ν} , we have $\mathfrak{g}_0 = \mathfrak{so}(p, q) \oplus \mathfrak{gl}(n, \mathbb{R})$ as a direct sum of subalgebras. Finally, a graded involution τ is given by

$$\tau(P^a_{\mu}) = \eta_{\mu\nu} K^{\nu}_a = -G^{a+m+n}_{\mu} + G^{a+m}_{\mu},$$

$$\begin{aligned} \tau(K^\mu_a) &= \eta^{\mu\nu} P_\nu^a = -G^\mu_{a+m+n} + G^\mu_{a+m}, \\ \tau(D^a_b) &= -D^b_a, \quad \tau(K_{ab}) = P^{ab}, \\ \tau(G^\mu_\nu) &= G^\mu_\nu, \quad \tau(P^{ab}) = K_{ab}. \end{aligned} \tag{4.11}$$

Thus \mathfrak{g}_{-1} is a generalized Jordan triple system with the triple product

$$\begin{aligned} (P_\mu^a P_\nu^b P_\rho^c) &= [[P_\mu^a, \tau(P_\nu^b)], P_\rho^c] = [[P_\mu^a, \eta_{\nu\lambda} K^\lambda_b], P_\rho^c] = -2\delta^{ab} \eta_{\mu\lambda} (\delta^\lambda_\rho P_\nu^c - \eta_{\nu\rho} \eta^{\lambda\kappa} P_\kappa^c) \\ &\quad + 2\delta^{bc} \eta_{\mu\nu} P_\rho^a = 2\delta^{ab} (\eta_{\nu\rho} P_\mu^c - \eta_{\mu\rho} P_\nu^c) + 2\delta^{bc} \eta_{\mu\nu} P_\rho^a. \end{aligned} \tag{4.12}$$

If we now insert (4.12) in (A29) (but rescale the elements in \mathfrak{g}_{-2} according to Ref. 14) and use the isomorphism (A20) so that we identify any operator f with the vector field $-f^\mu_a \partial_\mu^a - f_{ab} \delta^{ab}$, then we get the realization

$$\begin{aligned} P^{ab} &= -2\delta^{ab}, \\ P_\mu^a &= \partial_\mu^a - 2x_{\mu b} \delta^{ab}, \\ G^\mu_\nu &= x_{\nu a} \partial^{\mu a} - x^\mu_a \partial_\nu^a, \\ D^a_b &= x^\mu_b \partial_\mu^a + 2x_{bc} \delta^{ac}, \end{aligned} \tag{4.13}$$

$$K^\mu_a = -2x^\nu_\alpha x^\mu_b \partial_\nu^b + x^\nu_\alpha x_{\nu b} \delta^{\mu b} - x_{ab} \delta^{\mu b} - 2x^\nu_\alpha x^\mu_b x_{\nu c} \delta^{bc} + 2x_{ab} x^\mu_c \delta^{bc},$$

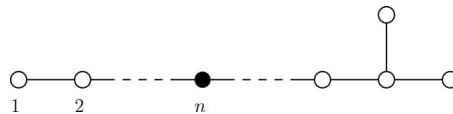
$$K_{ab} = x^\mu_\alpha x^\nu_b x_{\mu c} \partial_\nu^c - x^\mu_b x^\nu_\alpha x_{\mu c} \partial_\nu^c - x_{ac} x^\mu_b \partial_\mu^c + x_{bc} x^\mu_\alpha \partial_\mu^c + 2x^\mu_\alpha x^\nu_b x_{\mu c} x_{\nu d} \delta^{cd} - 2x_{ac} x_{bd} \delta^{cd}.$$

Straightforward calculations show that these generators indeed satisfy the commutation relations (4.10). When $n=1$, the $\mathfrak{gl}(n, \mathbb{R})$ indices a, b, \dots take only one value, so we can suppress them, and everything antisymmetric in these indices vanishes. We are then left with the conformal realization

$$\begin{aligned} P_\mu &= \partial_\mu \quad (\text{translations}), \\ G^\mu_\nu &= x_\nu \partial^\mu - x^\mu \partial_\nu \quad (\text{Lorentz transformations}), \\ D &= x^\mu \partial_\mu \quad (\text{dilatations}), \end{aligned} \tag{4.14}$$

$$K^\mu = -2x^\nu x^\mu \partial_\nu + x^\nu x_\nu \partial^\mu \quad (\text{special conformal transformations}).$$

We will now show that the 5-grading of $\mathfrak{so}(p+n, q+n)$ in this section is generated (as described in Sec. II A) by the simple root corresponding to node n in the Dynkin diagram below of \mathfrak{d}_r for $p+q=2r$. We will then show that the cases $n=1$ and $n>1$ are related in the way that we described in Sec. II B.



For this, we must relate the P_μ basis of \mathfrak{g}_{-1} used in this section to the basis consisting of root vectors. The relation will of course be different for different Lie algebras $\mathfrak{so}(p+n, q+n)$. We consider first the case $n=1$. Below we give explicitly the relations (with a suitable choice of Cartan–Weyl generators) for two examples, $(p, q)=(5, 5)$ in the first table and $(p, q)=(1, 9)$ in the

second table:

| μ | e_μ | f_μ |
|-------|-------------------------|-------------------------|
| 0 | | |
| 10000 | $\frac{1}{2}(P_5+P_0)$ | $\frac{1}{2}(K^5+K^0)$ |
| 0 | | |
| 11000 | $\frac{1}{2}(P_6+P_1)$ | $\frac{1}{2}(K^6+K^1)$ |
| 0 | | |
| 11100 | $\frac{1}{2}(P_7+P_2)$ | $\frac{1}{2}(K^7+K^2)$ |
| 0 | | |
| 11110 | $\frac{1}{2}(P_8+P_3)$ | $\frac{1}{2}(K^8+K^3)$ |
| 0 | | |
| 11110 | $\frac{1}{2}(P_9+P_4)$ | $\frac{1}{2}(K^9+K^4)$ |
| 1 | | |
| 11110 | $\frac{1}{2}(P_9-P_4)$ | $\frac{1}{2}(K^9-K^4)$ |
| 1 | | |
| 11111 | $\frac{1}{2}(P_3-P_8)$ | $\frac{1}{2}(K^3-K^8)$ |
| 1 | | |
| 11121 | $\frac{1}{2}(P_7-P_2)$ | $\frac{1}{2}(K^7-K^2)$ |
| 1 | | |
| 11221 | $\frac{1}{2}(P_1-P_6)$ | $\frac{1}{2}(K^1-K^6)$ |
| 1 | | |
| 12221 | $\frac{1}{2}(P_5-P_0)$ | $\frac{1}{2}(K^5-K^0)$ |
| | | |
| μ | e_μ | f_μ |
| 0 | | |
| 10000 | $\frac{1}{2}(P_5+P_0)$ | $\frac{1}{2}(K^5+K^0)$ |
| 0 | | |
| 11000 | $\frac{1}{2}(P_6-iP_1)$ | $\frac{1}{2}(K^6+iK^1)$ |
| 0 | | |
| 11100 | $\frac{1}{2}(P_7-iP_2)$ | $\frac{1}{2}(K^7+iK^2)$ |
| 0 | | |
| 11110 | $\frac{1}{2}(P_8-iP_3)$ | $\frac{1}{2}(K^8-iK^3)$ |
| 0 | | |
| 11111 | $\frac{1}{2}(P_9-iP_4)$ | $\frac{1}{2}(K^9+iK^4)$ |
| 1 | | |
| 11110 | $\frac{1}{2}(P_9+iP_4)$ | $\frac{1}{2}(K^9-iK^4)$ |
| 1 | | |
| 11111 | $\frac{1}{2}(iP_3+P_8)$ | $\frac{1}{2}(iK^3-K^8)$ |
| 1 | | |
| 11121 | $\frac{1}{2}(P_7+iP_2)$ | $\frac{1}{2}(K^7-iK^2)$ |
| 1 | | |
| 11221 | $\frac{1}{2}(iP_1+P_6)$ | $\frac{1}{2}(iK^1-K^6)$ |
| 1 | | |
| 12221 | $\frac{1}{2}(P_5-P_0)$ | $\frac{1}{2}(K^5-K^0)$ |

We have indicated the roots by their coefficients in the basis of simple roots, corresponding to the nodes in the Dynkin diagram above. (For example, $e_\mu=[e_1, e_2]$ in the second row and $e_\mu=[[e_1, e_2], e_3]$ in the third.) It is evident from these tables how to generalize them to arbitrary values of p, q (with $p+q$ even). It follows that the bilinear form associated with the involution

$$\tau(P_\mu) = \eta_{\mu\nu} K^\nu \tag{4.15}$$

is given in the P_μ basis by $(P_\mu, P_\nu) = 2\eta_{\mu\nu}$.

When we extend $\mathfrak{so}(p+1, q+1)$ to $\mathfrak{g} = \mathfrak{so}(p+n, q+n)$ for $n > 1$, we put a superscript 1 on P_μ in the expressions for the root vectors e_μ of the subalgebra $\mathfrak{h} = \mathfrak{so}(p+1, q+1)$ in \mathfrak{h}_{-1} (and a subscript 1 on K^μ). Then the graded involution (4.11) on \mathfrak{g} is indeed the extension of the original involution (4.15) on \mathfrak{h} that we described in Theorem 2.1, and we get

$$\begin{aligned}
(P_\mu^a P_\nu^b P_\rho^c) &= 2\delta^{ab}\eta_{\nu\rho}P_\mu^c - 2\delta^{ab}\eta_{\mu\rho}P_\nu^c + 2\delta^{bc}\eta_{\mu\nu}P_\rho^a = 2\delta^{ab}\eta_{\nu\rho}P_\mu^c - 2\delta^{ab}\eta_{\mu\rho}P_\nu^c + 2\delta^{ab}\eta_{\mu\nu}P_\rho^c \\
&\quad - 2\delta^{ab}\eta_{\mu\nu}P_\rho^c + 2\delta^{bc}\eta_{\mu\nu}P_\rho^a = \delta^{ab}(P_\mu P_\nu P_\rho)^c - 2\delta^{ab}\eta_{\mu\nu}P_\rho^c + 2\delta^{bc}\eta_{\mu\nu}P_\rho^a \\
&= \delta^{ab}[[P_\mu, \tau(P_\nu)], P_\rho]^c - \delta^{ab}(P_\mu, P_\nu)P_\rho^c + \delta^{bc}(P_\mu, P_\nu)P_\rho^a,
\end{aligned} \tag{4.16}$$

as we should, according to Theorem 2.1.

A. Connection to Jordan algebras

When $n=1$, the triple product (4.12) becomes

$$(P_\mu P_\nu P_\rho) = 2\eta_{\nu\rho}P_\mu - 2\eta_{\mu\rho}P_\nu + 2\eta_{\mu\nu}P_\rho. \tag{4.17}$$

If we introduce an inner product in the vector space \mathfrak{g}_{-1} by $P_\mu \cdot P_\nu = \eta_{\mu\nu}$, then this can be written as

$$(xyz) = 2(z \cdot y)x - 2(z \cdot x)y + 2(x \cdot y)z. \tag{4.18}$$

Let U be the subspace of \mathfrak{g}_{-1} spanned by P_i for $i=1, 2, \dots, p+q-1$. Then we can consider \mathfrak{g}_{-1} as the Jordan algebra $J(U)$, with the product

$$P_i \circ P_j = (P_i \cdot P_j)P_0 \tag{4.19}$$

for $i, j=1, 2, \dots, p+q-1$ and P_0 as identity element. If we introduce a linear map

$$J(U) \rightarrow J(U), \quad z \mapsto \tilde{z}, \tag{4.20}$$

which changes sign on P_0 but otherwise leaves the basis elements P_μ unchanged, then we can write the inner product as

$$2(u \cdot v) = u \circ \tilde{v} + v \circ \tilde{u}. \tag{4.21}$$

Inserting (4.21) in (4.18), we get

$$\begin{aligned}
(xyz) &= (\tilde{z} \circ y) \circ x - (\tilde{z} \circ x) \circ y + (\tilde{x} \circ y) \circ z + (z \circ \tilde{y}) \circ x - (z \circ \tilde{x}) \circ y + (x \circ \tilde{y}) \circ z \\
&= [y, \tilde{z}, x] + [y, \tilde{x}, z] + (z \circ \tilde{y}) \circ x + (x \circ \tilde{y}) \circ z.
\end{aligned} \tag{4.22}$$

It is easy to see that the *associators* in the last line remain unchanged if we move the tilde from one element to another. Thus, we get

$$\begin{aligned}
(xyz) &= [y, \tilde{z}, x] + [y, \tilde{x}, z] + (z \circ \tilde{y}) \circ x + (x \circ \tilde{y}) \circ z \\
&= [\tilde{y}, z, x] + [\tilde{y}, x, z] + (z \circ \tilde{y}) \circ x + (x \circ \tilde{y}) \circ z = 2(z \circ \tilde{y}) \circ x - 2(z \circ x) \circ \tilde{y} + 2(x \circ \tilde{y}) \circ z.
\end{aligned} \tag{4.23}$$

If we instead use the involution given by

$$\tau(\tilde{P}_\mu) = \eta_{\mu\nu}K^\nu, \tag{4.24}$$

then we can remove the tilde,

$$(xyz) = 2(z \circ y) \circ x - 2(z \circ x) \circ y + 2(x \circ y) \circ z. \tag{4.25}$$

Any Jordan algebra J is also a Jordan triple system with this triple product. The associated Lie algebra, defined by the construction in the Appendix or in (3.1), is its conformal algebra $\text{con } J$.

Consider now the case $p=1$ (and still $n=1$). Then U is a Euclidean space, and $J(U)$ is a *formally real* Jordan algebra. For $q=2, 3, 5, 9$, there is an isomorphism from $J(U)$ to $H_2(\mathbb{K})$, where $\mathbb{K}=\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$, respectively, given by

$$P_0 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad P_1 \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad P_{i+1} \mapsto \begin{pmatrix} 0 & -e_i \\ e_i & 0 \end{pmatrix}, \quad P_{p+q-1} \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (4.26)$$

for $i=1, 2, \dots, p+q-3$. (Here, e_i are the “imaginary units” that anticommute and square to -1 .) The involution (4.24) becomes $\tau(P_\mu)=K^\mu$, and we see from the tables that the associated bilinear form on \mathfrak{g}_{-1} has the simple form $(x, y)=\text{tr}(x \circ y)$. If we instead consider the split form ($p=q$) for $p=3, 5, 9$, then (4.26) is still an isomorphism if we replace $\mathbb{K}=\mathbb{C}, \mathbb{H}, \mathbb{O}$ by the split algebra \mathbb{K}^s which is obtained by changing the square of, respectively, 1, 2, and 4 imaginary units from -1 to 1 but otherwise leaving the multiplication table unchanged. Furthermore, the bilinear form on \mathfrak{g}_{-1} , associated with the graded involution (4.24), still has the form $(x, y)=\text{tr}(x \circ y)$.

To sum up, we have in this section given the 3-grading of $\mathfrak{so}(p+1, q+1)$ and shown that it is generated by the simple root corresponding to the leftmost node in the Dynkin diagram. If we add $n-1 \geq 1$ nodes to the left, then this simple root will instead generate a 5-grading of the resulting algebra $\mathfrak{so}(p+n, q+n)$. Theorem 2.1 tells us how the two triple systems, associated with the 5-graded Lie algebra $\mathfrak{so}(p+n, q+n)$ and its 3-graded subalgebra $\mathfrak{so}(p+1, q+1)$, respectively, are related to each other. It follows that $\mathfrak{so}(p+n, q+n)$ for $(p, q)=(1, 2), (1, 3), (1, 5), (1, 9)$ is the Lie algebra associated with the generalized Jordan triple system $H_2(\mathbb{K})^n$ for $\mathbb{K}=\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$, with the triple product

$$(x^a y^b z^c) = 2\delta^{ab}((z \circ y) \circ x)^c - 2\delta^{ab}((z \circ x) \circ y)^c + 2\delta^{ab}((x \circ y) \circ z)^c - \delta^{ab}(x, y)z^c + \delta^{bc}(x, y)z^a, \quad (4.27)$$

where $a, b, c=1, 2, \dots, n$ and $(x, y)=\text{tr}(x \circ y)$. The same holds if we replace these pseudo-orthogonal algebras by the split forms of the corresponding complex Lie algebras, and $\mathbb{C}, \mathbb{H}, \mathbb{O}$ by $\mathbb{C}^s, \mathbb{H}^s, \mathbb{O}^s$.

V. CONCLUSIONS

In this paper, we have shown that any Jordan algebra $H_2(\mathbb{K})$ or $H_3(\mathbb{K})$, via (generalized) Jordan triple systems, leads to an infinite sequence of Kac–Moody algebras, labeled by a positive integer n . The generalized Jordan triple product is given by (4.27) where $a, b, c=1, 2, \dots, n$, and (x, y) is the bilinear form associated with the graded involution on the Lie algebra in the $n=1$ case. We have shown that it is given by $(x, y)=\text{tr}(x \circ y)$ in the $H_2(\mathbb{K})$ case but not checked it for $H_3(\mathbb{K})$. However, the bilinear form is well defined by $(e_\mu, \tau(f_\nu))=\delta_{\mu\nu}$ and possible to determine if one would like to study the $H_3(\mathbb{K})$ case in detail, as we have done for $H_2(\mathbb{K})$. This would be an interesting subject of future research.

An important difference between the $H_2(\mathbb{K})$ and $H_3(\mathbb{K})$ cases is that the Lie algebra associated with $H_2(\mathbb{K})^n$ is 3-graded for $n=1$ and then 5-graded for all $n \geq 2$, while the Lie algebra associated with $H_3(\mathbb{K})^n$ is 3-graded for $n=1$ but 7-graded for $n=2$, and for $n=3, 4, 5, \dots$, we get infinitely many subspaces in the grading, since these Lie algebras are infinite dimensional. In the affine case, we only get the corresponding current algebra directly in this construction, which means that the central element and the derivation must be added by hand. The construction might be related to the “affinization” of generalized Jordan triple systems used in Refs. 21 and 22 (see also Ref. 23). Finally, concerning the hyperbolic case and further extensions, we hope that our new construction can give more information about these indefinite Kac–Moody algebras, for example, ϵ_{10} and ϵ_{11} , which both (but in different approaches) are conjectured to be symmetries underlying M -theory.^{24,25} In spite of great interest from both mathematicians and physicists, these algebras are not yet fully understood.

ACKNOWLEDGMENTS

I would like to thank Martin Cederwall, Axel Kleinschmidt, Daniel Mondoc, Hermann Nicolai, Bengt E. W. Nilsson, Daniel Persson, Christoffer Petersson, and Hidehiko Shimada for discussions.

APPENDIX: THE LIE ALGEBRA ASSOCIATED WITH A GENERALIZED JORDAN TRIPLE SYSTEM

In the end of Sec. II A, we saw that any graded Lie algebra with a graded involution gives rise to a generalized Jordan triple system. In this section, we will show the converse that any generalized Jordan triple system gives rise to a graded Lie algebra with a graded involution. The associated Lie algebra has been defined in a different (but equivalent) way by Kantor²⁶ and called the *Kantor algebra*.²⁷

We recall from Sec. II A that a generalized Jordan triple system is a triple system that satisfies the identity

$$(uv(xyz)) - (xy(uvz)) = ((uvx)yz) - (x(vuy)z). \quad (\text{A1})$$

For any pair of elements x, y in a generalized Jordan triple system T , we define the linear map

$$s_{xy}: T \rightarrow T, \quad s_{xy}(z) = (xyz). \quad (\text{A2})$$

Thus, (A1) (for all z) can be written as

$$[s_{uv}, s_{xy}] = s_{(uvx)y} - s_{x(vuy)}. \quad (\text{A3})$$

For any $x \in T$, we also define the linear map

$$v_x: T \rightarrow \text{End } T, \quad v_x(y) = s_{xy}, \quad (\text{A4})$$

which we will use in the following subsection.

1. Construction

Let T be a vector space and set $\tilde{U}_0 = \text{End } T$. For $k < 0$, define \tilde{U}_k recursively as the vector space of all linear maps from T to \tilde{U}_{k+1} . Let \tilde{U}_- be the direct sum of all these vector spaces,

$$\tilde{U}_- = \tilde{U}_{-1} \oplus \tilde{U}_{-2} \oplus \cdots, \quad (\text{A5})$$

and define a graded Lie algebra structure on \tilde{U}_- recursively by the relations

$$[u, v] = (\text{ad } u) \circ v - (\text{ad } v) \circ u. \quad (\text{A6})$$

Assume now that T is a generalized Jordan triple system. Let U_0 be the subspace of \tilde{U}_0 spanned by s_{uv} for all $u, v \in T$, and let U_- be the subspace of \tilde{U}_- generated by v_x for all $x \in T$. Furthermore, let U_+ be a Lie algebra isomorphic to U_- , with the isomorphism denoted by

$$*: U_- \rightarrow U_+, \quad u \mapsto u^*. \quad (\text{A7})$$

Thus, U_+ is generated by v_x^* for all $x \in T$. Consider the vector space

$$L(T) = U_- \oplus U_0 \oplus U_+. \quad (\text{A8})$$

We can extend the Lie algebra structures on each of these subspaces to a Lie algebra structure on the whole of $L(T)$ by the relations

$$[s_{xy}, v_z] = v_{(xyz)}, \quad [v_x, v_y^*] = s_{xy}, \quad [s_{xy}, v_z^*] = -v_{(yxz)}^*. \quad (\text{A9})$$

Furthermore, we can extend the isomorphism $*$ between the subalgebras U_- and U_+ to a graded involution on the Lie algebra $L(T)$. On U_+ , it is given by the inverse of the original isomorphism, $(u^*)^* = u$, and on U_0 by $s_{xy}^* = -s_{yx}$.

Theorem A.1:⁸ Let \mathfrak{g} be a graded simple Lie algebra generated by its subspaces $\mathfrak{g}_{\pm 1}$, with a graded involution τ . Let \mathfrak{g}_{-1} be the generalized Jordan triple system derived from \mathfrak{g} by

$$(uvw) = [[u, \tau(v)], w]. \quad (\text{A10})$$

Then the Lie algebra $L(\mathfrak{g}_{-1})$ is isomorphic to \mathfrak{g} .

Proof: Define the linear map $\varphi: \mathfrak{g} \rightarrow L(\mathfrak{g}_{-1})$, with $\mathfrak{g}_k \rightarrow U_k$ for all integers k , recursively by

$$\begin{aligned} u \in \mathfrak{g}_-: \varphi(u)(x) &= \varphi([u, \tau(x)]), \\ s \in \mathfrak{g}_0: \varphi(s)(x) &= [s, x], \end{aligned} \quad (\text{A11})$$

$$\tau(u) \in \mathfrak{g}_+: \varphi(\tau(u)) = \varphi(u)^*,$$

where $x \in \mathfrak{g}_{-1}$. We will show that φ is an isomorphism.

- φ is injective. Suppose that r and s are elements in \mathfrak{g}_0 such that $\varphi(r) = \varphi(s)$. Then $[r-s, x] = 0$ for all $x \in \mathfrak{g}_{-1}$, which means that $[r-s, \mathfrak{g}_-] = 0$ since \mathfrak{g}_- is generated by \mathfrak{g}_{-1} . But then the proper subspace

$$\sum_{k \in \mathbb{N}} (\text{ad}(\mathfrak{g}_0 + \mathfrak{g}_+))^k (r-s) \subset \mathfrak{g}_+ + \mathfrak{g}_0 \quad (\text{A12})$$

of \mathfrak{g} is an ideal. Since \mathfrak{g} is simple, it must be zero, but $r-s$ is an element of this subspace, so $r=s$. Suppose now that u and v are elements in \mathfrak{g}_- with $\varphi(u) = \varphi(v)$. Then $\varphi([u, \tau(x)]) = \varphi([v, \tau(x)])$ for all $x \in \mathfrak{g}_{-1}$, and by induction we can show that this implies $[u-v, \tau(x)] = 0$ for all $x \in \mathfrak{g}_{-1}$. Now we can use the same argument as before (but with \mathfrak{g}_+ replaced by \mathfrak{g}_-) to show that u and v must be equal. The case $u, v \in \mathfrak{g}_+$ then easily follows by

$$\varphi(\tau(u)) - \varphi(\tau(v)) = \varphi(u)^* - \varphi(v)^* = (\varphi(u) - \varphi(v))^*. \quad (\text{A13})$$

- φ is a homomorphism. It is sufficient to show this when $u \in \mathfrak{g}_i$ and $v \in \mathfrak{g}_j$ for all integers i, j and we will do it by induction over $|i|+|j|$. One easily checks that $\varphi([u, v]) = [\varphi(u), \varphi(v)]$ when $|i|+|j| \leq 1$. Thus, suppose that this is true if $|i|+|j| = p$ for some integer $p \geq 1$. For $i, j < 0$ we now have

$$\begin{aligned} [\varphi(u), \varphi(v)](x) &= [\varphi(u), \varphi(v)(x)] - [\varphi(v), \varphi(u)(x)] = [\varphi(u), \varphi([v, \tau(x)])] - [\varphi(v), \varphi([u, \tau(x)])] \\ &= \varphi([u, [v, \tau(x)]]) - [v, [u, \tau(x)]] = \varphi([u, v])(x) \end{aligned} \quad (\text{A14})$$

by the assumption of induction in the third step and by the Jacobi identity in the last one. We use this for the case $i, j > 0$, where we have

$$\varphi([\tau(u), \tau(v)]) = \varphi(\tau([u, v])) = \varphi([u, v])^* = [\varphi(u), \varphi(v)]^* = [\varphi(u)^*, \varphi(v)^*] = [\varphi(\tau(u)), \varphi(\tau(v))]. \quad (\text{A15})$$

Finally, we consider the case where $i \geq 0$ and $j \leq 0$. Again, we show it by induction over $|i|+|j|$, which means that $i-j$ in this case. One easily checks that it is true when $i=1$ and $j=-1$, so we can assume that $j \leq -2$ (or, analogously, $i \geq 2$). Then v can be written as a sum of elements $[x, y]$ where $x \in \mathfrak{g}_m$ and $y \in \mathfrak{g}_n$ for $j < m, n < 0$. We consider one such term and, using what we have already proven, we get

$$\begin{aligned} [\varphi(u), \varphi([x, y])] &= [\varphi(u), [\varphi(x), \varphi(y)]] = [[\varphi(u), \varphi(x)], \varphi(y)] - [[\varphi(u), \varphi(y)], \varphi(x)] \\ &= [\varphi([u, x]), \varphi(y)] - [\varphi([u, y]), \varphi(x)] = \varphi([[u, x], y] - [[u, y], x]) = \varphi([u, [x, y]]) \end{aligned} \quad (\text{A16})$$

by the assumption of induction in the third and fourth steps.

- φ is surjective. Since φ is a homomorphism, this follows from the fact that \mathfrak{g} and $L(\mathfrak{g}_{-1})$ are generated by $\mathfrak{g}_{\pm 1}$ and $U_{\pm 1}$, respectively. The proof is complete. \square

As we have seen, the theorem is useful when a generalized Jordan triple system T happens to be isomorphic to \mathfrak{g}_{-1} because it then tells us how to construct \mathfrak{g} from T .

2. Realization

The construction of the Lie algebra in the previous subsection may seem rather abstract, where the elements are linear operators acting on vector spaces of other linear operators, which in turn act on other vector spaces, and so on. However, once the Lie algebra is constructed, it can also be realized in a way such that the elements act on the same vector space, but in general nonlinearly, and there is a very simple formula for this, as we will see in this subsection.

Let V be the direct sum of (infinitely many) vector spaces V_1, V_2, \dots . We write an element $v \in V$ as $v = v_1 + v_2 + \dots$, where $v_k \in V_k$ for $k = 1, 2, \dots$. With an operator on V of order p we mean a map $f: V \rightarrow V$ such that for any $i = 1, 2, \dots$, there is a symmetric $(p_1 + p_2 + \dots)$ -linear map,

$$F_i: (V_1)^{p_1} \times (V_2)^{p_2} \times \dots \rightarrow V_i, \quad (\text{A17})$$

where $p_1 + 2p_2 + 3p_3 + \dots = i + p$, that satisfies

$$f(v)_i = F_i(v_1, v_1, \dots, v_1; v_2, v_2, \dots, v_2; \dots). \quad (\text{A18})$$

We define the composition $f \circ g$ of such an operator f and another operator g , of order q , as the operator of order $p + q$ given by

$$(f \circ g)_i(v) = p_1 F_i(g(v)_1, v_1, \dots, v_1; v_2, v_2, \dots, v_2; \dots) + p_2 F_i(v_1, v_1, \dots, v_1; g(v)_2, v_2, \dots, v_2; \dots) + \dots \quad (\text{A19})$$

for all $i = 1, 2, \dots$ and a Lie bracket as usual by $[f, g] = f \circ g - g \circ f$. Let M_p be the vector space of all operators on V of order p , and let $M(V)$ be the direct sum of all M_p for all integers p (note that they can also be negative). It follows that $M(V)$ is a graded Lie algebra. It is isomorphic to the Lie algebra of all vector fields $f^i \partial_i$ on V , where $f \in M(V)$, with an isomorphism given by

$$f \mapsto -f^i \partial_i. \quad (\text{A20})$$

Any graded Lie algebra $\mathfrak{g} = \mathfrak{g}_- + \mathfrak{g}_0 + \mathfrak{g}_+$ is isomorphic to a subalgebra of $M(\mathfrak{g}_-)$. It can be shown^{8,15} that an injective homomorphism $\chi: \mathfrak{g} \rightarrow M(\mathfrak{g}_-)$ is given by

$$\chi(u): x \mapsto \left(\frac{\text{ad } x}{1 - e^{-\text{ad } x}} P e^{-\text{ad } x} \right)(u), \quad (\text{A21})$$

where P is the projection onto U_- along $U_0 + U_+$, and the ratio should be considered as the power series

$$\frac{\text{ad } x}{1 - e^{-\text{ad } x}} = 1 + \frac{\text{ad } x}{2} + \frac{(\text{ad } x)^2}{12} - \frac{(\text{ad } x)^3}{720} + \dots \quad (\text{A22})$$

3. Examples

We will now illustrate the ideas in two cases, where the generalized Jordan triple system satisfies further conditions. First of all, we assume that the triple systems are such that if $(xyz) = 0$ for all y, z , then $x = 0$. This allows us to identify x with v_x for all x in the triple system, that is, we can identify U_{-1} with T and we can consider any element $[v_x, v_y] \in U_{-2}$ as a linear map on T , which we denote by $\langle x, y \rangle$. Since

$$[v_x, v_y](z) = [v_x, v_y(z)] - [v_y, v_x(z)] = [v_x, s_{yz}] - [v_y, s_{xz}] = v_{(xzy)} - v_{(yzx)}, \quad (\text{A23})$$

this linear map is given by

$$\langle x, y \rangle(z) = (xzy) - (yzx). \quad (\text{A24})$$

A generalized Jordan triple system is generalized in the sense that this linear map does not have to be zero—in a *Jordan triple system*,²⁸ the triple product (xyz) is by definition symmetric in x and z . Accordingly, the Lie algebra associated with a Jordan triple system is 3-graded, $\mathfrak{g} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$, and it can be realized on its subspace \mathfrak{g}_{-1} by applying the formula (A21). Everything that is left from the power series expansion (A22) is then the identity map

$$\frac{\text{ad } x}{1 - e^{-\text{ad } x}} = 1, \quad (\text{A25})$$

and we get

$$u \in \mathfrak{g}_{-1}: x \mapsto Pu = u$$

$$[u, \tau(v)] \in \mathfrak{g}_0: x \mapsto P([u, \tau(v)] - [x, [u, \tau(v)]]) = -[x, [\tau(u), v]] = [s_{uv}, x] = (uvx),$$

$$\tau(u) \in \mathfrak{g}_1: x \mapsto P(\tau(u) - [x, \tau(u)] + \frac{1}{2}[x, [x, \tau(u)]]) = \frac{1}{2}[x, [x, \tau(u)]] = -\frac{1}{2}[s_{xu}, x] = -\frac{1}{2}(xux). \quad (\text{A26})$$

This is the *conformal realization* of \mathfrak{g} on \mathfrak{g}_{-1} .

We now turn to *Kantor triple systems*²⁹ (or generalized Jordan triple systems of *second order*²⁶). These are generalized Jordan triple systems that in addition to the condition (A1) satisfy the identity

$$\langle \langle u, v \rangle(x), y \rangle = \langle (yxu), v \rangle - \langle (yxv), u \rangle. \quad (\text{A27})$$

It follows that the Lie algebra associated with a Kantor triple system is 5-graded, and the only part of (A22) that we have to keep is

$$\frac{\text{ad } x}{1 - e^{-\text{ad } x}} = 1 + \frac{\text{ad } x}{2}. \quad (\text{A28})$$

Then we get

$$[u, v] \in \mathfrak{g}_{-2}: z + Z \mapsto \langle u, v \rangle,$$

$$u \in \mathfrak{g}_{-1}: z + Z \mapsto u + \frac{1}{2}\langle u, z \rangle,$$

$$[u, \tau(v)] \in \mathfrak{g}_0: z + Z \mapsto (uvz) - \langle u, Z(v) \rangle, \quad (\text{A29})$$

$$\tau(u) \in \mathfrak{g}_1: z + Z \mapsto -\frac{1}{2}(zuz) - Z(u) + \frac{1}{12}\langle (zuz), z \rangle - \frac{1}{2}\langle Z(u), z \rangle,$$

$$\tau(u), \tau(v) \in \mathfrak{g}_2: z + Z \mapsto -\frac{1}{6}(z\langle u, v \rangle(z)z) - Z(\langle u, v \rangle(z)) + \frac{1}{24}\langle (z\langle u, v \rangle(z)z), z \rangle + \langle Z(u), Z(v) \rangle,$$

where $z \in \mathfrak{g}_{-1}$ and $Z \in \mathfrak{g}_{-2}$, which is the same realization as in Ref. 14 apart from a rescaling of the elements in \mathfrak{g}_{-2} by a factor of 2.

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