The $\mathfrak{su}(2|3)$ Undynamic Spin Chain

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The integrable spin chain found in perturbative planar $\mathcal{N} = 4$ supersymmetric gauge theory is dynamic. Here we propose a reformulation which removes the dynamic effects in order to make the model more accessible to an algebraic treatment.

§1. Introduction

The study of integrable structures in planar perturbative $\mathcal{N} = 4$ supersymmetric Yang–Mills theory following the works[1)–3)] has led to the discovery of an exciting integrable spin chain model. It displays several unusual and novel features with respect to the established integrable spin chains: First of all, the spin chain is perturbatively long-ranged.[4)] In other words, the Hamiltonian not only acts on nearest-neighbouring spins, but also on longer blocks of adjacent spins. The range is controlled by the perturbative order in a coupling constant $g \approx 0$. Moreover the chain is dynamic,[5)] that is, the Hamiltonian may dynamically change the number of spin sites of the chain. Finally, the Hamiltonian is an inseparable part of the symmetry algebra. Consequently, all the above features of the Hamiltonian apply to the symmetry generators as well. In addition it can be remarked that the spin module is non-compact and graded into bosons and fermions.

Despite these complications, it appears that the Hamiltonian is completely integrable.[1)–6)] Because it is homogeneous and acts locally, one can apply the asymptotic coordinate Bethe ansatz.[7),8)] The form of the asymptotic Bethe equations[9)] is fully constrained by symmetry considerations,[10)] merely one phase function remains undetermined. Imposing a further crossing symmetry[11),12)] together with inspiration from the dual superstring theory on $AdS_5 \times S^5$[13)] and its integrable structure[14)] one arrives at a viable proposal for the phase[15),16)] which has since passed several highly non-trivial tests.[17)–20)]

Note well that the above-mentioned asymptotic Bethe equations describe the spectrum only up to certain finite-size corrections, see Refs. 21), 22) and references therein, yet to be understood from the integrable model point of view. A conceivable path towards the exact finite-size spectrum is to fully understand the algebraic structure underlying the integrable spin chain model. One of the obstacles are the dynamic effects for which the conventional algebraic structures appear to be inapplicable.

In this note we consider the prototypical dynamic sector of the $\mathcal{N} = 4$ SYM spin chain with $\mathfrak{su}(2|3)$ symmetry.[5),*)] We shall propose an undynamic reformula-

*) The $\mathcal{N} = 6$ superconformal Chern-Simons theory[23)] with $\mathfrak{osp}(6|4, \mathbb{R})$ symmetry has an analogous $\mathfrak{su}(2|3)$ sector.[24)] The results of Ref. 5) and of this note are general and they also apply to

tion where length fluctuations are absent for a large part of the algebra including the Hamiltonian. This is meant to facilitate an eventual algebraic treatment the model. We will start with a review of the $\mathfrak{su}(2|3)$ sector, then propose the undynamic reformulation and finally discuss the implications and potential pitfalls.

§2. Dynamic chain

Let us start by reviewing the (apparently) integrable $\mathfrak{su}(2|3)$ dynamic spin chain constructed in Ref. 5).

2.1. *Hilbert space*

The spin at each site can be in three bosonic states $|\phi^a\rangle$ with $a = 1, 2, 3$, and two fermionic states $|\psi^\alpha\rangle$ with $\alpha = 1, 2$. Thus the graded spin module $\mathcal{V}$ is thus spanned by the five states

$$\mathcal{V} = \big\langle \phi^1, \phi^2, \phi^3 \big| \psi^1, \psi^2 \big\rangle. \qquad (2\cdot1)$$

The Hilbert space $\mathcal{H}$ of the spin chain model is given by the direct sum of cyclic chain spaces $\mathcal{H}_L$ of arbitrary length L

$$\mathcal{H} = \bigoplus_{L=1}^{\infty} \mathcal{H}_L, \qquad \mathcal{H}_L = \mathcal{V}^{\otimes L}\big|_{\text{cyclic}}. \qquad (2\cdot2)$$

The space $\mathcal{V}^{\otimes L}|_{\text{cyclic}}$ represents the subspace of $\mathcal{V}^{\otimes L}$ on which the graded cyclic shift operator acts as the identity. The dynamic nature of the model consists in the fact that the Hamiltonian (as well as the other symmetry generators) acts as an endomorphism of $\mathcal{H}$ and not of the individual $\mathcal{H}_L$'s, in other words, the length of the spin chain is a dynamic quantity. Furthermore our spin chain is homogeneous which entails the restriction to cyclic states: Homogeneous operators commute with the graded permutation whose spectrum $\exp(2\pi i\mathbb{Z}/L)$ crucially depends on the length. The only common eigenvalue on chains of L and $L+1$ is 1 and thus dynamic homogeneous models must be based on cyclic states.

2.2. *Symmetry Algebra.*

The symmetry of the dynamic chain is assumed to be $\mathfrak{su}(2|3)$. This algebra is spanned by the $\mathfrak{su}(3)$ generators $\mathfrak{R}^a{}_b$ ($\mathfrak{R}^a{}_a = 0$), the $\mathfrak{su}(2)$ generators $\mathfrak{L}^a{}_b$ ($\mathfrak{L}^a{}_a = 0$), the fermionic generators $\mathfrak{Q}^\alpha{}_b$ and $\mathfrak{S}^a{}_\beta$ and finally the Hamiltonian $\mathfrak{H}$. The Lie superalgebra is given by the canonical Lie brackets for $\mathfrak{su}(3)$ and $\mathfrak{su}(2)$ and the supercharges transform in (anti)fundamental representations, e.g.

$$[\mathfrak{R}^a{}_b, \mathfrak{Q}^\gamma{}_d] = -\delta^a_d \mathfrak{Q}^\gamma{}_b + \tfrac{1}{3}\delta^a_b \mathfrak{Q}^\gamma{}_d. \qquad (2\cdot3)$$

The non-trivial brackets among the supercharges are given by

$$\{\mathfrak{Q}^\alpha{}_b, \mathfrak{S}^c{}_\delta\} = \delta^\alpha_\delta \mathfrak{R}^c{}_b + \delta^c_b \mathfrak{L}^\alpha{}_\delta + \delta^\alpha_\delta \delta^c_b \mathfrak{H}. \qquad (2\cdot4)$$

Finally, the weights of the supercharges with respect to the Hamiltonian read

$$[\mathfrak{H}, \mathfrak{Q}^\alpha{}_b] = +\tfrac{1}{6}\mathfrak{Q}^\alpha{}_b, \qquad [\mathfrak{H}, \mathfrak{S}^a{}_\beta] = -\tfrac{1}{6}\mathfrak{S}^a{}_\beta. \qquad (2\cdot5)$$

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this model with some minor modifications regarding, e.g. the coupling constant and the embedding.

2.3. *Representation*

We want to construct a family of representations $\mathfrak{J}(g)$ of $\mathfrak{su}(2|3)$ on the Hilbert space $\mathcal{H}$ parametrised by a coupling constant g. The coupling constant g is assumed to be small and we shall treat the representation as a perturbation series around $g=0$

$$\mathfrak{J}(g) = \mathfrak{J}_0 + g\mathfrak{J}_1 + g^2\mathfrak{J}_2 + \ldots\ . \tag{2·6}$$

At leading order the representation $\mathfrak{J}_0$ is given by the standard tensor product of fundamental representations of $\mathfrak{su}(2|3)$

$$\begin{aligned} (\mathfrak{R}_0)^a{}_b &= \{{}^a_b\} - \tfrac{1}{3}\delta^a_b\{{}^c_c\}, & (\mathfrak{Q}_0)^\alpha{}_b &= \{{}^\alpha_b\}, \\ (\mathfrak{L}_0)^\alpha{}_\beta &= \{{}^\alpha_\beta\} - \tfrac{1}{2}\delta^\alpha_\beta\{{}^\gamma_\gamma\}, & (\mathfrak{S}_0)^a{}_\beta &= \{{}^a_\beta\}, \end{aligned} \qquad \mathfrak{H}_0 = \tfrac{1}{3}\{{}^a_a\} + \tfrac{1}{2}\{{}^\alpha_\alpha\}. \tag{2·7}$$

The interaction symbols $\{{}^\cdot_\cdot\}$ have the following meaning: For example, $\{{}^\beta_a\}$ picks any boson ϕ^a from the chain and replaces it by a fermion ψ^β. Here Latin and Greek indices refer to bosons and fermions, respectively. A homogeneous sum over all sites with proper grading is implicit in this notation.

The $\mathfrak{su}(3)$ and $\mathfrak{su}(2)$ representations are finite-dimensional and cannot be deformed continuously

$$\mathfrak{R}^a{}_b(g) = (\mathfrak{R}_0)^a{}_b, \qquad \mathfrak{L}^\alpha{}_\beta(g) = (\mathfrak{L}_0)^\alpha{}_\beta. \tag{2·8}$$

The representation of supercharges is deformed at all order in g, the first correction reads

$$(\mathfrak{Q}_1)^\alpha{}_b = \varepsilon^{\alpha\gamma}\varepsilon_{bde}\{{}^{de}_\gamma\}, \qquad (\mathfrak{S}_1)^a{}_\beta = \varepsilon^{acd}\varepsilon_{\beta\epsilon}\{{}^\epsilon_{cd}\}. \tag{2·9}$$

Symbols $\{{}^{\cdots}_{\cdots}\}$ with more than two indices refer to more complex interactions. For example, $\{{}^\epsilon_{cd}\}$ replaces a sequence of two bosons $\phi^c\phi^d$ by a single fermion ψ^ϵ. In the model the range of interactions is bounded by the perturbative order: At order g^n the interactions may consist of no more than $2+n$ spins (incoming plus outgoing), i.e. three in this case.

In fact, this is the leading appearance of dynamic effects within the model. The restriction to cyclic states simplifies the specification of interactions symbols: In cyclic states only the sequence of spins matters but not their overall position along the chain. Thus there is no need to specify how the final spins (ψ^ϵ) are aligned with respect to the initial spins ($\phi^c\phi^d$), e.g. left, right or centred.

These first corrections to the supercharges preserve the algebra. The possibility of such corrections is in fact very remarkable and related to a compatibility of the representation theory of cyclic chains of length L and $L+1$.

2.4. *Hamiltonian*

The role of the Hamiltonian is somewhat special. It is a Cartan generator of $\mathfrak{su}(2|3)$, but unlike the others its representation does receive corrections. Without loss of generality[5)] we may assume that (2·5) holds for $\mathfrak{H}_0$ instead of $\mathfrak{H}(g)$

$$[\mathfrak{H}_0, \mathfrak{Q}^\alpha{}_b(g)] = +\tfrac{1}{6}\mathfrak{Q}^\alpha{}_b(g), \qquad [\mathfrak{H}_0, \mathfrak{S}^a{}_\beta(g)] = -\tfrac{1}{6}\mathfrak{S}^a{}_\beta(g), \tag{2·10}$$

and consequently for $\delta\mathfrak{H}(g) = \mathfrak{H}(g) - \mathfrak{H}_0$

$$[\delta\mathfrak{H}(g), \mathfrak{Q}^\alpha{}_b(g)] = 0, \qquad [\delta\mathfrak{H}(g), \mathfrak{S}^a{}_\beta(g)] = 0. \tag{2·11}$$

In other words, the quantum corrections to the Hamiltonian are invariant under the full representation of $\mathfrak{su}(2|3)$. In particular, the leading correction to $\mathfrak{H}(g)$ must be invariant under the undeformed $\mathfrak{su}(2|3)$ representation. The simplest non-trivial such term is a graded permutation of two sites which can first appear at order g^2. Together with a two-site identity operator the second order contribution reads

$$\mathfrak{H}_2 = \left\{ {ab \atop ab} \right\} + \left\{ {\alpha b \atop \alpha b} \right\} + \left\{ {a\beta \atop a\beta} \right\} + \left\{ {\alpha\beta \atop \alpha\beta} \right\} - \left\{ {ba \atop ab} \right\} - \left\{ {b\alpha \atop \alpha b} \right\} - \left\{ {\beta a \atop a\beta} \right\} + \left\{ {\beta\alpha \atop \alpha\beta} \right\}. \tag{2·12}$$

The next correction to the Hamiltonian appears at order g^3

$$\mathfrak{H}_3 = -\varepsilon^{abc}\varepsilon_{\delta\epsilon}\left\{ {\delta\epsilon \atop abc} \right\} - \varepsilon^{\alpha\beta}\varepsilon_{cde}\left\{ {cde \atop \alpha\beta} \right\}. \tag{2·13}$$

It is compatible with the first corrections to the supercharges $\mathfrak{Q}_1$ and $\mathfrak{S}_1$. To some extent one can say that the Hamiltonian generally is shifted by two orders in g with respect to the remainder of the algebra.

2.5. *Beyond*

The higher orders of the Hamiltonian and the algebra have been constructed at orders $\mathcal{O}(g^6)$ and $\mathcal{O}(g^4)$, respectively in Ref. 5). The concrete expressions are long and little enlightening, but they appear to preserve integrability.

A dynamic charge which commutes with the whole algebra has been derived in Ref. 25) at order $\mathcal{O}(g^1)$ providing evidence for the compatibility of integrability with dynamic effects.

To make integrability rigorous one could construct the bi-local Yangian generators and show that they commute properly with the algebra and among themselves. The Yangian generators $\hat{\mathfrak{J}}$ are expected to take the generic form[6), 26)–28)]

$$\hat{\mathfrak{J}}^I{}_J \sim \{J^I{}_K|J^K{}_J\} - \{J^K{}_J|J^I{}_K\} + \text{local}, \tag{2·14}$$

where the vertical bar stands for arbitrarily many intermediate sites and the local terms represent a local regularisation of the bi-local insertions. For example, the Yangian generator $\hat{\mathfrak{Q}}$ corresponding to the supercharge $\mathfrak{Q}$ reads at leading order

$$(\hat{\mathfrak{Q}}_0)^\alpha{}_b \sim \left\{ {\alpha \atop c} \middle| {c \atop b} \right\} - \left\{ {c \atop b} \middle| {\alpha \atop c} \right\} + \left\{ {\alpha \atop \gamma} \middle| {\gamma \atop b} \right\} - \left\{ {\gamma \atop b} \middle| {\alpha \atop \gamma} \right\}. \tag{2·15}$$

The first correction is expected to take the form

$$(\hat{\mathfrak{Q}}_1)^\alpha{}_b \sim \varepsilon^{\alpha\gamma}\varepsilon_{def}\Big(\left\{ {de \atop \gamma} \middle| {f \atop b} \right\} - \left\{ {f \atop b} \middle| {de \atop \gamma} \right\}\Big) + \varepsilon^{\gamma\delta}\varepsilon_{bde}\Big(\left\{ {\alpha \atop \gamma} \middle| {de \atop \delta} \right\} - \left\{ {de \atop \delta} \middle| {\alpha \atop \gamma} \right\}\Big), \tag{2·16}$$

where in both expressions the local regularisation terms are very restricted and can merely be proportional to $\mathfrak{Q}_0$ and $\mathfrak{Q}_1$, respectively. It may be interesting to treat the realisation of the Yangian algebra explicitly. In particular, there may be complications[27)] due to the fact that the Hamiltonian is part of the algebra itself and because it is well-known that the Yangian is conserved only up to boundary terms.

§3. Undynamic chain

Dynamic spin chains as presented in the previous section have not been explored to a large extent yet. In this section we present an alternative formulation in terms of a chain with an undynamic Hamiltonian. The reformulation will show that the difficulties of this particular model cannot be attributed to the dynamic effects. They are rather due to the long-range nature of the interactions.

3.1. *Hilbert space*

The dynamic effects are essentially due to the degeneracy of quantum numbers for $\phi_{[1}\phi_2\phi_{3]}$ and $\psi_{[1}\psi_{2]}$. The trick of freezing out the dynamic effects consists in moving one of the bosons into the "background" and thus balancing the number of spins.

Let us single out one of the three bosons

$$\mathcal{Z} := \phi^3 \tag{3·1}$$

and restrict Latin indices to the range $a, b = 1, 2$ for the remainder of the paper. We now introduce composites as the fundamental spin degrees of freedom

$$\phi^a_n := \phi^a \underbrace{\mathcal{Z}\cdots\mathcal{Z}}_{n}, \qquad \psi^\alpha_n := \psi^\alpha \underbrace{\mathcal{Z}\cdots\mathcal{Z}}_{n}, \qquad \mathcal{V} = \bigoplus_{n=0}^{\infty} \big\langle \phi^1_n, \phi^2_n \big| \psi^1_n, \psi^2_n \big\rangle. \tag{3·2}$$

Every state of the above dynamic Hilbert space can obviously be translated to a state of an undynamic Hilbert space defined analogously to (2·2). One simply counts the number of $\mathcal{Z}$'s following any of the ϕ^a or ψ^α and puts as an additional index to the spin.*) Note that by this redefinition we trade in the dynamic effects for infinitely many spin degrees of freedom.

3.2. *Algebra decomposition*

Clearly the new notation breaks the manifest $\mathfrak{su}(3)$ symmetry of the bosons to $\mathfrak{su}(2)$. Together with the other $\mathfrak{su}(2)$ and some of the fermionic generators the residual symmetry algebra reduces to $\mathfrak{u}(2|2)$. This subalgebra is characterised by preserving the number of spin sites and it includes the Hamiltonian. The remaining generators are actually still dynamic but it in a controlled way: They either add or take away one site.

Let us decorate the residual $\mathfrak{u}(2|2)$ generators by a tilde. Their embedding into $\mathfrak{su}(2|3)$ is given by

$$\begin{aligned} &\tilde{\mathfrak{R}}^a{}_b = \mathfrak{R}^a{}_b + \tfrac{1}{2}\delta^a_b \mathfrak{R}^3{}_3, && \tilde{\mathfrak{Q}}^\alpha{}_b = \mathfrak{Q}^\alpha{}_b, && \tilde{\mathfrak{B}} = \tfrac{3}{2}\mathfrak{R}^3{}_3, \\ &\tilde{\mathfrak{L}}^\alpha{}_\beta = \mathfrak{L}^\alpha{}_\beta, && \tilde{\mathfrak{S}}^a{}_\beta = \mathfrak{S}^a{}_\beta, && \tilde{\mathfrak{C}} = \mathfrak{H} - \tfrac{1}{2}\mathfrak{R}^3{}_3. \end{aligned} \tag{3·3}$$

We shall call the remaining generators dynamic and distinguished them by a hat.

*) The only exceptions are the states made from $\mathcal{Z}$ alone. These states cannot be represented, but luckily they are trivial and can be ignored to a large extent.

Their embedding into $\mathfrak{su}(2|3)$ reads

$$\hat{\mathfrak{R}}^a = \mathfrak{R}^a{}_3, \qquad \hat{\mathfrak{R}}_a = \mathfrak{R}^3{}_a,$$
$$\hat{\mathfrak{Q}}^\alpha = \mathfrak{Q}^\alpha{}_3, \qquad \hat{\mathfrak{S}}_\alpha = \mathfrak{S}^3{}_\alpha. \tag{3·4}$$

The residual $\mathfrak{u}(2|2)$ algebra is determined by the following brackets:

$$[\tilde{\mathfrak{B}}, \tilde{\mathfrak{Q}}^\alpha{}_b] = +\tfrac{1}{2}\tilde{\mathfrak{Q}}^\alpha{}_b, \quad [\tilde{\mathfrak{B}}, \tilde{\mathfrak{S}}^a{}_\beta] = -\tfrac{1}{2}\tilde{\mathfrak{S}}^a{}_\beta, \qquad \{\tilde{\mathfrak{Q}}^\alpha{}_b, \tilde{\mathfrak{S}}^c{}_\delta\} = \delta^\alpha_\delta \tilde{\mathfrak{R}}^c{}_b + \delta^c_b \tilde{\mathfrak{L}}^\alpha{}_\delta + \delta^\alpha_\delta \delta^c_b \tilde{\mathfrak{C}}, \tag{3·5}$$

along with the obvious brackets of $\mathfrak{su}(2)\times\mathfrak{su}(2)$ generators and trivial brackets for the central charge $\tilde{\mathfrak{C}}$. The dynamical generators form two irreducible multiplets of $\mathfrak{u}(2|2)$: $(\hat{\mathfrak{R}}^a, \hat{\mathfrak{Q}}^\alpha)$ and $(\hat{\mathfrak{R}}_a, \hat{\mathfrak{S}}_\alpha)$. The non-obvious mixed brackets for the first multiplet take the form

$$[\tilde{\mathfrak{Q}}^\alpha{}_b, \hat{\mathfrak{R}}^c] = \delta^c_b \hat{\mathfrak{Q}}^\alpha, \qquad [\tilde{\mathfrak{B}}, \hat{\mathfrak{R}}^a] = -\tfrac{3}{2}\hat{\mathfrak{R}}^a, \qquad [\tilde{\mathfrak{C}}, \hat{\mathfrak{R}}^a] = +\tfrac{1}{2}\hat{\mathfrak{R}}^a,$$
$$\{\tilde{\mathfrak{S}}^a{}_\beta, \hat{\mathfrak{Q}}^\gamma\} = \delta^\gamma_\beta \hat{\mathfrak{R}}^a, \qquad [\tilde{\mathfrak{B}}, \hat{\mathfrak{Q}}^\alpha] = -\hat{\mathfrak{Q}}^\alpha, \qquad [\tilde{\mathfrak{C}}, \hat{\mathfrak{Q}}^\alpha] = +\tfrac{1}{2}\hat{\mathfrak{Q}}^\alpha. \tag{3·6}$$

The brackets for the conjugate multiplet essentially follow by conjugation. Finally, the non-trivial brackets between the dynamic generators yield

$$[\hat{\mathfrak{R}}^a, \hat{\mathfrak{R}}_b] = \tilde{\mathfrak{R}}^a{}_b - \delta^a_b \tilde{\mathfrak{B}}, \qquad [\hat{\mathfrak{Q}}^\alpha, \hat{\mathfrak{R}}_b] = \tilde{\mathfrak{Q}}^\alpha{}_b,$$
$$[\hat{\mathfrak{R}}^a, \hat{\mathfrak{S}}_\beta] = \tilde{\mathfrak{S}}^a{}_\beta, \qquad \{\hat{\mathfrak{Q}}^\alpha, \hat{\mathfrak{S}}_\beta\} = \tilde{\mathfrak{L}}^\alpha{}_\beta + \delta^\alpha_\beta(\tilde{\mathfrak{B}} + \tilde{\mathfrak{C}}). \tag{3·7}$$

3.3. *Representation of the residual algebra*

With the above decomposition relations it is straightforward to convert the representation of the previous section to the new basis. The leading order $\mathfrak{u}(2|2)$ algebra reads

$$\mathfrak{R}^a{}_b = \left\{ {a(n) \atop b(n)} \right\} - \tfrac{1}{2}\delta^a_b \left\{ {c(n) \atop c(n)} \right\}, \qquad (\mathfrak{Q}_0)^\alpha{}_b = \left\{ {\alpha(n) \atop b(n)} \right\}, \qquad \tilde{\mathfrak{C}}_0 = \tfrac{1}{2}\left\{ {I(n) \atop I(n)} \right\},$$
$$\mathfrak{L}^\alpha{}_\beta = \left\{ {\alpha(n) \atop \beta(n)} \right\} - \tfrac{1}{2}\delta^\alpha_\beta \left\{ {\gamma(n) \atop \gamma(n)} \right\}, \qquad (\mathfrak{S}_0)^a{}_\beta = \left\{ {a(n) \atop \beta(n)} \right\}, \qquad \tilde{\mathfrak{B}} = n\left\{ {I(n) \atop I(n)} \right\} - \tfrac{1}{2}\left\{ {a(n) \atop a(n)} \right\}. \tag{3·8}$$

Here we have extended the notation for interaction symbols in a hopefully evident way to the new states (3·2), where n stands for the number of trailing $\mathcal{Z}$'s. A repeated upper and lower index n is implicitly summed over all integers starting from 0. A capital Latin letter represents either a boson or fermion. For example, the symbols $\left\{ {I(n) \atop I(n)} \right\}$ and $n\left\{ {I(n) \atop I(n)} \right\}$ count the length of the new chain and the number of $\mathcal{Z}$'s, respectively.

The leading correction to the supercharges reads

$$(\tilde{\mathfrak{Q}}_1)^\alpha{}_b = \varepsilon^{\alpha\gamma}\varepsilon_{bd}\left(\left\{ {d(n+1) \atop \gamma(n)} \right\} - \left\{ {I(k+1),d(n) \atop I(k),\gamma(n)} \right\} \right),$$
$$(\tilde{\mathfrak{S}}_1)^a{}_\beta = \varepsilon^{ac}\varepsilon_{\beta\delta}\left(\left\{ {\delta(n) \atop c(n+1)} \right\} - \left\{ {I(k),\delta(n) \atop I(k+1),c(n)} \right\} \right). \tag{3·9}$$

While in (2·9) all interactions were one-to-two or two-to-one site, here we get one-to-one site or two-to-two site operators. In the case of the two-to-two site contributions the second site is merely needed to account for the change of leading $\mathcal{Z}$'s which cannot be represented otherwise.

A careful conversion of the leading interacting Hamiltonian (2·12) yields the new representation

$$\begin{aligned}\tilde{\mathfrak{C}}_2 = {}& \left\{\begin{smallmatrix} I(k),J(n+1) \\ I(k),J(n+1)\end{smallmatrix}\right\} - \left\{\begin{smallmatrix} I(k+1),J(n) \\ I(k),J(n+1)\end{smallmatrix}\right\} - \left\{\begin{smallmatrix} I(k),J(n+1) \\ I(k+1),J(n)\end{smallmatrix}\right\} + \left\{\begin{smallmatrix} I(k+1),J(n) \\ I(k+1),J(n)\end{smallmatrix}\right\} \\ &+ \left\{\begin{smallmatrix} I(0),J(n) \\ I(0),J(n)\end{smallmatrix}\right\} - \left\{\begin{smallmatrix} a(0),b(n) \\ b(0),a(n)\end{smallmatrix}\right\} - \left\{\begin{smallmatrix} \alpha(0),b(n) \\ b(0),\alpha(n)\end{smallmatrix}\right\} - \left\{\begin{smallmatrix} b(0),\alpha(n) \\ \alpha(0),b(n)\end{smallmatrix}\right\} + \left\{\begin{smallmatrix} \beta(0),\alpha(n) \\ \alpha(0),\beta(n)\end{smallmatrix}\right\}. \qquad (3{\cdot}10)\end{aligned}$$

Gladly, this is still a nearest-neighbour spin chain Hamiltonian. Note that the terms on the two above lines have a somewhat different meaning: The terms on the first row represent propagation terms of the magnons along the original chain, while the terms on the second row represent spin interactions of two adjacent magnons. The first correction to the interacting Hamiltonian (2·13) was showed the leading appearance of dynamic effects. In the new basis, however, the length remains fixed

$$\begin{aligned}\tilde{\mathfrak{C}}_3 = {}& \varepsilon_{cd}\varepsilon^{\alpha\beta}\left(-\left\{\begin{smallmatrix} c(0),d(n+1) \\ \alpha(0),\beta(n)\end{smallmatrix}\right\} + \left\{\begin{smallmatrix} c(1),d(n) \\ \alpha(0),\beta(n)\end{smallmatrix}\right\} - \left\{\begin{smallmatrix} I(k+1),c(0),d(n) \\ I(k),\alpha(0),\beta(n)\end{smallmatrix}\right\}\right) \\ &+ \varepsilon^{cd}\varepsilon_{\alpha\beta}\left(-\left\{\begin{smallmatrix} \alpha(0),\beta(n) \\ c(0),d(n+1)\end{smallmatrix}\right\} + \left\{\begin{smallmatrix} \alpha(0),\beta(n) \\ c(1),d(n)\end{smallmatrix}\right\} - \left\{\begin{smallmatrix} I(k),\alpha(0),\beta(n) \\ I(k+1),c(0),d(n)\end{smallmatrix}\right\}\right). \qquad (3{\cdot}11)\end{aligned}$$

3.4. *Representation of dynamic generators*

Note that $\mathfrak{C}_0$ measures half the length of the undynamic chain and thus the two brackets in (3·6) imply that the generators $\hat{\mathfrak{R}}^a$ and $\hat{\mathfrak{Q}}^\alpha$ add one site while $\hat{\mathfrak{R}}_a$ and $\hat{\mathfrak{S}}_\alpha$ remove one site. The leading-order representation takes the form

$$\begin{aligned}\hat{\mathfrak{R}}^a &= \left\{\begin{smallmatrix} I(k),a(n-1-k) \\ I(n)\end{smallmatrix}\right\}, & \hat{\mathfrak{R}}_a &= \left\{\begin{smallmatrix} I(n) \\ I(k),a(n-1-k)\end{smallmatrix}\right\}, \\ (\hat{\mathfrak{Q}}_0)^\alpha &= \left\{\begin{smallmatrix} I(k),\alpha(n-1-k) \\ I(n)\end{smallmatrix}\right\}, & (\hat{\mathfrak{S}}_0)_\alpha &= \left\{\begin{smallmatrix} I(n) \\ I(k),\alpha(n-1-k)\end{smallmatrix}\right\},\end{aligned} \qquad (3{\cdot}12)$$

which changes the length by one unit, because they replace a background spin $\mathcal{Z}$ by something else or vice versa.

Despite the length fluctuation, these generators close onto the one-to-one generators of the residual $\mathfrak{u}(2|2)$ representation. For example the non-manifest $\mathfrak{su}(3)$ brackets can be performed easily

$$\begin{aligned}[\hat{\mathfrak{R}}^a, \hat{\mathfrak{R}}_b] &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{n-1}\sum_{j=0}^{m-1}\left[\left\{\begin{smallmatrix} I(k),a(n-1-k) \\ I(n)\end{smallmatrix}\right\}, \left\{\begin{smallmatrix} J(m) \\ J(j),b(m-1-j)\end{smallmatrix}\right\}\right] \\ &= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\left\{\begin{smallmatrix} I(k),a(n) \\ I(k),b(n)\end{smallmatrix}\right\} - \sum_{n=0}^{\infty}\delta^a_b n\left\{\begin{smallmatrix} I(n) \\ I(n)\end{smallmatrix}\right\} = \tilde{\mathfrak{R}}^a{}_b - \delta^a_b\tilde{\mathfrak{B}}, \qquad (3{\cdot}13)\end{aligned}$$

as it should according to (3·7).

The first correction to the dynamic supercharges reads

$$(\hat{\mathfrak{Q}}_1)^\alpha = \varepsilon^{\alpha\beta}\varepsilon_{cd}\left\{\begin{smallmatrix} c(0),d(n) \\ \beta(n)\end{smallmatrix}\right\}, \qquad (\hat{\mathfrak{S}}_1)_\alpha = \varepsilon^{cd}\varepsilon_{\alpha\beta}\left\{\begin{smallmatrix} \beta(n) \\ c(0),d(n)\end{smallmatrix}\right\}. \qquad (3{\cdot}14)$$

Actually, it is not necessary to specify either of the pairs $\hat{\mathfrak{Q}}, \hat{\mathfrak{S}}$ or $\tilde{\mathfrak{Q}}, \tilde{\mathfrak{S}}$ explicitly because according to (3·6) and (3·7) one pair can simply be obtained from the other by commutation with the exact generators $\hat{\mathfrak{R}}$.

Dynamic (super)symmetries which relate conventional nearest-neighbour spin chain models at lengths differing by one unit are not unheard of: In particular they have appeared in various sectors of AdS/CFT.[9),29)–31)] They also exist for the XXX_1 chain,[32)] the $\mathrm{XXZ}_{1/2}$ chain with $q = e^{\pm 2\pi i/3}$ [33),34)] (or more generally XXZ_s with $q = e^{n\pi i/(1\pm s)}$), and a more exotic model.[35)] They all share the feature that Bethe roots at rapidity 0 induce the symmetry and that the symmetry can only exist for cyclic closed chains or for open chains.

§4. Comments

In this final section I would like to comment on the reformulation performed in the previous section and on the possibility of extending such a reformulation to the whole AdS/CFT spin chain with $\mathfrak{psu}(2,2|4)$ symmetry.

4.1. *Algebraic formulation*

It is fair to say that the picture presented in the previous section does not constitute an improvement of the situation per se. For example, the construction of Ref. 5) would not simplify in the new basis. In fact it would be somewhat worse, because the range of the interactions changes drastically between the pictures: The perturbative construction is expected to follow the range of the original spin chain, while the range in the new basis represents the number of magnon excitations involved in the interaction. Moreover, the manifest $\mathfrak{su}(3)$ symmetry reduces to merely $\mathfrak{su}(2)\times\mathfrak{u}(1)$. Finally, there is an unaesthetic asymmetry between leading and trailing background spins $\mathcal{Z}$. Nevertheless, there is a one-to-one map of interactions, and thus essentially nothing is lost by the change of picture.

The potential advantages of the new basis are of a more formal nature. There is some hope that the absence of dynamic effects for a large part of the algebra will make the model more accessible to conventional algebraic methods such as a Hopf algebra treatment. For example, to represent the action of symmetry generators through a coproduct appears reasonable only if the representation is undynamic. However, the problems introduced by long-range interactions certainly remain to be overcome. The remaining dynamic symmetry generators change the length by exactly one unit, which is much better than the arbitrariness in the original picture. In fact the various symmetry enhancements discussed in Refs. 33)–35), 9), 29) and 31) are of or can be brought to this form and they call for a more general story yet to be understood.

4.2. *Excitation picture*

The picture advertised here closely resembles what one obtains by performing the coordinate Bethe ansatz in Ref. 36).[*)] Namely, the spin $\mathcal{Z}$ is treated as a background spin and the magnons become sites of the reduced chain with $2|2$ spin orientations per site. The only difference is that magnons carry a definite momentum while here

[*)] A similar picture also underlies the NLIE approach [see e.g. Refs. 37) and 38)].

the spin orientation also specifies the distance between two adjacent excitations.[*] In that sense, these two pictures are essentially related by Fourier transformation.

In fact, the residual $\mathfrak{u}(2|2)$ algebra acting on the new basis coincides with the $\mathfrak{su}(2|2)$ algebra of the coordinate Bethe ansatz. In this context the difference between the pictures is that here UV effects, i.e. what happens whey two magnons come close (in the original model), can be honestly represented. This may be crucial for understanding finite-size effects. In the asymptotic coordinate Bethe ansatz such effects are largely ignored and collectively accounted for by the S-matrix. Conversely, here it is not possible to represent gauge transformations in a consistent manner. In the coordinate Bethe ansatz the gauge transformations alias the central extensions were crucial for success of the construction. Representing the $\mathfrak{su}(2|2)$ algebra within the coordinate Bethe ansatz is particularly simple because one only has to understand the single-magnon representation and how to assemble multi-magnon representation from that. The latter is achieved by a coproduct[39),40)] within the Hopf algebra framework. One might actually do the same here, at least to some approximation: Namely, find a representation of $\mathfrak{u}(2|2)$ on the infinite-dimensional spin module. A similar proposal has appeared recently for the closely related exceptional superalgebra $\mathfrak{d}(2,1;\alpha)$ in Ref. 41).

4.3. *Complete AdS/CFT spin chain*

It would be desirable to represent the complete AdS/CFT spin chain with $\mathfrak{psu}(2,2|4)$ symmetry. However, the generalisation is not straightforward: The spin module $\mathcal{V}$ contains not only the background spin and single excitations, but also multiple excitations. The decomposition of $\mathcal{V}$ in terms of the subalgebras $\mathfrak{psu}(2|2) \times \mathfrak{psu}(2|2)$ reads[42)]

$$\mathcal{V} = \bigoplus_{n=0}^{\infty} \mathcal{V}_n \otimes \mathcal{V}'_n, \qquad \mathcal{V}_n = (\mathcal{V}_1)^{\otimes n}\big|_{\text{antisym}}. \tag{4·1}$$

Here $\mathcal{V}_0$ is the trivial module spanned by the background spin $\mathcal{Z}$ and $\mathcal{V}_n$ is the n-fold graded anti-symmetric tensor product of $\mathcal{V}_1 = \langle \phi^1, \phi^2 | \psi^1, \psi^2 \rangle$. The $\mathcal{V}'_n$ denote the corresponding modules of the second $\mathfrak{psu}(2|2)$. There are now two ways in which one could attempt to proceed.

As before one could dress each of the components $\mathcal{V}_n \otimes \mathcal{V}'_n$ for $n \neq 0$ by an arbitrary number of background spins $\mathcal{Z}$. However this would not freeze the spin chain because only the overall number of excitations n is conserved. For example, two single excitations ($n = 1$) can be mapped to one double excitation ($n = 2$).

Instead one should work only with $\mathcal{V}_1 \otimes \mathcal{V}'_1$ trailed by arbitrarily many background spins $\mathcal{Z}$. The higher excitations would be represented by gluing together single excitations. For example the double excitation $\bar{\mathcal{Z}}$ can be thought of as being composed from $\phi^b \otimes \phi^{\dot{a}}$ and $\phi^d \otimes \phi^{\dot{c}}$:

$$\bar{\mathcal{Z}}_n \to \varepsilon_{bd}\varepsilon_{\dot{a}\dot{c}}\, (\phi^b \otimes \phi^{\dot{a}})_{-1} (\phi^d \otimes \phi^{\dot{c}})_n. \tag{4·2}$$

[*] It might also be worthwhile to investigate an absolute (instead of relative) position space picture, where, however, length fluctuations may become difficult to handle.

Here the number "−1" of trailing $\mathcal{Z}$'s is meant to indicate that the two consecutive single excitations reside on a single site (i.e. −1 sites in between) and thus form a double excitation. The problem with this representation is the graded anti-symmetrisation implicit for multiple excitations. Consequently the Hilbert space $\mathcal{H}_L$ of the model contains additional unphysical states.*) Therefore one has to ensure that the Hamiltonian and the symmetry generators do not map physical states to unphysical states. One could project out unphysical states from the Hilbert space from the start. This would lead to potential problems with the definition of interactions (they have to be compatible with the projection). Alternatively one could adjoin the Hamiltonian with a projection onto physical states. Unfortunately the latter are defined in a long-ranged fashion (an arbitrary number of adjacent spins has to be symmetrised). This apparently makes even the leading-order Hamiltonian long-ranged.

4.4. *Conclusions*

In conclusion, I have presented a reformulation of the $\mathfrak{su}(2|3)$ dynamic spin chain constructed in Ref. 5) where the dynamic effects are frozen out for a $\mathfrak{u}(2|2)$ subalgebra including the Hamiltonian. The other generators remain dynamic, but they merely change the length by precisely one unit as in Refs. 9), 29)–31) and 33)–35). The reformulation is intended to make the chain more accessible to a conventional algebraic treatment; it is merely the first step.

The change of picture works nicely for $\mathfrak{su}(2|3)$ where only single excitations of the ferromagnetic vacuum exist. A similar treatment of the complete AdS/CFT spin chain with $\mathfrak{psu}(2,2|4)$ symmetry and infinite-dimensional spin representations requires further insight due to the existence of multiple coincident excitations. However, if the proposed undynamic reformulation leads to a better understanding of the $\mathfrak{su}(2|3)$ model then there may well be a way to generalise those results to $\mathfrak{psu}(2,2|4)$.

Acknowledgements

I would like to thank the Galileo Galilei Institute (workshop "Non-Perturbative Methods in Strongly Coupled Gauge Theories"), Institut des Hautes Études Scientifiques and in particular Kyoto University (conference "30 Years of Mathematical Methods in High Energy Physics") for hospitality while part of this work was performed.

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*) The same problem arises in an absolute position space picture, cf. footnote ∗) on page 9, when the excitations are not well-ordered.

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