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# RENORMALIZATION GROUP AND EFFECTIVE POTENTIAL IN CLASSICALLY CONFORMAL THEORIES

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Making use of a general formula for the RG improved effective (Coleman–Weinberg) potential for classically conformal models and applying it to several examples of physical interest, and in particular a model of QCD coupled via quarks to a colorless scalar field, we discuss the range of validity of the effective potential as well as the issue of 'large logarithms' in a way different from previous such analyses. In all examples considered, convexity of the effective potential is restored by the RG improvement, or otherwise the potential becomes unstable. In the former case, symmetry breaking becomes unavoidable due to the appearance of an infrared barrier $\Lambda_{\rm IR}$, which hints at a so far unsuspected link between $\Lambda_{\rm QCD}$ and the scale of electroweak symmetry breaking.

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## 1. Introduction

The main purpose of this paper is to clarify some aspects of the Coleman–Weinberg (CW) mechanism of radiatively induced breaking of symmetry [1, 2] by means of examples, for which the renormalization group (RG) improved versions of the one-loop effective CW potential can be obtained in closed form. We believe that these results constitute further evidence in support of the scenario proposed in [3–5], according to which the CW mechanism can be implemented in the context of the full standard model with a classically conformal Lagrangian (*i.e.* without explicit mass terms in the scalar potential), provided all couplings remain bounded over a large range of energies. This may happen when there are non-trivial cancellations in the relevant $\beta$-functions, as a consequence of which Landau poles are shifted to very large scales. As we argued in [3], this may be the ultimate reason for the stability of the weak scale *vis-à-vis* the Planck scale $M_{\rm Planck}$, if the standard model couplings were to conspire in precisely such a way so as to make the model survive to very large scales (a similar scenario, but without radiative symmetry breaking, was proposed and elaborated in [6, 7]; see also [8, 9] for a different attempt to implement the CW mechanism).

RG methods to improve the effective potential have been applied and studied in previous work, and for a long time; we refer readers to [1, 2, 10] for the classic early work, and [11–19] for more recent treatments and comprehensive bibliographies. Making use of a general formula for the RG improved effective potential at one loop [1, 12], we show here how one can in principle arrive at closed form expressions for the effective potential at one loop by means of exact solutions of the corresponding one-loop $\beta$-functions. In cases, where closed form solutions of the latter are not available, the potentials can nevertheless be studied numerically in a convenient and effective manner by exploiting these general formulas. Closed form solutions of the $\beta$-function equations were presented for instance in [20] for a model of a scalar field coupled to fermions, and in [21] for a simplified version of the standard model. All examples that we discuss support our key assertion that *the CW potential can be trusted in the range where the running couplings (here expressed as functions of the classical field $\varphi$) stay small.* While this fact was already anticipated in [1], we emphasize here that this criterion can also be used to ascertain the consistency of the CW breaking in cases where the widely used 'rule of thumb', according to which the product of the logarithm of $\langle\varphi\rangle$ and the input coupling must be small, would indicate its failure. The latter requirement, when applied to the unimproved CW potential, is deficient inasmuch as it does not take into account possible cancellations in the $\beta$-functions. This is explicitly borne out by our results in Section 5, which show that the running couplings may stay small in the presence of such cancellations despite 'large logarithms'.

The present work also sheds new light on another long-standing issue, namely the apparent clash between symmetry breaking and convexity of the effective potential. As is well known on general grounds [10, 20, 22–24] the effective potential must be a *convex function* of the classical scalar field, a condition that is generically violated by the (unimproved) one-loop expressions derived in quantum field theory (but see [24] for a physical interpretation of the imaginary terms in the effective potential resulting from non-convexity). Remarkably, for the classically conformal theories we find here that, in all examples studied, either the convexity of the potential over its domain of definition is restored by the RG improvement, or otherwise the potential develops an instability[1]. The restoration of convexity is mainly due to the fact that the effective potential not only exhibits an ultraviolet (UV) barrier at the location $\Lambda_{\rm UV}$ of the Landau pole, but in general also an infrared (IR) barrier $\Lambda_{\rm IR}$ which arises through the couplings of the scalar field to the other fields via the coupled system of RG equations. The allowed regions for $\varphi$ are thus separated by a 'forbidden zone' $|\varphi| < \Lambda_{\rm IR}$[2]. In this way, the conflict between convexity on the one hand, and the existence of non-trivial vacua with $\langle\varphi\rangle \neq 0$ disappears. When the running coupling turns negative before this barrier is reached, the expression for the one-loop potential becomes unbounded from below for very small $\varphi$.

In the explicit examples we shall see that the Landau pole $\Lambda_{\rm UV}$ can be shifted to very large values. By contrast, in semi-realistic models involving the strong interactions, the IR barrier $\Lambda_{\rm IR}$ is unmovable because its value is in essence dictated by the known IR properties of $\alpha_{\rm s}$. *If $\Lambda_{\rm IR} > 0$, we must have $\langle\varphi\rangle \neq 0$, and symmetry breaking becomes unavoidable!* To be sure, our QCD-like example is not yet fully realistic in that the minimum is too close to $\Lambda_{\rm IR}$, whereas the scale of electroweak symmetry breaking is more than a hundred times larger than $\Lambda_{\rm QCD}$. Nevertheless, we find it most remarkable how the strong interactions — so far not thought to play any role in this context — may intervene to enforce breaking of electroweak symmetry (and conformal invariance). Readers may recall that symmetry breaking in the standard model is conventionally implemented by means of an explicit mass term $m^2\varphi^2$ — leaving us with the question why nature should prefer a negative value of $m^2$ over the equally consistent positive value!

Evidently, the poles at $\Lambda_{\rm UV}$ and $\Lambda_{\rm IR}$ both signal a breakdown of perturbation theory, where the one-loop approximation can no longer be trusted. By itself the UV Landau pole is not a problem if 'new physics' sets in well before this pole is reached. In keeping with the scenario of classical conformal invariance [3,4] we assume here that this new scale is actually the Planck scale (at which gravity becomes strong), and that the 'new physics' should therefore follow from a theory of quantum gravity and emergent space-time that can no longer be understood in terms of space-time based quantum field theory. Of course, it is only for a very small subset of possible Standard Model parameters that there is no UV Landau pole below the Planck

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[1] Although these statements seem not to apply in the presence of explicit mass terms, when the CW potential only represents a small correction to the classical potential, the point, in our opinion, even then requires further study.

[2] Of course, this does *not* mean that in a path integral treatment one should cut out a region around the origin in field space. The variable $\varphi$ here is the *effective classical field* defined as $\varphi = \delta W/\delta J$ from the generating functional $W[J]$ of connected Green's functions, and as such must obviously be distinguished from the scalar field over which one integrates in the path integral.

scale, and it was precisely this criterion that dictated the choice of parameters in [3–5]. On the other hand, the possible significance of scalar couplings becoming strong in the IR (as opposed to the Higgs coupling becoming strong in the UV) apparently has not been appreciated so far.

For simplicity we consider only a single real scalar field with couplings to different non-scalar fields (see *e.g.* [25, 26] for a discussion of the multi-field case); this has the advantage that the full result takes a rather simple form (*cf.* (1), (8), (9) and (10) below) which is completely determined by the $\beta$-function. The multi-field case, where such simplifications are presumably absent, will require separate study. After explaining some general features of the RG improvement procedure in Section 2, we first consider scalar QED in Section 3, not only recovering known results, but also to expose an IR instability that has gone unnoticed so far. In Section 4, we turn to our main example, QCD coupled to a colorless scalar field via Yukawa interactions. This example incorporates some essential features of the model investigated in [3, 5] in that the various running couplings keep each other under control over a large range of energies so as to ensure the survival of the theory up to some large scale. Finally, we present some numerical results and discuss the normalization of couplings in terms of physical parameters.

## 2. Generalities

In a classically conformally invariant theory[3] with one real scalar field that couples to any number of fermions and/or gauge fields, the effective CW potential is generally of the form[4]

$$W_{\text{eff}}(\varphi) = \varphi^4 f(L, g)\,, \tag{1}$$

where

$$L \equiv \ln \frac{\varphi^2}{v^2} \tag{2}$$

and $v$ is some mass scale required by regularization. The letter $g$ in (1) stands for a collection of coupling constants $\{g_1, g_2, \ldots\}$ corresponding to the quartic scalar self-coupling and various (dimensionless) couplings to and among other fields (fermions, gauge fields) in the theory. Below we will also use letters $u, x, y, z$ from the end of the alphabet to denote convenient combinations of these couplings, as they occur in the $\beta$-functions.

[3] See *e.g.* the recent article [27] for a comprehensive review of conformal invariance in field theory and quantum field theory. The derivation in this section is a slight generalization of arguments already given in [1, 2].

[4] We generally write $W_{\text{eff}}$ for the RG improved, and $V_{\text{eff}}$ for the unimproved effective potential at one loop.

The quantity $v$ is the only dimensionful parameter in the theory, but has no physical significance in itself; eventually, it should thus be replaced by a more physical parameter, such as the vacuum expectation value $\langle\varphi\rangle$. When one uses dimensional regularization (as we do), $v$ enters via the replacement

$$\int \frac{d^4k}{(2\pi)^4} \longrightarrow v^{2\epsilon} \int \frac{d^{4-2\epsilon}k}{(2\pi)^{4-2\epsilon}} \tag{3}$$

which is to be performed in all divergent integrals. The parameter $v$ breaks conformal invariance explicitly, and the question is then whether this breaking persists after renormalization, in which case the classical conformal symmetry is broken by anomalies. For the renormalization we employ the prescription of [4], according to which the local part of the effective action is to be kept conformally invariant throughout the regularization procedure: in this way, the structure of the anomalous Ward identity is preserved at every step (as originally suggested in [28]). As a consequence, the mass parameter $v$ appears in the effective action only via (2), *i.e.* logarithmically.

The function $f$ must be determined from a perturbative expansion [1,2]. A main problem then is to assess the reliability of such approximations, and to ascertain whether extrema found by minimizing the perturbative potential are within the perturbative range or not. On general grounds, the effective potential $W_{\rm eff}$ must satisfy the RG equation (see *e.g.* [10, 13, 29])

$$\left[v\frac{\partial}{\partial v} + \sum_j \beta_j(g)\frac{\partial}{\partial g_j} + \gamma(g)\varphi\frac{\partial}{\partial\varphi}\right] W_{\rm eff}(\varphi, g, v) = 0\,, \tag{4}$$

where $\beta_j(g) \equiv \beta_j(g_1, g_2, \ldots)$ and $\gamma(g) \equiv \gamma(g_1, g_2, \ldots)$ are the relevant $\beta$-functions and anomalous dimension, respectively; they depend on the theory under consideration. Substituting (1) into this formula, we obtain [1,12]

$$\left[-2\frac{\partial}{\partial L} + \sum_j \tilde\beta_j(g)\frac{\partial}{\partial g_j} + 4\tilde\gamma(g)\right] f(L, g_j) = 0\,, \tag{5}$$

where

$$\tilde\beta(g) \equiv \frac{\beta(g)}{1-\gamma(g)}\,, \qquad \tilde\gamma(g) \equiv \frac{\gamma(g)}{1-\gamma(g)}\,. \tag{6}$$

We note that the effect of the rescaling by the factor $(1-\gamma)$ will only appear in higher orders, and thus not play any role for the (one-loop) considerations in this paper. For this reason, we will omit the tildes in all formulas in the following sections. A perturbative analysis of these equations was already begun in [18, 19], and a closed form of the solution of the one-loop RG equations, albeit very complicated, was derived for a simplified version of the standard model in [21].

The general solution of (5) (to *any* order) is obtained by first solving the system of ordinary differential equations for the running couplings, *viz.*[5]

$$2\frac{d}{dL}\hat{g}_j(L) = \tilde{\beta}_j(\hat{g}(L))\,, \tag{7}$$

where the initial values $\hat{g}_j(0)$ are the input parameters from the classical Lagrangian. Given a solution (7), the partial differential equation (5) is solved by

$$f(L,g_j) \equiv F(\hat{g}_1(L),\hat{g}_2(L),\dots)\exp\left[2\int\limits_0^L \tilde{\gamma}(\hat{g}(t))dt\right]\,, \tag{8}$$

where $F$ is an *a priori arbitrary* function, which can be determined by matching it with the perturbative (loop) expansion of the effective action. The precise choice of $F$ together with the choice of $\beta$-functions fixes the renormalization scheme (recall that the $\beta$-functions are renormalization scheme dependent in higher loop order). We take

$$F(\hat{g}(L)) \equiv \hat{g}_1(L)\,, \tag{9}$$

where $g_1$ is the scalar self-coupling. In this way we arrive at the general formula

$$W_{\rm eff}(\varphi,g,v) = \hat{g}_1(L)\varphi^4\exp\left[2\int\limits_0^L \tilde{\gamma}(\hat{g}(t)dt\right]\,. \tag{10}$$

At any order in the loop expansion, (9) is the most natural choice because the effective potential (10) evidently reduces to the classical potential in the limit $\hbar \to 0$ (we recall that each factor of $L$ is accompanied by a factor $\hbar$, but we usually set $\hbar = 1$). Other schemes can differ from this one by higher powers of the (running) coupling constants on the r.h.s. of (9), such that

$$F(\hat{g}(L)) = \hat{g}_1(L) + \sum_{i,j}\alpha_{ij}\hat{g}_i(L)\hat{g}_j(L) + \dots \tag{11}$$

corresponding to a non-linear redefinition of the coupling constants. We note that whenever closed form solutions of (7) can be found, (10) yields a *completely explicit* formula for the RG improved potential. Moreover, even if such explicit solutions do not exist, (10) can be conveniently exploited to explore the RG improved potential numerically (as we do in Section 4).

[5] For notational clarity we always put hats on the $L$-dependent running couplings.

Obviously the above expression for the RG improved effective potential (and (10) in particular) can be trusted as long as the running couplings $\hat{g}_j(L)$ remain small (in which case the higher order corrections in (11) are likewise small). Imposing smallness of the running couplings in turn determines an admissible range of values for $L$, and thereby for the field $\varphi$ itself. In realistic applications we will seek to ensure by judicious choice of the couplings that this admissible range extends beyond the Planck mass for positive $L$, but we usually will also encounter a *lower* bound for negative $L$. With the replacement of the one-loop effective potentials by RG improved potentials, there is no more need to consider running coupling constants. Rather, the energy scale is now replaced by the classical field $\varphi$ itself, or more precisely, the ratio $\varphi/v$. Let us also recall that the one-loop RG improved effective action corresponds to a resummation of the perturbation series where only leading logarithms are kept.

To explain the main point in a nutshell, let us consider the textbook example of pure (*i.e.* massless) $\phi^4$ theory, for which the one-loop effective potential is [1]

$$V_{\text{eff}}(\varphi) = \frac{g}{4}\varphi^4 + \frac{9g^2}{64\pi^2}\varphi^4 L\,. \tag{12}$$

The RG improved potential is found by first solving the RG equation for the scalar self-coupling

$$2\frac{d\hat{y}}{dL} = \frac{9}{2}\hat{y}^2 \qquad \text{with} \qquad \hat{y}(L) \equiv \frac{\hat{g}(L)}{4\pi^2}\,. \tag{13}$$

The well known solution is [29]

$$\hat{y}(L) = \frac{y_0}{1-(9/4)y_0 L} \tag{14}$$

exhibiting the famous Landau pole at $\varphi = \Lambda_{\text{UV}} \equiv v\exp[2/(9y_0)]$. From (10), the RG improved potential at one loop is therefore given by [1]

$$W_{\text{eff}}(\varphi, g_0) = \frac{\pi^2 y_0 \varphi^4}{1-(9/4)y_0 L} = \pi^2 y_0 \varphi^4 \Big[1 + (9/4)y_0 L + \ \ldots\Big] \tag{15}$$

(the anomalous scaling can be neglected because $\gamma(g) = 0$ at one loop). This expression exhibits the very same Landau pole as the scalar self-coupling, and in this way limits the range of trustability to the values $y_0 L < 4/9$, or equivalently $|\varphi| < \Lambda_{\text{UV}}$ (there is no IR barrier, so $\varphi$ can become arbitrarily small). Because there is no non-trivial minimum of $W_{\text{eff}}$ in this range we recover the well-known result that the symmetry breaking minimum in the unimproved potential $V_{\text{eff}}$ (12) is spurious. The existence of a spurious minimum $\langle\varphi\rangle \neq 0$ in (12) is related to the fact that the function (12) is *not convex*. The RG improved version (15), on the other hand, *is* convex, and therefore the minimum is moved back to $\langle L\rangle = -\infty$, that is, $\langle\varphi\rangle = 0$. As we will see in our 'real world' example (and also in scalar QED), the latter possibility is precluded by an IR barrier, which entails $\langle L\rangle > -\infty$.

## 3. Massless scalar QED revisited

Next we turn to massless scalar QED, which describes the coupling of one complex scalar field to electromagnetism [1]. Introducing

$$y = \frac{g}{4\pi^2}\,, \qquad u = \frac{e^2}{4\pi^2} \tag{16}$$

for the scalar self-coupling and the electromagnetic coupling, respectively, we have the system of equations [1]

$$2\frac{d\hat{y}}{dL} = a_1\hat{y}^2 - a_2\hat{y}\hat{u} + a_3\hat{u}^2\,, \qquad 2\frac{d\hat{u}}{dL} = 2b\hat{u}^2\,, \tag{17}$$

where for the scalar QED $a_1 = \frac{5}{6},\ a_2 = 3,\ a_3 = 9$ and $b = \frac{1}{12}$. At this order, we also have

$$\gamma(y,u) = cu \tag{18}$$

with $c = \frac{3}{4}$. The solution was found in [1]

$$\begin{aligned} \hat{u}(L) &= \frac{u_0}{1 - bu_0 L}\,, \\ \hat{y}(L) &= \frac{(a_2+2b)\hat{u}(L)}{2a_1} + \frac{A\hat{u}(L)}{2a_1}\tan\left(\frac{A}{8b}(\ln(\hat{u}(L)) + C)\right) \end{aligned} \tag{19}$$

with (we assume that the constants give $A > 0$ as is the case for scalar QED)

$$A = \sqrt{4a_1a_3 - (a_2+2b)^2} \tag{20}$$

and $C$ should be chosen to satisfy $\hat{y}(0) = y_0$.

As we explained, $\hat{y}(L, y_0, u_0)$ then also solves the RG equation (5)

$$\left[-2\frac{\partial}{\partial L} + \beta_y\frac{\partial}{\partial y} + \beta_u\frac{\partial}{\partial u}\right] F(L,y,u) = 0 \tag{21}$$

with the proper classical limit $F(0,y,u) = y$. Hence, from (10) we obtain the *exact* RG improved effective potential at one loop

$$W_{\rm eff}(\varphi,y,u) = \pi^2\varphi^4\,\hat{y}(L)/(1-bu_0L)^{2c/b}\,. \tag{22}$$

Let us first recall the standard treatment of (22), as explained in [1]. By taking $y$ and $u$ small, with the additional relation $y = \alpha u^2$, and $\alpha = \mathcal{O}(1)$, one easily checks that the minimum occurs at $\langle L \rangle = O(1)$ independently of the input value of the coupling $u$. Because $u\langle L \rangle$ is thus small, one concludes that the symmetry breaking minimum is not spurious, unlike for pure $\phi^4$.

Inspection of the RG improved potential (22) reveals one feature of the full potential which is not visible in the unimproved potential. Because of the tangent function in the solution there are now *two* barriers — one is the usual UV Landau pole, whereas the other represents an IR barrier, disallowing small values of $\varphi$. As a consequence of the lower unboundedness of the one-loop potential at the IR barrier the minimum may only be metastable, even though it occurs inside the region of applicability of perturbation theory (where the running coupling is still small). This clearly indicates a breakdown of perturbation theory in the IR; it is conceivable that higher orders might re-stabilize the effective potential there, but one will have to resort to non-perturbative methods in order to settle the question.

The unboundedness of $W_{\text{eff}}$ in the IR can be traced back the strict positivity of $\hat{y}'(L)$ in (17). The IR instability can be avoided by pushing $\ln(\Lambda_{\text{IR}}/v)$ back to $-\infty$, as may happen for other choices of the input parameters, but then we are back to (15) and there would be no symmetry breaking. When the RG improved potential $W_{\text{eff}}$ is unstable, it also fails to be convex, exemplifying the claimed link between instability and lack of convexity.

## 4. QCD with one colorless scalar

Our main example is closely modeled on [3, 5], and thus incorporates features of the Standard Model. The Lagrangian can be written schematically as

$$\mathcal{L} = -\frac{1}{4}\text{Tr}\, F_{\mu\nu}F^{\mu\nu} + \bar{q}^i\gamma^\mu D_\mu q^i - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + g_{\text{Y}}\phi\bar{q}^i q^i - \frac{g}{4}\phi^4\,. \tag{23}$$

It is classically conformally invariant and depends on three couplings, the gauge coupling $g_s$, the Yukawa coupling $g_{\text{Y}}$ and the scalar self-coupling $g$. The (real) scalar field $\phi$ is not charged under Yang–Mills SU($N$) (hence colorless), but couples to color-charged quarks via the Yukawa term, much like the standard model Higgs. The main advantage of the model (23) is that the one-loop RG equations can again be solved exactly. This enables us to exhibit new phenomena that are not visible in the usual perturbative expansion, and which require a minimum of *three* independent couplings. An important feature is that the one-loop $\beta$-functions for the scalar and for the fermions both have two contributions of opposite sign, so that both couplings can remain stationary over a large range of energies with suitable initial conditions.

The one-loop $\beta$-function equations are now given by the system

$$2\frac{d\hat{y}}{dL} = a_1\hat{y}^2 + a_2\hat{x}\hat{y} - a_3\hat{x}^2\,, \qquad 2\frac{d\hat{x}}{dL} = b_1\hat{x}^2 - b_2\hat{x}\hat{z}\,, \qquad 2\frac{d\hat{z}}{dL} = -c\hat{z}^2 \quad (24)$$

for the functions $\hat{y} \equiv \hat{y}(L)$ , $\hat{x} \equiv \hat{x}(L)$ and $\hat{z} \equiv \hat{z}(L)$, where

$$x \equiv \frac{g_{\mathrm{Y}}^2}{4\pi^2}\,, \qquad y \equiv \frac{g}{4\pi^2}\,, \qquad z \equiv \frac{g_s^2}{4\pi^2} \equiv \frac{\alpha_{\mathrm{s}}}{\pi} \quad (25)$$

and the anomalous dimension of $\phi$ is equal to $\gamma(x,y,z) = -ax$ at this order.

For the Standard Model the values are [13]

$$a_1 = 6\,, \quad a_2 = 3\,, \quad a_3 = \frac{3}{2}\,, \quad b_1 = \frac{9}{4}\,, \quad b_2 = 4\,, \quad c = \frac{7}{2}\,, \quad a = \frac{3}{4} \quad (26)$$

and they are similar also to the model with massive neutrinos such as the one in [3]. The whole system of equations admits a stable solution with suitable initial values, because asymptotic freedom keeps the YM coupling under control, which in turn slows down the running of the Yukawa coupling, which in turn keeps the scalar self-coupling under control. This cascade of mutual cancellations is the mechanism invoked in [3,4] to avoid Landau poles or instabilities between the small scales and the Planck scale. Furthermore, the model shows that indeed the one-loop solution of RG equations for the effective potential has a minimum for a field much below the value of the field where the UV Landau pole occurs and the crucial point that we want to emphasize in this paper is that there exists an IR Landau pole so that the resulting effective potential is simultaneously convex and leads to symmetry breaking.

In the case of the Standard Model a closed form solution of (24) was presented in [21], with a (complicated) ratio of hypergeometric functions as the result. The generic feature of the solution is the presence of either a Landau pole or an instability on the UV side and a Landau pole on the IR side (caused by the strong coupling evolution), depending on the initial values of the coupling constants. Since the explicit expressions are not very illuminating let us here concentrate on the behavior of the solution near the IR Landau pole, which is the most interesting feature of this model. The solutions for $\hat{z}(L)$ and $\hat{x}(L)$ can be easily found

$$\begin{aligned} \hat{z}(L) &= \frac{z_0}{1 + cz_0L/2}\,, \\ \hat{x}(L) &= \frac{(b_2 - c)\hat{z}(L)}{b_1 - K\hat{z}(L)^{1-b_2/c}}\,, \end{aligned} \quad (27)$$

where $K$ should be chosen to satisfy $\hat{x}(0) = x_0$. We also need the formula

$$\int_0^L \hat{x}(t)dt = -2\ln\left[\frac{b_1\hat{z}(L)^{b_2/c-1} - K}{b_1 z_0^{b_2/c-1} - K}\right] . \tag{28}$$

For $b_2 > c$ (as is the case for (26)) the power of $\hat{z}(L)$ in the denominator of the expression for $\hat{x}(L)$ is negative so independently of the initial value $\hat{x}(0)$

$$\hat{x}(L) \to \beta\hat{z}(L) , \qquad \beta = \frac{b_2 - c}{b_1} \tag{29}$$

when $L \to \ln(\Lambda_{\mathrm{IR}}{}^2/v^2)$ (*i.e.* $\hat{z}(L) \to \infty$). The IR pole is obviously at

$$\Lambda_{\mathrm{IR}} = \exp\left(-\frac{1}{cz_0}\right) . \tag{30}$$

Plugging these solutions into the equations for $\hat{y}(L)$ we get in the same limit

$$\hat{y}(L) \to \alpha\hat{z}(L) , \qquad \alpha = \frac{-a_2\beta - c + \sqrt{(a_2\beta + c)^2 + 4a_1a_3\beta^2}}{2a_1} . \tag{31}$$

The effective potential has a correction due to the anomalous dimension; in the same limit it reads

$$W_{\mathrm{eff}}(L) \to \left(\frac{b_1\hat{z}(L)^{b_2/c-1} - K}{b_1 z_0^{b_2/c-1} - K}\right)^{2a} \alpha\hat{z}(L) . \tag{32}$$

Therefore the IR pole is a generic feature of this quasi-realistic system. The function $W(L)$ is convex due to the presence of two poles, and the symmetry breaking is unavoidable.

## 5. Numerics and normalization

As the solution of (24) is somewhat cumbersome, we have investigated it numerically for values (26) given by the Standard Model and a variety of values of the input parameters. In all cases, we find that the solution is either convex or unstable, in agreement with our general claim. Concentrating on the first case for its obvious physical interest a typical set of values is

$$x_0 = 0.120 , \qquad y_0 = 0.020 , \qquad z_0 = 0.249 . \tag{33}$$

For these parameters, we display the running coupling $\hat{y}(L)$ and the effective potential $W_{\mathrm{eff}}$ in Fig. 1 and Fig. 2, respectively; note that the scales for $L$ in the two figures are very different. As one can see, $\hat{y}(L)$ stays well behaved up to very large values $L \sim 100$. We thus see that the smallness of $\hat{y}(L)$ does not necessarily require the product $6\hat{y}(0)L$ to be small (near the Landau pole, we have $6\hat{y}(0)L \sim 15$), showing that the approximation can be trusted in spite of 'large logarithms'. On the IR side, we have a pole at $\ln(\Lambda_{\mathrm{IR}}/v) \sim -2.29$, while the minimum is located at $\langle L\rangle \sim -1.73$. This 'closeness' of the minimum and the IR barrier is the only non-realistic feature of our model. Our curves not only put in evidence the convexity of $W_{\mathrm{eff}}$ but also show that $\hat{y}(L)$ is very flat over a large range of values for $L$.

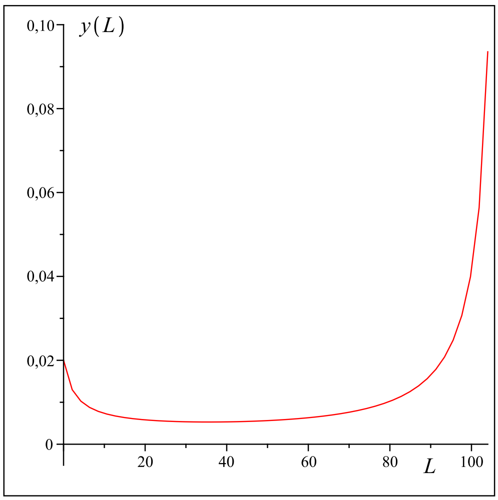


Fig. 1. The scalar self-coupling for the model (23) with (33).

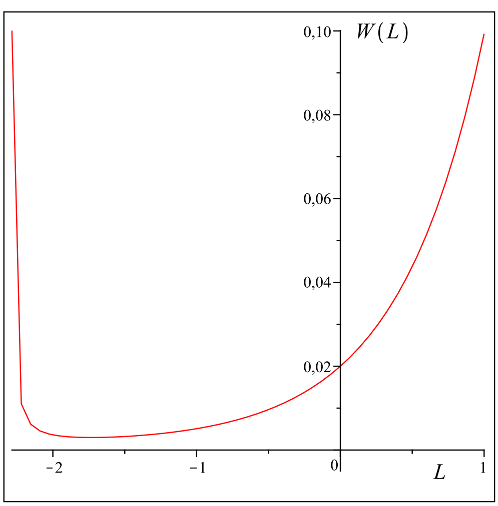


Fig. 2. The RG improved effective potential for (23) with (33).

Let us also comment on the question of how to normalize the couplings. Our choice (9) corresponds to normalizing all couplings at a fixed but arbitrary value $\varphi = v$ where $L = 0$, with the values $\hat{g}_j(0)$ as the input parameters from the classical Lagrangian. However, because the normalization parameter $v$ has no physical significance in itself, and having found a non-trivial minimum at $\varphi = \langle\varphi\rangle$, one would like to change this normalization *a posteriori* to one, where $v$ is traded for the actual value of $\varphi$ at the minimum (as is usually done for $\phi^4$ and scalar QED [1]). This is desirable in view of the fact that all physical quantities (masses, effective couplings) are obtained as derivatives of the effective potential at the minimum[6].

Quite generally, having determined $\langle\varphi\rangle$, we can evolve $\hat{y}(L)$ from $\hat{y}(0)$ to $\hat{y}(\langle L\rangle)$, and try to fix the latter 'backwards' to some given value by varying the input parameters. Alternatively, we can define the scalar self-coupling as the fourth derivative of $W_{\rm eff}$ at the minimum. Thanks to (1), (8) and (9), we can work out the relation between these two quantities explicitly. Neglecting the contributions from the anomalous rescaling, we get

$$\frac{1}{24\pi^2}\frac{d^4F(\varphi,L)}{d\varphi^4}\bigg|_{\varphi=\langle\varphi\rangle} = \hat{y}(\langle L\rangle) + \left\{\frac{25}{6}\frac{d\hat{y}(L)}{dL} + \frac{35}{6}\frac{d^2\hat{y}(L)}{d^2L} + \frac{10}{3}\frac{d^3\hat{y}(L)}{d^3L} + \frac{2}{3}\frac{d^4\hat{y}(L)}{d^4L}\right\}\bigg|_{L=\langle L\rangle}. \qquad (34)$$

Since, at the minimum, we have $(2\hat{y} + \hat{y}')|_{L=\langle L\rangle} = 0$, the difference could be appreciable, but there are cancellations since $\hat{y}' < 0$ while $\hat{y}'' > 4\hat{y}$ by convexity. Indeed, with the values (33) we find that the l.h.s. $\sim 0.156$, while $\hat{y}(\langle L\rangle) = 0.196$. We have checked for a range of input couplings that the difference between the two numbers does remain rather small.

Similar comments apply to the other couplings. Taking the Yukawa interaction as an example, the RG improved version of the corresponding term in the effective action takes the form

$$\mathit{\Gamma}_{\rm Y}(\varphi, q, \bar{q}) = h(L)\varphi\bar{q}q\,, \qquad (35)$$

where

$$h(L) \equiv \hat{g}_{\rm Y}(L)\exp\left[\int_0^L (\gamma(t) + 2\gamma_q(t))\,dt\right]. \qquad (36)$$

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[6] Clearly, it makes no sense to normalize the couplings at $\varphi = 0$, not only because the fourth derivative diverges at $\varphi = 0$ for the unimproved effective potential $V_{\rm eff}$, but also because this value is outside the domain of definition of $W_{\rm eff}$ if there are IR barriers.

As before, we can evolve this coupling from $\hat{g}_{\rm Y}(0)$ to $\hat{g}_{\rm Y}(\langle L\rangle)$ and compare with the relevant derivative. This gives

$$\begin{aligned} m_q &= \langle\varphi\rangle\,\hat{g}_{\rm Y}(\langle L\rangle)\,, \\ \frac{d}{d\varphi}\Big(\varphi h(L)\Big)\Big|_{\varphi=\langle\varphi\rangle} &= \hat{g}_{\rm Y}(\langle L\rangle) + 2\frac{d\hat{g}_{\rm Y}(L)}{d\varphi}\Bigg|_{L=\langle L\rangle}\,, \end{aligned} \tag{37}$$

respectively, for the fermion mass and the effective Yukawa coupling at the minimum. Again, we see that the difference between the latter and $\hat{g}_{\rm Y}(\langle L\rangle)$ can be small provided $\hat{g}'_{\rm Y}$ is small there. These considerations justify (to some extent) the approximation used [3, 5], where we effectively defined the couplings in terms of derivatives at the minimum of $V_{\rm eff}$, and took those values as the input parameters at $\varphi = \langle\varphi\rangle$ to analyze the evolution of couplings.

## 6. Discussion

Much of our discussion was based on exact solutions of the $\beta$-function equations, but the extension of the present considerations to more realistic examples is straightforward. In particular, the standard model with one Higgs doublet falls into the class of models investigated here, since there remains only one real scalar field after absorption of three scalar degrees of freedom into the massive vector vector bosons. Even when closed form solutions of the RG equations are no longer available, we can solve numerically for the running couplings and determine the RG improved effective potential from the general formulas (1) and (9). The latter can be analyzed numerically along the lines described here. As already pointed out, the only non-realistic feature here is the closeness of the minimum to the IR barrier $\Lambda_{\rm IR}$, which appears to be a generic feature of the one-field case. In realistic applications, on the other hand, we would have to arrange $\Lambda_{\rm IR} \sim \Lambda_{\rm QCD} = \mathcal{O}(1\,{\rm GeV})$ and $\langle\varphi\rangle = \mathcal{O}(200\,{\rm GeV})$. This may indicate the need for extra scalar fields (which are anyhow needed for the inclusion of massive neutrinos [3, 5–8]).

We believe that the present analysis strengthens the case for the applicability of the CW mechanism in a realistic context, if Landau poles can be shifted beyond the Planck scale. In addition, it puts in evidence the potential importance of IR poles in the scalar sector of the standard model, which may arise through the coupled RG equations. The phenomenon of the Higgs coupling becoming strong in the IR appears puzzling, and its physical consequences remain to be explored. Somewhat ironically, our analysis also shows that scalar QED, often cited as the showcase example of the CW mechanism, suffers from a potential IR instability (or at least from an IR breakdown of perturbation theory). It would be interesting to work out the consequences of the present results for cosmology, and in particular for models of scalar field inflation, where the effective potential (rather than the classical potential) should play a decisive role. We would not be surprised if our results can be used to rule out many of the currently popular 'designer potentials' for inflation.

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