Adv. Calc. Var. **2** (2009), 247–269

DOI 10.1515 / ACV.2009.010

Some applications of the isoperimetric inequality for integral varifolds

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Communicated by N. Fusco

Abstract. In this work the isoperimetric inequality for integral varifolds of locally bounded first variation is used to obtain sharp estimates for the size of the set where the density quotient is small and to generalise Calderón's and Zygmund's theory of first order differentiability for functions in Lebesgue spaces from Lebesgue measure to integral varifolds.

Keywords. Integral varifolds, locally bounded first variation, height-excess, tilt-excess.

AMS classification. Primary 49Q15; secondary 26B35.

Introduction

This work contributes to the study of weak notions of regularity for integral varifolds in an open subset of Euclidean space whose distributional first variation is given by either a Radon measure or by a function locally in L^p. As it is well known, see e.g. Allard [1, 8.1 (2)], the set where the support does not locally correspond to a submanifold of class $\mathcal{C}^1$ may have positive measure even if $p = \infty$. Therefore the notions of regularity studied here, and subsequently in [10] and [11], are decay rates of height-excess and tilt-excess near almost every point which provide a way to quantify the amount of flatness entailed by the conditions on the mean curvature. The main focus of the present paper is to investigate the effects of large mean curvature on the decay rates of the afore-mentioned excess quantities and also on the availability of an analogue for integral varifolds of Calderón's and Zygmund's theory of first order differentiability for functions in Lebesgue spaces. In both cases the results are accompanied by examples demonstrating the sharpness of the conditions on the mean curvature involved. The differentiability theory will turn out to be useful in the analysis of the decay rates.

Next, in order to precisely state the problem and the related results, some definitions are recalled mainly from Simon's book on geometric measure theory [17] which includes a list of basic notation on page (vii).

The author acknowledges financial support via the DFG Forschergruppe 469. The research was carried out at the University of Tübingen and put in its final form at the AEI Golm and the ETH Zürich. *AEI publication number:* AEI-2008-063.

Basic definitions and notation. To introduce the terminology, suppose $n, m \in \mathbb{N}$ and U is an open subset of $\mathbb{R}^{n+m}$. Using [17, Theorem 11.8] as a definition, μ is an integral n-varifold in U if and only if μ is a Radon measure on U and for μ almost all $x \in U$ there exists an approximate tangent plane $T_x\mu \in G(n+m, n)$ with multiplicity $\theta^n(\mu, x) \in \mathbb{N}$ of μ at x, $G(n+m, n)$ denoting the set of n-dimensional, unoriented planes in $\mathbb{R}^{n+m}$. The distributional first variation of mass of μ equals

$$(\delta\mu)(\eta) = \int \operatorname{div}_\mu \eta \, d\mu \quad \text{whenever } \eta \in C_c^1(U, \mathbb{R}^{n+m})$$

where $\operatorname{div}_\mu \eta(x)$ is the trace of $D\eta(x)$ with respect to $T_x\mu$. $\|\delta\mu\|$ denotes the total variation measure associated to $\delta\mu$ and μ is said to be of locally bounded first variation if and only if $\|\delta\mu\|$ is a Radon measure, in this case the generalised mean curvature vector $\vec{\mathbf{H}}_\mu(x) \in \mathbb{R}^{n+m}$ can be defined by the requirement

$$\vec{\mathbf{H}}_\mu(x) \bullet v = -\lim_{\varrho \downarrow 0} \frac{(\delta\mu)(\chi_{B_\varrho(x)} v)}{\mu(B_\varrho(x))} \quad \text{for } v \in \mathbb{R}^{n+m}$$

whenever these limits exist for $x \in U$; here $\bullet$ denotes the usual inner product on $\mathbb{R}^{n+m}$. Moreover, μ is said to satisfy condition (H_p), $1 \le p \le \infty$, if and only if it is of locally bounded first variation, $\vec{\mathbf{H}}_\mu \in L^p_{\mathrm{loc}}(\mu, \mathbb{R}^{n+m})$, and, in case $p > 1$, satisfies

$$(\delta\mu)(\eta) = -\int \vec{\mathbf{H}}_\mu \bullet \eta \, d\mu \quad \text{whenever } \eta \in C_c^1(U, \mathbb{R}^{n+m}). \qquad (H_p)$$

Also, adapting Anzellotti's and Serapioni's definition in [3], μ is called countably rectifiable of class $\mathscr{C}^2$, or for short $\mathscr{C}^2$-rectifiable, if and only if μ almost all of U can be covered by a countable collection of n-dimensional submanifolds of class $\mathscr{C}^2$.

The problem. The following questions arise.

(i) Suppose $n, m \in \mathbb{N}$, $1 \le p \le \infty$, $0 < \alpha \le 1$, and $1 \le q \le \infty$. Does the condition (H_p) on an integral n-varifold μ in U, U a nonempty, open subset of $\mathbb{R}^{n+m}$, imply

$$\limsup_{\varrho \downarrow 0} \varrho^{-1-\alpha-n/q} \| \operatorname{dist}(\cdot - x, T_x\mu) \|_{L^q(\mu \llcorner B_\varrho(x))} < \infty$$

for μ almost all $x \in U$?

(ii) Suppose $n, m \in \mathbb{N}$, $1 \le p \le \infty$, $0 < \alpha \le 1$, and $1 \le q \le \infty$. Does the condition (H_p) on an integral n-varifold μ in U, U a nonempty, open subset of $\mathbb{R}^{n+m}$, imply

$$\limsup_{\varrho \downarrow 0} \varrho^{-\alpha-n/q} \| T.\mu - T_x\mu \|_{L^q(\mu \llcorner B_\varrho(x))} < \infty$$

for μ almost all $x \in U$? Here $S \in G(n+m, n)$ is identified with the element of $\operatorname{Hom}(\mathbb{R}^{n+m}, \mathbb{R}^{n+m})$ given by the orthogonal projection of $\mathbb{R}^{n+m}$ onto S.

Clearly, the two questions are related by Caccioppoli type inequalities, see e.g. Brakke [4, 5.5], at least in the case $q = 2$ where the quantities considered agree with the classical height-excess and tilt-excess. Also note that an affirmative answer to one of the questions with α, q implies an affirmative answer to the same question for any $0 < \alpha' \leq 1$, $q < q' < \infty$ such that $\alpha q = \alpha' q'$ by use of the trivial L^∞ bounds of the functions involved. The case $\alpha = 1$ is of particular interest in both questions. A varifold satisfying the decay estimate in the first question with $\alpha = 1$ and $q = 1$ is $\mathscr{C}^2$-rectifiable, see Schätzle [16, Appendix A]. In the second question the case $\alpha = 1$ is related to the local computability of the mean curvature vector from the geometry of $\{x \in U : \theta^n(\mu, x) \geq 1\}$, see Schätzle in [14, Lemma 6.3] or [15, Proposition 6.1] or [16, Theorem 4.1]. On the other hand the quantity αq to some extend determines how well μ can be approximated by multivalued graphs near generic points, see the forthcoming paper [10]. Such kind of approximation of integral varifolds has been fundamental for regularity investigations, for example, in the work of Almgren in [2], Brakke in [4] and Schätzle in [15, 16]. It was introduced by Almgren in [2, 3.1–3.12] and extended by Brakke in [4, 5.4].

Known results. Brakke answers both questions in the affirmative for any n and m in case

$$\text{either} \quad p = 1, \ \alpha = 1/2, \ q = 2 \qquad \text{or} \quad p = 2, \ \alpha < 1, \ q = 2$$

in [4, 5.7]. Schätzle provides a positive answer in the case

$$m = 1, \ \ p > n, \ \ p \geq 2, \ \ \alpha = 1, \ \ q = \infty$$

for the first question and in the case

$$m = 1, \ \ p > n, \ \ p \geq 2, \ \ \alpha = 1, \ \ q = 2$$

for the second question, see [15, Proposition 4.1, Theorem 5.1]. Moreover, in subsequent work Schätzle showed for arbitrary dimensions that the decay rates occurring in the two questions hold if

$$p = 2, \alpha = 1, q = 2$$

provided μ is additionally assumed to be $\mathscr{C}^2$-rectifiable. Also note that Brakke's example in [4, 6.1] shows that the answer to the second question is in the negative for any m, p and α if $n \geq 2$ and $q = \infty$.

Results of the present paper. First, it is shown by an example of a unit density, $\mathscr{C}^2$-rectifiable n-varifold in $\mathbb{R}^{n+1}$ that the answers to both questions are in the negative if $p < n$ and $\alpha q > np/(n-p)$, see 1.2. In particular, in case $1 \leq p < 2n/(n+2)$ proving appropriate decay for the classical height-excess or tilt-excess, i.e. answering the first or second question in the affirmative for $\alpha = 1$, $q = 2$, cannot serve as an intermediate step in studying $\mathscr{C}^2$-rectifiability or local computability of the mean curvature vector. This was the original motivation to consider exponents $q \neq 2$.

Second, in order to provide new cases where the questions are answered in the affirmative, it will turn out to be useful in [11] to have a theory of first order differentiability for functions in $L^q(\mu)$, μ an integral n-varifold, similar to the one developed by Calderón and Zygmund in [5] for $L^q(\mathscr{L}^n)$, at one's disposal. Also, whenever the decay condition occurring in the second question with $\alpha = 1$, $q < \infty$ holds for some μ satisfying condition (H_p) for some p with $q \leq np/(n-p)$ if $p < n$, one may apply the Rademacher type result 3.9 to obtain a differential of $T.\mu$ in an $L^q(\mu)$ sense μ almost everywhere on U. In particular, this covers all cases for which the answer to the second question with $\alpha = 1$ might be in the affirmative.

The key to carry over the theory from the Lebesgue measure case to the case of integral varifolds is the following differentiation theorem which corresponds to [5, Theorem 10 (ii)] by Calderón and Zygmund but whose proof uses techniques employed by Mickle and Radó in [12, Theorem 1] and [13, Section 5], see also Federer [6, 2.9.17].

Theorem 3.1. *Suppose $m \in \mathbb{N}_0$, $n \in \mathbb{N}$, $1 \leq p \leq n$, U is an open subset of $\mathbb{R}^{n+m}$, μ is an integral n-varifold in U satisfying condition (H_p), ν measures U with $\nu(U \sim \operatorname{spt}\mu) = 0$, A is μ-measurable with $\nu(A) = 0$, and $1 \leq q < \infty$. In case $p < n$ additionally suppose for some $1 \leq r \leq \infty$ and some nonnegative function $f \in L^r_{\mathrm{loc}}(\mu)$ that*

$$\nu = f\mu \quad \text{and} \quad q \leq 1 + (1 - 1/r)\frac{p}{n-p}.$$

Then for $\mathscr{H}^n$ almost all $a \in A$

$$\limsup_{s \downarrow 0} \nu(\bar{B}_s(a)) \big/ s^{nq} \quad \text{equals either } 0 \text{ or } \infty.$$

The bound on q is sharp as demonstrated in 3.3, 3.4. Its occurrence is due to the fact that in case $p < n$ the number $n^2/(n-p)$ in the following proposition cannot be replaced by any larger number, see 1.2: *Suppose $m \in \mathbb{N}_0$, $n \in \mathbb{N}$, $1 \leq p < n$, μ is an integral n-varifold in $\mathbb{R}^{n+m}$ satisfying condition (H_p), then for μ almost all $a \in U$ there exists $\varepsilon > 0$ such that*

$$\lim_{r \downarrow 0} \frac{\mu\big(\bar{B}_r(a) \sim \{x : \mu(\bar{B}_\varrho(x)) \geq c_n \varrho^n \text{ for } 0 < \varrho < \varepsilon\}\big)}{r^{n^2/(n-p)}} = 0$$

where c_n is a positive, finite number depending only on n, see 2.9, 2.10. Similar propositions with $n^2/(n-p)$ replaced by any slightly smaller number can be obtained by use of [17, Theorem 17.6], an inequality derived via integration of the monotonicity formula. The optimal exponent is derived using the isoperimetric inequality. All these results will be proven under the weaker condition $\theta^n(\mu, x) \geq 1$ for μ almost every $x \in U$ replacing the integrality condition on μ.

Organisation of the paper. In the first section the example is constructed. In the second section the isoperimetric inequality is used to derive some sharp bounds on the size of the set where the n-density ratio is small and in the last section a theory of first order differentiation in Lebesgue spaces defined with respect to a varifold is presented.

Note that with the exception of 3.6–3.10 the work was part of the author's PhD thesis, see [9].

1 An example concerning height and tilt decays of integral varifolds

In this section a family of integral n-varifolds with prescribed decay rates of height and tilt quantities is constructed. In fact, the decay rate for tilt can be arranged to be slightly larger than the one of the height with the same exponent. However, this feature will only become relevant in [10].

Definition 1.1. Suppose $k \in \mathbb{N}$, $x \in \mathbb{R}^k$ and $0 < \varrho < \infty$.

Then $Q_\varrho(x) := \{y \in \mathbb{R}^k : |y_i - x_i| < \varrho \text{ for } i = 1, \dots, k\}$. To avoid ambiguity, $Q^k_\varrho(0)$ will be written instead of $Q_\varrho(0)$.

Example 1.2. Suppose $n \in \mathbb{N}$, $1 \le p < n$, $0 < \alpha_i \le 1$, $1 \le q_i < \infty$ for $i \in \{1, 2\}$, such that

$$\alpha_2 q_2 \le \alpha_1 q_1, \qquad \frac{1}{p} > 1 + \frac{\alpha_2 q_2}{\alpha_1 q_1}\Big(\frac{1}{n} + \frac{1}{\alpha_2 q_2} - 1\Big).$$

In case $\alpha_1 q_1 = \alpha_2 q_2$ the last condition reads $\alpha_2 q_2 > np/(n-p)$.

Then there exists an integral n-varifold μ in $\mathbb{R}^{n+1}$, $T \in G(n+1, n)$ and $0 < \Gamma < \infty$ with the following properties:

(i) The support of μ equals the disjoint union of T and an n-dimensional submanifold of $\mathbb{R}^{n+1}$ of class $\mathscr{C}^\infty$.

(ii) There holds $\theta^n(\mu, x) = 1$ for $x \in \operatorname{spt} \mu$ and $T_x \mu = T$ for $x \in T$.

(iii) There holds $\vec{\mathbf{H}}_\mu \in L^p_{\mathrm{loc}}(\mu, \mathbb{R}^{n+1})$ and $(\delta\mu)(\eta) = -\int \vec{\mathbf{H}}_\mu \bullet \eta \, d\mu$ whenever $\eta \in C^1_{\mathrm{c}}(\mathbb{R}^{n+1}, \mathbb{R}^{n+1})$.

(iv) Whenever $x \in T$ and $0 < \varrho \le 1$

$$\Gamma^{-1} \varrho^{\alpha_2 q_2} \le \varrho^{-n} \mu(\{\xi \in \bar{B}_\varrho(x) : \operatorname{dist}(\xi - x, T) \ge \varrho/\Gamma\}),$$

$$\varrho^{-n} \mu(\bar{B}_\varrho(x) \sim T) \le \Gamma \varrho^{\alpha_2 q_2},$$

$$\varrho^{-1-n/q_2} \Big(\int_{\bar{B}_\varrho(x)} \operatorname{dist}(\xi - x, T_x\mu)^{q_2} \, d\mu(\xi) \Big)^{1/q_2} \approx \varrho^{\alpha_2},$$

$$\varrho^{-n/q_1} \Big(\int_{\bar{B}_\varrho(x)} |T_\xi \mu - T_x \mu|^{q_1} \, d\mu(\xi) \Big)^{1/q_1} \approx \varrho^{\alpha_1},$$

here $a \approx b$ means that $a \le \Gamma_1 b$ and $b \le \Gamma_1 a$ for some positive, finite number Γ_1 depending only on n, and α_i, q_i for $i \in \{1, 2\}$.

(v) Whenever $1 < r < \infty$, $n + (1 - 1/r)\alpha_2 q_2 < s < \infty$ there exists a nonnegative function $f \in L^r_{\mathrm{loc}}(\mu)$ such that $f(x) = 0$ for $x \in T$, and

$$\varrho^s \approx \int_{\bar{B}_\varrho(x)} f \,\mathrm{d}\mu \quad \text{whenever } x \in T, 0 < \varrho \le 1,$$

here $a \approx b$ means $a \le \Gamma_2 b$ and $b \le \Gamma_2 a$ for some positive, finite number Γ_2 depending only on n and s.

Construction of example. Let $a := \alpha_2 q_2/n + 1$, $b := (\alpha_1 q_1 - \alpha_2 q_2)/a + 1 \ge 1$. Define for $i \in \mathbb{N}_0$

$$W_i := \big\{Q_{2^{-i-2}}(x) : 2^{i+1}x \in \mathbb{Z}^n\big\}.$$

Clearly, $\bigcup_{Q \in W_i} \overline{Q} = \mathbb{R}^n$ and W_i are pairwise disjoint. Let

$$F_i := \big\{]2^{-i-1}, 2^{-i}[\times W : W \in W_i\big\} \quad \text{for } i \in \mathbb{N}_0, \quad F := \bigcup_{i \in \mathbb{N}_0} F_i.$$

Clearly, $\bigcup_{S \in F} \overline{S} =]0, 1] \times \mathbb{R}^n$ and F is pairwise disjoint. Let $T := \{0\} \times \mathbb{R}^n$.

Next, it will be indicated how to construct for every $0 < \sigma \le \varrho < \infty$ a compact n-dimensional submanifold M of $\mathbb{R}^{n+1}$ of class $\mathscr{C}^\infty$ such that

$$M \subset Q^{n+1}_\varrho(0), \quad (\Gamma_0)^{-1}\varrho^n \le \mathscr{H}^n(M) \le \Gamma_0 \varrho^n, \quad |\vec{\mathbf{H}}_M| \le \Gamma_0 \sigma^{-1},$$

$$\mathscr{H}^n(\{x \in M : |T_x M - T| \ge 1\}) \ge (\Gamma_0)^{-1}\sigma\varrho^{n-1},$$

$$\mathscr{H}^n(\{x \in M : \vec{\mathbf{H}}_M(x) \ne 0 \text{ or } T_x M \ne T\}) \le \Gamma_0 \sigma \varrho^{n-1}$$

where $\vec{\mathbf{H}}_M$ denotes the mean curvature vector of M and Γ_0 is a positive, finite number depending only on n. To construct M, one may assume $\varrho = 1$. Choose a concave function $f : [-1/2, 1/2] \to [0, 1]$ and $0 < \Gamma_1 < \infty$ such that

$$f(-1/2) = \sigma/4 = f(1/2),$$

$$f(s) = \sigma/2 \quad \text{whenever } s \in [-1/2 + \sigma/4, 1/2 - \sigma/4]$$

and such that

$$N := \{(s, t) \in [-1/2, 1/2] \times \mathbb{R} : |t| = f(s)\} \cup (\{-1/2, 1/2\} \times [-\sigma/4, \sigma/4])$$

is a one-dimensional submanifold of class $\mathscr{C}^\infty$ with $|\vec{\mathbf{H}}_N| \le \Gamma_1 \sigma^{-1}$. Noting

$$\mathscr{H}^1(\text{graph } f|[-1/2, -1/2+\sigma/4] \cup [1/2-\sigma/4, 1/2]) \le \sigma$$

by [6, 3.2.27], one can take

$$M := \{(y,z) \in \mathbb{R} \times \mathbb{R}^n : (|z|, y) \in N\}.$$

For each $i \in \mathbb{N}_0$ and $Q \in F_i$ choose an n-dimensional submanifold M_Q of the type just constructed corresponding to $\varrho_i := 2^{-ia-2}$, $\sigma_i := 2^{-iba-2}$ contained in Q and let M be the union of those submanifolds. Take $\mu := \mathscr{H}^n \llcorner (T \cup M)$. (i) is now evident.

To prove the estimates, fix $x \in T$ and define for $i, j \in \mathbb{N}_0$

$$b_{i,j} := \#\big\{Q \in F_j : Q \cap Q_{2^{-i}}(x) \ne \emptyset\big\}, \quad c_{i,j} := \#\big\{Q \in F_j : Q \subset Q_{2^{-i}}(x)\big\}.$$

Clearly, $b_{i,j} = c_{i,j} = 0$ if $j < i$. If $j \ge i$, one estimates

$$b_{i,j} \le \big(2^{j-i+2}+1\big)^n \le \big(5 \cdot 2^{j-i}\big)^n, \quad c_{i,j} \ge \big(2^{j-i+2}-1\big)^n \ge \big(3 \cdot 2^{j-i}\big)^n.$$

One calculates

$$\mu(Q_{2^{-i}}(x) \sim T) \le \sum_{j=0}^{\infty} b_{i,j} \Gamma_0 (\varrho_j)^n \le (5/4)^n \Gamma_0 (2^{-i})^{an} (1 - 2^{n(1-a)})^{-1},$$

$$n - ba(1-p) + (1-n)a = -\alpha_1 q_1 + p(\alpha_1 q_1 - \alpha_2 q_2 + \alpha_2 q_2/n + 1) < 0,$$

$$\begin{aligned} &\int_{Q_{2^{-i}}(x) \sim T} |\vec{\mathbf{H}}_M|^p \, \mathrm{d}\mu \\ &\quad \le \sum_{j=0}^{\infty} b_{i,j} (\Gamma_0)^{p+1} (\sigma_j)^{1-p} (\varrho_j)^{n-1} \\ &\quad \le 5^n (\Gamma_0)^{p+1} (2^{-i})^{ba(1-p)+(n-1)a} (1 - 2^{n-ba(1-p)+(1-n)a})^{-1} < \infty, \end{aligned}$$

$$\int_{Q_{2^{-i}}(x)} \operatorname{dist}(\xi - x, T)^{q_2} \, \mathrm{d}\mu(\xi) \le 2^{-iq_2} \mu(Q_{2^{-i}}(x) \sim T),$$

$$\begin{aligned} &\int_{Q_{2^{-i}}(x)} |T_\xi \mu - T|^{q_1} \, \mathrm{d}\mu(\xi) \\ &\qquad \le (2n)^{q_1} \sum_{j=0}^{\infty} b_{i,j} \Gamma_0 \sigma_j (\varrho_j)^{n-1} \\ &\qquad \le (2n)^{q_1} (5/4)^n \Gamma_0 (2^{-i})^{ba+a(n-1)} (1 - 2^{n-ba-a(n-1)})^{-1}, \end{aligned}$$

$$
\begin{aligned}
2^{(i+1)q_2} \int_{Q_{2^{-i}}(x)} & \operatorname{dist}(\xi - x, T)^{q_2} \, d\mu(\xi) \\
& \geq \mu(\{\xi \in Q_{2^{-i}}(x) : \operatorname{dist}(\xi - x, T) \geq 2^{-i-1}\}) \\
& \geq (\Gamma_0)^{-1} (\varrho_i)^n = (4^n \Gamma_0)^{-1} 2^{-ian},
\end{aligned}
$$

$$
\int_{Q_{2^{-i}}(x)} |T_\xi \mu - T|^{q_1} \mu(\xi) \geq (\Gamma_0)^{-1} \sigma_i (\varrho_i)^{n-1} = (4^n \Gamma_0)^{-1} (2^{-i})^{ab + a(n-1)}.
$$

Therefore (iii) and (iv) are proven and the second estimate of (iv) implies (ii).

To prove (v), define f by $f(y) := 2^{(na-s)i}$ if $y \in \bigcup_{S \in F_i} S$ for some $i \in \mathbb{N}_0$ and $f(y) = 0$ else. Then for $i \in \mathbb{N}_0$

$$
\begin{aligned}
\int_{Q_{2^{-i}}(x)} |f| \, d\mu & \leq \sum_{j=0}^{\infty} b_{i,j} 2^{(na-s)j} \Gamma_0 (\varrho_j)^n \leq (5/4)^n \Gamma_0 (2^{-i})^s (1 - 2^{n-s})^{-1}, \\
\int_{Q_{2^{-i}}(x)} |f|^r \, d\mu & \leq \sum_{j=0}^{\infty} b_{i,j} 2^{(na-s)rj} \Gamma_0 (\varrho_j)^n \\
& \leq (5/4)^n \Gamma_0 (2^{-i})^{(s-na)r+na} (1 - 2^{n+(na-s)r-na})^{-1} < \infty
\end{aligned}
$$

because

$$
n + (na - s)r - an = \alpha_2 q_2 (r - 1) + r(n - s) < 0.
$$

The estimate from below is similar to the one from above. □

Remark 1.3. The integral n-varifold μ constructed depends only on n and the products $\alpha_i q_i$ for $i \in \{1, 2\}$. Moreover, the assumption $\alpha_i \leq 1$ for $i \in \{1, 2\}$ could be replaced by $\alpha_i < \infty$ for $i \in \{1, 2\}$.

Remark 1.4. Taking $p = 1$, $\alpha_1 = \alpha_2$, and $q_1 = q_2 = 2$ in the last two estimates of (iv) shows that for every $n \in \mathbb{N}$, $n > 1$, $1/2 + (2(n-1))^{-1} < \alpha \leq 1$ there exists an integral n-varifold μ of $\mathbb{R}^{n+1}$ of locally bounded first variation such that for some A with $\mu(A) > 0$

$$
\lim_{\varrho \downarrow 0} \varrho^{-2\alpha} \operatorname{heightex}_\mu(x, \varrho, T_x \mu) = \infty, \quad \lim_{\varrho \downarrow 0} \varrho^{-2\alpha} \operatorname{tiltex}_\mu(x, \varrho, T_x \mu) = \infty
$$

for $x \in A$. In [4, 5.7] Brakke showed in arbitrary codimension that the above limits equal 0 almost everywhere with respect to μ if $\alpha = 1/2$.

Remark 1.5. Similarly to the preceding remark, taking $\alpha_1 = \alpha_2 = 1$, $q_1 = q_2 = q$ and noting (i), one obtains for every $p^* = np/(n-p) < q < \infty$ an integral n-varifold μ satisfying condition (H_p) which is countably rectifiable of class $\mathscr{C}^2$ such that for some A with $\mu(A) > 0$

$$\lim_{\varrho \downarrow 0} \varrho^{-2-n/q} \Big(\int_{B_\varrho(x)} \operatorname{dist}(\xi - x, T_x\mu)^q \, \mathrm{d}\mu(\xi) \Big)^{1/q} = \infty,$$
$$\lim_{\varrho \downarrow 0} \varrho^{-1-n/q} \Big(\int_{B_\varrho(x)} |T_\xi\mu - T_x\mu|^q \, \mathrm{d}\mu(\xi) \Big)^{1/q} = \infty$$

for $x \in A$. In particular, if $p < 2n/(n+2)$ then countable rectifiability of class $\mathscr{C}^2$ does not imply quadratic decay of neither $\operatorname{tiltex}_\mu$ nor $\operatorname{heightex}_\mu$. If $p = 2$, countable rectifiability of class $\mathscr{C}^2$ is equivalent to quadratic decay of both quantities, see [16, Theorem 3.1] by Schätzle.

2 The size of the set where the *n*-density quotient is small

In this section the isoperimetric inequality is used to derive basic facts on the size of the set where the n-density quotient is small. Although the general procedure of such estimates is clearly known, see 2.5, it appears to be rarely used in literature. The sharpness of the results is necessary to determine the precise limiting exponent up to which the differentiation theory in the next section can be developed. Similarly, the accuracy of the bounds obtained in [10] depends on the results of this section.

2.1. The following situation will be studied:

Suppose $m \in \mathbb{N}_0$, $n \in \mathbb{N}$, $1 \le p \le n$, U is an open subset of $\mathbb{R}^{n+m}$, μ is a rectifiable n-varifold[1] in U of locally bounded first variation, $\theta^n(\mu, x) \ge 1$ for μ almost all $x \in U$, and, in case $p > 1$,

$$(\delta\mu)(\eta) = -\int \vec{\mathbf{H}}_\mu \bullet \eta \, \mathrm{d}\mu \quad \text{whenever } \eta \in C^1_{\mathrm{c}}(U, \mathbb{R}^{n+m})$$

and $\vec{\mathbf{H}}_\mu \in L^p_{\mathrm{loc}}(\mu, \mathbb{R}^{n+m})$. In doing so, the following abbreviation will be used:

$$\psi = \|\delta\mu\| \quad \text{if } p = 1, \qquad \psi = |\vec{\mathbf{H}}_\mu|^p \mu \quad \text{else.}$$

Theorem 2.2 (Isoperimetric inequality for varifolds). *Suppose $m \in \mathbb{N}_0$, $n \in \mathbb{N}$, μ is a rectifiable n-varifold in $\mathbb{R}^{n+m}$ with $\mu(\mathbb{R}^{n+m}) < \infty$ and $\|\delta\mu\|(\mathbb{R}^{n+m}) < \infty$.*

Then for some positive, finite number γ depending only on n

$$\mu\big(\{x \in \mathbb{R}^{n+m} : \theta^n(\mu, x) \ge 1\}\big) \le \gamma\, \mu(\mathbb{R}^{n+m})^{1/n} \|\delta\mu\|(\mathbb{R}^{n+m}).$$

[1] Note that a definition of a rectifiable n-varifold results from the definition of an integral n-varifold through replacement of the condition $\theta^n(\mu, x) \in \mathbb{N}$ by $0 < \theta^n(\mu, x) < \infty$.

Proof. This follows from [1, Theorem 7.1] by Allard with a constant γ depending on $n + m$ (which would be sufficient for the purpose of this work). A slight modification of [17, Lemma 18.7, Theorem 18.6] yields the stated result. □

Definition 2.3. For $n \in \mathbb{N}$ let γ_n denote the best constant γ in 2.2.

Remark 2.4. Taking $m = 0$, $\mu = \mathscr{L}^n \llcorner \bar{B}_1^n(0)$ yields

$$\gamma_n \geq \omega_n^{-1/n}/n.$$

Does equality hold?

2.5. An important consequence of the isoperimetric inequality 2.2 and the starting point for the estimates in the present section is the following fact which can be derived by a variant of [6, 5.1.6] or Allard [1, 8.3], see Leonardi and Masnou [8, Proposition 3.1] or [9, A.8, A.9].

Suppose n, m, $p = 1$, $U = B_r(a)$ for some $a \in \mathbb{R}^{n+m}$ and $0 < r < \infty$, and μ are as in 2.1, $a \in \operatorname{spt} \mu$, $0 < \varepsilon < \gamma_n^{-1}$, and

$$\|\delta\mu\|(\bar{B}_\varrho(a)) \leq \varepsilon\, \mu(\bar{B}_\varrho(a))^{1-1/n} \quad \textit{whenever } 0 < \varrho < r,$$

then

$$\mu(\bar{B}_\varrho(a)) \geq ((\gamma_n^{-1} - \varepsilon)/n)^n \varrho^n \quad \textit{whenever } 0 < \varrho < r.$$

Also note, if $p = n > 1$ or $p = n = 1$ and $\|\delta\mu\|(\{a\}) < \varepsilon$, then

$$\|\delta\mu\|(\bar{B}_\varrho(x)) \leq \varepsilon\, \mu(\bar{B}_\varrho(x))^{1-1/n}$$

whenever $0 < \varrho < r$, $x \in \operatorname{spt} \mu \cap \bar{B}_r(a)$ is satisfied for all sufficiently small positive radii r. In the present paper the preceding statements will only be applied with $\varepsilon = (2\gamma_n)^{-1}$, hence $(\gamma_n^{-1} - \varepsilon)/n = (2n\gamma_n)^{-1}$.

Lemma 2.6. *Suppose $m \in \mathbb{N}_0$, $n \in \mathbb{N}$, and $\delta > 0$.*

Then there exists a positive number ε with the following property.

If $a \in \mathbb{R}^{n+m}$, $0 < r < \infty$, m, n, p, U, and μ are related as in 2.1 with $U = B_r(a)$, $p = 1$, $a \in \operatorname{spt} \mu$, and

$$\|\delta\mu\|(\bar{B}_\varrho(a)) \leq (2\gamma_n)^{-1} \mu(\bar{B}_\varrho(a))^{1-1/n} \quad \textit{for } 0 < \varrho < r,$$
$$\|\delta\mu\|(B_r(a)) \leq \varepsilon\, \mu(B_r(a))^{1-1/n},$$

then

$$\mu(B_r(a)) \geq (1 - \delta)\omega_n r^n.$$

Proof. A variant of the argument used in [1, 8.4] by Allard with 2.5 replacing [1, 8.3] yields the stated result. □

Remark 2.7. The following proposition is implied by 2.6.

If m, n, p, U, μ and ψ are as in 2.1, $p = n$, then

$$\theta_*^n(\mu, a) \geq 1 \quad \textit{whenever } a \in \operatorname{spt} \mu \textit{ and } \psi(\{a\}) = 0.$$

Clearly, the condition $\psi(\{a\}) = 0$ is redundant in case $\|\delta\mu\|$ is absolutely continuous with respect to μ (i.e. $\delta\mu$ has no singular part with respect to μ). In case $n = 2$, this was proven together with existence and upper semicontinuity of $\theta^2(\mu, \cdot)$ in [7, Appendix A] by Kuwert and Schätzle. It is not known to the author whether or not the latter two properties hold for general n.

Definition 2.8. For $k \in \mathbb{N}$ denote by $N(k)$ the best constant in Besicovitch's covering theorem in $\mathbb{R}^k$.

Theorem 2.9. *Suppose m, n, p, U, μ, and ψ are as in 2.1, $p < n$, $0 \leq s < \infty$, $0 < \varepsilon \leq (2\gamma_n)^{-p/(n-p)}$, $4\gamma_n n < \Gamma < \infty$,*

$$A = \left\{ x \in U : \theta^{*n-p}(\psi, x) < (\varepsilon/\Gamma)^{n-p}/\omega_{n-p} \right\},$$

denote by B_i for $i \in \mathbb{N}$ the set of all $x \in U$ such that either $\bar{B}_{1/i}(x) \not\subset U$ or

$$\psi(\bar{B}_\varrho(x)) > \varepsilon^{n-p}\, \mu(\bar{B}_\varrho(x))^{1-p/n} \quad \textit{for some } 0 < \varrho < 1/i,$$

and denote by X_i for $i \in \mathbb{N}$ the set of all $a \in U$ such that

$$\lim_{r \downarrow 0} \mu\big(B_i \cap \bar{B}_r(a)\big)/r^{sn/(n-p)} = 0.$$

Then $\left\{ x \in B_i : \bar{B}_{1/i}(x) \subset U \right\}$ are open sets, X_i are Borel sets and

$$\mathscr{H}^s\Big(A \sim \bigcup_{i \in \mathbb{N}} X_i\Big) = 0.$$

Proof. Clearly, $B_{i+1} \subset B_i$, $X_i \subset X_{i+1}$ and X_i is a Borel set for $i \in \mathbb{N}$. The sets $\{x \in B_i : \bar{B}_{1/i}(x) \subset U\}$ are open, as may be obtained by adapting [6, 2.9.14].

Define for $i \in \mathbb{N}$ the set A_i of all $x \in U$ such that $B_{1/i}(x) \subset U$ and

$$\psi(\bar{B}_\varrho(x)) \leq (\varepsilon/\Gamma)^{n-p} \varrho^{n-p} \quad \text{whenever } 0 < \varrho < 1/i.$$

The sets A_i are closed (cp. [6, 2.9.14]) and satisfy $A \subset \bigcup_{i \in \mathbb{N}} A_i$. Let C denote the set of all $x \in \operatorname{spt} \mu$ such that

$$\limsup_{\varrho \downarrow 0} \frac{\psi(\bar{B}_\varrho(x))}{\mu(\bar{B}_\varrho(x))^{1-p/n}} < \varepsilon^{n-p}$$

and note $\mu(U \sim C) = 0$ by [6, 2.9.5]. By [6, 2.10.6, 2.10.19 (4)] it is enough to prove $a \in X_{2i}$ for a point $a \in A_i$ with $\theta^s(\psi \llcorner U \sim A_i, a) = 0$.

For this purpose the following assertion will be proven. *For each* $x \in B_{2i} \cap B_{1/(2i)}(a) \cap C$ *there exists* $0 < \varrho < \infty$ *with*

$$\bar{B}_\varrho(x) \subset B_{2|x-a|}(a) \sim A_i, \quad \mu(\bar{B}_\varrho(x)) < \varepsilon^{-n} \psi(\bar{B}_\varrho(x))^{n/(n-p)}.$$

Choose $y \in A_i$ with $|y - x| = \operatorname{dist}(x, A_i)$ and let J be the set of all $0 < \varrho < 1/(2i)$ with

$$\mu(\bar{B}_\varrho(x)) < \varepsilon^{-n} \psi(\bar{B}_\varrho(x))^{n/(n-p)}.$$

Then $J \neq \emptyset$, because $x \in B_{2i}$, $\bar{B}_{1/(2i)}(x) \subset B_{1/i}(a) \subset U$, and, since $x \in C$, $\inf J > 0$. Therefore $t := \inf J$ satisfies

$$0 < t < 1/(2i), \quad \mu(\bar{B}_t(x)) \le \varepsilon^{-n} \psi(\bar{B}_t(x))^{n/(n-p)},$$
$$\mu(\bar{B}_\varrho(x)) \ge \varepsilon^{-n} \psi(\bar{B}_\varrho(x))^{n/(n-p)} \quad \text{for } 0 < \varrho < t.$$

Noting

$$|y - x| = \operatorname{dist}(x, A_i) \le |x - a| \le 1/(2i), \quad t + |y - x| < 1/i,$$
$$\bar{B}_t(x) \subset \bar{B}_{t+|y-x|}(y) \subset B_{1/i}(y) \subset U,$$

one estimates

$$\begin{aligned} \psi(\bar{B}_t(x))^{n/(n-p)} &\le \psi(\bar{B}_{t+|y-x|}(y))^{n/(n-p)} \\ &\le (\varepsilon/\Gamma)^n (t + |y - x|)^n < \varepsilon^n 2^{-n} (1 + |y - x|/t)^n (2n\gamma_n)^{-n} t^n \end{aligned}$$

and, using the inequalities derived from the definition of t and 2.5,

$$\mu(\bar{B}_t(x)) \le \varepsilon^{-n} \psi(\bar{B}_t(x))^{n/(n-p)} < 2^{-n} (1 + |y - x|/t)^n \mu(\bar{B}_t(x)),$$

hence

$$(1 + |y - x|/t)^n > 2^n, \quad |y - x| > t$$

and the assertion follows by taking $\varrho \in J$ slightly larger than t.

Let $0 < r < 1/(2i)$. Then the preceding assertion in conjunction with Besicovitch's covering theorem implies the existence of countable, pairwise disjoint collections of closed balls $F_1, \dots, F_{N(n+m)}$ satisfying

$$B_{2i} \cap \bar{B}_r(a) \cap C \subset \bigcup_{j=1}^{N(n+m)} \bigcup_{S \in F_j} S \subset B_{2r}(a) \sim A_i,$$

$$\mu(S) < \varepsilon^{-n} \psi(S)^{n/(n-p)} \quad \text{for } S \in \bigcup_{j=1}^{N(n+m)} F_j.$$

Hence

$$\begin{aligned}\mu(B_{2i} \cap \bar{B}_r(a)) &= \mu(B_{2i} \cap \bar{B}_r(a) \cap C)\\ &\le \sum_{j=1}^{N(n+m)} \sum_{S \in F_j} \mu(S) \le \varepsilon^{-n} \sum_{j=1}^{N(n+m)} \sum_{S \in F_j} \psi(S)^{n/(n-p)}\\ &\le \varepsilon^{-n} \sum_{j=1}^{N(n+m)} \Big(\sum_{S \in F_j} \psi(S)\Big)^{n/(n-p)}\\ &\le \varepsilon^{-n} N(n+m) \psi(B_{2r}(a) \sim A_i)^{n/(n-p)}\end{aligned}$$

and the conclusion follows by taking the limit $r \downarrow 0$. □

Remark 2.10. This theorem deserves some explanations.

First, note that *if* $\|\delta\mu\|$ *is absolutely continuous with respect to* μ, *then*

$$\mathscr{H}^{n-p}(U \sim A) = 0$$

and *if* $p = 1$, *then*

$$\mathscr{H}^{n-1}(X \sim A) \le (\Gamma/\varepsilon)^{n-1} \omega_{n-1} \|\delta\mu\|(X \sim A) \quad \textit{for } X \subset U$$

by [6, 2.10.6, 2.10.19 (3)]. These estimates for the size of $U \sim A$ suggest that the theorem is most useful if $n - p \le s \le n$.

Clearly, *if* $a \in (\operatorname{spt} \mu) \sim B_i$, *then* $\bar{B}_{1/i}(a) \subset U$ *and*

$$(2n\gamma_n)^{-n} \varrho^n \le \mu(\bar{B}_\varrho(a)) \quad \textit{for } 0 < \varrho < 1/i$$

by 2.5. On the other hand, since the sets $\{x \in B_i : \bar{B}_{1/i}(x) \subset U\}$ are open and $B_{i+1} \subset B_i$, $X_i \subset X_{i+1}$ for $i \in \mathbb{N}$, one infers that $\mathscr{H}^s$ *almost all* $a \in A \cap \bigcap_{i \in \mathbb{N}} B_i$ *satisfy*

$$\lim_{r \downarrow 0} \mu(\bar{B}_r(a)) \big/ r^{sn/(n-p)} = 0.$$

Remark 2.11. Similar to the preceding remark one obtains using 2.6 instead of 2.5 that $\mathscr{H}^n$ almost all $x \in U$ satisfy

$$\text{either} \quad \theta^n_*(\mu, x) \ge 1 \quad \text{or} \quad \theta^{n^2/(n-p)}(\mu, x) = 0$$

and, in case $\|\delta\mu\|$ is absolutely continuous with respect to μ, that $\mathscr{H}^{n-p}$ almost all $x \in U$ satisfy

$$\text{either} \quad \theta^n_*(\mu, x) \ge 1 \quad \text{or} \quad \theta^n(\mu, x) = 0.$$

Moreover, the exponent $n^2/(n-p)$ cannot be replaced by any larger number as may be seen by taking $\mu \llcorner \mathbb{R}^{n+1} \sim T$ with μ and T as in 1.2. Hence, the same holds for the exponent $sn/(n-p)$ in the last equality of 2.10 if $s = n$.

Since $n^2/(n-1) \geq n+1$ if $n > 1$, this property of the n-densities seems to be the underlying fact used in the proof of [8, Theorem 3.4] by Leonardi and Masnou.

Remark 2.12. It can happen that $\mathscr{H}^n\big(A \cap (\operatorname{spt}\mu) \cap \bigcap_{i\in\mathbb{N}} B_i\big) > 0$. In fact taking $\mu \llcorner \mathbb{R}^{n+1} \sim T$ with μ and T as in 1.2 one sees from 2.10 and 1.2 (iv) that $T \subset \bigcap_{i\in\mathbb{N}} B_i$.

3 A differentiation theorem

In this section the theory of first order differentiation of functions in Lebesgue spaces defined with respect to a rectifiable varifold, similar to the one of Calderón and Zygmund in [5] for the special case of Lebesgue measure, is developed. First, an abstract differentiation theorem for measures, 3.1, is proven which then allows to establish the differentiation theorem for functions, 3.7. The first part of the latter theorem states an approximability result by functions which are Hölder continuous with exponent α which, in the particular case $\alpha = 1$ implies a Rademacher type theorem for differentiability in Lebesgue spaces, see 3.9. The second part of 3.7 may in fact be regarded as an application of this theory and is designed for use in [11].

Theorem 3.1. *Suppose m, n, p, U, and μ are as in 2.1, ν measures U, $\nu(U \sim \operatorname{spt}\mu) = 0$, A is μ-measurable with $\nu(A) = 0$, and $1 \leq q < \infty$. In case $p < n$ additionally suppose for some $1 \leq r \leq \infty$ and some nonnegative function $f \in L^r_{\mathrm{loc}}(\mu)$ that*

$$\nu = f\mu \quad \text{and} \quad q \leq 1 + (1 - 1/r)\frac{p}{n-p}.$$

Then for $\mathscr{H}^n$ almost all $a \in A$

$$\limsup_{s\downarrow 0} \nu(\bar{B}_s(a))\big/s^{nq} \quad \text{equals either } 0 \text{ or } \infty.$$

Proof. For $i \in \mathbb{N}$ let B_i denote the set of all $x \in U$ such that either $\bar{B}_{1/i}(x) \not\subset U$ or

$$\|\delta\mu\|(\bar{B}_\varrho(x)) > (2\gamma_n)^{-1}\mu(\bar{B}_\varrho(x))^{1-1/n} \quad \text{for some } 0 < \varrho < 1/i.$$

First, the case $A \subset \{x \in U : \theta^{*n}(\mu, x) > 0\}$ will be treated. In this case A is measurable and σ finite with respect to $\mathscr{H}^n$ by [6, 2.10.19 (1) (3)]. Hence one may assume A to be compact. Define

$$A_i = \{a \in A : \nu(\bar{B}_s(a))/ \leq i\, s^{nq} \text{ for } 0 < s < 1/i\}$$

whenever $i \in \mathbb{N}$, $1/i < \operatorname{dist}(A, \mathbb{R}^{n+m} \sim U)$. The sets A_i are compact (cp. [6, 2.9.14]) and their union equals

$$\bigl\{a \in A : \limsup_{s \downarrow 0} \nu(\bar{B}_s(a))/s^{nq} < \infty\bigr\}.$$

It therefore suffices to show for each $i \in \mathbb{N}$ with $1/i < \operatorname{dist}(A, \mathbb{R}^{n+m} \sim U)$

$$\lim_{s \downarrow 0} \nu(\bar{B}_s(a))\big/s^{nq} = 0 \quad \text{for } \mathscr{H}^n \text{ almost all } a \in A_i.$$

In fact, this equality will be proven for all $a \in A_i$ satisfying

$$\|\delta\mu\|(\{a\}) = 0, \quad \theta^n(\mu \llcorner U \sim A_i, a) = 0, \quad \theta^n(f^r\mu, a) = 0 \quad \text{if } r < \infty,$$
$$\limsup_{s \downarrow 0} \mu(B_j \cap \bar{B}_s(a))\big/s^{n^2/(n-p)} = 0 \quad \text{for some } j \in \mathbb{N},\ j \geq 2i, \text{ if } p < n$$

as $\mathscr{H}^n$ almost all $a \in A_i$ do according to [6, 2.10.19 (3) (4)] and 2.9.

In case $p = n$ one chooses $j \in \mathbb{N}$, $j \geq 2i$, using 2.5 such that

$$B_j \cap \bar{B}_{1/j}(a) = \emptyset.$$

Let $0 < s < 1/j$. For $x \in \bar{B}_s(a) \cap (\operatorname{spt}\mu) \sim (B_j \cup A_i)$ there exists $y \in A_i$ with $|x - y| = \operatorname{dist}(x, A_i)$, hence

$$t := |x - y| \leq |x - a| \leq s < 1/j \leq 1/(2i),$$
$$\bar{B}_{|x-y|/2}(x) \subset \bar{B}_{3|x-y|/2}(y) \cap \bar{B}_{2s}(a) \sim A_i,$$
$$\nu(\bar{B}_{t/2}(x)) \leq \nu(\bar{B}_{3t/2}(y)) \leq i\,3^{nq}(t/2)^{nq} \leq c\,\mu(\bar{B}_{t/2}(x))^q$$

where $c = i\,3^{nq}(2\gamma_n n)^{nq}$. Therefore one infers from Besicovitch's covering theorem the existence of countable, pairwise disjoint collections $F_1, \ldots, F_{N(n+m)}$ of closed balls such that

$$\bar{B}_s(a) \cap (\operatorname{spt}\mu) \sim (B_j \cup A_i) \subset \bigcup_{k=1}^{N(n+m)} \bigcup_{S \in F_k} S \subset \bar{B}_{2s}(a) \sim A_i,$$
$$\nu(S) \leq c\,\mu(S)^q \quad \text{whenever } S \in \bigcup_{k=1}^{N(n+m)} F_k,$$

hence

$$\nu(\bar{B}_s(a) \sim B_j) = \nu(\bar{B}_s(a) \cap (\operatorname{spt}\mu) \sim (B_j \cup A_i)) \leq c\,N(n+m)\,\mu(\bar{B}_{2s}(a) \sim A_i)^q,$$
$$\lim_{s \downarrow 0} \nu(\bar{B}_s(a) \sim B_j)\big/s^{nq} = 0.$$

To conclude the proof of the first case, one observes

$$\nu(B_j \cap \bar{B}_s(a)) = 0 \quad \text{if } p = n,$$
$$\nu(B_j \cap \bar{B}_s(a)) \le \mu(B_j \cap \bar{B}_s(a))^{1-1/r} \| f \|_{L^r(\mu \llcorner \bar{B}_s(a))} \quad \text{if } p < n$$

implying

$$\lim_{s \downarrow 0} \nu(B_j \cap \bar{B}_s(a)) / s^{nq} = 0$$

because $(1 - 1/r)n/(n-p) + 1/r \ge q$ in case $p < n$.

It remains to treat the case $A \subset \{x \in U : \theta^n(\mu, x) = 0\}$. Using 2.5 and 2.11 one obtains

$$A \cap \operatorname{spt} \mu \text{ is countable} \quad \text{if } p = n,$$
$$\theta^{n^2/(n-p)}(\mu, a) = 0 \quad \text{for } \mathscr{H}^n \text{ almost all } a \in A \quad \text{if } p < n$$

and the claim follows by using Hölder's inequality as in the preceding paragraph noting by [6, 2.10.19 (4)]

$$\theta^n(f^r \mu, a) = 0 \quad \text{for } \mathscr{H}^n \text{ almost all } a \in A \quad \text{if } r < \infty. \qquad \square$$

Remark 3.2. This theorem combines two lines of development. The first deals with the case $q = 1$ and focuses on the possibility to allow for arbitrary measures ν, see [12, Theorem 1] and [13, Section 5] by Mickle and Radó. The second focuses on arbitrary q while restricting ν to be absolutely continuous with respect to Lebesgue measure, see [5, Theorem 10 (ii)] by Calderón and Zygmund.

The current approach adapts the presentation of Mickle's and Radó's results by Federer in [6, 2.9.17]. The condition on the mean curvature replaces, via the use of 2.9, a diametric regularity condition employed by Federer. A measure ϕ on a metric space X satisfies this condition if and only if for some $R > 0$ and $\lambda < \infty$ there holds $\phi(\bar{B}_{5r}(x)) < \lambda\, \phi(\bar{B}_r(x))$ whenever $x \in X$ and $0 < r < R$.

Remark 3.3. If $q = 1$, the condition $\nu(U \sim \operatorname{spt} \mu) = 0$ cannot be omitted as may be seen from [6, 2.9.18 (2)].

Remark 3.4. If $p < n$ the condition $q \le 1 + (1 - 1/r)p/(n-p)$ cannot be omitted as can be shown using 1.2. In fact given μ and T as in 1.2 a counterexample is provided by $\nu := \mu \llcorner \mathbb{R}^{n+1} \sim T$ in case $r = \infty$ and if $1 < r < \infty$ applying 1.2 (v) with $s = nq$ and $\alpha_1 q_1 = \alpha_2 q_2$ slightly larger than $np/(n-p)$ yields a function f such that $\nu := f\mu$ does not satisfy the conclusion of 3.1. Finally, if $r = 1$ the condition is also violated for a slightly larger r, hence reducing this case to the previous one.

Remark 3.5. Note that the preceding two remarks remain valid if $\mathscr{H}^n$ is replaced by μ in the conclusion of 3.1.

Definition 3.6. Whenever A is ϕ-measurable set with $0 < \phi(A) < \infty$ and $f \in L^1(\phi \llcorner A)$ one defines $\fint_A f \, \mathrm{d}\phi = \phi(A)^{-1} \int_A f \, \mathrm{d}\phi$.

Theorem 3.7. *Suppose n, m, p, U, and μ are as in 2.1, Z is a separable Banach space, $f : U \to Z$ is μ-measurable, $0 < \alpha \leq 1$, $1 \leq q < \infty$, and A is the set of all $x \in \operatorname{spt} \mu$ such that*

$$\limsup_{\varrho \downarrow 0} \varrho^{-\alpha q} \fint_{\bar{B}_\varrho(x)} |f(\xi) - z|^q \, \mathrm{d}\mu(\xi) < \infty \quad \textit{for some } z \in Z.$$

In case $p < n$ additionally suppose that $f \in L^r_{\mathrm{loc}}(\mu, Z)$ for some $1 \leq r \leq \infty$ satisfying

$$\alpha q / n \leq \Big(1 - \frac{q}{r}\Big) \frac{p}{n-p}.$$

Then A is a Borel set and the following two statements hold:

(i) For every $\varepsilon > 0$ there exists a function $g : U \to Z$ which locally satisfies a Hölder condition with exponent α such that

$$\mu(A \cap \{x : f(x) \neq g(x)\}) \leq \varepsilon.$$

Moreover, for every function g which locally satisfies a Hölder condition with exponent α there holds

$$\lim_{\varrho \downarrow 0} \varrho^{-\alpha q} \fint_{\bar{B}_\varrho(x)} |f(\xi) - g(\xi)|^q \, \mathrm{d}\mu(\xi) = 0$$

for μ almost all $x \in A$ with $f(x) = g(x)$.

(ii) If $\varepsilon > 0$, $D_i(a)$ denotes for $a \in \operatorname{domain} f$, $i \in \mathbb{N}$ the set of all $x \in U$ such that either $\bar{B}_{1/i}(x) \not\subset U$ or

$$\int_{\bar{B}_\varrho(x)} |f(\xi) - f(a)|^q \, \mathrm{d}\mu(\xi) > \varepsilon \, \mu(\bar{B}_\varrho(x)) \quad \textit{for some } 0 < \varrho < 1/i,$$

Y_i denotes for $i \in \mathbb{N}$ the set of all $a \in \operatorname{domain} f$ such that

$$\lim_{r \downarrow 0} \mu(D_i(a) \cap \bar{B}_r(a)) / r^{n + \alpha q} = 0,$$

then the sets Y_i are μ-measurable and

$$\mu\big(A \sim \bigcup \{Y_i : i \in \mathbb{N}\}\big) = 0.$$

Proof of (i). Let $\pi : \mathbb{R}^{n+m} \times Z \to \mathbb{R}^{n+m}$ denote the projection and for $i \in \mathbb{N}$ let E_i denote the set of all $(x, z) \in \operatorname{spt} \mu \times Z$ such that $B_{1/i}(x) \subset U$ and

$$\fint_{\bar{B}_\varrho(x)} |f(\xi) - z|^q \, \mathrm{d}\mu(\xi) \le i \varrho^{\alpha q} \quad \text{whenever } 0 < \varrho < 1/i.$$

Then E_i is closed (cp. [6, 2.9.14]), $\pi | E_i$ is univalent, and both $\pi(E_i)$ and $A = \bigcup \{\pi(E_i) : i \in \mathbb{N}\}$ are Borel sets by [6, 2.2.10].

To prove the first part of (i), the problem is reduced to the case $\mu = \mathscr{L}^n \llcorner K$ for some compact set K (not necessarily satisfying a condition on $\delta\mu$) via [6, 3.2.18]. This case can then be treated by adapting [6, 3.1.8, 3.1.14], see also [18, VI.2.2.2].

Concerning the second half of (i), one observes that every such function g satisfies

$$\limsup_{\varrho \downarrow 0} \varrho^{-\alpha q} \fint_{\bar{B}_\varrho(x)} |f(\xi) - g(\xi)|^q \, \mathrm{d}\mu(\xi) < \infty$$

for μ almost all $x \in A$ with $f(x) = g(x)$ by [6, 2.9.9] and 3.1 may be applied with ν, r, q, A replaced by $|f - g|^q \mu$, r/q, $1 + \alpha q/n$, $\{x \in A : f(x) = g(x)\}$ if $p < n$ and $|f - g|^q \mu$, ∞, $1 + \alpha q/n$, $\{x \in A : f(x) = g(x)\}$ else. □

Proof of (ii). For any $0 < \varrho < \infty$, $x \in \mathbb{R}^{n+m}$ denote by $b_{x,\varrho}$ the characteristic function of $\bar{B}_\varrho(x)$, define $U_i = \{x \in U : \operatorname{dist}(x, \mathbb{R}^{n+m} \sim U) > 1/i\}$ and observe that the function mapping $(a, x, \xi) \in (\operatorname{domain} f) \times U \times (\operatorname{domain} f)$ onto

$$b_{x,\varrho}(\xi) |f(\xi) - f(a)|^q - \varepsilon b_{x,\varrho}(\xi)$$

is $(\mu \times \mu \times \mu)$-measurable for every $0 < \varrho < \infty$. Applying Fubini's theorem, one infers that the function mapping $(a, x) \in (\operatorname{domain} f) \times U_i$ onto

$$\sup \left\{ \int_{\bar{B}_\varrho(x)} |f(\xi) - f(a)|^q \, \mathrm{d}\mu(\xi) - \varepsilon \, \mu(\bar{B}_\varrho(x)) : 0 < \varrho < 1/i \right\}$$

is $(\mu \times (\mu \llcorner U_i))$-measurable for each $i \in \mathbb{N}$, since the supremum may be restricted to a countable, dense subset of $\{\varrho : 0 < \varrho < 1/i\}$. For the same reason

$$\sup \left\{ r^{-n-\alpha q} \mu(D_i(a) \cap \bar{B}_r(a)) : 0 < r < 1/j \right\}$$

depends μ-measurably on a for each $i, j \in \mathbb{N}$. Therefore the sets Y_i are μ-measurable.

For $i \in \mathbb{N}$ let A_i denote the set of all $a \in (\operatorname{domain} f) \cap (\operatorname{spt} \mu)$ such that $B_{1/i}(a) \subset U$ and whenever $0 < \varrho < 1/i$

$$\mu(\bar{B}_\varrho(a)) \le i \varrho^n, \quad \fint_{\bar{B}_\varrho(a)} |f(\xi) - f(a)|^q \, \mathrm{d}\mu(\xi) \le i \varrho^{\alpha q}.$$

Since the last condition in the definition of the sets A_i could be replaced by the two conditions

$$a \in \pi(E_i), \quad \lim_{\varrho \downarrow 0} \fint_{\bar{B}_\varrho(a)} f(\xi)\, \mathrm{d}\mu(\xi) = f(a),$$

the μ-measurability of the sets A_i may be verified using the first paragraph of the proof of (i). Note $\mu(A \sim \bigcup\{A_i : i \in \mathbb{N}\}) = 0$ by [6, 2.9.9]. For $i \in \mathbb{N}$ let C_i denote the set of all $x \in \operatorname{spt} \mu$ such that either $\bar{B}_{1/i}(x) \not\subset U$ or

$$\|\delta\mu\|\, \bar{B}_\varrho(x) > (2\gamma_n)^{-1} \mu(\bar{B}_\varrho(x))^{1-1/n} \quad \text{for some } 0 < \varrho < 1/i.$$

Moreover, define

$$X_i = \left\{x \in U : \theta^{n+\alpha q}(\mu \llcorner C_i, x) = 0\right\} \quad \text{for } i \in \mathbb{N},$$

note $n + \alpha q \le n^2/(n-p)$ if $p < n$, and observe by 2.9 in case $p < n$, by 2.5 in case $p = n$, that

$$\mu\big(U \sim \bigcup\{X_i : i \in \mathbb{N}\}\big) = 0.$$

Using (i), one constructs sequences K_i of compact subsets of U and $g_i : U \to Z$ such that

$$K_i \subset A_j \text{ for some } j \in \mathbb{N}, \quad f|K_i = g_i|K_i,$$
$$g_i \text{ locally satisfies a Hölder condition with exponent } \alpha,$$
$$\mu\big(A \sim \bigcup\{K_i : i \in \mathbb{N}\}\big) = 0.$$

Also note $A_i \subset A_{i+1}$, $C_i \supset C_{i+1}$, and $X_i \subset X_{i+1}$ for $i \in \mathbb{N}$.

From the observations of the preceding paragraph, [6, 2.10.6, 2.10.19 (4)] and (i) it follows that it is enough to prove $a \in \bigcup\{Y_j : j \in \mathbb{N}\}$ whenever $a \in A$ satisfies for some $i \in \mathbb{N}$, some compact set K, and some $g : U \to Z$

$$a \in X_i, \quad a \in K \subset A_i, \quad \theta^n(\mu \llcorner U \sim K, a) = 0, \quad g|K = f|K,$$
$$g \text{ locally satisfies a Hölder condition with exponent } \alpha,$$
$$r^{-n-\alpha q} \int_{\bar{B}_r(a)} |f(\xi) - g(\xi)|^q\, \mathrm{d}\mu(\xi) \to 0 \quad \text{as } r \downarrow 0.$$

For this purpose define $h = \sup\{|g(x) - g(y)|/|x-y|^\alpha : x, y \in K, x \ne y\}$, choose $j \in \mathbb{N}$, $j \ge 2i$, and $0 < R < 1/(2i)$ satisfying

$$2^{q-1} i^2 \big((1/j + R)^{\alpha q} + h^q (2R)^{\alpha q}\big) \le \varepsilon 2^{-n} (2\gamma_n n)^{-n}.$$

Next, it will be shown

$$\int_{\bar{B}_\varrho(x)} |f(\xi) - f(a)|^q \, \mathrm{d}\mu(\xi) \le \varepsilon 2^{-n} (1 + |\zeta - x|/\varrho)^n \mu(\bar{B}_\varrho(x))$$

whenever $x \in \operatorname{spt} \mu \cap \bar{B}_r(a) \sim C_i$, $\zeta \in K$, $|\zeta - x| = \operatorname{dist}(x, K)$, $0 < r \le R$, *and* $0 < \varrho < 1/j$. Noting

$$\varrho + |\zeta - x| < 1/j + |x - a| \le 1/j + R < 1/i,$$
$$\bar{B}_\varrho(x) \subset \bar{B}_{\varrho + |\zeta - x|}(\zeta) \subset B_{1/i}(\zeta) \subset U,$$
$$|\zeta - a| \le |\zeta - x| + |x - a| \le 2|x - a| \le 2R,$$
$$2^{q-1} i^2 \big((\varrho + |\zeta - x|)^{\alpha q} + h^q |\zeta - a|^{\alpha q} \big) \le \varepsilon 2^{-n} (2\gamma_n n)^{-n},$$

one estimates

$$\begin{aligned}
&\int_{\bar{B}_\varrho(x)} |f(\xi) - f(a)|^q \, \mathrm{d}\mu(\xi) \\
&\quad \le \int_{\bar{B}_{\varrho + |\zeta - x|}(\zeta)} |f(\xi) - f(a)|^q \, \mathrm{d}\mu(\xi) \\
&\quad \le 2^{q-1} \Big(\int_{\bar{B}_{\varrho + |\zeta - x|}(\zeta)} |f(\xi) - f(\zeta)|^q \, \mathrm{d}\mu(\xi) + |f(\zeta) - f(a)|^q \mu(\bar{B}_{\varrho + |\zeta - x|}(\zeta)) \Big) \\
&\quad \le 2^{q-1} i \big((\varrho + |\zeta - x|)^{\alpha q} + h^q |\zeta - a|^{\alpha q} \big) \mu(\bar{B}_{\varrho + |\zeta - x|}(\zeta)) \\
&\quad \le \varepsilon 2^{-n} (2\gamma_n n)^{-n} (1 + |\zeta - x|/\varrho)^n \varrho^n
\end{aligned}$$

and 2.5 implies the assertion. Therefore, if

$$\int_{\bar{B}_\varrho(x)} |f(\xi) - f(a)|^q \, \mathrm{d}\mu(\xi) > \varepsilon \, \mu(\bar{B}_\varrho(x)),$$

then

$$(1 + |\zeta - x|/\varrho)^n > 2^n, \quad \varrho < |\zeta - x| \le |x - a| \le r, \quad |x - a| + \varrho < 2r,$$
$$\bar{B}_\varrho(x) \subset B_{2r}(a) \sim K \subset U.$$

This implies that for each $x \in \operatorname{spt} \mu \cap \bar{B}_r(a) \cap D_j(a) \sim C_i$ with $0 < r \le R$ there exists $0 < \varrho < 1/j$ such that

$$\bar{B}_\varrho(x) \subset B_{2r}(a) \sim K \subset U, \quad \int_{\bar{B}_\varrho(x)} |f(\xi) - f(a)|^q \, \mathrm{d}\mu(\xi) > \varepsilon \, \mu(\bar{B}_\varrho(x)),$$

because $a \in A_i$, $x \in \bar{B}_r(a)$ implies $\bar{B}_{1/j}(x) \subset U$. Hence one infers from Besicovitch's covering theorem

$$\mu(\bar{B}_r(a) \cap D_j(a) \sim C_i) \leq N(n+m)\varepsilon^{-1} \int_{B_{2r}(a) \sim K} |f(\xi) - f(a)|^q \, d\mu(\xi)$$

for $0 < r \leq R$. Recalling $a \in X_i$, the proof may be concluded by showing

$$r^{-n-\alpha q} \int_{B_{2r}(a) \sim K} |f(\xi) - f(a)|^q \, d\mu(\xi) \to 0 \quad \text{as } r \downarrow 0$$

which is a consequence of

$$\begin{aligned} \int_{B_{2r}(a) \sim K} |f(\xi) - f(a)|^q \, d\mu(\xi) \leq 2^{q-1} \Big(& \int_{B_{2r}(a)} |f(\xi) - g(\xi)|^q \, d\mu(\xi) \\ & + \int_{B_{2r}(a) \sim K} |g(\xi) - g(a)|^q \, d\mu(\xi) \Big), \end{aligned}$$

$$\int_{B_{2r}(a) \sim K} |g(\xi) - g(a)|^q \, d\mu(\xi) \leq \mu(B_{2r}(a) \sim K)(h_0)^q (2r)^{\alpha q}$$

for $0 < r \leq R$ with $h_0 = \sup\{|g(x) - g(y)|/|x-y|^\alpha : x, y \in \bar{B}_R(a), x \neq y\}$. □

Remark 3.8. If $p < n$ the assumption $\alpha q/n \leq (1-q/r)p/(n-p)$ cannot be omitted in order to obtain the second part of (i) as may be seen from the family of examples constructed in 1.2; in fact one can take $\alpha_1 = \alpha_2 = \alpha$, $q_1 = q_2 = q$, and $f = \chi_{\mathbb{R}^{n+1} \sim T}$ in case $r = \infty$, and in case $r < \infty$ one can assume $q < r$ and apply 1.2 (v) with r, s, $\alpha_1 = \alpha_2$, $q_1 = q_2$ replaced by r/q, $\alpha q + n$, 1 and a number slightly larger than $np/(n-p)$ to obtain a function $f \in L^{r/q}_{\mathrm{loc}}(\mu)$ such that the second statement of (i) does not hold for f, g replaced by $f^{1/q}$, 0.

Remark 3.9. If $\dim Z < \infty$ and $\alpha = 1$, (i) in conjunction with [6, 3.2.18, 3.1.16] implies, adapting the use of g in [6, 4.5.9 (26) (II)], that for μ almost all $a \in A$

$$\lim_{\varrho \downarrow 0} ⨍_{\bar{B}_\varrho(a)} (|f(\xi) - f(a) - \langle (T_a\mu)(\xi - a), (\mu, n) \operatorname{ap} Df(a) \rangle| / |\xi - a|)^q \, d\mu(\xi) = 0$$

where the notion of approximate differentials, see [6, 3.2.16], is employed.

Remark 3.10. Part (ii) can be seen in two ways as a refinement of the simple fact that

$$\begin{aligned} & \theta^{*n+\alpha q}(\mu \llcorner \{x \in U : |f(x) - f(a)|^q > \varepsilon\}, a) \\ & \qquad \leq \varepsilon^{-1} \theta^{*n+\alpha q}(|f(\cdot) - f(a)|^q \mu, a) < \infty \end{aligned}$$

whenever $a \in A$. Firstly, $|f(x) - f(a)|^q > \varepsilon$ is replaced in the definition of $D_i(a)$ by $\fint_{\bar{B}_\varrho(x)} |f(\xi) - f(a)|^q \, d\mu(\xi) > \varepsilon$ for some $0 < \varrho < 1/i$. Secondly, in the conclusion $\theta^{n+\alpha q}(\mu \llcorner D_i(a), a) = 0$ occurs instead of $\theta^{*n+\alpha q}(\mu \llcorner D_i(a), a) < \infty$. Whereas the first improvement is vital for the applications in [11], the second one is only used under the stronger assumption

$$\lim_{r \downarrow 0} r^{-n-\alpha q} \int_{\bar{B}_r(x)} |f(\xi) - z|^q \, d\mu(\xi) = 0 \quad \text{for some } z \in Z$$

for μ almost all $x \in U$.

Acknowledgments. The author offers his thanks to Professor Reiner Schätzle under whose guidance the underlying PhD thesis was written. The author also would like to thank Professor Tom Ilmanen for inviting him to the ETH Zürich in 2006. The author thanks the referee for his or her careful reading of this paper.

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Received November 28, 2008.

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