

# SOME APPLICATIONS OF THE ISOPERIMETRIC INEQUALITY FOR INTEGRAL VARIFOLDS

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ABSTRACT. In this work the Isoperimetric Inequality for integral varifolds is used to obtain sharp estimates for the size of the set where the density quotient is small and to generalise Calderón's and Zygmund's theory of first order differentiability for functions in Lebesgue spaces from Lebesgue measure to integral varifolds.

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## INTRODUCTION

In this work, and also in the work in [Men08b] and [Men08c] depending on it, weak notions of regularity for integral varifolds in an open subset of Euclidian space whose distributional first variation is given by either a Radon measure or a locally to the  $p$ -th power summable function,  $1 < p \leq \infty$ , are investigated. As it is well known, see e.g. [All72, 8.1 (2)], even in the second case with  $p = \infty$  the singular set where the support does not locally correspond to a submanifold of class  $C^1$  may have positive measure. Therefore the notions of regularity studied here are decay rates of height-excess and tilt-excess which provide a way to quantify the amount of flatness entailed by the conditions on the mean curvature near almost every point.

Next, in order to precisely state the problem and the results, some definitions will be recalled. Suppose throughout the introduction that  $n, m \in \mathbb{N}$ ,  $U$  is an open subset of  $\mathbb{R}^{n+m}$ , and  $\mu$  is an integral  $n$  varifold in  $U$ , i.e., using [Sim83, Theorem 11.8] as a definition,  $\mu$  is a Radon measure on  $U$  and for  $\mu$  almost all  $x \in U$  there exists an approximate tangent plane  $T_x\mu \in G(n+m, n)$  with multiplicity  $\theta^n(\mu, x) \in \mathbb{N}$  of  $\mu$  at  $x$ ,  $G(n+m, n)$  denoting the set of  $n$  dimensional, unoriented planes in  $\mathbb{R}^{n+m}$ . The distributional first variation of mass of  $\mu$  equals

$$(\delta\mu)(\eta) = \int \operatorname{div}_\mu \eta \, d\mu \quad \text{whenever } \eta \in C_c^1(U, \mathbb{R}^{n+m})$$

where  $\operatorname{div}_\mu \eta(x)$  is the trace of  $D\eta(x)$  with respect to  $T_x\mu$ .  $\|\delta\mu\|$  denotes the total variation measure associated to  $\delta\mu$  and  $\mu$  is said to be of locally bounded first

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variation if and only if  $\|\delta\mu\|$  is a Radon measure, in this case the generalised mean curvature vector  $\vec{\mathbf{H}}_\mu(x) \in \mathbb{R}^{n+m}$  can be defined by the requirement

$$\vec{\mathbf{H}}_\mu(x) \bullet v = -\lim_{\varrho \downarrow 0} \frac{(\delta\mu)(\chi_{B_\varrho(x)}v)}{\mu(B_\varrho(x))} \quad \text{for } v \in \mathbb{R}^{n+m}$$

whenever these limits exist for  $x \in U$ ; here  $\bullet$  denotes the usual inner product on  $\mathbb{R}^{n+m}$ . Moreover,  $\mu$  is said to satisfy  $(H_p)$ ,  $1 \leq p \leq \infty$ , if and only if it is of locally bounded first variation,  $\vec{\mathbf{H}}_\mu \in L^p_{\text{loc}}(\mu, \mathbb{R}^{n+m})$ , and, in case  $p > 1$ , satisfies

$$(H_p) \quad (\delta\mu)(\eta) = -\int \vec{\mathbf{H}}_\mu \bullet \eta \, d\mu \quad \text{whenever } \eta \in C_c^1(U, \mathbb{R}^{n+m}).$$

Also, adapting Anzellotti's and Serapioni's definition in [AS94],  $\mu$  is called countably rectifiable of class  $\mathcal{C}^2$ , or for short  $\mathcal{C}^2$  rectifiable, if and only if  $\mu$  almost all of  $U$  can be covered by a countable collection of  $n$  dimensional submanifolds of class  $\mathcal{C}^2$ . The notation follows [Sim83] which includes a list of basic notation on page (vii).

The following questions arise.

**Question 1.** Suppose  $n, m \in \mathbb{N}$ ,  $1 \leq p \leq \infty$ ,  $0 < \alpha \leq 1$ , and  $1 \leq q \leq \infty$ . Does the condition  $(H_p)$  on an integral  $n$  varifold  $\mu$  in  $U$ ,  $U$  a nonempty, open subset of  $\mathbb{R}^{n+m}$ , imply

$$\limsup_{\varrho \downarrow 0} \varrho^{-1-\alpha-n/q} \|\text{dist}(\cdot - x, T_x\mu)\|_{L^q(\mu \llcorner B_\varrho(x))} < \infty \quad \text{for } \mu \text{ almost all } x \in U?$$

**Question 2.** Suppose  $n, m \in \mathbb{N}$ ,  $1 \leq p \leq \infty$ ,  $0 < \alpha \leq 1$ , and  $1 \leq q \leq \infty$ . Does the condition  $(H_p)$  on an integral  $n$  varifold  $\mu$  in  $U$ ,  $U$  a nonempty, open subset of  $\mathbb{R}^{n+m}$ , imply

$$\limsup_{\varrho \downarrow 0} \varrho^{-\alpha-n/q} \|T_\cdot\mu - T_x\mu\|_{L^q(\mu \llcorner B_\varrho(x))} < \infty \quad \text{for } \mu \text{ almost all } x \in U?$$

Here  $S \in G(n+m, n)$  is identified with the element of  $\text{Hom}(\mathbb{R}^{n+m}, \mathbb{R}^{n+m})$  given by the orthogonal projection of  $\mathbb{R}^{n+m}$  onto  $S$ .

Clearly, the two questions are related by Caccioppoli type inequalities, see e.g. in [Bra78, 5.5], at least in the case  $q = 2$  where the quantities considered agree with the classical tilt and height excess. Also note that an affirmative answer to one of the questions with  $\alpha, q$  implies an affirmative answer to the same question for any  $0 < \alpha' \leq 1$ ,  $q < q' < \infty$  such that  $\alpha q = \alpha' q'$  by use of the trivial  $L^\infty$  bounds of the functions involved. The case  $\alpha = 1$  is of particular interest in both questions. A varifold satisfying the decay estimate in the first question with  $\alpha = 1$  and  $q = 1$  is  $\mathcal{C}^2$  rectifiable, see [Sch04b, Appendix A]. In the second question the case  $\alpha = 1$  is related to the local computability of the mean curvature vector from the geometry of  $\{x \in U : \theta^n(\mu, x) \geq 1\}$ , see [Sch01, Lemma 6.3] (or [Sch04a, Prop. 6.1] or [Sch04b, Theorem 4.1]). On the other hand the quantity  $\alpha q$  to some extent determines how well  $\mu$  can be approximated by multivalued graphs near generic points, see Almgren [Alm00, Chapter 3] and Brakke [Bra78, Chapter 5] and also the forthcoming paper [Men08b]. Such kind of approximation has been fundamental for regularity investigations, for example, in the work of Almgren in [Alm00].

Next, an overview of results concerning these two questions will be given. Brakke answers both questions in the affirmative for any  $n, m \in \mathbb{N}$  if

$$\text{either } p = 1, \alpha = 1/2, q = 2 \quad \text{or } p = 2, \alpha < 1, q = 2$$

in [Bra78, 5.7]. Schätzle provides a positive answer in the case

$$m = 1, p > n, p \geq 2, \alpha = 1, q = \infty$$

for the first question and in the case

$$m = 1, p > n, p \geq 2, \alpha = 1, q = 2$$

for the second question, see [Sch04a, Prop. 4.1, Thm. 5.1]. Moreover, in subsequent work Schätzle showed for arbitrary dimensions that the decay rates occurring in the two questions hold if

$$p = 2, \alpha = 1, q = 2$$

provided  $\mu$  is additionally assumed to be  $\mathcal{C}^2$  rectifiable.

In this paper, it is shown by an example of a unit density,  $\mathcal{C}^2$  rectifiable  $n$  varifold in  $\mathbb{R}^{n+1}$  that the answers to both questions is in the negative if  $p < n$  and  $\alpha q > \frac{np}{n-p}$ , see 1.2. In particular, in case  $1 \leq p < \frac{2n}{n+2}$  proving appropriate decay for the classical height-excess or tilt-excess, i.e. answering the first or second question in the affirmative for  $\alpha = 1, q = 2$ , cannot serve as an intermediate step in studying  $\mathcal{C}^2$  rectifiability or local computability of the mean curvature vector. This was the original motivation to consider exponents  $q \neq 2$ .

In order to provide new cases where the questions are answered in the affirmative, it will turn out to be useful in [Men08b, Men08c] to have a theory of first order differentiability for functions in  $L^p(\mu)$ ,  $\mu$  an integral  $n$  varifold, similar to the one developed by Calderón and Zygmund in [CZ61] for  $L^p(\mathcal{L}^n)$ , at one's disposal. However, other kinds of applications may occur in the future. The key to carry over this theory from the Lebesgue measure case to the case of integral varifolds is the following differentiation theorem which corresponds to [CZ61, Theorem 10 (ii)] but whose proof uses techniques employed by Federer in [Fed69, 2.9.17].

**Theorem 3.1.** *Suppose  $n, m \in \mathbb{N}$ ,  $1 \leq p \leq n$ ,  $U$  is an open subset of  $\mathbb{R}^{n+m}$ ,  $\mu$  is an integral  $n$  varifold in  $U$  satisfying  $(H_p)$ ,  $\nu$  measures  $U$  with  $\nu(U \setminus \text{spt } \mu) = 0$ ,  $A$  is  $\mu$  measurable with  $\nu(A) = 0$ , and  $1 \leq q < \infty$ . In case  $p < n$  additionally suppose for some  $1 \leq r \leq \infty$  and some nonnegative function  $f \in L^r_{\text{loc}}(\mu)$  that*

$$\nu = f\mu \quad \text{and} \quad q \leq 1 + (1 - 1/r) \frac{p}{n-p}.$$

*Then for  $\mathcal{H}^n$  almost all  $a \in A$*

$$\limsup_{s \downarrow 0} \nu(\bar{B}_s(a)) / s^{nq} \quad \text{equals either } 0 \text{ or } \infty.$$

The bound on  $q$  is sharp as demonstrated in 3.3, 3.4. Its occurrence is due to the fact that in case  $p < n$  the number  $n^2/(n-p)$  in the following proposition cannot be replaced by any larger number, see 1.2: *Suppose  $n, m \in \mathbb{N}$ ,  $1 \leq p < n$ ,  $\mu$  is an integral  $n$  varifold in  $\mathbb{R}^{n+m}$  satisfying  $(H_p)$ , then for  $\mu$  almost all  $a \in U$  there exists  $\varepsilon > 0$  such that*

$$\lim_{r \downarrow 0} \frac{\mu(\bar{B}_r(a) \setminus \{x : \mu(\bar{B}_\varrho(x)) \geq c_n \varrho^n \text{ for } 0 < \varrho < \varepsilon\})}{r^{n^2/(n-p)}} = 0$$

where  $c_n$  is a positive, finite number depending only on  $n$ , see 2.9, 2.10. Similar propositions with  $n^2/(n-p)$  replaced by any slightly smaller number can be obtained via integration of the monotonicity formula, see [Sim83, Theorem 17.6]. The optimal bound is derived using the Isoperimetric Inequality. All these results will be proved under the weaker condition  $\theta^n(\mu, x) \geq 1$  for  $\mu$  almost every  $x \in U$  replacing the integrality condition on  $\mu$ .

The work is organised as follows. In the first section the example is constructed. In the second section the Isoperimetric Inequality is used to derive some sharp bounds on the size of the set where the  $n$  density ratio is small and in the last section a theory of first order differentiation in Lebesgue spaces defined with respect to a varifold is presented. Note that with the exception of 3.6–3.10 the work was part of the author's PhD thesis, see [Men08a].

For notation of geometric measure theory see [Sim83, Fed69].

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### 1. AN EXAMPLE CONCERNING HEIGHT AND TILT DECAYS OF INTEGRAL VARIFOLDS

In this section a family of integral  $n$  varifolds with prescribed decay rates of height and tilt quantities is constructed. In fact, the decay rate for tilt can be arranged to be slightly larger than the one of the height with the same exponent. However, this feature that will only become relevant in [Men08b].

1.1. **Definition.** Suppose  $x \in \mathbb{R}^{n+m}$  and  $0 < \varrho < \infty$ .

Then  $Q_\varrho(x) := \{y \in \mathbb{R}^{n+m} : |y_i - x_i| \leq \varrho \text{ for } i = 1, \dots, n+m\}$ . To avoid ambiguity,  $Q_\varrho^{n+m}(0)$  will be written instead of  $Q_\varrho(0)$ .

1.2. *Example.* Suppose  $n \in \mathbb{N}$ ,  $1 \leq p < n$ ,  $0 < \alpha_i \leq 1$ ,  $1 \leq q_i < \infty$  for  $i \in \{1, 2\}$ , such that

$$\alpha_2 q_2 \leq \alpha_1 q_1, \quad \frac{1}{p} > 1 + \frac{\alpha_2 q_2}{\alpha_1 q_1} \left( \frac{1}{n} + \frac{1}{\alpha_2 q_2} - 1 \right)$$

In case  $\alpha_1 q_1 = \alpha_2 q_2$  the last condition reads  $\alpha_2 q_2 > \frac{np}{n-p}$ .

Then there exists a rectifiable  $n$  varifold  $\mu$  in  $\mathbb{R}^{n+1}$ ,  $T \in G(n+1, n)$  and  $0 < \Gamma < \infty$  with the following properties:

- (1)  $T \subset \text{spt } \mu$  and  $(\text{spt } \mu) \sim T$  is an  $n$  dimensional manifold of class  $\mathcal{C}^\infty$ .
- (2)  $\theta^n(\mu, x) = 1$  for  $x \in \text{spt } \mu$  and  $T_x \mu = T$  for  $x \in T$ .
- (3) For some  $\vec{\mathbf{H}}_\mu \in L^p_{\text{loc}}(\mu, \mathbb{R}^{n+1})$  there holds  $(\delta\mu)(\eta) = -\int \vec{\mathbf{H}}_\mu \bullet \eta \, d\mu$  whenever  $\eta \in C^1_c(\mathbb{R}^{n+1}, \mathbb{R}^{n+1})$ .
- (4) Whenever  $x \in T$  and  $0 < \varrho \leq 1$

$$\Gamma^{-1} \varrho^{\alpha_2 q_2} \leq \varrho^{-n} \mu(\{\xi \in \bar{B}_\varrho(x) : \text{dist}(\xi - x, T) \geq \varrho/\Gamma\}),$$

$$\varrho^{-n} \mu(\bar{B}_\varrho(x) \sim T) \leq \Gamma \varrho^{\alpha_2 q_2},$$

$$\varrho^{-1-n/q_2} \left( \int_{\bar{B}_\varrho(x)} \text{dist}(\xi - x, T_x \mu)^{q_2} \, d\mu(\xi) \right)^{1/q_2} \approx \varrho^{\alpha_2},$$

$$\varrho^{-n/q_1} \left( \int_{\bar{B}_\varrho(x)} |T_\xi \mu - T_x \mu|^{q_1} \, d\mu(\xi) \right)^{1/q_1} \approx \varrho^{\alpha_1},$$

here  $a \approx b$  means that  $a \leq \Gamma_1 b$  and  $b \leq \Gamma_1 a$  for some positive, finite number  $\Gamma_1$  depending only on  $n$ , and  $\alpha_i, q_i$  for  $i \in \{1, 2\}$ .

- (5) Whenever  $1 < r < \infty$ ,  $n + (1 - 1/r)\alpha_2 q_2 < s < \infty$  there exists a nonnegative function  $f \in L^r_{\text{loc}}(\mu)$  such that  $f(x) = 0$  for  $x \in T$ , and

$$\varrho^s \approx \int_{\bar{B}_\varrho(x)} f \, d\mu \quad \text{whenever } x \in T, 0 < \varrho \leq 1,$$

here  $a \approx b$  means  $a \leq \Gamma_2 b$  and  $b \leq \Gamma_2 a$  for some positive, finite number  $\Gamma_2$  depending only on  $n$  and  $s$ .

*Construction of example.* Let  $a := \alpha_2 q_2 / n + 1$ ,  $b := (\alpha_1 q_1 - \alpha_2 q_2) / a + 1 \geq 1$ . Define for  $i \in \mathbb{N}_0$

$$W_i := \{Q_{2^{-i-2}}(x) : 2^{i+1}x \in \mathbb{Z}^n\}.$$

Clearly,  $\bigcup_{Q \in W_i} \bar{Q} = \mathbb{R}^n$  and  $W_i$  is pairwise disjoint. Let

$$F_i := \{[2^{-i-1}, 2^{-i}] \times W : W \in W_i\} \quad \text{for } i \in \mathbb{N}_0, \quad F := \bigcup_{i \in \mathbb{N}_0} F_i.$$

Clearly,  $\bigcup_{S \in F} \bar{S} = ]0, 1] \times \mathbb{R}^n$  and  $F$  is pairwise disjoint. Let  $T := \{0\} \times \mathbb{R}^n$ .

Next, it will be indicated how to construct for every  $0 < \sigma \leq \varrho < \infty$  a compact  $n$  dimensional submanifold  $M$  of  $\mathbb{R}^{n+1}$  of class  $\mathcal{C}^\infty$  such that

$$\begin{aligned} M &\subset Q_\varrho^{n+1}(0), \quad (\Gamma_0)^{-1}\varrho^n \leq \mathcal{H}^n(M) \leq \Gamma_0\varrho^n, \quad |\vec{\mathbf{H}}_M| \leq \Gamma_0\sigma^{-1}, \\ \mathcal{H}^n(\{x \in M : |T_x M - T| \geq 1\}) &\geq (\Gamma_0)^{-1}\sigma\varrho^{n-1}, \\ \mathcal{H}^n(\{x \in M : \vec{\mathbf{H}}_M(x) \neq 0 \text{ or } T_x M \neq T\}) &\leq \Gamma_0\sigma\varrho^{n-1} \end{aligned}$$

where  $\Gamma_0$  is a positive, finite number depending only on  $n$ . To construct  $M$ , one may assume  $\varrho = 1$ . Choose a concave function  $f : [-1/2, 1/2] \rightarrow [0, 1]$  and  $0 < \Gamma_1 < \infty$  such that

$$\begin{aligned} f(-1/2) &= \sigma/4 = f(1/2), \\ f(s) &= \sigma/2 \quad \text{whenever } s \in [-1/2 + \sigma/4, 1/2 - \sigma/4] \end{aligned}$$

and such that

$$N := \{(s, t) \in [-1/2, 1/2] \times \mathbb{R} : |t| = f(s)\} \cup (\{-1/2, 1/2\} \times [-\sigma/4, \sigma/4])$$

is a 1 dimensional submanifold of class  $\mathcal{C}^\infty$  with  $|\vec{\mathbf{H}}_N| \leq \Gamma_1\sigma^{-1}$ . Noting

$$\mathcal{H}^1(\text{graph } f|_{[-1/2, -1/2 + \sigma/4] \cap [1/2 - \sigma/4, 1/2]}) \leq \sigma,$$

one can take

$$M := \{(y, z) \in \mathbb{R} \times \mathbb{R}^n : (|z|, y) \in N\}.$$

For each  $i \in \mathbb{N}_0$  and  $Q \in F_i$  choose a  $n$  dimensional submanifold  $M_Q$  of the type just constructed corresponding to  $\varrho_i := 2^{-ia-2}$ ,  $\sigma_i := 2^{-iba-2}$  contained in  $Q$  and let  $M$  be the union those submanifolds. Take  $\mu := \mathcal{H}^n \llcorner (T \cup M)$ . (1) is now evident.

To prove the estimates, fix  $x \in T$  and define for  $i, j \in \mathbb{N}_0$

$$b_{i,j} := \#\{Q \in F_j : Q \cap Q_{2^{-i}}(x) \neq \emptyset\}, \quad c_{i,j} := \#\{Q \in F_j : Q \subset Q_{2^{-i}}(x)\}.$$

Clearly,  $b_{i,j} = c_{i,j} = 0$  if  $j < i$ . If  $j \geq i$ , one estimates

$$b_{i,j} \leq (2^{j-i+2} + 1)^n \leq (5 \cdot 2^{j-i})^n, \quad c_{i,j} \geq (2^{j-i+2} - 1)^n \geq (3 \cdot 2^{j-i})^n.$$

One calculates

$$\begin{aligned} \mu(Q_{2^{-i}}(x) \sim T) &\leq \sum_{j=0}^{\infty} b_{i,j} \Gamma_0(\varrho_j)^n \leq (5/4)^n \Gamma_0(2^{-i})^{an} (1 - 2^{n(1-a)})^{-1}, \\ n - ba(1-p) + (1-n)a &= -\alpha_1 q_1 + p(\alpha_1 q_1 - \alpha_2 q_2 + \alpha_2 q_2/n + 1) < 0, \\ \int_{Q_{2^{-i}}(x) \sim T} |\vec{\mathbf{H}}_M|^p d\mu &\leq \sum_{j=0}^{\infty} b_{i,j} (\Gamma_0)^{p+1} (\sigma_j)^{1-p} (\varrho_j)^{n-1} \\ &\leq 5^n (\Gamma_0)^{n+1} (2^{-i})^{ba(1-p)+n-1} (1 - 2^{n-ba(1-p)+(1-n)a})^{-1} < \infty, \\ \int_{Q_{2^{-i}}(x)} \text{dist}(\xi - x, T)^{q_2} d\mu(\xi) &\leq 2^{-iq_2} \mu(Q_{2^{-i}}(x) \sim T), \\ \int_{Q_{2^{-i}}(x)} |T_\xi \mu - T|^{q_1} d\mu(\xi) &\leq (2n)^{q_1} \sum_{j=0}^{\infty} b_{i,j} \Gamma_0 \sigma_j (\varrho_j)^{n-1} \\ &\leq (2n)^{q_1} (5/4)^n \Gamma_0 (2^{-i})^{ba+a(n-1)} (1 - 2^{n-ba-a(n-1)})^{-1}, \\ &\quad 2^{(i+1)q_2} \int_{Q_{2^{-i}}(x)} \text{dist}(\xi - x, T)^{q_2} d\mu(\xi) \\ &\geq \mu(\{\xi \in Q_{2^{-i}}(x) : \text{dist}(\xi - x, T) \geq 2^{-i-1}\}) \geq (\Gamma_0)^{-1} (\varrho_i)^n = (4^n \Gamma_0)^{-1} 2^{-ian}, \\ \int_{Q_{2^{-i}}(x)} |T_\xi \mu - T|^{q_1} d\mu(\xi) &\geq (\Gamma_0)^{-1} \sigma_i (\varrho_i)^{n-1} = (4^n \Gamma_0)^{-1} (2^{-i})^{ab+a(n-1)}. \end{aligned}$$

Therefore (3) and (4) are proved and the first estimate of (4) implies (2).

To prove (5), define  $f$  by  $f(y) := 2^{(na-s)i}$  if  $y \in \bigcup_{S \in F_i} S$  for some  $i \in \mathbb{N}_0$  and  $f(y) = 0$  else. Then for  $i \in \mathbb{N}_0$

$$\begin{aligned} \int_{Q_{2^{-i}}(x)} |f| \, d\mu &\leq \sum_{j=0}^{\infty} b_{i,j} 2^{(na-s)j} \Gamma_0(\varrho_j)^n \leq (5/4)^n \Gamma_0(2^{-i})^s (1 - 2^{n-s})^{-1}, \\ \int_{Q_{2^{-i}}(x)} |f|^r \, d\mu &\leq \sum_{j=0}^{\infty} b_{i,j} 2^{(na-s)rj} \Gamma_0(\varrho_j)^n \\ &\leq (5/4)^n \Gamma_0(2^{-i})^{(s-na)r+na} (1 - 2^{n+(na-s)r-na})^{-1} < \infty \end{aligned}$$

because

$$n + (na - s)r - an = \alpha_2 q_2 (r - 1) + r(n - s) < 0.$$

The estimate from below is similar to the one from above.  $\square$

1.3. *Remark.* The integral  $n$  varifold  $\mu$  constructed depends only on  $n$  and the products  $\alpha_i q_i$  for  $i \in \{1, 2\}$ . Moreover, the assumption  $\alpha_i \leq 1$  for  $i \in \{1, 2\}$  could be replaced by  $\alpha_i < \infty$  for  $i \in \{1, 2\}$ .

1.4. *Remark.* Taking  $p = 1$ ,  $\alpha_1 = \alpha_2$ , and  $q_1 = q_2 = 2$  in the last two estimates of (4) shows that for every  $n \in \mathbb{N}$ ,  $n > 1$ ,  $1/2 + (2(n-1))^{-1} < \alpha \leq 1$ , there exists an integral  $n$  varifold  $\mu$  of  $\mathbb{R}^{n+1}$  of locally bounded first variation such that for some  $A$  with  $\mu(A) > 0$

$$\lim_{\varrho \downarrow 0} \varrho^{-2\alpha} \text{heightex}_\mu(x, \varrho, T_x \mu) = \infty, \quad \lim_{\varrho \downarrow 0} \varrho^{-2\alpha} \text{tiltex}_\mu(x, \varrho, T_x \mu) = \infty$$

for  $x \in A$ . In [Bra78, 5.7] Brakke showed in arbitrary codimension that the above limits equal 0 almost everywhere with respect to  $\mu$  if  $\alpha = 1/2$ .

1.5. *Remark.* Similarly to the preceding remark, taking  $\alpha_1 = \alpha_2 = 1$ ,  $q_1 = q_2 = q$  and noting (1), one obtains for every  $p^* = \frac{np}{n-p} < q < \infty$  an integral  $n$  varifold  $\mu$  satisfying  $(H_p)$  which is countably rectifiable of class  $\mathcal{C}^2$  such that for some  $A$  with  $\mu(A) > 0$

$$\begin{aligned} \lim_{\varrho \downarrow 0} \varrho^{-2-n/q} \left( \int_{B_\varrho(x)} \text{dist}(\xi - x, T_x \mu)^q \, d\mu(\xi) \right)^{1/q} &= \infty, \\ \lim_{\varrho \downarrow 0} \varrho^{-1-n/q} \left( \int_{B_\varrho(x)} |T_\xi \mu - T_x \mu|^q \, d\mu(\xi) \right)^{1/q} &= \infty \end{aligned}$$

for  $x \in A$ . In particular, if  $p < \frac{2n}{n+2}$  then countable rectifiability of class  $\mathcal{C}^2$  does not imply quadratic decay of neither  $\text{tiltex}_\mu$  nor  $\text{heightex}_\mu$ . If  $p = 2$ , countable rectifiability of class  $\mathcal{C}^2$  is equivalent to quadratic decay of both quantities, see [Sch04b, Theorem 3.1].

## 2. THE SIZE OF THE SET WHERE THE $n$ DENSITY QUOTIENT IS SMALL

In this section the Isoperimetric Inequality is used to derive basic facts on the size of the set where the  $n$  density quotient is small. Although the general procedure of such estimates is clearly known, see 2.5, it appears to be rarely used in literature. The sharpness of the results is necessary to determine the precise limiting exponent up to which the differentiation theory in the next section can be developed. Similarly, the accuracy of the bounds obtained in [Men08b] depends on the results of this section.

2.1. The following situation will be studied:

$m, n \in \mathbb{N}$ ,  $1 \leq p \leq n$ ,  $U$  is an open subset of  $\mathbb{R}^{n+m}$ ,  $\mu$  is a rectifiable  $n$  varifold<sup>1</sup> in  $U$  of locally bounded first variation,  $\theta^n(\mu, x) \geq 1$  for  $\mu$  almost all  $x \in U$ , and, in case  $p > 1$ ,

$$(\delta\mu)(\eta) = -\int \vec{\mathbf{H}}_\mu \bullet \eta \, d\mu \quad \text{whenever } \eta \in C_c^1(U, \mathbb{R}^{n+m})$$

for some  $\vec{\mathbf{H}}_\mu \in L_{\text{loc}}^p(\mu, \mathbb{R}^{n+m})$ . In doing so, the following abbreviation will be used:

$$\psi = \|\delta\mu\| \quad \text{if } p = 1, \quad \psi = |\vec{\mathbf{H}}_\mu|^p \mu \quad \text{else.}$$

**2.2. Theorem** (Isoperimetric Inequality for varifolds). *Suppose  $m \in \mathbb{N}_0$ ,  $n \in \mathbb{N}$ ,  $\mu$  is a rectifiable  $n$  varifold in  $\mathbb{R}^{n+m}$  with  $\mu(\mathbb{R}^{n+m}) < \infty$  and  $\|\delta\mu\|(\mathbb{R}^{n+m}) < \infty$ .*

*Then for some positive, finite number  $\gamma$  depending only on  $n$*

$$\mu(\{x \in \mathbb{R}^{n+m} : \theta^n(\mu, x) \geq 1\}) \leq \gamma \mu(\mathbb{R}^{n+m})^{1/n} \|\delta\mu\|(\mathbb{R}^{n+m}).$$

*Proof.* This follows from [All72, Theorem 7.1] with a constant  $\gamma$  depending on  $n + m$  (what would be sufficient for the purpose of this work). A slight modification of [Sim83, Lemma 18.7, Theorem 18.6] yields the stated result.  $\square$

**2.3. Definition.** For  $n \in \mathbb{N}$  let  $\gamma_n$  denote the best constant  $\gamma$  in 2.2.

**2.4. Remark.** Taking  $m = 0$ ,  $\mu = \mathcal{L}^n \llcorner \bar{B}_1^n(0)$  yields

$$\gamma_n \geq \omega_n^{-1/n}/n.$$

Does equality hold?

**2.5.** An important consequence of the Isoperimetric Inequality 2.2 and the starting point for the estimates in the present section is the following fact which can be derived by a variant of [Fed69, 5.1.6] or [All72, 8.3], see [LM08, Prop. 3.1] or [Men08a, A.8, A.9].

*Suppose  $n, m, p = 1$ ,  $U = B_r(a)$  for some  $a \in \mathbb{R}^{n+m}$  and  $0 < r < \infty$ , and  $\mu$  are as in 2.1,  $a \in \text{spt } \mu$ , and*

$$\|\delta\mu\|(\bar{B}_\varrho(a)) \leq (2\gamma_n)^{-1} \mu(\bar{B}_\varrho(a))^{1-1/n} \quad \text{whenever } 0 < \varrho < r,$$

*then*

$$\mu(\bar{B}_\varrho(a)) \geq (2n\gamma_n)^{-n} \varrho^n \quad \text{whenever } 0 < \varrho < r.$$

Also note, if  $p = n > 1$  or  $p = n = 1$  and  $\|\delta\mu\|(\{a\}) < (2\gamma_n)^{-n}$ , then

$$\|\delta\mu\|(\bar{B}_\varrho(x)) \leq (2\gamma_n)^{-1} \mu(\bar{B}_\varrho(x))^{1-1/n}$$

whenever  $0 < \varrho < r$ ,  $x \in \text{spt } \mu \cap \bar{B}_r(a)$  is satisfied for all sufficiently small positive radii  $r$ .

**2.6. Lemma.** *Suppose  $m, n \in \mathbb{N}$ , and  $\delta > 0$ .*

*Then there exists a positive number  $\varepsilon$  with the following property.*

*If  $a \in \mathbb{R}^{n+m}$ ,  $0 < r < \infty$ ,  $m, n, p, U$ , and  $\mu$  are related as in 2.1 with  $U = B_r(a)$ ,  $p = 1$ ,  $a \in \text{spt } \mu$ , and*

$$\|\delta\mu\|(\bar{B}_\varrho(a)) \leq (2\gamma_n)^{-1} \mu(\bar{B}_\varrho(a))^{1-1/n} \quad \text{for } 0 < \varrho < r,$$

$$\|\delta\mu\|(B_r(a)) \leq \varepsilon \mu(B_r(a))^{1-1/n},$$

*then*

$$\mu(B_r(a)) \geq (1 - \delta) \omega_n r^n.$$

*Proof* (cf. [Men08a, A.10]). If the lemma were false, using 2.5, a compactness (see e.g. [Sim83, Corollary 17.8, Theorem 42.7]) argument would lead to a contradiction to the monotonicity formula (see e.g. [Sim83, (17.5)]).  $\square$

<sup>1</sup>Note that a definition of a rectifiable  $n$  varifold results from the definition of an integral  $n$  varifold through replacement of the condition  $\theta^n(\mu, x) \in \mathbb{N}$  by  $0 < \theta^n(\mu, x) < \infty$ .

2.7. *Remark.* 2.5 and 2.6 imply the following proposition.

If  $m, n, p, U, \mu$  and  $\psi$  are as in 2.1,  $p = n$ , then

$$\theta_*^n(\mu, a) \geq 1 \quad \text{whenever } a \in \text{spt } \mu \text{ and } \psi(\{a\}) = 0.$$

Clearly, the condition  $\psi(\{a\}) = 0$  is redundant in case  $\|\delta\mu\|$  is absolutely continuous with respect to  $\mu$  (i.e.  $\delta\mu$  has no singular part with respect to  $\mu$ ).

2.8. **Definition.** For  $k \in \mathbb{N}$  denote by  $N(k)$  the best constant in Besicovitch's covering theorem in  $\mathbb{R}^k$ .

2.9. **Theorem.** Suppose  $m, n, p, U, \mu$ , and  $\psi$  are as in 2.1,  $p < n$ ,  $0 \leq s < \infty$ ,  $0 < \varepsilon \leq (2\gamma_n)^{-p/(n-p)}$ ,  $4\gamma_n n < \Gamma < \infty$ ,

$$A = \{x \in U : \theta^{*n-p}(\psi, x) < (\varepsilon/\Gamma)^{n-p}/\omega_{n-p}\},$$

denote by  $B_i$  for  $i \in \mathbb{N}$  the set of all  $x \in U$  such that either  $\bar{B}_{1/i}(x) \not\subset U$  or

$$\psi(\bar{B}_\varrho(x)) > \varepsilon^{n-p} \mu(\bar{B}_\varrho(x))^{1-p/n} \quad \text{for some } 0 < \varrho < 1/i,$$

and denote by  $X_i$  for  $i \in \mathbb{N}$  the set of all  $a \in U$  such that

$$\lim_{r \downarrow 0} \mu(B_i \cap \bar{B}_r(a)) / r^{sn/(n-p)} = 0.$$

Then  $\{x \in B_i : \bar{B}_{1/i}(x) \subset U\}$  are open sets,  $X_i$  are Borel sets and

$$\mathcal{H}^s(A \sim \bigcup_{i \in \mathbb{N}} X_i) = 0.$$

*Proof.* Clearly,  $B_{i+1} \subset B_i$ ,  $X_i \subset X_{i+1}$  and  $X_i$  is a Borel set for  $i \in \mathbb{N}$ . The sets  $\{x \in B_i : \bar{B}_{1/i}(x) \subset U\}$  are open, as may be obtained by adapting [Fed69, 2.9.14].

Define for  $i \in \mathbb{N}$  the set  $A_i$  of all  $x \in U$  such that  $B_{1/i}(x) \subset U$  and

$$\psi(\bar{B}_\varrho(x)) \leq (\varepsilon/\Gamma)^{n-p} \varrho^{n-p} \quad \text{whenever } 0 < \varrho < 1/i.$$

The sets  $A_i$  are closed (cp. [Fed69, 2.9.14]) and satisfy  $A \subset \bigcup_{i \in \mathbb{N}} A_i$ . Let  $C$  denote the set of all  $x \in \text{spt } \mu$  such that

$$\limsup_{\varrho \downarrow 0} \frac{\psi(\bar{B}_\varrho(x))}{\mu(\bar{B}_\varrho(x))^{1-p/n}} < \varepsilon^{n-p}$$

and note  $\mu(U \sim C) = 0$  by [Fed69, 2.9.5]. By [Fed69, 2.10.6, 2.10.19 (4)] it is enough to prove  $a \in X_{2i}$  for a point  $a \in A_i$  with  $\theta^s(\psi \llcorner U \sim A_i, a) = 0$ .

For this purpose the following assertion will be proved. For each  $x \in B_{2i} \cap B_{1/(2i)}(a) \cap C$  there exists  $0 < \varrho < \infty$  with

$$\bar{B}_\varrho(x) \subset B_{2|x-a|}(a) \sim A_i, \quad \mu(\bar{B}_\varrho(x)) < \varepsilon^{-n} \psi(\bar{B}_\varrho(x))^{n/(n-p)}.$$

Choose  $y \in A_i$  with  $|y - x| = \text{dist}(x, A_i)$  and let  $J$  be the set of all  $0 < \varrho < 1/(2i)$  with

$$\mu(\bar{B}_\varrho(x)) < \varepsilon^{-n} \psi(\bar{B}_\varrho(x))^{n/(n-p)}.$$

Then  $J \neq \emptyset$ , because  $x \in B_{2i}$ ,  $\bar{B}_{1/(2i)}(x) \subset B_{1/i}(a) \subset U$ , and, since  $x \in C$ ,  $\inf J > 0$ . Therefore  $t := \inf J$  satisfies

$$\begin{aligned} 0 < t < 1/(2i), \quad \mu(\bar{B}_t(x)) &\leq \varepsilon^{-n} \psi(\bar{B}_t(x))^{n/(n-p)}, \\ \mu(\bar{B}_\varrho(x)) &\geq \varepsilon^{-n} \psi(\bar{B}_\varrho(x))^{n/(n-p)} \quad \text{for } 0 < \varrho < t. \end{aligned}$$

Noting

$$\begin{aligned} |y - x| = \text{dist}(x, A_i) &\leq |x - a| \leq 1/(2i), \quad t + |y - x| < 1/i, \\ \bar{B}_t(x) &\subset \bar{B}_{t+|y-x|}(y) \subset B_{1/i}(y) \subset U, \end{aligned}$$



one estimates

$$\begin{aligned} \psi(\bar{B}_t(x))^{n/(n-p)} &\leq \psi(\bar{B}_{t+|y-x|}(y))^{n/(n-p)} \\ &\leq (\varepsilon/\Gamma)^n (t+|y-x|)^n < \varepsilon^n 2^{-n} (1+|y-x|/t)^n (2n\gamma_n)^{-n} t^n \end{aligned}$$

and, using the inequalities derived from the definition of  $t$  and 2.5,

$$\mu(\bar{B}_t(x)) \leq \varepsilon^{-n} \psi(\bar{B}_t(x))^{n/(n-p)} < 2^{-n} (1+|y-x|/t)^n \mu(\bar{B}_t(x)),$$

hence

$$(1+|y-x|/t)^n > 2^n, \quad |y-x| > t$$

and the assertion follows by taking  $\varrho \in J$  slightly larger than  $t$ .

Let  $0 < r < 1/(2i)$ . Then the preceding assertion in conjunction with Besicovitch's covering theorem implies the existence of countable, pairwise disjoint collections of closed balls  $F_1, \dots, F_{N(n+m)}$  satisfying

$$\begin{aligned} B_{2i} \cap \bar{B}_r(a) \cap C &\subset \bigcup_{j=1}^{N(n+m)} \bigcup_{S \in F_j} S \subset B_{2r}(a) \sim A_i, \\ \mu(S) &< \varepsilon^{-n} \psi(S)^{n/(n-p)} \quad \text{for } S \in \bigcup_{j=1}^{N(n+m)} F_j. \end{aligned}$$

Hence

$$\begin{aligned} \mu(B_{2i} \cap \bar{B}_r(a)) &= \mu(B_{2i} \cap \bar{B}_r(a) \cap C) \\ &\leq \sum_{j=1}^{N(n+m)} \sum_{S \in F_j} \mu(S) \leq \varepsilon^{-n} \sum_{j=1}^{N(n+m)} \sum_{S \in F_j} \psi(S)^{n/(n-p)} \\ &\leq \varepsilon^{-n} \sum_{j=1}^{N(n+m)} (\sum_{S \in F_j} \psi(S))^{n/(n-p)} \leq \varepsilon^{-n} N(n+m) \psi(B_{2r}(a) \sim A_i)^{n/(n-p)} \end{aligned}$$

and the conclusion follows by taking the limit  $r \downarrow 0$ .  $\square$

2.10. *Remark.* This theorem deserves some explanations.

First, note that if  $\|\delta\mu\|$  is absolutely continuous with respect to  $\mu$ , then

$$\mathcal{H}^{n-p}(U \sim A) = 0$$

and if  $p = 1$ , then

$$\mathcal{H}^{n-1}(X \sim A) \leq (\Gamma/\varepsilon)^{n-1} \omega_{n-1} \|\delta\mu\|(X \sim A) \quad \text{for } X \subset U.$$

by [Fed69, 2.10.6, 2.10.19 (3)]. These estimates for the size of  $U \sim A$  suggest that the theorem is most useful if  $n-p \leq s \leq n$ .

Clearly, if  $a \in (\text{spt } \mu) \sim B_i$ , then  $\bar{B}_{1/i}(a) \subset U$  and

$$(2n\gamma_n)^{-n} \varrho^n \leq \mu(\bar{B}_\varrho(a)) \quad \text{for } 0 < \varrho < 1/i$$

by 2.5. On the other hand, since the sets  $\{x \in B_i : \bar{B}_{1/i}(x) \subset U\}$  are open and  $B_{i+1} \subset B_i$ ,  $X_i \subset X_{i+1}$  for  $i \in \mathbb{N}$ , one infers that  $\mathcal{H}^s$  almost all  $a \in A \cap \bigcap_{i \in \mathbb{N}} B_i$  satisfy

$$\lim_{r \downarrow 0} \mu(\bar{B}_r(a)) / r^{sn/(n-p)} = 0.$$

2.11. *Remark.* Similar to the preceding remark one obtains using 2.6 instead of 2.5 that  $\mathcal{H}^n$  almost all  $x \in U$  satisfy

$$\text{either } \theta_*^n(\mu, x) \geq 1 \quad \text{or} \quad \theta^{n^2/(n-p)}(\mu, x) = 0$$

and, in case  $\|\delta\mu\|$  is absolutely continuous with respect to  $\mu$ , that  $\mathcal{H}^{n-p}$  almost all  $x \in U$  satisfy

$$\text{either } \theta_*^n(\mu, x) \geq 1 \quad \text{or} \quad \theta^n(\mu, x) = 0.$$

Moreover, the exponent  $n^2/(n-p)$  cannot be replaced by any larger number as may be seen by taking  $\mu \ll \mathbb{R}^{n+1} \sim T$  with  $\mu$  as in 1.2. Hence, the same holds for the exponent  $sn/(n-p)$  in the last equality of 2.10 if  $s = n$ .

Since  $n^2/(n-1) \geq n+1$  if  $n > 1$ , this remark seems to be the underlying fact used in the proof of [LM08, Theorem 3.4].

2.12. *Remark.* It can happen that  $\mathcal{H}^n(A \cap (\text{spt } \mu) \cap \bigcap_{i \in \mathbb{N}} B_i) > 0$ . In fact taking  $\mu \ll \mathbb{R}^{n+1} \sim T$  with  $\mu$  as in 1.2 one sees from 2.10 and 1.2 (4) that  $T \subset \bigcap_{i \in \mathbb{N}} B_i$ .

### 3. A DIFFERENTIATION THEOREM

In this section the theory of first order differentiation of functions in Lebesgue spaces defined with respect to a rectifiable varifold, similar to the one of Calderón and Zygmund in [CZ61] for the special case of Lebesgue measure, is developed. First, an abstract differentiation theorem for measures, 3.1, is proved which then allows to establish the differentiation theorem for functions, 3.7. The first part of that theorem states an approximability result by functions which are Hölder continuous with exponent  $\alpha$  which, in the particular case  $\alpha = 1$  implies a Rademacher type theorem for differentiability in Lebesgue spaces, see 3.9. The second part of 3.7 may in fact be regarded as an application of this theory and is designed for use in [Men08b].

3.1. **Theorem.** *Suppose  $m, n, p, U$ , and  $\mu$  are as in 2.1,  $\nu$  measures  $U$  with  $\nu(U \setminus \text{spt } \mu) = 0$ ,  $A$  is  $\mu$  measurable with  $\nu(A) = 0$ , and  $1 \leq q < \infty$ . In case  $p < n$  additionally suppose for some  $1 \leq r \leq \infty$  and some nonnegative function  $f \in L^r_{\text{loc}}(\mu)$  that*

$$\nu = f\mu \quad \text{and} \quad q \leq 1 + (1 - 1/r) \frac{p}{n-p}.$$

Then for  $\mathcal{H}^n$  almost all  $a \in A$

$$\limsup_{s \downarrow 0} \nu(\bar{B}_s(a)) / s^{nq} \quad \text{equals either } 0 \text{ or } \infty.$$

*Proof.* For  $i \in \mathbb{N}$  let  $B_i$  denote the set of all  $x \in U$  such that either  $\bar{B}_{1/i}(x) \not\subset U$  or

$$\|\delta\mu\|(\bar{B}_\varrho(x)) > (2\gamma_n)^{-1} \mu(\bar{B}_\varrho(x))^{1-1/n} \quad \text{for some } 0 < \varrho < 1/i.$$

First, the case  $A \subset \{x \in U : \theta^{*n}(\mu, x) > 0\}$  will be treated. In this case  $A$  is measurable and  $\sigma$  finite with respect to  $\mathcal{H}^n$  by [Fed69, 2.10.19 (1) (3)]. Hence one may assume  $A$  to be compact. Define

$$A_i = \{a \in A : \nu(\bar{B}_s(a)) / \leq i s^{nq} \text{ for } 0 < s < 1/i\}$$

whenever  $i \in \mathbb{N}$ ,  $1/i < \text{dist}(A, \mathbb{R}^{n+m} \setminus U)$ . The sets  $A_i$  are compact (cp. [Fed69, 2.9.14]) and their union equals

$$\{a \in A : \limsup_{s \downarrow 0} \nu(\bar{B}_s(a)) / s^{nq} < \infty\}.$$

It therefore suffices to show for each  $i \in \mathbb{N}$  with  $1/i < \text{dist}(A, \mathbb{R}^{n+m} \setminus U)$

$$\lim_{s \downarrow 0} \nu(\bar{B}_s(a)) / s^{nq} = 0 \quad \text{for } \mathcal{H}^n \text{ almost all } a \in A_i.$$

In fact, this equality will be proved for all  $a \in A_i$  satisfying

$$\begin{aligned} \|\delta\mu\|(\{a\}) &= 0, \quad \theta^n(\mu \llcorner U \sim A_i, a) = 0, \quad \theta^n(f^r \mu, a) = 0 \quad \text{if } r < \infty, \\ \limsup_{s \downarrow 0} \mu(B_j \cap \bar{B}_s(a)) / s^{n^2/(n-p)} &= 0 \quad \text{for some } j \in \mathbb{N}, j \geq 2i, \text{ if } p < n \end{aligned}$$

as  $\mathcal{H}^n$  almost all  $a \in A_i$  do according to [Fed69, 2.10.19 (3) (4)] and 2.9.

In case  $p = n$  one chooses  $j \in \mathbb{N}$ ,  $j \geq 2i$ , using 2.5 such that

$$B_j \cap \bar{B}_{1/j}(a) = \emptyset.$$

Let  $0 < s < 1/j$ . For  $x \in \bar{B}_s(a) \cap (\text{spt } \mu) \sim (B_j \cup A_i)$  there exists  $y \in A_i$  with  $|x - y| = \text{dist}(x, A_i)$ , hence

$$\begin{aligned} t &:= |x - y| \leq |x - a| \leq s < 1/j \leq 1/(2i), \\ \bar{B}_{|x-y|/2}(x) &\subset \bar{B}_{3|x-y|/2}(y) \cap \bar{B}_{2s}(a) \sim A_i, \\ \nu(\bar{B}_{t/2}(x)) &\leq \nu(\bar{B}_{3t/2}(y)) \leq i3^{nq}(t/2)^{nq} \leq c\mu(\bar{B}_{t/2}(x))^q \end{aligned}$$

where  $c = i3^{nq}(2\gamma_n n)^{nq}$ . Therefore one infers from Besicovitch's covering theorem the existence of countable, pairwise disjoint collections  $F_1, \dots, F_{N(n+m)}$  of closed balls such that

$$\begin{aligned} \bar{B}_s(a) \cap (\text{spt } \mu) \sim (B_j \cup A_i) &\subset \bigcup_{k=1}^{N(n+m)} \bigcup_{S \in F_k} S \subset \bar{B}_{2s}(a) \sim A_i, \\ \nu(S) &\leq c\mu(S)^q \quad \text{whenever } S \in \bigcup_{k=1}^{N(n+m)} F_k, \end{aligned}$$

hence

$$\begin{aligned} \nu(\bar{B}_s(a) \sim B_j) &= \nu(\bar{B}_s(a) \cap (\text{spt } \mu) \sim (B_j \cup A_i)) \leq cN(n+m)\mu(\bar{B}_{2s}(a) \sim A_i)^q, \\ \lim_{s \downarrow 0} \nu(\bar{B}_s(a) \sim B_j) / s^{nq} &= 0. \end{aligned}$$

To conclude the proof of the first case, one observes

$$\begin{aligned} \nu(B_j \cap \bar{B}_s(a)) &= 0 \quad \text{if } p = n, \\ \nu(B_j \cap \bar{B}_s(a)) &\leq \mu(B_j \cap \bar{B}_s(a))^{1-1/r} \|f\|_{L^r(\mu \llcorner \bar{B}_s(a))} \quad \text{if } p < n \end{aligned}$$

implying

$$\lim_{s \downarrow 0} \nu(B_j \cap \bar{B}_s(a)) / s^{nq} = 0$$

because  $(1 - 1/r)\frac{n}{n-p} + 1/r \geq q$  in case  $p < n$ .

It remains to treat the case  $A \subset \{x \in U : \theta^n(\mu, x) = 0\}$ . Using 2.5 and 2.11 one obtains

$$A \cap \text{spt } \mu \text{ is countable} \quad \text{if } p = n,$$

$$\theta^{n^2/(n-p)}(\mu, a) = 0 \quad \text{for } \mathcal{H}^n \text{ almost all } a \in A \quad \text{if } p < n$$

and the claim follows by using Hölder's inequality as in the preceding paragraph noting by [Fed69, 2.10.19 (4)]

$$\theta^n(f^r \mu, a) = 0 \quad \text{for } \mathcal{H}^n \text{ almost all } a \in A \quad \text{if } r < \infty. \quad \square$$

**3.2. Remark.** This theorem generalises [Fed69, 2.9.17] and [CZ61, Theorem 10 (ii)]. The case treated by Federer roughly corresponds to the case  $p = n$ ,  $q = 1$  with  $\mu$  satisfying a doubling condition. The case treated by Calderón and Zygmund corresponds to  $p = n$ ,  $m = 0$ ,  $\mu = \mathcal{L}^{n+m}$  and  $\nu$  absolutely continuous with respect to  $\mu$ . The method of proof is based on Federer's proof and 2.9 is used because of the absence of a doubling condition.

**3.3. Remark.** If  $q = 1$ , the condition  $\nu(U \sim \text{spt } \mu) = 0$  cannot be omitted as may be seen from [Fed69, 2.9.18 (2)].

**3.4. Remark.** If  $p < n$  the condition  $q \leq 1 + (1 - 1/r)p/(n - p)$  cannot be omitted as can be shown using 1.2. In fact given  $\mu$  and  $T$  as in 1.2 a counterexample is provided by  $\nu := \mu \llcorner \mathbb{R}^{n+1} \sim T$  in case  $r = \infty$  and if  $1 < r < \infty$  applying 1.2 (5) with  $s = nq$  and  $\alpha_1 q_1 = \alpha_2 q_2$  slightly larger than  $\frac{np}{n-p}$  yields a function  $f$  such that  $\nu := f\mu$  does not satisfy the conclusion of 3.1. Finally, if  $r = 1$  the condition is also violated for a slightly larger  $r$ , hence reducing this case to the previous one.

**3.5. Remark.** Note that the preceding two remarks remain valid if  $\mathcal{H}^n$  is replaced by  $\mu$  in the conclusion of 3.1.

**3.6. Definition.** Whenever  $A$  is  $\phi$  measurable set with  $0 < \phi(A) < \infty$  and  $f \in L^1(\phi \llcorner A)$  one defines  $f_A f d\phi = \phi(A)^{-1} \int_A f d\phi$ .

**3.7. Theorem.** Suppose  $n, m, p, U$ , and  $\mu$  are as in 2.1,  $Z$  is a separable Banach space,  $f : U \rightarrow Z$  is  $\mu$  measurable,  $0 < \alpha \leq 1$ ,  $1 \leq q < \infty$ , and  $A$  is the set of all  $x \in \text{spt } \mu$  such that

$$\limsup_{\varrho \downarrow 0} \varrho^{-\alpha q} f_{\bar{B}_\varrho(x)} |f(\xi) - z|^q d\mu(\xi) < \infty \quad \text{for some } z \in Z.$$

In case  $p < n$  additionally suppose that  $f \in L^r_{\text{loc}}(\mu, Z)$  for some  $1 \leq r \leq \infty$  satisfying

$$\alpha q/n \leq \left(1 - \frac{q}{r}\right) \frac{p}{n-p}.$$

Then  $A$  is a Borel set and the following two statements hold:

- (1) For every  $\varepsilon > 0$  there exists a function  $g : U \rightarrow Z$  which locally satisfies a Hölder condition with exponent  $\alpha$  such that

$$\mu(A \cap \{x : f(x) \neq g(x)\}) \leq \varepsilon.$$

Moreover, for every function  $g$  which locally satisfies a Hölder condition with exponent  $\alpha$  there holds

$$\lim_{\varrho \downarrow 0} \varrho^{-\alpha q} f_{\bar{B}_\varrho(x)} |f(\xi) - g(\xi)|^q d\mu(\xi) = 0$$

for  $\mu$  almost all  $x \in A$  with  $f(x) = g(x)$ .

- (2) If  $\varepsilon > 0$ ,  $D_i(a)$  denotes for  $a \in \text{dmn } f$ ,  $i \in \mathbb{N}$  the set of all  $x \in U$  such that either  $\bar{B}_{1/i}(x) \not\subset U$  or

$$\int_{\bar{B}_\varrho(x)} |f(\xi) - f(a)|^q d\mu(\xi) > \varepsilon \mu(\bar{B}_\varrho(x)) \quad \text{for some } 0 < \varrho < 1/i,$$

$Y_i$  denotes for  $i \in \mathbb{N}$  the set of all  $a \in U$  such that

$$\lim_{r \downarrow 0} \mu(D_i(a) \cap \bar{B}_r(a)) / r^{n+\alpha q} = 0,$$

then the sets  $Y_i$  are  $\mu$  measurable and

$$\mu(A \sim \bigcup \{Y_i : i \in \mathbb{N}\}) = 0.$$

*Proof of (1).* Let  $\pi : \mathbb{R}^{n+m} \times Z \rightarrow \mathbb{R}^{n+m}$  denote the projection and for  $i \in \mathbb{N}$  let  $E_i$  denote the set of all  $(x, z) \in \text{spt } \mu \times Z$  such that  $B_{1/i}(x) \subset U$  and

$$f_{\bar{B}_\varrho(x)} |f(\xi) - z|^q d\mu(\xi) \leq i \varrho^{\alpha q} \quad \text{whenever } 0 < \varrho < 1/i.$$

Then  $E_i$  is closed (cp. [Fed69, 2.9.14]),  $\pi|_{E_i}$  is univalent, and both  $\pi(E_i)$  and  $A = \bigcup \{\pi(E_i) : i \in \mathbb{N}\}$  are Borel sets by [Fed69, 2.2.10].

To prove the first part of (1), the problem is reduced to the case  $\mu = \mathcal{L}^n \llcorner K$  for some compact set  $K$  (not necessarily satisfying a condition on  $\delta\mu$ ) via [Fed69, 3.2.18]. This case can then be treated by adapting [Fed69, 3.1.8, 3.1.14], see also [Ste70, VI.2.2.2].

Concerning the second half of (1), one observes that every such function  $g$  satisfies

$$\limsup_{\varrho \downarrow 0} \varrho^{-\alpha q} f_{\bar{B}_\varrho(x)} |f(\xi) - g(\xi)|^q d\mu(\xi) < \infty$$

for  $\mathcal{L}^n$  almost all  $x \in A$  with  $f(x) = g(x)$  and 3.1 may be applied with  $\nu, r, q, A$  replaced by  $|f - g|^q \mu, r/q, 1 + \alpha q/n, \{x \in A : f(x) = g(x)\}$  if  $p < n$  and  $|f - g|^q \mu, \infty, 1 + \alpha q/n, \{x \in A : f(x) = g(x)\}$  else.  $\square$

*Proof of (2).* For any  $0 < \varrho < \infty$ ,  $x \in \mathbb{R}^{n+m}$  denote by  $b_{x,\varrho}$  the characteristic function of  $\bar{B}_\varrho(x)$ , define  $U_i = \{x \in U : \text{dist}(x, \mathbb{R}^{n+m} \setminus U) > 1/i\}$  and observe that the function mapping  $(a, x, \xi) \in (\text{dmn } f) \times U \times (\text{dmn } f)$  onto

$$b_{x,\varrho}(\xi)|f(\xi) - f(a)|^q - \varepsilon b_{x,\varrho}(\xi)$$

is  $\mu \times \mu \times \mu$  measurable for every  $0 < \varrho < \infty$ . Applying Fubini's theorem, one infers that the function mapping  $(a, x) \in (\text{dmn } f) \times U_i$  onto

$$\sup \left\{ \int_{\bar{B}_\varrho(x)} |f(\xi) - f(a)|^q d\mu(\xi) - \varepsilon \mu(\bar{B}_\varrho(x)) : 0 < \varrho < 1/i \right\}$$

is  $\mu \times (\mu \llcorner U_i)$  measurable for each  $i \in \mathbb{N}$ , since the supremum by restricted to a countable, dense subset of  $\{\varrho : 0 < \varrho < 1/i\}$ . For the same reason

$$\sup \{r^{-m-\alpha q} \mu(X_i(a) \cap \bar{B}_r(a)) : 0 < r < 1/j\}$$

depends  $\mu$  measurably on  $a$  for each  $i, j \in \mathbb{N}$ . Therefore the sets  $Y_i$  are  $\mu$  measurable.

For  $i \in \mathbb{N}$  let  $A_i$  denote the set of all  $a \in (\text{dmn } f) \cap (\text{spt } \mu)$  such that  $B_{1/i}(a) \subset U$  and whenever  $0 < \varrho < 1/i$

$$\mu(\bar{B}_\varrho(a)) \leq i\varrho^n, \quad \int_{\bar{B}_\varrho(a)} |f(\xi) - f(a)|^q d\mu(\xi) \leq i\varrho^{\alpha q}.$$

$A_i$  are  $\mu$  measurable sets as may be verified using the first paragraph of the proof of (1) and noting the fact that the last condition may be replaced by the two conditions

$$a \in \pi(E_i), \quad \lim_{\varrho \downarrow 0} \int_{\bar{B}_\varrho(a)} f(\xi) d\mu(\xi) = f(a).$$

Note  $\mu(A \sim \bigcup \{A_i : i \in \mathbb{N}\}) = 0$ . For  $i \in \mathbb{N}$  let  $C_i$  denote the set of all  $x \in \text{spt } \mu$  such that either  $\bar{B}_{1/i}(x) \not\subset U$  or

$$\|\delta\mu\| \bar{B}_\varrho(x) > (2\gamma_n)^{-1} \mu(\bar{B}_\varrho(x))^{1-1/n} \quad \text{for some } 0 < \varrho < 1/i.$$

Moreover, define

$$X_i = \{x \in U : \theta^{n+\alpha q}(\mu \llcorner C_i, x) = 0\} \quad \text{for } i \in \mathbb{N},$$

note  $n + \alpha q \leq n^2/(n-p)$  if  $p < n$ , and observe by 2.9 in case  $p < n$ , by 2.5 in case  $p = n$ , that

$$\mu(U \sim \bigcup \{X_i : i \in \mathbb{N}\}) = 0.$$

Using (1), one constructs sequences  $K_i$  of compact subsets of  $U$  and  $g_i : U \rightarrow Z$  such that

$$K_i \subset A_j \text{ for some } j \in \mathbb{N}, \quad f|K_i = g_i|K_i,$$

$g_i$  locally satisfies a Hölder condition with exponent  $\alpha$ ,

$$\mu(A \sim \bigcup \{K_i : i \in \mathbb{N}\}) = 0.$$

Also note  $A_i \subset A_{i+1}$ ,  $C_i \supset C_{i+1}$ , and  $X_i \subset X_{i+1}$  for  $i \in \mathbb{N}$ .

From the observations of the preceding paragraph, [Fed69, 2.10.6, 2.10.19 (4)] and (1) it follows that it is enough to prove  $a \in \bigcup \{X_j : j \in \mathbb{N}\}$  whenever  $a \in A$  satisfies for some  $i \in \mathbb{N}$ , some compact set  $K$ , and some  $g : U \rightarrow Z$

$$a \in X_i, \quad a \in K \subset A_i, \quad \theta^n(\mu \llcorner U \sim K, a) = 0, \quad g|K = f|K,$$

$g$  locally satisfies a Hölder condition with exponent  $\alpha$ ,

$$r^{-n-\alpha q} \int_{\bar{B}_r(a)} |f(\xi) - g(\xi)|^q d\mu(\xi) \rightarrow 0 \quad \text{as } r \downarrow 0.$$

For this purpose define  $h = \sup\{|g(x) - g(y)|/|x - y|^\alpha : x, y \in K, x \neq y\}$ , choose  $j \in \mathbb{N}$ ,  $j \geq 2i$ , and  $0 < R < 1/(2i)$  satisfying

$$2^{q-1}i^2((1/j + R)^{\alpha q} + h^q(2R)^{\alpha q}) \leq \varepsilon 2^{-n}(2\gamma_n n)^{-n}.$$

Next, it will be shown

$$\int_{\bar{B}_\varrho(x)} |f(\xi) - f(a)|^q d\mu(\xi) \leq \varepsilon 2^{-n} (1 + |\zeta - x|/\varrho)^n \mu(\bar{B}_\varrho(x))$$

whenever  $x \in \text{spt } \mu \cap \bar{B}_r(a) \sim C_i$ ,  $\zeta \in K$ ,  $|\zeta - x| = \text{dist}(x, K)$ ,  $0 < r \leq R$ ,  $0 < \varrho < 1/j$ . Noting

$$\begin{aligned} \varrho + |\zeta - x| &< 1/j + |x - a| \leq 1/j + R < 1/i, \\ \bar{B}_\varrho(x) &\subset \bar{B}_{\varrho+|\zeta-x|}(\zeta) \subset B_{1/i}(\zeta) \subset U, \\ |\zeta - a| &\leq |\zeta - x| + |x - a| \leq 2|x - a| \leq 2R, \\ 2^{q-1}i^2((\varrho + |\zeta - x|)^{\alpha q} + h^q|\zeta - a|^{\alpha q}) &\leq \varepsilon 2^{-n} (2\gamma_n n)^{-n}, \end{aligned}$$

one estimates

$$\begin{aligned} \int_{\bar{B}_\varrho(x)} |f(\xi) - f(a)|^q d\mu(\xi) &\leq \int_{\bar{B}_{\varrho+|\zeta-x|}(\zeta)} |f(\xi) - f(a)|^q d\mu(\xi) \\ &\leq 2^{q-1} \left( \int_{\bar{B}_{\varrho+|\zeta-x|}(\zeta)} |f(\xi) - f(\zeta)|^q d\mu(\xi) + |f(\zeta) - f(a)|^q \mu(\bar{B}_{\varrho+|\zeta-x|}(\zeta)) \right) \\ &\leq 2^{q-1}i((\varrho + |\zeta - x|)^{\alpha q} + h^q|\zeta - a|^{\alpha q}) \mu(\bar{B}_{\varrho+|\zeta-x|}(\zeta)) \\ &\leq \varepsilon 2^{-n} (2\gamma_n n)^{-n} (1 + |\zeta - x|/\varrho)^n \varrho^n \end{aligned}$$

and 2.5 implies the assertion. Therefore, if

$$\int_{\bar{B}_\varrho(x)} |f(\xi) - f(a)|^q d\mu(\xi) > \varepsilon \mu(\bar{B}_\varrho(x)),$$

then

$$(1 + |\zeta - x|/\varrho)^n > 2^n, \quad \varrho < |\zeta - x| \leq |x - a| \leq r, \quad |x - a| + \varrho < 2r, \\ \bar{B}_\varrho(x) \subset B_{2r}(a) \sim K \subset U.$$

This implies that for each  $x \in \text{spt } \mu \cap \bar{B}_r(a) \cap D_j(a) \sim C_i$  with  $0 < r \leq R$  there exists  $0 < \varrho < 1/j$  such that

$$\bar{B}_\varrho(x) \subset B_{2r}(a) \sim K \subset U, \quad \int_{\bar{B}_\varrho(x)} |f(\xi) - f(a)|^q d\mu(\xi) > \varepsilon \mu(\bar{B}_\varrho(x)),$$

because  $a \in A_i$ ,  $x \in \bar{B}_r(a)$  implies  $\bar{B}_{1/j}(x) \subset U$ . Hence one infers from Besicovitch's covering theorem

$$\mu(\bar{B}_r(a) \cap D_j(a) \sim C_i) \leq N(n+m)\varepsilon^{-1} \int_{B_{2r}(a) \sim K} |f(\xi) - f(a)|^q d\mu(\xi)$$

for  $0 < r \leq R$ . Recalling  $a \in X_i$ , the proof may be concluded by showing

$$r^{-n-\alpha q} \int_{B_{2r}(a) \sim K} |f(\xi) - f(a)|^q d\mu(\xi) \rightarrow 0 \quad \text{as } r \downarrow 0$$

which is a consequence of

$$\begin{aligned} &\int_{B_{2r}(a) \sim K} |f(\xi) - f(a)|^q d\mu(\xi) \\ &\leq 2^{q-1} \left( \int_{B_{2r}(a)} |f(\xi) - g(\xi)|^q d\mu(\xi) + \int_{B_{2r}(a) \sim K} |g(\xi) - g(a)|^q d\mu(\xi) \right), \\ &\int_{B_{2r}(a) \sim K} |g(\xi) - g(a)|^q d\mu(\xi) \leq \mu(B_{2r}(a) \sim K) (h_0)^q (2r)^{\alpha q} \end{aligned}$$

for  $0 < r \leq R$  with  $h_0 = \sup\{|g(x) - g(y)|/|x - y|^\alpha : x, y \in \bar{B}_R(a), x \neq y\}$ .  $\square$

**3.8. Remark.** If  $p < n$  the assumption  $\alpha q/n \leq (1 - q/r)p/(n - p)$  cannot be omitted in order to obtain the second part of (1) as may be seen from the family of examples constructed in 1.2; in fact one can take  $\alpha_1 = \alpha_2 = \alpha$ ,  $q_1 = q_2 = q$ , and  $f = \chi_{\mathbb{R}^{n+1}} \sim_T$  in case  $r = \infty$ , and in case  $r < \infty$  one can assume  $q < r$  and apply 1.25 with  $r$ ,  $s$ ,  $\alpha_1 = \alpha_2$ ,  $q_1 = q_2$  replaced by  $r/q$ ,  $\alpha q + n$ , 1 and a number slightly larger than  $np/(n - p)$  to obtain a function  $f \in L_{\text{loc}}^{r/q}(\mu)$  such that the second statement of (1) does not hold for  $f, g$  replaced by  $f^{1/q}, 0$ .

3.9. *Remark.* If  $\dim Z < \infty$  and  $\alpha = 1$ , (1) in conjunction with [Fed69, 3.2.18, 3.1.16] implies that for  $\mu$  almost all  $a \in A$

$$\lim_{\varrho \downarrow 0} \int_{\bar{B}_\varrho(a)} (|f(\xi) - f(a) - \langle (T_a \mu)(\xi - a), (\mu, n) \rangle \text{ap } Df(a) \rangle| / |\xi - a|^q) d\mu(\xi) = 0$$

where the notion of approximate differentials, see [Fed69, 3.2.16], is employed.

3.10. *Remark.* (2) can be seen in two ways as a refinement of the simple fact that

$$\theta^{*n+\alpha q}(\mu \llcorner \{x \in U : |f(x) - f(a)|^q > \varepsilon\}, a) \leq \varepsilon^{-1} \theta^{*n+\alpha q}(|f(\cdot) - f(a)|^q \mu, a) < \infty$$

whenever  $a \in A$ . Firstly,  $|f(x) - f(a)|^q > \varepsilon$  is replaced in the definition of  $D_i(a)$  by  $\int_{\bar{B}_\varrho(x)} |f(\xi) - f(a)|^q d\mu(\xi) > \varepsilon$  for some  $0 < \varrho < 1/i$ . Secondly, in the conclusion  $\theta^{*n+\alpha q}(\mu \llcorner D_i(a), a) = 0$  occurs instead of  $\theta^{*n+\alpha q}(\mu \llcorner D_i(a), a) < \infty$ . Whereas the first improvement is vital for the applications in [Men08c], the second one is only used under the stronger assumption

$$\lim_{r \downarrow 0} r^{-n-\alpha q} \int_{\bar{B}_r(x)} |f(\xi) - z|^q d\mu(\xi) = 0 \quad \text{for some } z \in Z$$

for  $\mu$  almost all  $x \in U$ .

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