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Analytic three-loop solutions for $\mathcal{N}=4$ SYM twist operators

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Abstract

We introduce a method to obtain the analytic solution of the higher-order Baxter equation for twist-two and twist-three operators of planar $\mathcal{N}=4$ SYM. Our result proofs the conjectured formula for the three-loop anomalous dimension of twist-two operators. As such we derive the maximally transcendental part of the corresponding three-loop QCD result from the maximal supersymmetric gauge theory in four dimension purely by methods of integrability.

1. Introduction

The exploration of the AdS/CFT correspondence using the methods of integrability has led to a first insight into its interpolating property between weak and strong coupling. Since the discovery of integrable structures on both sides of the correspondence [1] many techniques have been introduced and developed which will hopefully lead to the complete solution of the planar spectral problem in a finite volume.

Integrable structures in four-dimensional quantum field theories are not new and have also been found in non-supersymmetric theories before, see for example [2–4] and [5] and references therein. However, superconformal invariance preserves integrability of the maximal supersym-

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metric planar $\mathcal{N}=4$ SYM to higher, possibly all, orders. The factorized scattering of elementary excitations of a super-spin chain [6] describing $\mathcal{N}=4$ SYM at one-loop is extended to higher-loop order [7] and allows the determination of the asymptotic spectrum in terms of long-range Bethe ansatz equations [8]. The ambiguity of a phase factor of the S-matrix, determined by symmetry solely [9], is also fixed by a constraining crossing-symmetry [10] supplemented by an unequivocally specification of its solution [11,13].

The operators which gained the most attention so far are twist operators with lowest possible anomalous dimension, whose thermodynamical behavior is governed by the so-called universal scaling function [12,13]. It was shown that the strong coupling expansion of this scaling function, see [14] and [15] and references therein, correctly reproduces the know string results [16] of the corresponding dual state to two-loop order in perturbation theory.

To this class of operators belong the shortest possible local composite operators of the $\mathcal{N}=4$ theory, the twist-two operators. For a simple representative of these one starts from the protected half-BPS states Tr $\mathcal{Z}^2$ and inserts M covariant derivatives $\mathcal{D}$

$$\mathrm{Tr}\big(\mathcal{Z}\mathcal{D}^M\mathcal{Z}\big)+\cdots. \tag{1.1}$$

In the spin chain picture this is a non-compact $\mathfrak{sl}(2)$ spin $=-1/2$ length-two Heisenberg magnet with M magnons. The dots indicate the mixing of all states where the covariant derivatives may act on any of the two fields. For each even M there is precisely one highest weight non-BPS state whose total scaling dimension is

$$\Delta=2+M+\gamma(g,M),\quad \text{with}\quad \gamma(g,M)=\sum_{\ell=1}^{\infty}\gamma_{2\ell}(M)g^{2\ell}, \tag{1.2}$$

where $\gamma(g,M)$ is the anomalous part of the dimension that depends on the coupling constant $g^2=\lambda/16\pi^2$ and $\lambda=Ng_{\mathrm{YM}}^2$ is the 't Hooft coupling constant.

For twist-two closed expressions for the anomalous part $\gamma(g,M)$ of the dimension are known to two-loop order from explicit field-theory calculations [17], and at three-loops from a solid conjecture [18] extracted by the principle of maximum transcendentality [19] from the QCD splitting functions at three-loops [20]. To this order the anomalous part $\gamma(g,M)$ of the dimension can also be reliably computed by the asymptotic Bethe ansatz [7] for fixed values of M. Up to relatively high values of M it was checked [7] that it coincides with the two- and three-loop anomalous dimensions of the twist-two operators, which are known in terms of nested harmonic sums as obtained in [17,18]. Due to wrapping effects that appear at four-loop order, these operators have also been used to show the incorrectness of the asymptotic Bethe ansatz at this order of perturbation theory [21].

It had not been known how to obtain these closed results in a rigorous analytical way from the Bethe ansatz. This problem will finally be solved in the present paper. By assuming the principle of maximum transcendentality it is possible to make a general ansatz that contains harmonic sums of the proper degree with unknown coefficients and to fit these with a high numerical precision from the Bethe ansatz, but this principle is not granted in general for other natural operators [22].

Therefore, we introduce a systematic approach to derive these results from the Baxter equation that can be obtained from the $\mathfrak{sl}(2)$ Bethe ansatz. As such we derive the maximally transcendental part of the corresponding QCD computation from the maximal supersymmetric Yang–Mills theory in four dimensions solely by means of integrability. As an application of our method we also derive the three-loop anomalous dimension of twist-three operators.

The method presented in this paper is based on deforming the one-loop solution. The precise structure of this deformations can either be obtained by taking the Mellin transformation of the perturbative Baxter equation or simply by an educated guess, based on comparing the higher-loop Baxter equation with the difference equation of a general deformation of the one-loop solution. In the main text we follow the latter, while the aspects of the solution in Mellin space are given in the appendix.

We begin our analysis by introducing the perturbative Baxter equation, which has been proposed in [23]. After constructing the solution for twist-two operators in an pedagogical way, we will apply the same techniques to twist-three operators but only state the final result. Lengthy and detailed computations are shifted to the appendix as well as the complete calculation of anomalous dimension. We close with a summary and give some ideas of further applications.

2. Baxter equation

The Bethe roots that parametrize a solution of the Bethe ansatz equation obey the following expansion in the coupling constant g^2 up to a loop order ℓ

$$u_k(g^2) = \sum_{i=0}^{\ell-1} g^{2i} u_k^{(i)}. \tag{2.1}$$

They are given by zeros of the corresponding Baxter function $Q(u)$ which also exhibits a similar expansion

$$Q(u) = \prod_{k=1}^{M}\bigl(u - u_k(g^2)\bigr) = \sum_{i=0}^{\ell-1} g^{2i} Q^{(i)}. \tag{2.2}$$

To three-loop-order $Q(u)$ satisfies the Baxter equation in form of a second-order finite difference equation [23]

$$\Delta_+(x^+)Q(u+i) + \Delta_-(x^-)Q(u-i) = t_L(x)Q(u). \tag{2.3}$$

The $x^\pm$ variables are defined through the spectral parameter u in the following way [24]

$$x^\pm(u) = x\left(u \pm \frac{i}{2}\right), \qquad x(u) = \frac{u}{2}\left(1 + \sqrt{1 - \frac{4g^2}{u^2}}\right). \tag{2.4}$$

To the first three orders of perturbation theory, $\ell = 2$, the functions $\Delta_\pm$ have the following form

$$\Delta_\pm(x) = x^L \exp\left(\frac{ig^2}{x}\gamma_\pm^{(0)} + \frac{ig^4}{x}\gamma_\pm^{(1)} - \frac{g^4}{x^2}\frac{d^2}{du^2}\log Q^{(0)}(u)\bigg|_{u=\pm\frac{i}{2}}\right). \tag{2.5}$$

The auxiliary transfer matrix $t_L(x)$ for twist-two and three operators is given by[2]

$$t_L(x) = 2u^L + q_2(g^2)u^{L-2}, \qquad L = 2, 3, \tag{2.6}$$

where the charge q_2 is also to be expanded in g^2. Using (2.2) one can read off the coefficient functions $q_2^{(i)}$, see below. The resulting expressions for the anomalous dimensions are given by

$$\gamma(g^2) = \gamma_+(g^2) - \gamma_-(g^2) \tag{2.7}$$

[2] Note, that q_3 is zero for the twist-three ground state [21].

and

$$\gamma_\pm(g^2) = \left(2g^2\frac{d}{du} + g^4\frac{d^3}{du^3} + \frac{g^6}{6}\frac{d^5}{du^5}\right)\left(i\log Q(u)\right)\Big|_{u=\pm\frac{i}{2}} = \sum_{i=0}^{\ell-1} g^{2(i+1)}\gamma_\pm^{(i)}. \tag{2.8}$$

3. Twist-two operators

We will now explain how to find the higher order Baxter function for operators that have been analyzed to a large extend in AdS/CFT, twist-two operators. After restating the one-loop problem with its known solution which for completeness is also given in Appendix A, we focus on the explicit construction of the higher loop solution. The main idea of the construction is based on deforming the one-loop solution in a suitable manner. As mentioned before, one can also perform the complete computation in Mellin space and verify the result, as included in Appendix A.

The leading order Baxter equation is given by

$$\left(u+\frac{i}{2}\right)^2 Q^{(0)}(u+i) + \left(u-\frac{i}{2}\right)^2 Q^{(0)}(u-i) - t_2^{(0)}(u)Q^{(0)}(u) = 0, \tag{3.1}$$

with the leading order transfer matrix given by

$$t_2^{(0)}(u) = 2u^2 + q_2^{(0)}, \quad q_2^{(0)} = -M(M+1) - \frac{1}{2}. \tag{3.2}$$

The solution to this equation is given in terms of the continuous Hahn polynomial

$$Q^{(0)}(u) = {}_3F_2\left(\left.\begin{matrix}-M,\ M+1,\ \frac{1}{2}+iu\\ 1,\ 1\end{matrix}\right|1\right). \tag{3.3}$$

It is rather straightforward to compute the anomalous dimension using the above explicit formula for the Baxter function

$$\gamma^{(0)} = 8S_1(M). \tag{3.4}$$

At next-to-leading order the Baxter equation can be obtained by expanding the corresponding asymptotic all-loop equation, see [23], and is given by

$$\begin{aligned}&\left(u+\frac{i}{2}\right)^2 Q^{(1)}(u+i) + \left(u-\frac{i}{2}\right)^2 Q^{(1)}(u-i) - t_2^{(0)}(u)Q^{(1)}(u)\\ &\quad= \left(2 - i\frac{\gamma^{(0)}}{2}\left(u+\frac{i}{2}\right)\right)Q^{(0)}(u+i) + \left(2 + i\frac{\gamma^{(0)}}{2}\left(u-\frac{i}{2}\right)\right)Q^{(0)}(u-i)\\ &\qquad+ t_2^{(1)}(u)Q^{(0)}(u),\end{aligned} \tag{3.5}$$

with $t_2^{(1)} = -(4 + \frac{1}{2}\gamma^{(0)}(2M+1))$. In order to solve the above equation one is lead to consider the following class of continuous Hahn polynomials

$$y(u) = {}_3F_2\left(\left.\begin{matrix}-n,\ n+a+b+c+d-1,\ a+iu\\ a+c,\ a+d\end{matrix}\right|1\right), \tag{3.6}$$

which satisfy the difference equation [25]

$$\begin{aligned}n(n+a+b+c+d-1)y(u) &= B(u)y(u+i)\\ &\quad- \left[B(u)+D(u)\right]y(u) + D(u)y(u-i)\end{aligned} \tag{3.7}$$

where

$$B(u) = (c - iu)(d - iu), \qquad D(u) = (a + iu)(b + iu).$$

By a deformation of the one-loop solution (3.3) we understand a suitable choice of a, b, c and d, such that when the deformation parameters are set to zero, the function (3.6) coincides with (3.3). One of such deformations is given by

$$Q_\delta^A(u) = {}_3F_2\left(\left. \begin{matrix} -M, \ M+1+2\delta, \ \frac{1}{2}+iu \\ 1+\delta, \ 1 \end{matrix} \right| 1 \right). \tag{3.8}$$

Upon differentiating once w.r.t. δ at $\delta = 0$ the resulting difference equation reads

$$\begin{aligned} &\left(u+\frac{i}{2}\right)^2 Q_0^{A\prime}(u+i) + \left(u-\frac{i}{2}\right)^2 Q_0^{A\prime}(u-i) - t_2^{(0)}(u) Q_0^{A\prime}(u) \\ &\quad = -i\left(u+\frac{i}{2}\right) Q^{(0)}(u+i) + i\left(u-\frac{i}{2}\right) Q^{(0)}(u-i) - (2M+1) Q^{(0)}(u). \end{aligned} \tag{3.9}$$

In the above formula we have used the following notation

$$\left.\frac{\partial}{\partial \delta} Q_\delta^A(u)\right|_{\delta=0} = Q_0^{A\prime}(u) \quad \text{and} \quad Q_\delta^A(u)|_{\delta=0} = Q^{(0)}. \tag{3.10}$$

Thus using the linearity of (3.5) one can multiply (3.8) with $\frac{\gamma^{(0)}}{2}$ to obtain the part of $Q^{(1)}(u)$ which is proportional to the one-loop anomalous dimension.

The missing parts of the full solution can be found by considering the following deformation of the one-loop solution

$$Q_\delta^B(u) = {}_3F_2\left(\left. \begin{matrix} -M, \ M+1, \ \frac{1}{2}+iu \\ 1+\delta, \ 1-\delta \end{matrix} \right| 1 \right). \tag{3.11}$$

The second derivative of the resulting difference equation evaluated at $\delta = 0$ takes the form

$$\begin{aligned} &\left(u+\frac{i}{2}\right)^2 Q_0^{B\prime\prime}(u+i) + \left(u-\frac{i}{2}\right)^2 Q_0^{B\prime\prime}(u-i) - t_2^{(0)}(u) Q_0^{B\prime\prime}(u) \\ &\quad = 2Q^{(0)}(u) - 2Q^{(0)}(u+i). \end{aligned} \tag{3.12}$$

To complete the solution one must note that a third type of the deformation must be introduced such that the argument of the r.h.s. of the above equation is shifted $u \to (u - i)$. From (3.7) one infers that this is obtained by interchanging $a \leftrightarrow c$ and $b \leftrightarrow d$. Thus, the last term reads

$$Q_\delta^C(u) = {}_3F_2\left(\left. \begin{matrix} -M, \ M+1, \ \frac{1}{2}+iu+\delta \\ 1+\delta, \ 1+\delta \end{matrix} \right| 1 \right). \tag{3.13}$$

Since the one-loop solution (3.3) is also a solution to the homogeneous part of the two-loop Baxter equation (3.5) one can add it to $Q^{(1)}(u)$ with an arbitrary coefficient function of M. To fix the function uniquely one notices that if the leading order Baxter function is a polynomial of degree M, then the higher loop corrections are polynomials of degree $M - 1$ and for even distribution of roots, as in the case considered,[3] $M - 2$. Hence, one has to add the term $a(M)Q^{(0)}(u)$, where

[3] This follows from expanding $u_k(g^2)$ in (2.2) in a power series in g.

$-a(M)$ is the ratio of the highest order term u^M of $Q^{(1)}(u)$ and $Q^{(0)}(u)$ given by

$$a(M) = 4\big(S_2(M) + 4S_1(M)^2 - 2S_1(M)S_1(2M)\big). \tag{3.14}$$

Finally, the full next-to-leading order Baxter function with the appropriate normalization[4] is given by

$$\begin{aligned} Q^{(1)}(u) = {} & a(M)\,{}_3F_2\left(\begin{matrix} -M,\ M+1,\ \frac{1}{2}+iu \\ 1,\ 1 \end{matrix}\middle|1\right) \\ & + \frac{\gamma^{(0)}}{2}\frac{\partial}{\partial\delta}\,{}_3F_2\left(\begin{matrix} -M,\ M+1+2\delta,\ \frac{1}{2}+iu \\ 1+\delta,\ 1 \end{matrix}\middle|1\right)\Bigg|_{\delta=0} \\ & - \frac{\partial^2}{\partial\delta^2}\,{}_3F_2\left(-M,\ M+1,\ \tfrac{1}{2}+iu\ \ 1+\delta,\ 1-\delta \middle|1\right)\Bigg|_{\delta=0} \\ & - \frac{\partial^2}{\partial\delta^2}\,{}_3F_2\left(\begin{matrix} -M,\ M+1,\ \frac{1}{2}+iu+\delta \\ 1+\delta,\ 1+\delta \end{matrix}\middle|1\right)\Bigg|_{\delta=0}. \end{aligned} \tag{3.15}$$

With these results it is rather straightforward to compute the two-loop anomalous dimension in a closed form, see Appendix B. It is given by the following known expression, where all harmonic sums are evaluated at argument M

$$\gamma^{(1)}(M) = -16\big(S_3 + S_{-3} - 2S_{-2,1} + 2S_1(S_2 + S_{-2})\big). \tag{3.16}$$

At three-loop order one finds the following perturbative Baxter equation for $Q^{(2)}(u)$

$$\begin{aligned} & \left(u+\frac{i}{2}\right)^2 Q^{(2)}(u+i) + \left(u-\frac{i}{2}\right)^2 Q^{(2)}(u-i) - t_2^{(0)}(u)Q^{(2)}(u) \\ & = \left(2 - i\frac{\gamma^{(0)}}{2}\left(u+\frac{i}{2}\right)\right)Q^{(1)}(u+i) + \left(2 + i\frac{\gamma^{(0)}}{2}\left(u-\frac{i}{2}\right)\right)Q^{(1)}(u-i) \\ & \quad + t_2^{(1)}(u)Q^{(1)}(u) + P(u)Q^{(0)}(u+i) + P^*(u)Q^{(0)}(u-i) \\ & \quad - \left(\frac{\gamma^{(1)}}{2}(2M+1) + 2K_2 + \left(u+\frac{i}{2}\right)^{-2} + \left(u-\frac{i}{2}\right)^{-2}\right)Q^{(0)}(u), \end{aligned} \tag{3.17}$$

where $P(u)$ and $K_2 = K_2(M)$ are given by

$$\begin{aligned} P(u) &= \left(\frac{1}{(u+\frac{i}{2})^2} + \frac{i\gamma^{(0)}}{2(u+\frac{i}{2})} + K_2 - \frac{i\gamma^{(1)}}{2}\left(u+\frac{i}{2}\right)\right), \\ K_2(M) &= \frac{\gamma^{(0)}(M)^2}{8} - 4S_{-2}(M). \end{aligned} \tag{3.18}$$

It is instructive to split the solution into five different sub-classes, which are independent of each other due to the transcendental composition of the corresponding inhomogeneities and can be analyzed separately as presented in Table 1.

The classes I, II_a, II_b and II_c can be obtained by using the same arguments as for the next-to-leading order analysis. Terms polynomial in Q_0 lead to

[4] The global normalization of the full Baxter function $Q(u) = Q^{(0)}(u) + g^2 Q^{(1)}(u)$ can be chosen arbitrarily.

Table 1
Classification of solutions.

Index	Classification
I	polynomial parts of Q_0
II_a	terms of order $(\gamma^{(0)})^0$
II_b	terms of order $(\gamma^{(0)})^1$
II_c	terms of order $(\gamma^{(0)})^2$
III	non-polynomial parts of Q_0
IV	one-loop normalization

$$Q_{\mathrm{I}}^{(2)}(u) = \frac{\gamma^{(1)} + a(M)\gamma^{(0)}}{2} \frac{\partial}{\partial\delta} {}_3F_2\left(\left.\begin{matrix} -M,\ M+1+2\delta,\ \frac{1}{2}+iu \\ 1+\delta,\ 1 \end{matrix}\right| 1\right)\bigg|_{\delta=0}$$
$$- \frac{K_2(M)+2a(M)}{2} \frac{\partial^2}{\partial\delta^2} {}_3F_2\left(\left.\begin{matrix} -M,\ M+1,\ \frac{1}{2}+iu \\ 1+\delta,\ 1-\delta \end{matrix}\right| 1\right)\bigg|_{\delta=0}$$
$$- \frac{K_2(M)+2a(M)}{2} \frac{\partial^2}{\partial\delta^2} {}_3F_2\left(\left.\begin{matrix} -M,\ M+1,\ \frac{1}{2}+iu+\delta \\ 1+\delta,\ 1+\delta \end{matrix}\right| 1\right)\bigg|_{\delta=0}. \tag{3.19}$$

The II_a part of the solution is given by

$$Q_{\mathrm{II}_a}^{(2)}(u) = \frac{1}{6} \frac{\partial^4}{\partial\delta^4} {}_3F_2\left(\left.\begin{matrix} -M,\ M+1,\ \frac{1}{2}+iu \\ 1+\delta,\ 1-\delta \end{matrix}\right| 1\right)\bigg|_{\delta=0}$$
$$+ \frac{1}{6} \frac{\partial^4}{\partial\delta^4} {}_3F_2\left(\left.\begin{matrix} -M,\ M+1,\ \frac{1}{2}+iu+\delta \\ 1+\delta,\ 1+\delta \end{matrix}\right| 1\right)\bigg|_{\delta=0}$$
$$+ \frac{\partial^2}{\partial\alpha^2} \frac{\partial^2}{\partial\beta^2} {}_3F_2\left(\left.\begin{matrix} -M,\ M+1,\ \frac{1}{2}+iu+\beta \\ 1+\alpha+\beta,\ 1-\alpha+\beta \end{matrix}\right| 1\right)\bigg|_{\alpha,\beta=0}. \tag{3.20}$$

Polynomial inhomogeneities linear in $\gamma^{(0)}$ redound to the result

$$Q_{\mathrm{II}_b}^{(2)}(u) = -\frac{\gamma^{(0)}}{2}\left(\frac{\partial}{\partial\alpha} \frac{\partial^2}{\partial\beta^2} {}_3F_2\left(\left.\begin{matrix} -M,\ M+1+2\alpha,\ \frac{1}{2}+iu+\beta \\ 1+\alpha+\beta,\ 1+\beta \end{matrix}\right| 1\right)\bigg|_{\alpha,\beta=0}\right.$$
$$+ \frac{1}{3} \frac{\partial^3}{\partial\beta^3} {}_3F_2\left(\left.\begin{matrix} -M,\ M+1,\ \frac{1}{2}+iu+\beta \\ 1+\beta,\ 1+\beta \end{matrix}\right| 1\right)\bigg|_{\beta=0}$$
$$\left. + \frac{\partial}{\partial\alpha} \frac{\partial^2}{\partial\beta^2} {}_3F_2\left(\left.\begin{matrix} -M,\ M+1+2\alpha,\ \frac{1}{2}+iu \\ 1+\alpha+\beta,\ 1-\beta \end{matrix}\right| 1\right)\bigg|_{\alpha,\beta=0}\right). \tag{3.21}$$

The term of $\mathcal{O}((\gamma^{(0)})^2)$ is given by

$$Q_{\mathrm{II}_c}^{(2)}(u) = \frac{(\gamma^{(0)})^2}{8} \frac{\partial^2}{\partial\delta^2} {}_3F_2\left(\left.\begin{matrix} -M,\ M+1+2\delta,\ \frac{1}{2}+iu \\ 1+\delta,\ 1 \end{matrix}\right| 1\right)\bigg|_{\delta=0}. \tag{3.22}$$

To find the solution for class III, these are all the terms that contain the denominators $1/(u \pm i/2)$, one needs to proceed more generally. The difference equation satisfied by continuous Hahn polynomials does not allow to generate $1/(u \pm i/2)$ terms by the deformations previously introduced. However, all terms of type III add up to a polynomial and one can make use of a general statement formulated in Lemma 1 to find the solution, see Appendix C where also details of the calculation can be found. When brought together, the terms in class III can be expressed by a sum of two polynomials. The first part is given by a hypergeometric polynomial, and the corresponding solution can be easily found using the techniques discussed in this section

$$Q^{(2)}_{\mathrm{III}_a}(u) = \frac{\gamma^{(0)}}{4}\frac{\partial^2}{\partial\alpha^2}\frac{\partial}{\partial\beta}\,{}_4F_3\left(\begin{matrix}-M,\ M+1,\ \frac{1}{2}+iu,\ 1\\ 1+\alpha,\ 1-\alpha+\beta,\ 1-\beta\end{matrix}\middle|1\right)\Bigg|_{\alpha,\beta=0} + \frac{\gamma^{(0)}}{4}\frac{\partial^2}{\partial\alpha^2}\frac{\partial}{\partial\beta}\,{}_4F_3\left(\begin{matrix}-M,\ M+1,\ \frac{1}{2}-iu,\ 1\\ 1+\alpha,\ 1-\alpha+\beta,\ 1-\beta\end{matrix}\middle|1\right)\Bigg|_{\alpha,\beta=0}. \tag{3.23}$$

On the other hand, the second polynomial is much more complicated and to find the solution one has to apply Lemma 1 from Appendix C

$$Q^{(2)}_{\mathrm{III}_b}(u) = \sum_{k=0}^{M}\frac{m_k R(k,M)}{k!}\left(\left(\frac{1}{2}+iu\right)_k + \left(\frac{1}{2}-iu\right)_k\right),$$
$$m_k = \sum_{j=1}^{k}\frac{a_{j-1}(j-1)!}{j^2 R(j,M)}. \tag{3.24}$$

See Appendices A and C for the definitions of $R(j,M)$ and a_j. Finally the solution to the homogeneous equation, i.e. the leading order solution with the correct normalization is fixed again by imposing the proper degree reduction of $Q_2(u)$ such that $Q_2(u)$ is a polynomial of degree $M-2$. The normalization is computed in the same fashion as in the two-loop case. Hence $Q^{(2)}_{\mathrm{IV}}$ is given by

$$Q^{(2)}_{\mathrm{IV}}(u) = a^{(2)}(M)\,{}_3F_2\left(\begin{matrix}-M,\ M+1,\ \frac{1}{2}+iu\\ 1,\ 1\end{matrix}\middle|1\right), \tag{3.25}$$

with

$$\begin{aligned}a^{(2)}(M) = \frac{8}{3}\big(&52S_1(M)^4 - S_{-3}(M)\big(9S_1(M) - 6S_1(2M)\big) - 48S_1(M)^3S_1(2M)\\ &- 3S_{-2}(M)\big(7S_1(M)^2 - 4S_1(M)S_1(2M) + S_2(M)\big)\\ &+ 3S_1(M)^2\big(4S_1(2M)^2 - 5S_2(M) + 4S_2(2M)\big) - 6S_4(M)\\ &+ 3\big(S_2(M)^2 + 2S_1(2M)\big(S_3(M) - 2S_{-2,1}(M)\big)\big)\\ &- 2S_1(M)\big(11S_3(M) - 9S_{-2,1}(M)\big)\big) - 2m_M.\end{aligned} \tag{3.26}$$

The anomalous dimension obtained from this result in terms of nested harmonic sums with argument M is given by

$$\begin{aligned}\gamma^{(2)}(M) = 64\big(&2S_{-5} + 2S_5 - 4S_{-4,1} - 2S_{-3,-2} - S_{-3,2} - 2S_{-2,-3}\\ &-8S_{1,-4} - 4S_{1,4} - 9S_{2,-3} - 5S_{2,3} - 2S_{3,-2} - 5S_{3,2}\\ &+ 2\big(-2S_{4,1} + S_{-2,-2,1} + S_{-2,1,-2} + 4S_{1,-3,1} + S_{1,-2,-2} + S_{1,-2,2}\\ &+ 6S_{1,1,-3} + 2S_{1,1,3} + 2S_{1,2,-2} + 2S_{1,2,2} + 2S_{1,3,1} + 3S_{2,-2,1}\\ &+ 2(S_{2,1,-2} + S_{2,1,2} + S_{2,2,1} + S_{3,1,1} - 2S_{1,1,-2,1})\big)\big).\end{aligned} \tag{3.27}$$

It agrees precisely with the conjecture of [18], upon changing the basis of the harmonic sums.

4. Twist-three operators

A special class of maximal helicity twist-three operators in QCD has also found to be integrable to the first loop correction [26] and the relation of the solution to Wilson polynomials has been established by the Baxter approach in [27].

In this section we will apply the methods of the previous one to obtain the three-loop expression for the Baxter function of twist-three operators of $\mathcal{N}=4$ SYM

$$\mathrm{Tr}\big(\mathcal{D}^{s_1}\mathcal{Z}\mathcal{D}^{s_2}\mathcal{Z}\mathcal{D}^{s_3}\mathcal{Z}\big)+\cdots, \qquad s_1+s_2+s_3=M. \tag{4.1}$$

The leading order Baxter function is given by the Wilson polynomial [21]

$$Q^{(0)}(u)={}_4F_3\left(\begin{matrix}-\frac{M}{2},\ \frac{M}{2}+1,\ \frac{1}{2}+iu,\ \frac{1}{2}-iu\\ 1,\ 1,\ 1\end{matrix}\middle|1\right), \tag{4.2}$$

from which one computes the anomalous dimension

$$\gamma_3^{(0)}=8S_1\left(\frac{M}{2}\right). \tag{4.3}$$

The Baxter equation for the twist-three operators (4.1) to two-loops is given by

$$\begin{aligned}&\left(u+\frac{i}{2}\right)^3Q^{(1)}(u+i)+\left(u-\frac{i}{2}\right)^3Q^{(1)}(u-i)-t_3^{(0)}(u)Q^{(1)}(u)\\&\quad=t_3^{(1)}Q^{(0)}(u)+\left(3-i\frac{\gamma_3^{(0)}}{2}\left(u+\frac{i}{2}\right)\right)\left(u+\frac{i}{2}\right)Q^{(0)}(u+i)\\&\qquad+\left(3+i\frac{\gamma_3^{(0)}}{2}\left(u-\frac{i}{2}\right)\right)\left(u-\frac{i}{2}\right)Q^{(0)}(u-i),\end{aligned} \tag{4.4}$$

where

$$t_3^{(0)}(u)=2u^3-\left(M^2+2M+\frac{3}{2}\right)u, \qquad t_3^{(1)}(u)=\big(-\gamma_3^{(0)}(M+1)-6\big)u. \tag{4.5}$$

Using the same techniques as for the twist-two case one finds the final result to be given by

$$\begin{aligned}Q^{(1)}(u)&=c(M)\,{}_4F_3\left(\begin{matrix}-\frac{M}{2},\ \frac{M}{2}+1,\ \frac{1}{2}+iu,\ \frac{1}{2}-iu\\ 1,\ 1,\ 1\end{matrix}\middle|1\right)\\&\quad+\frac{\gamma_3^{(0)}}{4}\frac{\partial}{\partial\delta}\,{}_4F_3\left(\begin{matrix}-\frac{M}{2},\ \frac{M}{2}+1+2\delta,\ \frac{1}{2}+iu,\ \frac{1}{2}-iu\\ 1+\delta,\ 1+\delta,\ 1\end{matrix}\middle|1\right)\bigg|_{\delta=0}\\&\quad-\frac{3}{2}\frac{\partial^2}{\partial\delta^2}\,{}_4F_3\left(\begin{matrix}-\frac{M}{2},\ \frac{M}{2}+1,\ \frac{1}{2}+iu,\ \frac{1}{2}-iu\\ 1+\delta,\ 1-\delta,\ 1\end{matrix}\middle|1\right)\bigg|_{\delta=0},\end{aligned} \tag{4.6}$$

with the normalization function $c(M)$ given by

$$c(M)=\frac{\gamma_3^{(0)}}{4}\left(4S_1\left(\frac{M}{2}\right)-2S_1(M)\right)+3S_2\left(\frac{M}{2}\right). \tag{4.7}$$

The anomalous dimension computed from these closed expressions, see Appendix B.2, is given by

$$\gamma_3^{(1)}(M) = -8\left(S_3\left(\frac{M}{2}\right) + 2S_1\left(\frac{M}{2}\right)S_2\left(\frac{M}{2}\right)\right). \tag{4.8}$$

as has been guessed in [21,28].

The NNLO correction to the Baxter function can be found as the solution of

$$\begin{aligned}
&\left(u+\frac{i}{2}\right)^3 Q^{(2)}(u+i) + \left(u-\frac{i}{2}\right)^3 Q^{(2)}(u-i) - t_3^{(0)}(u)Q^{(2)}(u) \\
&\quad = \left(3 - i\frac{\gamma_3^{(0)}}{2}\left(u+\frac{i}{2}\right)\right)\left(u+\frac{i}{2}\right)Q^{(1)}(u+i) \\
&\qquad + \left(3 + i\frac{\gamma_3^{(0)}}{2}\left(u-\frac{i}{2}\right)\right)\left(u-\frac{i}{2}\right)Q^{(1)}(u-i) \\
&\qquad + t_3^{(1)}(u)Q^{(1)}(u) + P_3(u)Q^{(0)}(u+i) + P_3^*(u)Q^{(0)}(u-i) \\
&\qquad - \left(\gamma_3^{(1)}(M+1) + 2K_3\right)Q^{(0)}(u),
\end{aligned} \tag{4.9}$$

where $P_3(u)$ and $K_3 = K_3(M)$ are given by

$$\begin{aligned}
P_3(u) &= \left(i\gamma_3^{(0)} + K_3\left(u+\frac{i}{2}\right) - \frac{i\gamma_3^{(1)}}{2}\left(u+\frac{i}{2}\right)^2\right), \\
K_3(M) &= \frac{\gamma_3^{(0)}(M)^2}{8}.
\end{aligned} \tag{4.10}$$

Following the same method outlined in Section 3 and considering each part of the solution separately according to Table 1 it is straightforward to find the complete solution. The normalization function $c^{(2)}$ for the homogeneous solution is again fixed by degree reduction i.e. $Q^{(2)}$ is a polynomial of degree $(M-2)$,

$$\begin{aligned}
c^{(2)}(M) = {}& 32S_1\left(\frac{M}{2}\right)^4 - 32S_1\left(\frac{M}{2}\right)^3 S_1(M) + \frac{9}{2}S_2\left(\frac{M}{2}\right)^2 + 8S_1\left(\frac{M}{2}\right)^2\big(S_1(M)^2 \\
&+ S_2(M)\big) + 4S_1(M)S_3\left(\frac{M}{2}\right) - 4S_1\left(\frac{M}{2}\right)\left(S_1(M)S_2\left(\frac{M}{2}\right)\right. \\
&\left. + 3S_3\left(\frac{M}{2}\right)\right) - \frac{9}{2}S_4\left(\frac{M}{2}\right).
\end{aligned} \tag{4.11}$$

The three-loop Baxter function is given by

$$\begin{aligned}
Q^{(2)}(u) = {}& c^{(2)}(M)\,{}_4F_3\left(\left.\begin{matrix} -\frac{M}{2},\ \frac{M}{2}+1,\ \frac{1}{2}+iu,\ \frac{1}{2}-iu \\ 1,\ 1,\ 1 \end{matrix}\right| 1\right) \\
&+ \frac{\gamma_3^{(1)} + c(M)\gamma_3^{(0)}}{4}\frac{\partial}{\partial\delta}\,{}_4F_3\left(\left.\begin{matrix} -\frac{M}{2},\ \frac{M}{2}+1+2\delta,\ \frac{1}{2}+iu,\ \frac{1}{2}-iu \\ 1+\delta,\ 1+\delta,\ 1 \end{matrix}\right| 1\right)\Bigg|_{\delta=0} \\
&- \frac{K_3(M) + 3c(M)}{2}\frac{\partial^2}{\partial\delta^2}\,{}_4F_3\left(\left.\begin{matrix} -\frac{M}{2},\ \frac{M}{2}+1,\ \frac{1}{2}+iu,\ \frac{1}{2}-iu \\ 1+\delta,\ 1-\delta,\ 1 \end{matrix}\right| 1\right)\Bigg|_{\delta=0} \\
&+ \frac{3}{8}\frac{\partial^4}{\partial\delta^4}\,{}_4F_3\left(\left.\begin{matrix} -\frac{M}{2},\ \frac{M}{2}+1,\ \frac{1}{2}+iu,\ \frac{1}{2}-iu \\ 1+\delta,\ 1-\delta,\ 1 \end{matrix}\right| 1\right)\Bigg|_{\delta=0} \\
&- \frac{3}{8}\gamma_3^{(0)}\frac{\partial}{\partial\alpha}\frac{\partial^2}{\partial\beta^2}\,{}_4F_3\left(\left.\begin{matrix} -\frac{M}{2},\ \frac{M}{2}+1+2\alpha,\ \frac{1}{2}+iu,\ \frac{1}{2}-iu \\ 1+\alpha,\ 1+\alpha+\beta,\ 1-\beta \end{matrix}\right| 1\right)\Bigg|_{\alpha,\beta=0}
\end{aligned}$$

$$+\frac{1}{8}\gamma_3^{(0)}\frac{\partial^2}{\partial\alpha^2}\frac{\partial}{\partial\beta}\,{}_4F_3\left(\begin{matrix}-\frac{M}{2},\ \frac{M}{2}+1,\ \frac{1}{2}+iu,\ \frac{1}{2}-iu\\ 1+\alpha,\ 1-\alpha+\beta,\ 1-\beta\end{matrix}\middle|1\right)\Bigg|_{\alpha,\beta=0}$$
$$+\frac{(\gamma_3^{(0)})^2}{32}\frac{\partial^2}{\partial\delta^2}\,{}_4F_3\left(\begin{matrix}-\frac{M}{2},\ \frac{M}{2}+1+2\delta,\ \frac{1}{2}+iu,\ \frac{1}{2}-iu\\ 1+\delta,\ 1+\delta,\ 1\end{matrix}\middle|1\right)\Bigg|_{\delta=0}$$
$$+\frac{(\gamma_3^{(0)})^2}{32}\frac{\partial^2}{\partial\delta^2}\,{}_4F_3\left(\begin{matrix}-\frac{M}{2},\ \frac{M}{2}+1,\ \frac{1}{2}+iu,\ \frac{1}{2}-iu\\ 1+\delta,\ 1-\delta,\ 1\end{matrix}\middle|1\right)\Bigg|_{\delta=0}. \tag{4.12}$$

For the three-loop anomalous dimension one obtains, again see Appendix B.2 for details,

$$\gamma^{(2)}(M)=8\big(S_5-2S_{1,4}-6S_{2,3}-10S_{3,2}-6S_{4,1}$$
$$+8(S_{1,2,2}+S_{2,1,2}+S_{2,2,1}+S_{1,3,1}+S_{3,1,1})\big), \tag{4.13}$$

with all sums evaluated at $M/2$. The result coincides[5] with the conjecture of [21,28].

It is important to note, that the three-loop Baxter equation for the twist-three operators, in contradistinction to the case of twist-two operators, *does not* contain superficially non-polynomial parts and thus the solution can be found in terms of deformations of the one-loop solution only. Although the wrapping problem for twist-two operators starts from the four-loop order, this superficial breakdown of the polynomiality of the Baxter equation may signalize its incorrectness at the next order. The same applies for the twist-three operators, where the corresponding four-loop Baxter equation contains rational functions, though the solution should correctly reproduce the corresponding anomalous dimension [21].

5. Summary and outlook

We have shown how to solve the two- and three-loop Baxter equation for a special subset of operators. With the explicit form of the Baxter function we were able to reproduce the known results based on Feynman calculus [17] and to prove the three-loop conjecture [18] for twist two-operators that has e.g. been used to check the field theory solution for the three-loop planar dilatation generator obtained by algebraic methods in [29]. Likewise, we gave a proof for the anomalous dimensions of twist-three operators that were conjectured in [21,28]. Besides our approach to find analytic solutions to the Baxter equation there are also techniques to obtain such solutions directly from Bethe ansatz equations, see [22] and especially [30] and references therein.

Due to the nature of the mechanism one can trace back all different contributions to the anomalous dimension to the corresponding inhomogeneity of the perturbative Baxter equation. As such it would be of great importance to generalize the successful application of TBA [31] of the four-loop Konishi [32], i.e. twist-two $M=2$, operator to all twist-two operators. The knowledge of the correct four-loop result of twist-two operators in a finite volume could than be used to analyze and, if possible, to fix the four-loop Baxter and hence Bethe equations. A detailed application of our methods to the next order should also reveal a different structure appearing in the Baxter equation that renders the twist-two solution incorrect but, in turn, should give the right result for twist-three.

[5] Note, that we choose to write the harmonic sums in the canonical basis.

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Appendix A. Solution in Mellin space

In this appendix we will show how to derive the one and two-loop solution of the Baxter equation for twist two operators in Mellin space following [4]. We consider the Mellin transform $Q^{(i)}(\omega)$ of the Baxter function $Q^{(i)}(u)$

$$Q^{(i)}(u) = \int_0^\infty d\omega\, \omega^{iu-1} Q^{(i)}(\omega). \tag{A.1}$$

In what follows we will make use of the relations

$$\begin{aligned} Q^{(i)}(u+ai) &= \int_0^\infty d\omega\, \omega^{iu-1}\left\{\omega^{-a} Q^{(i)}(\omega)\right\}, \\ (u+bi)^J Q^{(i)}(u+ai) &= \int_0^\infty d\omega\, \omega^{iu-1}\omega^{-b}\left(i\omega\frac{d}{d\omega}\right)^J \left\{\omega^{b-a} Q^{(i)}(\omega)\right\}. \end{aligned} \tag{A.2}$$

One-loop. The differential equation we obtain from the leading order Baxter equation (3.1) reads

$$\left\{\frac{(\omega-1)^2}{\omega}\left(i\omega\frac{d}{d\omega}\right)^2 + i\frac{\omega^2-1}{\omega}\left(i\omega\frac{d}{d\omega}\right) - q_2^{(0)} - \frac{\omega^2+1}{4\omega}\right\} Q^{(0)}(\omega) = 0. \tag{A.3}$$

By means of the variable transformation $z = 1/(1-\omega)$ and by applying the Faddeev–Korchemsky substitution, see [4], $Q^{(i)}(\omega) \to Q^{(i)}(-\omega)$ in (A.1) we replace (A.1) and (A.3) by

$$Q^{(i)}(u) = \frac{-i\pi}{\Gamma(iu)\Gamma(1-iu)} \int_0^1 dz\,(1-z)^{iu-1} z^{-iu-1} Q^{(i)}(z), \tag{A.4}$$

$$\left\{z(1-z)\left(\frac{d}{dz}\right)^2 - q_2^{(0)} + \frac{1-2z(1-z)}{4z(1-z)}\right\} Q^{(0)}(z) = 0, \tag{A.5}$$

respectively. It is convenient to introduce new functions $\bar{Q}^{(i)}(z)$

$$Q^{(i)}(z) = \sqrt{z(1-z)}\,\bar{Q}^{(i)}(z), \tag{A.6}$$

for which (A.5) takes a simpler form

$$\left\{z(1-z)\left(\frac{d}{dz}\right)^2+(1-2z)\frac{d}{dz}-q_2^{(0)}-\frac{1}{2}\right\}\bar{Q}^{(0)}(z)=0, \tag{A.7}$$

where we have omitted the factor $\sqrt{z(1-z)}$. It should be noted, that both $\bar{Q}^{(0)}(z)$ and $\bar{Q}^{(0)}(1-z)$ satisfy the above equation. Being a differential equation of the second order, Eq. (A.7) has two algebraically independent solutions and both, $\bar{Q}^{(0)}(z)$ and $\bar{Q}^{(0)}(1-z)$, can be written as linear combination of these two basis vectors. In particular, one may impose symmetry properties on the solutions. We choose the two independent solutions to obey[6] $Q^{(0)}(1-z)=Q^{(0)}(z)$ and $Q_B^{(0)}(1-z)=-Q_B^{(0)}(z)$. The first solution can be found with the ansatz

$$\bar{Q}^{(0)}(z)=\sum_{k=0}^{\infty}C_k z^{k+\alpha} \tag{A.8}$$

with arbitrary coefficients C_k and α. The beginning of the resulting recurrence leads to the consistency condition $\alpha=0$ and comparing terms with same powers of z, we find the relation

$$(k+1)^2C_{k+1}=(k-M)(k+M+1)C_k. \tag{A.9}$$

This first order recurrence is solved by

$$C_k=R(k,M)C_0, \tag{A.10}$$

where

$$R(k,M)=\frac{\Gamma(k+M+1)\Gamma(k-M)}{(k!)^2\Gamma(M+1)\Gamma(-M)}=\frac{(-1)^k\Gamma(k+M+1)}{(k!)^2\Gamma(M+1-k)}. \tag{A.11}$$

Thus, one solution to the Baxter equation (A.7) reads

$$\bar{Q}^{(0)}(z)=C_0\,{}_2F_1\left(\left.\begin{matrix}-M,\ M+1\\ 1\end{matrix}\right|z\right). \tag{A.12}$$

Using (A.4) we obtain[7] the following final result in the variable u

$$Q^{(0)}(u)=C_0\frac{i\pi\Gamma(\frac{1}{2}+iu)\Gamma(\frac{1}{2}-iu)}{\Gamma(iu)\Gamma(1-iu)}\,{}_3F_2\left(\left.\begin{matrix}-M,\ M+1,\ \frac{1}{2}+iu\\ 1,\ 1\end{matrix}\right|1\right). \tag{A.13}$$

Note, that the factor containing gamma functions is just a phase and hence negligible.

The second solution can be found by including logarithmic terms in addition to (A.8). Imposing the antisymmetry, we are looking for a solution of the following form

$$\bar{Q}_B^{(0)}(z)=\sum_{k=0}^{\infty}\big(a_k\log z-a_k\log(1-z)+b_k\big)R(k,M)z^k. \tag{A.14}$$

Terms of power $\log(z)$ and $-\log(1-z)$ lead to the relation

$$(k+1)^2R(k+1,M)a_{k+1}=(k-M)(k+M+1)R(k,M)a_k, \tag{A.15}$$

6 In what follows we assume even values of M.

7 For the integration we make use of the invariance $\bar{Q}^{(0)}(z)=\bar{Q}^{(0)}(1-z)$.

which by definition of $R(k,M)$ in (A.11) simply leads to

$$a_{k+1} = a_k \equiv a_0. \tag{A.16}$$

All terms of z^k lead to the intertwining relation for the coefficients a_k and b_k

$$\begin{aligned} &2(k+1)R(k+1,M)a_{k+1} + (k+1)^2 R(k+1,M)b_{k+1} \\ &\quad = (k-M)(k+M+1)R(k,M)b_k, \end{aligned} \tag{A.17}$$

which can be solved straightforwardly

$$b_k = b_0 - 2a_0 S_1(k). \tag{A.18}$$

Please note, that one can make use of the antisymmetry of the solution to fix b_0 to be

$$b_0 = b_0(M) = 2S_1(M)a_0. \tag{A.19}$$

Upon integration of (A.14) by means of taking a derivative of the Beta integral

$$\int_0^1 dz\, z^{iu-1}(1-z)^{-iu-1}\sqrt{z(1-z)}z^k = \frac{1}{k!}\left(\frac{1}{2}+iu\right)_k \Gamma\left(\frac{1}{2}+iu\right)\Gamma\left(\frac{1}{2}-iu\right), \tag{A.20}$$

the second solution is given by

$$\begin{aligned} Q_B^{(0)}(u) = \sum_{k=0}^{M} &\frac{(-M)_k(1+M)_k(\frac{1}{2}+iu)_k}{(1)_k(1)_k k!} \\ &\times \left(\Psi_0\left(\frac{1}{2}+iu+k\right) - \Psi_0\left(\frac{1}{2}-iu\right) - 2S_1(k) + 2S_1(M)\right). \end{aligned} \tag{A.21}$$

The relation of this solution to the one already obtained can be seen using the following identity for the Polygamma function

$$\Psi_0\left(\frac{1}{2}+iu\right) - \Psi_0\left(\frac{1}{2}-iu\right) = i\pi\tanh(\pi u). \tag{A.22}$$

It can then be written in the following form

$$\begin{aligned} Q_B^{(0)}(u) = &\frac{d}{d\delta}\, {}_3F_2\left(\left.\begin{matrix} -M,\ M+1,\ \frac{1}{2}+iu+\delta \\ 1+\delta,\ 1+\delta \end{matrix}\right| 1\right)\Bigg|_{\delta=0} \\ &+ \big(2S_1(M) + i\pi\tanh(\pi u)\big)Q^{(0)}(u). \end{aligned} \tag{A.23}$$

The first part of this solution agrees with (A.13) when multiplied by the normalization factor $-2S_1(M)$ while the second part is nothing but (A.13) multiplied by a periodic function and a coefficient function of M.

Note, that there is still another representation for $Q^{(0)}$. Making a different choice of variables $p = 4z(1-z)$ to rewrite (A.5) and performing the same steps of the computation one finds

$$Q^{(0)}(u) = C_0 \frac{i\pi\Gamma(\frac{1}{2}+iu)\Gamma(\frac{1}{2}-iu)}{\Gamma(iu)\Gamma(1-iu)}\, {}_4F_3\left(\left.\begin{matrix} -\frac{M}{2},\ \frac{M+1}{2},\ \frac{1}{2}-iu,\ \frac{1}{2}+iu \\ 1,\ 1,\ \frac{1}{2} \end{matrix}\right| 1\right). \tag{A.24}$$

This representation is equal to (A.13), see for example [33].

Two-loops. The differential equation that corresponds to the two-loop Baxter equation (3.5) for the same choice of variables (A.6) reads

$$\left\{z(1-z)\left(\frac{d}{dz}\right)^2+(1-2z)\frac{d}{dz}-q_2^{(0)}-\frac{1}{2}\right\}\bar{Q}^{(1)}(z)$$
$$=-\left\{\frac{\gamma^{(0)}}{2}\left((1-2z)\frac{d}{dz}+2M\right)+\frac{2}{z(1-z)}\right\}\bar{Q}^{(0)}(z). \tag{A.25}$$

As the acting differential operator is linear we will analyze the two inhomogeneous terms separately, starting with the part proportional to $\gamma^{(0)}$. It can be rewritten as

$$-\left\{\frac{\gamma^{(0)}}{2}\left((1-2z)\frac{d}{dz}+2M\right)\right\}\bar{Q}^{(0)}(z)=\frac{\gamma^{(0)}}{2}\sum_{k=0}^{M}B_k z^k, \tag{A.26}$$

where the coefficients B_k are given by

$$B_k=\frac{(k-M)(1+k-M)}{k+1}R(k,M)C_0. \tag{A.27}$$

To find the first part of the solution $\bar{Q}_A^{(1)}(z)$ we make the ansatz

$$\bar{Q}_A^{(1)}(z)=\sum_{k=0}^{M}D_k R(k,M)z^k, \tag{A.28}$$

which leads to the condition

$$D_{k+1}=D_k+\frac{\gamma^{(0)}}{2}\frac{1+k-M}{(1+k)(1+k+M)}C_0. \tag{A.29}$$

The solution is given by

$$D_k=D_0+\frac{\gamma^{(0)}}{2}\big(2S_1(k+M)-2S_1(M)-S_1(k)\big)C_0. \tag{A.30}$$

Thus, the first part of the result can be written in a compact form, noting that

$$\frac{d}{d\delta}\frac{(1+M+2\delta)_k}{(1+\delta)_k}\bigg|_{\delta=0}=\frac{(1+M)_k}{(1)_k}\big(2S_1(k+M)-2S_1(M)-S_1(k)\big), \tag{A.31}$$

it is given by

$$\bar{Q}_A^{(1)}(z)=D_0\,{}_2F_1\left(\begin{matrix}-M,\ M+1\\ 1\end{matrix}\bigg|z\right)$$
$$+C_0\frac{\gamma^{(0)}}{2}\frac{d}{d\delta}\,{}_2F_1\left(\begin{matrix}-M,\ M+1+2\delta\\ 1+\delta\end{matrix}\bigg|z\right)\bigg|_{\delta=0}. \tag{A.32}$$

In u-space the result is obtained according to (A.4) and hence reads

$$Q_A^{(1)}(u)=\Lambda D_0\,{}_3F_2\left(\begin{matrix}-M,\ M+1,\ \frac{1}{2}+iu\\ 1,\ 1\end{matrix}\bigg|1\right)$$
$$+\Lambda C_0\frac{\gamma^{(0)}}{2}\frac{d}{d\delta}\,{}_3F_2\left(\begin{matrix}-M,\ M+1+2\delta,\ \frac{1}{2}+iu\\ 1+\delta,\ 1\end{matrix}\bigg|1\right)\bigg|_{\delta=0}, \tag{A.33}$$

with, for completeness, the phase Λ given by

$$\Lambda = \frac{i\pi\Gamma(\frac{1}{2}+iu)\Gamma(\frac{1}{2}-iu)}{\Gamma(iu)\Gamma(1-iu)}. \tag{A.34}$$

Finally, let us focus on the second term to complete the solution, i.e.

$$\left\{z(1-z)\left(\frac{d}{dz}\right)^2 + (1-2z)\frac{d}{dz} + M(M+1)\right\}\bar{Q}^{(1)}(z) = -\frac{2C_0}{z(1-z)}\sum_{k=0}^{M} R(k,M)z^k. \tag{A.35}$$

One realizes that for this specific differential operator, we have to make the ansatz that $\bar{Q}^{(1)}_{B_1}(z) \sim \log^2 z \sum_{k=0}^{M} a_k z^k$ to obtain any expression that contains $1/z$ terms. However, the resulting expression also includes terms of order $\log^2 z$, $\log z$, z^k such that we need three consistency conditions to meet the requirement of the r.h.s. of (A.35). Therefore we should consider an ansatz of the following form

$$\bar{Q}^{(1)}_B(z) = \sum_{k=0}^{\infty}\Big(a_k\big(\log z - \log(1-z)\big)^2 + b_k\big(\log z - \log(1-z)\big) + c_k\Big)R(k,M)z^k. \tag{A.36}$$

Comparing the powers of $\log^2 z$, $\log z$ and z^k respectively leads to the following recurrences

$$a_{k+1} = a_k \equiv a_0,$$
$$b_{k+1} = b_k - \frac{4}{k+1}a_0,$$
$$c_{k+1} = c_k - \frac{2}{k+1}b_{k+1} \quad \text{and} \quad a_0 = -C_0. \tag{A.37}$$

The solutions to (A.37) are given by

$$b_k = b_0 - 4a_0 S_1(k), \qquad c_k = c_0 - 2b_0 S_1(k) + 8a_0 S_{1,1}(k). \tag{A.38}$$

Note, that (A.36) is the natural transcendental generalization of the second solution of the leading order Baxter function (A.14) and as such its $b_0(1-2S_1(k))$ part leads to (A.23) and is not of importance since the one-loop part will be fixed in the end by requirement of degree reduction. For the same reason it is also dispensable to fix c_0 by symmetry requirements. In addition writing the double index sum as $2S_{1,1}(k) = S_1(k)^2 + S_2(k)$ the result is given by

$$Q^{(1)}_B(u) = \sum_{k=0}^{M}\frac{(-M)_k(1+M)_k(\frac{1}{2}+iu)_k}{(1)_k(1)_k k!}\Bigg(\Psi_1\left(\frac{1}{2}+iu+k\right) + \Psi_1\left(\frac{1}{2}-iu\right) + 2S_2(k) + \left(\Psi_0\left(\frac{1}{2}+iu+k\right) - \Psi_0\left(\frac{1}{2}-iu\right) - 2S_1(k)\right)^2 + 2S_2(k)\Bigg). \tag{A.39}$$

One of the terms containing $2S_2(k)$ can be written in a compact form by absorbing it into a deformation of the Pochhammer symbol

$$\frac{d^2}{d\delta^2}\frac{1}{(1+\delta)_k(1-\delta)_k}\bigg|_{\delta=0} = \frac{2}{(1)_k(1)_k}S_2(k). \tag{A.40}$$

Likewise as in the case of the second solution at leading order we use the identity (A.22) and

$$\Psi_1\left(\frac{1}{2}-iu\right)+\Psi_1\left(\frac{1}{2}+iu\right) = \frac{\pi^2}{\cosh^2(\pi u)}, \tag{A.41}$$

to obtain the following form

$$\begin{aligned} Q_B^{(1)}(u) = {} & -C_0\frac{d^2}{d\delta^2}\,{}_3F_2\left(\begin{matrix}-M,\ M+1,\ \frac{1}{2}+iu+\delta \\ 1+\delta,\ 1+\delta\end{matrix}\bigg|1\right)\bigg|_{\delta=0} \\ & - C_0\frac{d^2}{d\delta^2}\,{}_3F_2\left(\begin{matrix}-M,\ M+1,\ \frac{1}{2}+iu \\ 1+\delta,\ 1-\delta\end{matrix}\bigg|1\right)\bigg|_{\delta=0}. \end{aligned} \tag{A.42}$$

We have neglected all one-loop parts and phases and fix the overall leading order influence to the two-loop Baxter function by the fact that its degree should be $M-2$. The final result is given in (3.15).

Appendix B. Anomalous dimensions

To compute the anomalous dimension from the closed expressions obtained, it is important to note that the twist-two and twist-three Baxter functions are real and hence invariant under the map $u \to -u$. Furthermore the leading-order Q-functions evaluated at $\pm i/2$ are equal to one, i.e. $Q^{(0)}(\pm i/2) = 1$.

B.1. Twist-two operators

Two-loops. From (2.8) one infers the following form of $\gamma^{(1)}$

$$\gamma^{(1)}(M) = A(M) + B(M), \tag{B.1}$$

where

$$A(M) = i\big(Q^{(0)\prime\prime\prime}(u) + 2Q^{(0)\prime}(u)^3 - 3Q^{(0)\prime}(u)Q^{(0)\prime\prime}(u)\big)\big|_{u=-\frac{i}{2}}^{u=+\frac{i}{2}} = A_+ - A_-, \tag{B.2}$$

$$B(M) = 2i\big(Q^{(1)\prime}(u) - Q^{(1)}(u)Q^{(0)\prime}(u)\big)\big|_{u=-\frac{i}{2}}^{u=+\frac{i}{2}} = B_+ - B_-. \tag{B.3}$$

It is convenient to express $Q^{(0)}(u)$ in the following way

$$Q^{(0)}(u) = \sum_{k=0}^{M}\frac{b_{k,M}}{k!}\left(\frac{1}{2}+iu\right)_k, \quad \text{with} \quad b_{k,M} = (-1)^k\binom{M}{k}\binom{M+k}{k}. \tag{B.4}$$

Taking the derivative and using some identities for nested harmonic numbers, see [34], one finds the closed expression[8] for the terms entering $A_-(M)$

$$Q^{(0)\prime}\left(-\frac{i}{2}\right) = i\sum_{k=1}^{M} b_{k,M} S_1(k) = 2i S_1(M), \tag{B.5}$$

$$Q^{(0)\prime\prime}\left(-\frac{i}{2}\right) = i^2\sum_{k=1}^{M} b_{k,M}\big(2S_{1,1}(k) - 2S_2(k)\big) = 4i^2(2S_{1,1} - S_2 + S_{-2}), \tag{B.6}$$

$$\begin{aligned} Q^{(0)\prime\prime\prime}\left(-\frac{i}{2}\right) &= i^3\sum_{k=1}^{M} b_{k,M}\big(6S_{1,1,1}(k) - 6S_{1,2}(k) - 6S_{2,1}(k) + 6S_3(k)\big) \\ &= 24i^3(2S_{1,1,1} - S_{1,2} - S_{2,1} + S_{1,-2} - S_{-2,1}). \end{aligned} \tag{B.7}$$

Hence, A_- is given by

$$A_-(M) = 8\big(3S_{-3}(M) - S_3(M) - 6S_{-2,1}(M)\big). \tag{B.8}$$

One of the sums that needs to be evaluated for B_- is

$$Q^{(1)}\left(-\frac{i}{2}\right) = a(M) + 4S_{-2}(M) - 8S_{1,1}(M) + 4S_2(M). \tag{B.9}$$

To find the second part we are in need of the following auxiliary formula

$$\sum_{k=1}^{M} b_{k,M} S_1(k+M) S_1(k) = (-1)^M\big(8S_{1,1}(M) - 5S_2(M)\big). \tag{B.10}$$

After some algebra $Q^{(1)\prime}(-\frac{i}{2})$ is given by, harmonic sums evaluated with argument M,

$$\begin{aligned} Q^{(1)\prime}\left(-\frac{i}{2}\right) &= 2i\Big(a(M)S_1 - 4\big(2S_3 + S_{-2,1} - S_{1,-2} + S_1(S_{-2} + S_2) \\ &\quad - 3(S_{1,2} + S_{2,1} - 2S_{1,1,1})\big)\Big), \end{aligned} \tag{B.11}$$

such that B_- reads

$$B_-(M) = 16\big(S_3 - S_{-3} + S_1(S_2 + S_{-2}) + 2S_{-2,1}\big). \tag{B.12}$$

Due to the symmetry of the Baxter function, $A_+ = -A_-$ and $B_+ = -B_-$, the two-loop anomalous dimension of twist-two operators is given by (3.16).

Three-loops. It is straightforward to expand the formula for $\gamma^{(2)}$ in terms of the perturbative Baxter functions. In order to find its close form we need to evaluate the following sums of the leading-order result

$$\begin{aligned} Q^{(0)(4)}\left(-\frac{i}{2}\right) &= i^4\sum_{k=1}^{M} b_{k,M}\big(24S_{1,1,1,1}(k) - 12S_1(k)^2 S_2(k) - 12S_4(k)\big) \\ &= 96i^4\big(2(2S_{1,1,1,1} - S_{1,1,2} + S_{1,1,-2} - S_{1,2,1} - S_{1,-2,1} \\ &\quad - S_{2,1,1} + S_{-2,1,1}) + S_{2,2} - S_{2,-2} - S_{-2,2} + S_{-2,-2}\big), \end{aligned} \tag{B.13}$$

[8] Please, be reminded that all states of $\mathfrak{sl}(2)$ have even M.

$$Q^{(0)(5)}\left(-\frac{i}{2}\right) = 960i^5\big(2(2S_{1,1,1,1,1} - S_{2,1,1,1} - S_{1,2,1,1} - S_{1,1,2,1} - S_{1,1,1,2} + S_{1,1,1,-2} - S_{1,1,-2,1} + S_{1,-2,1,1} - S_{-2,1,1,1}) + S_{1,2,2} + S_{2,1,2} + S_{2,2,1} - S_{1,-2,2} - S_{1,2,-2} + S_{1,-2,-2} + S_{-2,1,2} - S_{2,1,-2} - S_{-2,1,-2} + S_{-2,2,1} + S_{2,-2,1} + S_{-2,-2,1}\big). \tag{B.14}$$

The NLO result contributes with the following terms

$$Q^{(1)\prime\prime}\left(-\frac{i}{2}\right) = 4i^2\big(a(M)(2S_{1,1} - S_2 + S_{-2}) - 8\big(4(3S_{1,1,1,1} - S_{2,1,1} - S_{1,2,1} - S_{1,1,2}) + 2(S_{1,1,-2} + S_{1,-2,1} - S_{-2,1,1}) + S_{1,3} + S_{3,1} - S_{1,-3} + S_{-3,1} + S_{2,2} - S_{2,-2} + S_{-2,2} - S_{-2,-2}\big)\big), \tag{B.15}$$

$$Q^{(1)\prime\prime\prime}\left(-\frac{i}{2}\right) = 24i^3\big(a(M)(2S_{1,1,1} - S_{1,2} - S_{2,1} + S_{1,-2} - S_{-2,1}) - 8\big(20S_{1,1,1,1,1} - 7(S_{1,1,1,2} + S_{1,1,2,1} + S_{1,2,1,1} + S_{2,1,1,1}) + 5S_{1,1,1,-2} + S_{1,1,-2,1} - 3S_{1,-2,1,1} + S_{-2,1,1,1} + 2S_{1,2,2} + 2S_{2,1,2} + 2S_{2,2,1} - 2S_{1,2,-2} + S_{1,-2,2} - S_{1,-2,-2} - 2S_{2,1,-2} - S_{-2,1,2} + S_{-2,1,-2} - S_{-2,2,1} - S_{-2,-2,1} + S_{1,1,3} + S_{1,3,1} + S_{3,1,1} - S_{1,1,-3} + S_{1,-3,1} - S_{-3,1,1} + S_{3,-2} + S_{-3,2}\big)\big). \tag{B.16}$$

$Q^{(2)}$ results in the contributions

$$Q^{(2)}\left(-\frac{i}{2}\right) = a^{(2)}(M) + 4\big(6S_{-4} + a(M)(S_{-2} + S_2 - 2S_{1,1})\big) + 2\big(4S_4 - 7S_{-3,1} - 3S_{-2,2} - 5S_{1,-3} - 9S_{1,3} - S_{2,-2} - 10S_{2,2} - 9S_{3,1} + 2\big(3S_{-2,1,1} + S_{1,1,-2} + 6(S_{1,1,2} + S_{1,2,1} + S_{2,1,1} - 4S_{1,1,1,1})\big)\big),$$

$$Q^{(2)\prime}\left(-\frac{i}{2}\right) = 2i\big(8S_{-5} + a^{(2)}(M)S_1 + 4a(M)\big(S_{-3} - S_3 + 2(S_{2,1} + S_{1,2} - S_{-2,1} - 3S_{1,1,1})\big) - 8\big(4S_5 + 4S_{-4,1} + 2S_{-3,-2} + 7S_{-3,2} - S_{-2,3} - 13S_{1,4} - 7S_{2,-3} - 20S_{2,3} - S_{3,-2} - 20S_{3,2} - 15S_{4,1} - 4S_{-3,1,1} + 3S_{-2,1,2} + 3S_{-2,2,1} + 4S_{1,-3,1} - 4S_{1,-2,-2} - S_{1,-2,2} + 16S_{1,1,-3} + 34S_{1,1,3} + 7S_{1,2,-2} + 42S_{1,2,2} + 34S_{1,3,1} + 7S_{2,-2,1} + 7S_{2,1,-2} - 2\big(-21S_{2,1,2} - 21S_{2,2,1} - 17S_{3,1,1} + 7S_{-2,1,1,1} + 5S_{1,-2,1,1} + 7S_{1,1,-2,1} + 9S_{1,1,1,-2} + 28(S_{1,1,1,2} + S_{1,1,2,1} + S_{1,2,1,1} + S_{2,1,1,1}) - 20S_{1,1,1,1,1}\big)\big)\big). \tag{B.17}$$

Combining these contributions together results in (3.27).

B.2. Twist-three operators

In analogy to the twist-two operators we choose to write the leading-order Baxter function as

$$Q^{(0)}(u)=\sum_{k=0}^{\frac{M}{2}}\frac{b_{k,M/2}}{(k!)^2}\left(\frac{1}{2}+iu\right)_k\left(\frac{1}{2}-iu\right)_k. \tag{B.18}$$

To obtain the expressions that contribute to the anomalous dimension we have to introduce a regulator, i.e. we analyze the derivatives of $Q^{(i)}(u)$ at $u=\pm i(1/2+\epsilon)$ and take the limit $\epsilon\to 0$. All sums now have the argument $M/2$.

$$Q^{(0)\prime}\left(-\frac{i}{2}\right)=i\sum_{k=1}^{\frac{M}{2}}\frac{b_{k,M/2}}{k}=2iS_1, \tag{B.19}$$

$$Q^{(0)\prime\prime}\left(-\frac{i}{2}\right)=i^2\sum_{k=1}^{\frac{M}{2}}\frac{b_{k,M/2}}{k^2}=4i^2(2S_{1,1}-S_2), \tag{B.20}$$

$$Q^{(0)\prime\prime\prime}\left(-\frac{i}{2}\right)=6i^3\sum_{k=1}^{\frac{M}{2}}\frac{b_{k,M}}{k}S_2(k-1)=24i^3(2S_{1,1,1}-S_{1,2}-S_{2,1}). \tag{B.21}$$

One easily verifies that

$$Q^{(1)}\left(-\frac{i}{2}\right)=c(M), \tag{B.22}$$

$$Q^{(1)\prime}\left(-\frac{i}{2}\right)=2i\big(c(M)S_1-2S_{1,2}-2S_{2,1}-S_3\big). \tag{B.23}$$

According to (B.2) and (B.3) the building blocks of the anomalous dimension become

$$A_-(M)=-8S_3, \tag{B.24}$$

$$B_-(M)=12S_3+8S_1S_2, \tag{B.25}$$

such that $\gamma^{(1)}$ is given by (4.8).

Three-loops. The NNLO anomalous dimension requires the additional derivatives of $Q^{(0)}$

$$\begin{aligned}Q^{(0)(4)}\left(-\frac{i}{2}\right)&=24i^4\sum_{k=1}^{\frac{M}{2}}\frac{b_{k,M/2}}{k^2}S_2(k-1)\\&=96i^4\big(2(2S_{1,1,1,1}-S_{1,1,2}-S_{1,2,1}-S_{2,1,1})+S_{2,2}+S_{3,1}\big),\end{aligned} \tag{B.26}$$

$$\begin{aligned}Q^{(0)(5)}\left(-\frac{i}{2}\right)&=60i^5\sum_{k=1}^{\frac{M}{2}}\frac{b_{k,M/2}}{k}\big(S_4(k-1)-S_2(k-1)^2\big)\\&=960i^5\big(2(2S_{1,1,1,1,1}-S_{1,1,1,2}-S_{1,1,2,1}-S_{1,2,1,1}-S_{2,1,1,1})\\&\quad+S_{1,2,2}+S_{2,1,2}+S_{2,2,1}+S_{1,3,1}\big).\end{aligned} \tag{B.27}$$

Higher derivatives of the NLO Baxter function are given by

$$Q^{(1)\prime\prime}\left(-\frac{i}{2}\right) = 4i^2\big((2S_{1,1} - S_2)c(M) - 8(S_{1,1,2} + S_{1,2,1} + S_{2,1,1}) + 4S_{1,3} + 10S_{3,1} + 8S_{2,2} - 3S_4\big), \tag{B.28}$$

$$Q^{(1)\prime\prime\prime}\left(-\frac{i}{2}\right) = 24i^3\big((2S_{1,1,1} - S_{1,2} - S_{2,1})c(M) - 12(S_{1,1,1,2} + S_{1,1,2,1} + S_{1,2,1,1} + S_{2,1,1,1}) + 2(5S_{1,1,3} + 6S_{1,3,1} + S_{3,1,1} + 6S_{1,2,2} + 6S_{2,1,2} + 6S_{2,2,1}) - 3S_{1,4} + S_{4,1} - 7S_{2,3} - 3S_{3,2}\big). \tag{B.29}$$

The three-loop correction contributes with the terms

$$Q^{(2)}\left(-\frac{i}{2}\right) = c^{(2)}(M),$$

$$Q^{(2)\prime}\left(-\frac{i}{2}\right) = 2i\big(c^{(2)}(M)S_1 - c(M)(S_3 + 2S_{1,2} + 2S_{2,1}) + 2(11S_{1,4} + 19S_{4,1} - 3S_{2,3} + 5S_{3,2}) + 8(S_{1,2,2} + S_{2,1,2} + S_{2,2,1} - 2S_{1,3,1} - 4S_{3,1,1}) - 15S_5\big). \tag{B.30}$$

Plugging all terms into the expansion formula for $\gamma^{(2)}$ and transforming to the canonical basis one obtains (4.13).

Appendix C. A general method for solving an inhomogeneous Baxter equation

In this appendix we formulate a method for solving *any* consistent inhomogeneous Baxter equation. For simplicity we confine ourselves to the case of twist-two operators, where the transfer matrix is unambiguously determined, but the generalization to more complicated cases (upon knowing the solution to the homogeneous equation) should be straightforward.

Lemma 1. *A minimal polynomial solution to the inhomogeneous Baxter equation of the form*

$$\left(u + \frac{i}{2}\right)^2 Q^{(\ell)}(u+i) + \left(u - \frac{i}{2}\right)^2 Q^{(\ell)}(u-i) - t_2^{(0)}(u)Q^{(\ell)}(u) = \sum_{k=0}^{M-1} \alpha_k \left(\frac{1}{2} \pm iu\right)_k, \tag{C.1}$$

with the twist-two transfer matrix $t_2^{(0)}(u) = 2u^2 - (M^2 + M + \frac{1}{2})$ and arbitrary coefficients α_k that are independent of the spectral parameter u is given by

$$Q^{(\ell)} = \sum_{k=0}^{M} \frac{\beta_k R(k,M)}{k!} \left(\frac{1}{2} \pm iu\right)_k - \frac{\beta_M R(M,M) i^M}{M!} Q^{(0)}(u) \tag{C.2}$$

with

$$\beta_k = \sum_{j=1}^{k} \frac{\alpha_{j-1}(j-1)!}{j^2 R(j,M)}, \qquad R(k,M) = \frac{(-1)^k \Gamma(k+M+1)}{(k!)^2 \Gamma(M+1-k)}, \tag{C.3}$$

and $Q^{(0)}(u)$ being the solution to the homogeneous equation with the coefficient in front of the highest power of u normalized to one.

The proof is easily obtained using the methods of Appendix A.

This lemma allows to write down the solution to the class III terms discussed in Section 3, as well as to all other inhomogeneities considered in this paper. It must be noted, however, that this method does not lead necessarily to the simplest representation of the solution.

Below we will show how to bring the above-mentioned inhomogeneities of class III to a form, for which Lemma 1 is directly applicable. The terms in question are a real combination of the following function

$$\left[\frac{1}{(u+\frac{i}{2})^2}+\frac{i\gamma^{(0)}}{2(u+\frac{i}{2})}\right]Q^{(0)}(u+i)-\frac{1}{(u+\frac{i}{2})^2}Q^{(0)}(u)=A(u). \tag{C.4}$$

Expanding around $u=-\frac{i}{2}$ allows one to check that $A(u)$ is a polynomial of degree $\deg A(u)=\deg Q^{(0)}(u)-1$. Moreover the real combination $A(u)+A^*(u)=B(u)$ is a real polynomial of degree $\deg B(u)=(\deg Q^{(0)}(u)-2)$. In order to find the closed form of $A(u)$ in (C.4) we note that

$$Q^{(0)}(u)=i\left(u+\frac{i}{2}\right)M(M+1)\,{}_4F_3\left(\begin{matrix}1-M,\ M+2,\ \frac{3}{2}-iu,\ 1\\ 2,\ 2,\ 2\end{matrix}\,\middle|\,1\right)+1. \tag{C.5}$$

The key fact that makes $A(u)$ a polynomial is that $\gamma^{(0)}$ can be written in a similar fashion, since

$$S_1(M)=\frac{M(M+1)}{2}\,{}_4F_3\left(\begin{matrix}1-M,\ M+2,\ 1,\ 1\\ 2,\ 2,\ 2\end{matrix}\,\middle|\,1\right). \tag{C.6}$$

This allows to write $A(u)$ as

$$\begin{aligned}A(u)={}&\frac{\gamma^{(0)}}{2}M(M+1)\,{}_4F_3\left(\begin{matrix}1-M,\ M+2,\ \frac{1}{2}+iu,\ 1\\ 2,\ 2,\ 2\end{matrix}\,\middle|\,1\right)\\ &-\frac{iM(M+1)}{u+\frac{i}{2}}\sum_{k=0}^{M-1}\frac{(1-M)_k(M+2)_k}{(2)_k^3}\\ &\times\left(\left(\frac{1}{2}+iu\right)_k+\left(\frac{3}{2}-iu\right)_k-2(1)_k\right).\end{aligned} \tag{C.7}$$

Subsequently, using the following identities

$$(x)_k=\sum_{n=0}^{k}(-1)^{k-n}s_1(k,n)(x)^n,\qquad (x)^k=\sum_{n=0}^{k}(-1)^{k-n}s_2(k,n)(x)_n, \tag{C.8}$$

where s_1 and s_2 denote Stirling numbers of, respectively, first and second kind, one finds the final result

$$\begin{aligned}A(u)={}&\frac{\gamma^{(0)}}{2}M(M+1)\,{}_4F_3\left(\begin{matrix}1-M,\ M+2,\ \frac{1}{2}+iu,\ 1\\ 2,\ 2,\ 2\end{matrix}\,\middle|\,1\right)\\ &+\sum_{k=0}^{M}a_k\left(\frac{1}{2}-iu\right)_k.\end{aligned} \tag{C.9}$$

The coefficients a_k in the above formula are given by

$$a_k = \sum_{n=k+1}^{M} \frac{(-M)_{n+1}(M+1)_{n+1}}{(2)_n^3} \sum_{j=k+1}^{n} (-1)^{n-j} s_1(n,j) \times \sum_{m=k+1}^{j} \frac{((-1)^{m-1}-1)}{m!}(-j)_m(-1)^{m-k} s_2(m-1,k). \tag{C.10}$$

The hypergeometric part of (C.9) can be treated with the presented method of orthogonal polynomials. On the other hand, the explicit form of the a_k coefficients in (C.10) allows for the application of Lemma 1. The final result is given by (3.23) together with (3.24).

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