arxiv:0812.3367
AEI-2008-090

Scattering in Mass-Deformed $\mathcal{N} \geq 4$ Chern–Simons Models

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Abstract

We investigate the scattering matrix in mass-deformed $\mathcal{N} \geq 4$ Chern–Simons models including as special cases the BLG and ABJM theories of multiple M2 branes. Curiously the structure of this scattering matrix in three spacetime dimensions is equivalent to (a) the two-dimensional worldsheet matrix found in the context of AdS/CFT integrability and (b) the R-matrix of the one-dimensional Hubbard model. The underlying reason is that all three models are based on an extension of the $\mathfrak{psu}(2|2)$ superalgebra which constrains the matrix completely. We also compute scattering amplitudes in one-loop field theory and find perfect agreement with scattering unitarity.

1 Introduction

The calculation of scattering matrices in supersymmetric gauge theories has seen significant recent progress. Exciting new insights, such as connections between scattering amplitudes and Wilson loops [1–5], and the discovery of novel hidden symmetries, namely the dual superconformal invariance [6, 7], have led to a deeper understanding of the gauge theory. Many of these results have been possible because of the development of powerful new tools, such as twistor methods [8], recursion relations [9, 10] and generalized unitarity [11–14], for calculating on-shell quantities in Yang–Mills theories. Further, in the specific case of maximally supersymmetric four-dimensional $\mathcal{N} = 4$ Yang–Mills theory (YM) the existence of a string dual has provided a tractable strong-coupling description and has resulted in several impressive results and conjectures, for a recent review of the subject see [15].

It is interesting to see if one can extend these results to a broader class of theories particularly those with less supersymmetry. One such class is the $\mathcal{N} \geq 4$ supersymmetric Chern–Simons (SCS) matter theories constructed by Hosomichi et al. (HLLLP) in [16, 17] which builds upon the construction of Gaiotto and Witten [18]. In the construction of Gaiotto and Witten, the gauge group was chosen to be a particular subgroup of the symplectic group $\mathrm{Sp}(2n)$, with no particular restrictions imposed on the representations of the matter fields and where there is an $\mathfrak{su}(2) \times \mathfrak{su}(2) = \mathfrak{so}(4)$ R-symmetry which is required for the $\mathcal{N} = 4$ supersymmetry. In [16, 17], the matter content was augmented by twisted hypermultiplets where the action of the $\mathfrak{su}(2)$'s on the bosonic and fermionic degrees of freedom is interchanged relative to the untwisted case. In the absence of further constraints on the representations of the two matter multiplets, this construction also results in $\mathcal{N} = 4$ supersymmetry. It was further shown that if both matter multiplets are in the same representation the supersymmetry extends to $\mathcal{N} = 5$. In the special case, where the representations of the matter multiplets can be decomposed into a complex representation and its conjugate, such as bifundamental representations of $\mathrm{SU}(N) \times \mathrm{SU}(M)$, the supersymmetry was shown to be enhanced to $\mathcal{N} = 6$. When the representations are furthermore real corresponds to $\mathcal{N} = 8$.

The theories with $\mathcal{N} = 6$ and 8 had been previously found in the context of the low energy effective action of multiple membranes by Bagger, Lambert, Gustavsson (BLG) and Aharony, Bergman, Jafferis, Maldacena (ABJM) [19, 20]. The $\mathcal{N} = 6$ theory, of which there is an infinite $\mathrm{SU}(N) \times \mathrm{SU}(N)$ family [20], has been conjectured for finite values of N and k, the Chern–Simons level number, to describe the low energy dynamics of N M2 branes on $\mathbb{R}^{1,2} \times \mathbb{C}^4/\mathbb{Z}_k$. The gauge theory also possesses a well defined large N limit, which is obtained by taking both N and k to be large while $\lambda = N/k$ is held fixed. In this limit, the theory is expected to be dual to string theory on $AdS_4 \times \mathbb{CP}^3$, which shares many similarities with the well studied case of string theory on $AdS_5 \times S^5$. Indeed, the conformal $\mathcal{N} = 6$ SCS gauge theory also shares some vital qualitative features with its four dimensional $\mathcal{N} = 4$ YM counterpart, including the fact that its spectrum at weak coupling is described by an integrable quantum spin chain [21]. Furthermore, for the $\mathcal{N} = 4$ Yang–Mills theory the planar integrability, which may be thought of as feature of the 'world sheet physics' of the gauge theory, is intimately tied to dual superconformal symmetry: a property of the spacetime scattering matrix of the gauge theory [22, 23]. It is interesting to ask if similar relationships and structures appear in the $\mathcal{N} = 6$ Chern–Simons scattering amplitudes.

With these motivations in mind, we explore the scattering amplitudes for a class of SCS theories with $\mathcal{N} \geq 4$ supersymmetry which includes the $\mathcal{N} = 6$ and 8 theories as special cases. In fact we will study the one parameter family of supersymmetry preserving massive deformations of the $4 \leq \mathcal{N} \leq 8$ superconformal theories. As in the study of most conformal field theories, the three dimensional SCS models are plagued with problems of infra-red divergences, which typically manifest themselves in the low momentum behavior of massless propagators. One might hope that such problems can be remedied by making the theory massive. In extended supersymmetric gauge theories, making the dynamical matter fields massive is impossible to achieve without violating some or all the supersymmetries of the massless cases. However, in the special case of superconformal Chern–Simons models, it is indeed possible to add masses to the matter fields while preserving all the supersymmetries of the massless models, at the cost of losing (super)conformal invariance. This was established for $\mathcal{N} \geq 4$ SCS theories in [16, 17] and in previous work for the mass-deformation of the $\mathcal{N} = 8$ M2 brane theory, [24] (more recent analyses of the mass deformations of SCS and M2 brane theories theory can be found in [25, 26]). Unfortunately, the only degree of freedom contributed by the gauge filed, namely its zero mode, does lead to residual mild infra-red divergences. To the order at which we carry out the computations in this paper, these additional potential divergences are largely irrelevant, however we expect them to play in important role at higher orders in perturbation theory, and elaborate upon this issue later in the paper.

Rendering the SCS theories massive amounts to adding non-central extensions of the supersymmetry algebras, which generically take on the form $\{\mathfrak{Q}, \mathfrak{Q}\} \sim \mathfrak{P} + m\mathfrak{R}$: $\mathfrak{R}$ denoting the internal symmetry generators. The massive theories that we study, typically have the mass deformed Poincaré algebras [26]

$$\mathrm{SL}(2, \mathbb{R}) \ltimes \mathrm{PSU}(2|2) \ltimes \mathbb{R}^3 \tag{1.1}$$

or

$$\mathrm{SL}(2, \mathbb{R}) \ltimes \mathrm{PSU}(2|2)^2 \ltimes \mathbb{R}^3 \tag{1.2}$$

as their underlying symmetries which are among the exceptional super-Poincaré algebras discussed in [27]. The appearance of the mass m in the supersymmetry algebra adds a new parameter to the theory. Being part of the fundamental anti-commutation relations of the supercharges, prevents the mass from 'running', in the sense of renormalization group flows. One may be tempted to view the mass-deformed theories as a one parameter family of models extending each of the conformal $\mathcal{N} \geq 4$ SCS theories, to which they reduce in the massless limit. However, note that m is the only mass scale in the theory and thus all models which differ only in m only are expected to be related by an overall rescaling of dimensionful quantities. In this sense the massless limit is singular and it involves enhancement to superconformal symmetry. For physical quantities the limit may nevertheless be smooth as we shall observe in this paper. Still one has to keep in mind that the IR singular behavior of some quantities may be different in the limit and one has to replace the mass by an alternative IR regulator.

It is worth noting that these massive supersymmetry algebras have played an important role in a number of recent studies of supersymmetric gauge theories. For instance, in the case of $\mathcal{N}=4$ supersymmetric Yang–Mills theory in four dimensions, $\mathfrak{su}(2|2)$ played a crucial role as the symmetry of the scattering matrix [28] of the spin chain describing the planar limit of the gauge theory [29]. It was shown that the symmetry algebra uniquely fixes the spin chain scattering matrix up to an overall prefactor. This S-matrix is, by the AdS/CFT correspondence, the worldsheet S-matrix for the dual string theory [30, 31] but it was also shown that it is equivalent to Shastry's R-matrix for the one-dimensional Hubbard model [32]. In the case of the spin chain, the $\mathfrak{sl}(2)$ automorphism of its symmetry algebra, does not translate into a real symmetry of the system, as the quantum spin chain is not relativistically invariant. As was pointed out in [33, 30], the $\mathfrak{su}(2|2)$ in conjunction with its $\mathfrak{sl}(2)$ automorphism, is nothing but a mass-deformed supersymmetric extension of the three dimensional Lorentz algebra. From the point of view of the three dimensional algebra, the non-Lorentz invariance implies that the physical spectrum of the spin chain corresponds to a preferred reference frame, see [30].

Various other interesting supersymmetric three dimensional Yang–Mills theories with mass deformed super-Poincaré algebras as their symmetries have also recently been studied. In particular, the D2 brane worldvolume theory, namely, $\mathcal{N}=8$ super Yang–Mills on $\mathbb{R}\times S^2$ and its spectrum was studied in [33].[1] In the same paper, supersymmetric Yang–Mills Chern–Simons theories, with various degrees of supersymmetry, were also formulated. A salient feature of these gauge theories is that they have massive spectra, as well as propagating gluonic degrees of freedom. The gluons are rendered massive by either requiring the spacial geometry to be S^2, or by introducing Chern–Simons terms in their actions. In this regard, the theories studied in the present paper depart substantially from the examples of the super Yang–Mills theories mentioned above. In our case, there are no propagating gluons, the 'gauge' part of the theories being described by pure Chern–Simons terms. Furthermore, the spacial part of the geometries underlying the gauge theories will be taken to be $\mathbb{R}^2$.

In this paper, we study the $2\leftrightarrow 2$ scattering processes in all the massive SCS theories mentioned above. One of the main observations is that, as in the case of scattering processes in the spin chain corresponding to $\mathcal{N}=4$ Yang–Mills theory in four dimensions, the matrix structure of the two particle (spacetime) scattering matrix is completely fixed by supersymmetry. For pure $\mathcal{N}=4$ SCS theories, without twisted hypermultiplets, this means that relevant scattering matrix is completely determined by supersymmetry up to a single undetermined function. Indeed the scattering matrix for the Chern–Simons theory is formally identical to the spin chain S-matrix. For the more general case of mixed $\mathcal{N}=4$ supersymmetry, i.e. SCS theories with twisted hypermultiplets, supersymmetry leaves one with three undetermined functions. As shown later in the paper, supersymmetry enhancement to $5\leq\mathcal{N}\leq 8$ can be obtained by imposing suitable constraints on the mixed $\mathcal{N}=4$ theories. The number of undetermined functions reduces from three to two or one upon supersymmetry enhancement. Importantly, being a direct consequence of the supersymmetry algebra, the structure of the scattering matrix derived in this fashion is expected to hold to *all* orders in perturbation theory.[2] This result has a parallel in the spacetime scattering matrix

[1] This theory can be viewed as a dimensional reduction of $\mathcal{N}=8$ super Yang–Mills on $\mathbb{R}\times S^3$ to $\mathbb{R}\times S^2$

[2] We should point out that divergences in scattering amplitudes may potentially deform the supersymmetry transformation laws in analogy to what happens for conformal theories. For example, the

of $\mathcal{N} = 4$ super Yang–Mills theory in four dimensions, where all the four particle scattering matrix elements can be determined in terms of a single function [34].

Apart from establishing the matrix structure of the scattering matrix, we compute the undetermined functions for the mass-deformed SCS theories at the tree and one-loop level. The perturbative calculations also lead to independent checks that the relations between the various elements of the scattering matrix predicted by supersymmetry are indeed satisfied. We perform the one-loop correction to the scattering matrix in two different ways, i.e. by the use of standard Feynman rules as well as by using unitarity. As is well known, perturbative corrections to scattering amplitudes, can be computed efficiently by 'gluing' lower order amplitudes together using relations derived from unitarity. However, in principle this only determines the piece of the amplitude with branch cuts and so suffers from the 'polynomial ambiguity' whereby there are undetermined rational functions of the kinematical variables. We demonstrate explicitly, by calculating specific elements using standard off-shell methods, that the amplitudes can indeed be completely evaluated using the discontinuities across the cuts of the integrands, and that there are no rational functions unrelated to terms with branch cuts. We then use this simplifying feature to compute all the two particle scattering matrix elements, at the one loop order, using unitarity. Interestingly, while our calculations yield non-trivial one loop corrections to the two particle scattering matrix of $\mathcal{N} = 4$ SCS theories, with or without additional hypermultiplets, we find that all such corrections vanish identically for the cases of $\mathcal{N} \geq 5$ supersymmetry.

The paper is organized as follows. In Section 2 we elaborate upon the realization of extended supersymmetry algebras and their mass-deformations in supersymmetric Chern–Simons theories. In particular, we discuss the realization on the supersymmetry algebra on the asymptotic/scattering states of the gauge theories in question. We also introduce a particular basis for spinors in three dimensions that closely resembles the often employed spinor-helicity basis in the case of four dimensional theories. Following this discussion, Section 3, we set-up the four particle scattering picture in terms of the asymptotic states and derive the constraints imposed upon the scattering matrix elements by supersymmetry. We show that the constraints can be solved, leading to a complete determination of the matrix structure of the $2 \leftrightarrow 2$ scattering matrix of all the massive $\mathcal{N} \geq 4$ SCS theories. Further, the explicit correspondence between two-dimensional worldsheet/spin chain scattering matrix and the Chern–Simons S-matrix is described. Section 4 is devoted to the analysis of color structure of the scattering amplitudes, which is left unspecified in the sections outlined above. Specifically, we focus on the interpretations of color ordering and planarity, while leaving the choice of the gauge group to be as general as possible. In the final two sections, Section 5 and Section 6, perturbative calculations that verify the predictions of the supersymmetry algebra as well as compute the unspecified functions at the tree and one-loop order, are presented. As mentioned before, the perturbative computations are carried out using both Feynman rules as well as unitarity methods whose one-loop validity is established. We end the paper with an elaborate appendix, where most of the details relevant to the Lagrangian formulation of the massive SCS theories as well as useful details regarding the supersymmetry algebra are contained.

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supersymmetry algebra requires the dimension of spacetime to be exactly three, while in dimensional regularization it is $3 - 2\epsilon$. Subleading terms in the ϵ expansion may not have the same structure.

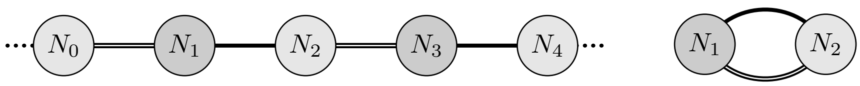

Figure 1: The quiver structure of a generic $\mathcal{N}=4$ Chern–Simons theory with unitary gauge groups (left) and of a $\mathcal{N}=5,6,8$ Chern–Simons theory (right). The solid and double lines represent untwisted and twisted matter, respectively. The circles represent gauge groups $\mathrm{U}(N_k)$ and gauge fields.

2 Supersymmetry and Asymptotic States

2.1 Extended Supersymmetry in Chern–Simons Theories

Let us start with a brief review of $\mathcal{N}\geq 4$ supersymmetry in three-dimensional quantum field theory coupled to a Chern–Simons gauge field.

The study of the conformal case with $\mathrm{OSp}(\mathcal{N}|4,\mathbb{R})$ symmetry was initiated for $\mathcal{N}=4$ supersymmetry in [18]; it was extended to include additional twisted matter and $\mathcal{N}=5,6$ supersymmetry by [16, 17], see also. This is in addition to the very many earlier and parallel developments in the $\mathcal{N}=6,8$ case briefly described earlier [20, 19, 35]. It was shown that there is a correspondence between the permissible field content of such a model and the classification of Lie superalgebras: In general, the even part of the superalgebra specifies the gauge symmetry for the Chern–Simons fields while the odd part specifies the matter content. For $\mathcal{N}\geq 5$ supersymmetry there is only one type of matter multiplet and a simple Lie superalgebra fixes the model completely. For $\mathcal{N}=4$ supersymmetry, however, there are two types of matter multiplets, so-called untwisted and twisted hypermultiplets. The field content in each of the two matter sectors is specified by a semi-simple Lie superalgebra. The even part of the two superalgebras must coincide in order for the Chern–Simons sector to be defined consistently. In particular, this leads to certain quivers of simple Lie superalgebras, see [16, 17]. The general structure of these quivers is illustrated in Fig. 1 for the example of unitary gauge groups: Considering only the untwisted matter fields one finds a direct sum of superalgebras $\mathfrak{su}(N_{2k-1}|N_{2k})$. Likewise the twisted matter fields define a direct sum of superalgebras $\mathfrak{su}(N_{2k}|N_{2k+1})$. For orthosymplectic superalgebras the nodes must alternate between orthogonal and symplectic algebras.[3] Globally, the alternating chain of nodes can be either open or closed. If the odd parts of the Lie superalgebras coincide as well, the supersymmetry enhances to $\mathcal{N}\geq 5$. This is equivalent to a closed quiver of length 2, see Fig. 1.

For $\mathcal{N}\geq 5$ supersymmetry the level $\mathcal{N}$ merely depends on which particular basic Lie superalgebra the model is based upon. The cases are summarized as follows

$$\left.\begin{array}{c}\mathfrak{osp}(n|2m)\\ \mathfrak{d}(2,1;\alpha)\\ \mathfrak{g}(3)\\ \mathfrak{f}(4)\end{array}\right\}\mathcal{N}=5,\qquad \left.\begin{array}{c}\mathfrak{sl}(n|m)\\ \mathfrak{osp}(2|2m)\end{array}\right\}\mathcal{N}=6,\qquad \mathfrak{psl}(2|2)\colon \mathcal{N}=8. \tag{2.1}$$

The representation of the even part on the odd part distinguishes the three types of superalgebras: An irreducible representation leads to $\mathcal{N}=5$ supersymmetry. If it can

[3]There may also be some more exotic quivers using bosonic algebras of lower rank which may allow to switch between the orthosymplectic and unitary series. We also do not consider explicitly the $\mathfrak{u}(1)$ factors which are present in the unitary superalgebras $\mathfrak{su}(N|M)$, see [16] for more details.

be reduced into two conjugate representations supersymmetry enhances to $\mathcal{N}=6$. If furthermore the two representations are isomorphic we obtain $\mathcal{N}=8$ supersymmetry.

This classification can be translated into a classification of continuous automorphisms of the superalgebras. The $\mathfrak{psl}(2|2)$ superalgebra is the only superalgebra with an Sp(1) outer automorphism. For $\mathfrak{sl}(n|m)$ and $\mathfrak{osp}(2|2m)$ there exist U(1) automorphisms whose action coincides with the $\mathfrak{gl}(1)$ and $\mathfrak{so}(2)$ parts of the superalgebra, respectively. The remaining basic superalgebras have no continuous outer automorphisms.[4] The classification in terms of automorphisms is natural when one views $\mathcal{N}$-extended supersymmetry from the point of view of $\mathcal{N}=4$ supersymmetry: When breaking $\mathrm{OSp}(\mathcal{N}|4,\mathbb{R})$ to manifest $\mathrm{OSp}(4|4,\mathbb{R})$ notation there must be an additional $\mathrm{SO}(\mathcal{N}-4)$ flavor symmetry. For $\mathcal{N}=5$ this requires no automorphism while for $\mathcal{N}=6$ the required automorphism is $\mathrm{SO}(2)\simeq\mathrm{U}(1)$. Finally, for $\mathcal{N}=8$ we need an $\mathrm{SO}(4)\simeq\mathrm{Sp}(1)\times\mathrm{Sp}(1)$ automorphism. Each of the two $\mathfrak{psl}(2|2)$ superalgebras provides one copy of Sp(1). Here one could also consider a manifest $\mathrm{OSp}(5|4,\mathbb{R})$ notation where the single $\mathfrak{psl}(2|2)$ superalgebra provides the $\mathrm{SO}(\mathcal{N}-5)=\mathrm{SO}(3)\simeq\mathrm{Sp}(1)$ automorphism.

Let us now turn to the massive case, which was investigated in [16,17] (see [24–26] for related work particularly in the context of massive M2-brane theories). There appears to be a one-to-one correspondence between the massive and conformal $\mathcal{N}\geq 4$ supersymmetric Chern–Simons theories; for each conformal model there is a mass deformation with the same amount of supersymmetry and for each massive model there is a conformal limit. The only additional parameter in the massive models is one overall mass scale m. The classification in terms of superalgebras remains the same. This result is somewhat curious because the massive models preserve less bosonic and only half of the fermionic symmetry and might, in principle, be less restrictive. We define the general mass-deformed $\mathcal{N}=4$ Chern–Simons theory in App. B.

In the massive case, the bosonic spacetime symmetry reduces to the Poincaré group $\mathrm{SL}(2,\mathbb{R})\ltimes\mathbb{R}^3$. Supersymmetry will be specified by some supergroup G which enters the full super-Poincaré algebra as $\mathrm{SL}(2,\mathbb{R})\ltimes\mathrm{G}\ltimes\mathbb{R}^3$. This means that the Lorentz algebra $\mathrm{SL}(2,\mathbb{R})$ acts as an automorphism on G and that the algebra of supercharges closes onto the momenta $\mathbb{R}^3$. For $\mathcal{N}=4$ supersymmetry the internal bosonic symmetry is SO(4). It joins with the supersymmetry generators into the supergroup $\mathrm{G}=\mathrm{PSU}(2|2)$. Altogether the mass-deformed $\mathcal{N}=4$ super-Poincaré group is (1.1)

$$\mathrm{SL}(2,\mathbb{R})\ltimes\mathrm{PSU}(2|2)\ltimes\mathbb{R}^3. \tag{2.2}$$

It is one of the exceptional cases of super-Poincaré algebras discussed in [27]. It is exceptional, because the supercharges close not only onto the momentum generators, but also onto the internal symmetries. This type of closure of spacetime supersymmetry is otherwise known only from the superconformal cases. Here it requires the introduction of a mass scale m to give the relation $\{\mathfrak{Q},\mathfrak{Q}\}\sim\mathfrak{P}+m\mathfrak{R}$ a consistent dimension.

For $\mathcal{N}\geq 5$ supersymmetry the supergroup G splits into two pieces, $\mathrm{G}=\mathrm{G_A}\times\mathrm{G_B}$ with $\mathrm{G_A}=\mathrm{PSU}(2|2)$ and $\mathrm{G_B}$ a supergroup of odd dimension $2(\mathcal{N}-4)$. The Lorentz group $\mathrm{SL}(2,\mathbb{R})$ must act on $\mathrm{G_B}$ as an automorphism and $\mathrm{G_B}$ must close onto the

[4] One can thus identify the three types of superalgebras in (2.1) with the fields $\mathbb{R},\mathbb{C},\mathbb{H}$, respectively.

momenta $\mathbb{R}^3$. The three cases are given by $\mathrm{G_B} = \mathrm{G}_{\mathcal{N}-4}$ with

$$\mathrm{G}_1 = \mathbb{R}^{0|2}, \qquad \mathrm{G}_2 = \mathrm{U}(1) \ltimes \mathrm{PSU}(1|1)^2 \rtimes \mathrm{U}(1), \qquad \mathrm{G}_4 = \mathrm{PSU}(2|2). \tag{2.3}$$

We shall discuss the associated superalgebras in more detail below. Here the role of the automorphisms of the superalgebra defining the field content is even more evident: They serve as the even part of $\mathrm{G}_{\mathcal{N}-4}$. For $\mathcal{N}=6$ the two U(1) automorphisms appear in G_2 while for $\mathcal{N}=8$ the two Sp(1) automorphism are equivalent to the two SU(2) factors in G_4.[5] The supergroup G_2 resembles G_4 in that the mass appears in the anticommutation relation $\{\mathfrak{Q},\mathfrak{Q}\} \sim \mathfrak{P} + m\mathfrak{R}$. However, this particular U(1) generator will commute with the remaining algebra unlike what happens in G_4. Conversely, the supergroup G_1 is almost trivial and the mass does not appear there.

Our main concern in this paper will be the case of $\mathcal{N}=4$ supersymmetry where the super group is a single copy of G_4. Let us nevertheless close this part with some remarks on $\mathcal{N} \le 4$. There appears to be the possibility of combining any two of the superalgebras $\mathrm{G}_{1,2,4}$. For instance there could in principle be a massive $\mathcal{N}=4$ supersymmetric model which preserves only $\mathrm{G}_2 \times \mathrm{G}_2$ instead of G_4. In fact the former is a subgroup of the latter and thus one can expect it to be less constraining. In the massless limit, however, both types of modes would result in the same supergroup $\mathrm{OSp}(4|4,\mathbb{R})$.

2.2 Mass-Deformed $\mathcal{N}=4$ Super-Poincaré Algebra

The mass-deformed $D=3$, $\mathcal{N}=4$ super-Poincaré algebra of (2.2) consists of the bosonic Poincaré generators $\mathfrak{L}_{\alpha\beta} = \mathfrak{L}_{\beta\alpha}$, $\mathfrak{P}_{\alpha\beta} = \mathfrak{P}_{\beta\alpha}$, the internal $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ generators $\mathfrak{R}_{ab} = \mathfrak{R}_{ba}$, $\dot{\mathfrak{R}}_{\dot a\dot b} = \dot{\mathfrak{R}}_{\dot b\dot a}$ and eight supercharges $\mathfrak{Q}_{\alpha b\dot c}$. The Lorentz and internal algebra is specified by its action on spinor indices ($|\mathcal{X}_{\dots}\rangle$ denotes any state with the indicated indices)

$$\begin{aligned}
\mathfrak{L}_{\alpha\beta}|\mathcal{X}_\gamma\rangle &= \tfrac{1}{2}\varepsilon_{\beta\gamma}|\mathcal{X}_\alpha\rangle + \tfrac{1}{2}\varepsilon_{\alpha\gamma}|\mathcal{X}_\beta\rangle,\\
\mathfrak{R}_{ab}|\mathcal{X}_c\rangle &= \tfrac{i}{2}\varepsilon_{bc}|\mathcal{X}_a\rangle + \tfrac{i}{2}\varepsilon_{ac}|\mathcal{X}_b\rangle,\\
\dot{\mathfrak{R}}_{\dot a\dot b}|\mathcal{X}_{\dot c}\rangle &= \tfrac{i}{2}\varepsilon_{\dot b\dot c}|\mathcal{X}_{\dot a}\rangle + \tfrac{i}{2}\varepsilon_{\dot a\dot c}|\mathcal{X}_{\dot b}\rangle.
\end{aligned} \tag{2.4}$$

It remains to specify the anticommutator of supercharges

$$\{\mathfrak{Q}_{\alpha b\dot c}, \mathfrak{Q}_{\delta e\dot f}\} = \varepsilon_{be}\varepsilon_{\dot c\dot f}\mathfrak{P}_{\alpha\delta} - 2m\varepsilon_{\alpha\delta}\varepsilon_{\dot c\dot f}\mathfrak{R}_{be} + 2m\varepsilon_{\alpha\delta}\varepsilon_{be}\dot{\mathfrak{R}}_{\dot c\dot f}. \tag{2.5}$$

In addition to the standard momentum generator $\mathfrak{P}_{\alpha\delta}$ it contains the internal rotation generators $\mathfrak{R}_{be}$ and $\dot{\mathfrak{R}}_{\dot c\dot f}$ which otherwise only appear in superconformal algebras. These dimensionless generators are multiplied by a common mass m for the correct mass dimension. The constant m is physical and it sets the mass scale of this model. Due to its appearance in the supersymmetry algebra it is protected from running.

For completeness we shall write the reality conditions of the generators in the relevant real form of the supersymmetry algebra. For the $\mathfrak{sl}(2)$ Lorentz and $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ internal rotations we require

$$(\mathfrak{L}_{\alpha\beta})^* = \mathfrak{L}_{\alpha\beta}, \qquad (\mathfrak{R}_{ab})^* = \varepsilon^{ac}\varepsilon^{bd}\mathfrak{R}_{cd}, \qquad (\dot{\mathfrak{R}}_{\dot a\dot b})^* = \varepsilon^{\dot a\dot c}\varepsilon^{\dot b\dot d}\dot{\mathfrak{R}}_{\dot c\dot d}. \tag{2.6}$$

[5] It is curious to observe that mass-deformed $\mathcal{N}=8$ Chern–Simons theory is, in some sense, constructed upon four copies of the superalgebra PSU(2|2).

The supersymmetry and momentum generators obey

$$(\mathfrak{Q}_{\alpha b\dot{c}})^* = -\varepsilon^{bd}\varepsilon^{\dot{c}\dot{e}}\mathfrak{Q}_{\alpha d\dot{e}}, \qquad (\mathfrak{P}_{\alpha\beta})^* = \mathfrak{P}_{\alpha\beta}. \tag{2.7}$$

2.3 $\mathcal{N} = 4$ Asymptotic Particle Representation

We can now turn to the transformation properties of the asymptotic particles. These particles can belong to any $D = 3$ quantum field theory whose spacetime symmetry is the above mass deformed $\mathcal{N} = 4$ super-Poincaré algebra. We assume that the particles are on shell, gauge-invariant and do not interact. In particular, this means that the action of the supercharges is linear (i.e. the symmetry is not spontaneously broken) and that the supersymmetry algebra closes exactly without additional gauge terms or terms proportional to the equations of motion. In particular it applies to the mass-deformed $\mathcal{N} = 4$ Chern–Simons theories outlined in App. B at arbitrary coupling. At weak coupling it is furthermore safe to identify the asymptotic particles with the fields.

For a fixed time-like momentum $p_{\alpha\beta} = p_\mu \sigma^\mu{}_{\alpha\beta}$, the stabilizer of the mass-deformed super-Poincaré algebra is $\mathfrak{u}(2|2)$. The smallest non-trivial particle multiplet thus corresponds to the (anti) fundamental representation of $\mathfrak{u}(2|2)$ consisting of two bosons and two fermions. On shell these can be identified with the (twisted) hypermultiplets of massive $\mathcal{N} = 4$ super Chern–Simons theory:

$$\text{(untwisted) hypermultiplet: } |\phi_a\rangle, |\psi_{\dot{a}}\rangle, \qquad \text{twisted hypermultiplet: } |\tilde{\phi}_{\dot{a}}\rangle, |\tilde{\psi}_a\rangle. \tag{2.8}$$

These both transform under $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ according to the general rule (2.4) but we note that the roles of the two different $\mathfrak{su}(2)$ indices are switched in the twisted case relative to the untwisted case. The most general representation of the supercharges on the hypermultiplets compatible with $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ symmetry is given by

$$\begin{aligned} \mathfrak{Q}_{\alpha b\dot{c}}|\phi_d\rangle &= \varepsilon_{bd}u_\alpha|\psi_{\dot{c}}\rangle, & \mathfrak{Q}_{\alpha b\dot{c}}|\tilde{\phi}_{\dot{d}}\rangle &= \varepsilon_{\dot{c}\dot{d}}v_\alpha|\tilde{\psi}_b\rangle, \\ \mathfrak{Q}_{\alpha b\dot{c}}|\psi_{\dot{d}}\rangle &= \varepsilon_{\dot{c}\dot{d}}v_\alpha|\phi_b\rangle, & \mathfrak{Q}_{\alpha b\dot{c}}|\tilde{\psi}_d\rangle &= \varepsilon_{bd}u_\alpha|\tilde{\phi}_{\dot{c}}\rangle. \end{aligned} \tag{2.9}$$

Closure of the supersymmetry algebra (2.5) implies the constraint

$$v_\alpha u_\beta = -p_{\alpha\beta} - im\varepsilon_{\alpha\beta}. \tag{2.10}$$

Note that (u_α, v_α) and (v_α, u_α) are effectively the incoming/outgoing polarizations for the massive spinors ψ and $\tilde{\psi}$, cf. the oscillator representation of free fermions in (B.15).

The mass of the asymptotic particles is constrained by the atypicality condition of the fundamental representation of $\mathfrak{u}(2|2)$ to equal the mass m appearing in the supersymmetry algebra (2.5). In particular, this implies that the mass of the hypermultiplets cannot run in this model.[6]

Let us investigate the relation (2.10) in some more detail. It implies that the particle momentum $p_{\alpha\beta} = p_\mu\sigma^\mu{}_{\alpha\beta}$ is a function of the spinors u_α and v_α. Therefore the representation of the stabilizer is specified through a pair of spinors (u_α, v_α) obeying the constraint

$$\varepsilon^{\alpha\beta}v_\alpha u_\beta = -2im. \tag{2.11}$$

[6]Depending on the renormalization scheme the bare and physical masses can differ by a finite amount.

The solutions of this constraint form the three-dimensional group manifold SL(2). Conversely, the mass shell in three dimensions is merely the two-dimensional hyperbolic space $H^2 = \mathrm{SL}(2)/\mathrm{U}(1)$. Thus representations of the little supersymmetry algebra carry one additional U(1) label as compared to the bosonic subalgebra. This label can be adjusted by changing phases of the spinors

$$u_\alpha \to e^{+i\alpha} u_\alpha, \qquad v_\alpha \to e^{-i\alpha} v_\alpha, \tag{2.12}$$

which does not change the relations (2.10,2.11). In (2.9) it can be seen to determine the relative phase between the bosons and fermions. The U(1) degree of freedom turns out to be inessential and it can in principle be fixed by restricting to a particular choice of phase $u(p), v(p)$ for each momentum p, e.g. (A.9). This is possible because the mass shell is topologically trivial and there is no global obstruction in choosing a U(1) element at each point of the SL(2)/U(1). Nevertheless, it is not always advisable to do this for two reasons: The spinors $u(p), v(p)$ are not covariant under Lorentz transformations, they are merely covariant up to phase. Secondly, it is sometimes convenient to complexify momenta. This however leads to a non-trivial topology of the above U(1) fibration and there is no globally consistent choice $u(p), v(p)$. In particular, this leads to potential sign ambiguities if one tries to define the spinors $u(p), v(p)$ for the two mass shells with positive and negative energies with a single analytic formula. Therefore we prefer to specify representations through the spinors u, v. However, in particular if the sign of the particle energy p_0 is well-known, it is safe to specify the spinors through the particle momentum p as in (A.9).

Finally we would like to discuss unitarity conditions of the representation. According to (2.7) and (2.9) hermiticity of the supersymmetry generators implies the unitarity condition

$$u_\alpha^* = +v_\alpha. \tag{2.13}$$

This also leads to a real momentum p_μ according to (2.10). Moreover, the energy p_0 is positive definite as usual in supersymmetry algebras. Conversely, particle multiplets with negative energy obey

$$u_\alpha^* = -v_\alpha \tag{2.14}$$

and they transform in a graded unitary representation where all bosonic generators are hermitian and the supercharges are anti-hermitian.

The two types of representations discussed above are just the simplest non-trivial representations of the mass-deformed $\mathcal{N} = 4$ super-Poincaré group (2.2). The representation theory follows closely the one of $\mathfrak{su}(2|2)$, cf. [32] and references therein: There are short/atypical representations $\langle k, l\rangle$ of dimension $4(k+1)(l+1) + 4kl$. The corresponding particles have an algebraically fixed mass $m = k+l+1$. The fundamental representations discussed above are the special case $\langle 0, 0\rangle$. Additionally there are long representations $\{k, l\}$ of dimension $16(k+1)(l+1)$. Their mass is unconstrained.[7] It would be interesting to study the spectrum of composite states in supersymmetric Chern–Simons theories.

[7]If one picks the particular value $m = k + l + 2$, however, the representation reduces to $\langle k+1, l\rangle$ and $\langle k, l+1\rangle$. In other words, one can combine two particle multiplets $\langle k+1, l\rangle$ and $\langle k, l+1\rangle$ to form a long multiplet whose mass is henceforth unconstrained in close analogy to the Higgs mechanism.

2.4 $\mathcal{N} = 5, 6, 8$ Supersymmetry and Multiplets

Let us also discuss the algebras and representations for $\mathcal{N} = 5, 6, 8$ supersymmetry. This is applicable to Chern–Simons theories with coinciding representations for untwisted and twisted matter (see App. B) which have manifest $\mathcal{N} = 5, 6, 8$ supersymmetry. In general the higher supersymmetries anticommute with the $\mathcal{N} = 4$ supercharges and thus they have to transform between untwisted and twisted $\mathcal{N} = 4$ hypermultiplets.

In the simplest $\mathcal{N} = 5$ case there are two additional supercharges $\tilde{\mathfrak{Q}}_\alpha$. Their algebra $\mathfrak{g}_1$ closes onto the momentum generators

$$\{\tilde{\mathfrak{Q}}_\alpha, \tilde{\mathfrak{Q}}_\delta\} = 2\mathfrak{P}_{\alpha\delta}. \tag{2.15}$$

The additional supercharges $\mathfrak{Q}_\alpha$ must act like

$$\begin{aligned}
\tilde{\mathfrak{Q}}_\alpha|\phi_b\rangle &= +v_\alpha(p)|\tilde{\psi}_b\rangle, & \tilde{\mathfrak{Q}}_\alpha|\psi_{\dot{b}}\rangle &= -v_\alpha(p)|\tilde{\phi}_{\dot{b}}\rangle,\\
\tilde{\mathfrak{Q}}_\alpha|\tilde{\psi}_b\rangle &= -u_\alpha(p)|\phi_b\rangle, & \tilde{\mathfrak{Q}}_\alpha|\tilde{\phi}_{\dot{b}}\rangle &= +u_\alpha(p)|\psi_{\dot{b}}\rangle.
\end{aligned} \tag{2.16}$$

For $\mathcal{N} = 6$ supersymmetry the additional algebra $\mathfrak{g}_2$ consists of four supersymmetries $\tilde{\mathfrak{Q}}^\pm_\alpha$ and two bosonic generators $\tilde{\mathfrak{B}}$ and $\tilde{\mathfrak{C}}$. Their non-trivial commutation relations are

$$[\tilde{\mathfrak{B}}, \tilde{\mathfrak{Q}}^\pm_\alpha] = \pm\tilde{\mathfrak{Q}}^\pm, \qquad \{\tilde{\mathfrak{Q}}^+_\alpha, \tilde{\mathfrak{Q}}^-_\beta\} = \mathfrak{P}_{\alpha\beta} - im\varepsilon_{\alpha\beta}\tilde{\mathfrak{C}}. \tag{2.17}$$

There are two types of multiplets with opposite eigenvalue of the central charge $\tilde{\mathfrak{C}}$. For $\tilde{\mathfrak{C}} \simeq -1$ the action of the supercharges reads

$$\begin{aligned}
\tilde{\mathfrak{Q}}^+_\alpha|\phi_{b-}\rangle &= +v_\alpha(p)|\tilde{\psi}_{b-}\rangle, & \tilde{\mathfrak{Q}}^+_\alpha|\psi_{\dot{b}-}\rangle &= -v_\alpha(p)|\tilde{\phi}_{\dot{b}-}\rangle,\\
\tilde{\mathfrak{Q}}^-_\alpha|\tilde{\psi}_{b-}\rangle &= -u_\alpha(p)|\phi_{b-}\rangle, & \tilde{\mathfrak{Q}}^-_\alpha|\tilde{\phi}_{\dot{b}-}\rangle &= +u_\alpha(p)|\psi_{\dot{b}-}\rangle,\\
\tilde{\mathfrak{Q}}^+_\alpha|\tilde{\psi}_{b-}\rangle &= 0, & \tilde{\mathfrak{Q}}^+_\alpha|\tilde{\phi}_{\dot{b}-}\rangle &= 0,\\
\tilde{\mathfrak{Q}}^-_\alpha|\phi_{b-}\rangle &= 0, & \tilde{\mathfrak{Q}}^-_\alpha|\psi_{\dot{b}-}\rangle &= 0.
\end{aligned} \tag{2.18}$$

The action on the conjugate multiplet with $\tilde{\mathfrak{C}} \simeq +1$ reads

$$\begin{aligned}
\tilde{\mathfrak{Q}}^-_\alpha|\phi_{b+}\rangle &= +v_\alpha(p)|\tilde{\psi}_{b+}\rangle, & \tilde{\mathfrak{Q}}^-_\alpha|\psi_{\dot{b}+}\rangle &= -v_\alpha(p)|\tilde{\phi}_{\dot{b}+}\rangle,\\
\tilde{\mathfrak{Q}}^+_\alpha|\tilde{\psi}_{b+}\rangle &= -u_\alpha(p)|\phi_{b+}\rangle, & \tilde{\mathfrak{Q}}^+_\alpha|\tilde{\phi}_{\dot{b}+}\rangle &= +u_\alpha(p)|\psi_{\dot{b}+}\rangle,\\
\tilde{\mathfrak{Q}}^-_\alpha|\tilde{\psi}_{b+}\rangle &= 0, & \tilde{\mathfrak{Q}}^-_\alpha|\tilde{\phi}_{\dot{b}+}\rangle &= 0,\\
\tilde{\mathfrak{Q}}^+_\alpha|\phi_{b+}\rangle &= 0, & \tilde{\mathfrak{Q}}^+_\alpha|\psi_{\dot{b}+}\rangle &= 0.
\end{aligned} \tag{2.19}$$

In supersymmetric Chern–Simons theories the splitting into the $\tilde{\mathfrak{C}} \simeq \pm 1$ multiplets originates from the structure of the superalgebra which defines the field content, cf. Sec. 2.1.

Finally, for $\mathcal{N} = 8$ supersymmetry there is a complete copy the the $\mathcal{N} = 4$ superalgebra $\mathfrak{g}_4$ consisting of the generators $\tilde{\mathfrak{R}}_{\tilde{a}\tilde{b}}$, $\hat{\mathfrak{R}}_{\hat{a}\hat{b}}$ and $\tilde{\mathfrak{Q}}_{\alpha\tilde{b}\hat{c}}$. The $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ generators act on spinor indices as usual

$$\begin{aligned}
\tilde{\mathfrak{R}}_{\tilde{a}\tilde{b}}|X_{\tilde{c}}\rangle &= \tfrac{i}{2}\varepsilon_{\tilde{b}\tilde{c}}|X_{\tilde{a}}\rangle + \tfrac{i}{2}\varepsilon_{\tilde{a}\tilde{c}}|X_{\tilde{b}}\rangle,\\
\hat{\mathfrak{R}}_{\hat{a}\hat{b}}|X_{\hat{c}}\rangle &= \tfrac{i}{2}\varepsilon_{\hat{b}\hat{c}}|X_{\hat{a}}\rangle + \tfrac{i}{2}\varepsilon_{\hat{a}\hat{c}}|X_{\hat{b}}\rangle.
\end{aligned} \tag{2.20}$$

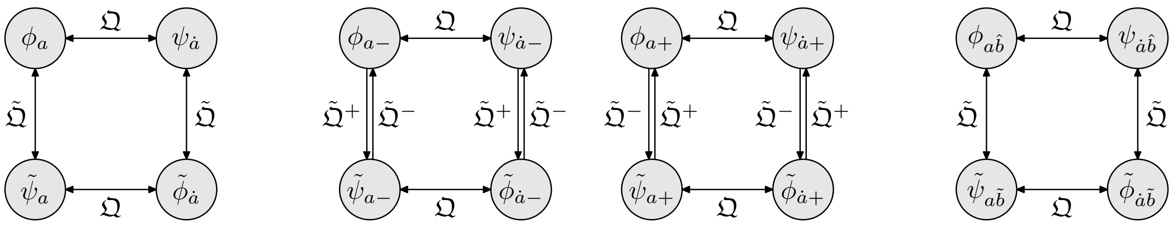

Figure 2: The action of the extended $\mathcal{N} = 5, 6, 8$ supersymmetries (left, middle two, right) on one-particle asymptotic states.

Furthermore the supercharges obey the same relation as above

$$\{\tilde{\mathfrak{Q}}_{\alpha\tilde b\hat c}, \tilde{\mathfrak{Q}}_{\delta\tilde e\hat f}\} = \varepsilon_{\tilde b\tilde e}\varepsilon_{\hat c\hat f}\mathfrak{P}_{\alpha\delta} - 2m\varepsilon_{\alpha\delta}\varepsilon_{\hat c\hat f}\tilde{\mathfrak{R}}_{\tilde b\tilde e} + 2m\varepsilon_{\alpha\delta}\varepsilon_{\tilde b\tilde e}\hat{\mathfrak{R}}_{\hat c\hat f}. \tag{2.21}$$

The representation on the fields is much like the one discussed in Sec. 2.3

$$\begin{aligned}
\tilde{\mathfrak{Q}}_{\alpha\tilde b\hat c}|\phi_{d\hat e}\rangle &= +\varepsilon_{\hat c\hat e}v_\alpha(p)|\tilde\psi_{d\tilde b}\rangle, &\qquad \tilde{\mathfrak{Q}}_{\alpha\tilde b\hat c}|\psi_{\dot d\hat e}\rangle &= -\varepsilon_{\hat c\hat e}v_\alpha(p)|\tilde\phi_{\dot d\tilde b}\rangle,\\
\tilde{\mathfrak{Q}}_{\alpha\tilde b\hat c}|\tilde\psi_{d\tilde e}\rangle &= +\varepsilon_{\tilde b\tilde e}u_\alpha(p)|\phi_{d\hat c}\rangle, &\qquad \tilde{\mathfrak{Q}}_{\alpha\tilde b\hat c}|\tilde\phi_{\dot d\tilde e}\rangle &= -\varepsilon_{\tilde b\tilde e}u_\alpha(p)|\psi_{\dot d\hat c}\rangle.
\end{aligned} \tag{2.22}$$

Again, the additional indices $\hat e$ and $\tilde e$ on the untwisted and twisted multiplets, respectively, originate from the structure of the defining superalgebra.

Note that $\mathcal{N} = 6$ Chern–Simons models at levels $k = 1$ or $k = 2$ are expected to have $\mathcal{N} = 8$ enhanced supersymmetry [20]. This may appear impossible considering that the Sp(1) automorphism required for $\mathcal{N} = 8$ supersymmetry cannot act on a single $\mathcal{N} = 6$ particle representation. Here one has to bear in mind that the points $k = 1, 2$ are strongly coupled. Particles can form bound states which may turn out to have degenerate energies such that an Sp(1) automorphism appears.

To conclude, we summarize the action of the $\mathcal{N} = 5, 6, 8$ supersymmetry generators $\mathfrak{Q}$ and $\tilde{\mathfrak{Q}}$ on the untwisted and twisted hypermultiplets in Fig. 2.

3 Scattering Amplitudes from Supersymmetry

In this section we shall derive the form of scattering amplitudes by means of supersymmetry and compare these predictions to field theory calculations.

3.1 Pure Scattering

We will now set up the amplitudes for a scattering process of four hypermultiplets. The processes described here account for scattering of purely untwisted hypermultiplets in models with $\mathcal{N} \geq 4$ supersymmetry. We assume that all four particle momenta p_k, $k = 1, 2, 3, 4$, are incoming and on shell. The polarization spinors (u_k, v_k) are on-shell according to (2.10).

The amplitudes will be represented by an operator $\langle\mathcal{T}|$ acting on four-particle states and returning the corresponding amplitude[8]

$$\langle\mathcal{T}|1234\rangle = A_{1234}. \tag{3.1}$$

[8]Alternatively we may provide give an invariant four-particle state or an invariant two-to-two scattering operator, cf. Sec. 3.5.

The most general ansatz for the scattering matrix elements with manifest $\mathfrak{su}(2)\oplus\mathfrak{su}(2)$ symmetry reads

$$
\begin{aligned}
\langle\mathcal{T}|\phi_a\phi_b\phi_c\phi_d\rangle &= \big(+\tfrac{1}{2}(A+B)\varepsilon_{ad}\varepsilon_{bc}+\tfrac{1}{2}(A-B)\varepsilon_{ac}\varepsilon_{bd}\big)\,\delta^3(p_1+p_2+p_3+p_4),\\
\langle\mathcal{T}|\psi_{\dot a}\psi_{\dot b}\psi_{\dot c}\psi_{\dot d}\rangle &= \big(+\tfrac{1}{2}(D+E)\varepsilon_{\dot a\dot d}\varepsilon_{\dot b\dot c}+\tfrac{1}{2}(D-E)\varepsilon_{\dot a\dot c}\varepsilon_{\dot b\dot d}\big)\,\delta^3(p_1+p_2+p_3+p_4),\\
\langle\mathcal{T}|\phi_a\psi_{\dot b}\phi_c\psi_{\dot d}\rangle &= -G\varepsilon_{ac}\varepsilon_{\dot b\dot d}\,\delta^3(p_1+p_2+p_3+p_4),\\
\langle\mathcal{T}|\psi_{\dot a}\phi_b\psi_{\dot c}\phi_d\rangle &= -L\varepsilon_{\dot a\dot c}\varepsilon_{bd}\,\delta^3(p_1+p_2+p_3+p_4),\\
\langle\mathcal{T}|\phi_a\phi_b\psi_{\dot c}\psi_{\dot d}\rangle &= -\tfrac{1}{2}C\varepsilon_{ab}\varepsilon_{\dot c\dot d}\,\delta^3(p_1+p_2+p_3+p_4),\\
\langle\mathcal{T}|\phi_a\psi_{\dot b}\psi_{\dot c}\phi_d\rangle &= -H\varepsilon_{ad}\varepsilon_{\dot b\dot c}\,\delta^3(p_1+p_2+p_3+p_4),\\
\langle\mathcal{T}|\psi_{\dot a}\psi_{\dot b}\phi_c\phi_d\rangle &= -\tfrac{1}{2}F\varepsilon_{\dot a\dot b}\varepsilon_{cd}\,\delta^3(p_1+p_2+p_3+p_4),\\
\langle\mathcal{T}|\psi_{\dot a}\phi_b\phi_c\psi_{\dot d}\rangle &= -K\varepsilon_{\dot a\dot d}\varepsilon_{bc}\,\delta^3(p_1+p_2+p_3+p_4).
\end{aligned}
\tag{3.2}
$$

At this stage there are 10 independent matrix elements $A,\ldots,L$ of the scattering amplitude.

We would now like to impose invariance under supersymmetry on the amplitude leading to further constraints on the matrix elements. This is conveniently done by imposing invariance conditions of the sort $\langle\mathcal{T}|\mathfrak{Q}_{11}|\phi_1\phi_2\phi_2\psi_2\rangle = 0$. From these we obtain altogether 32 constraints which are collected in the following 16 spinor-valued equations

$$
\begin{aligned}
0 &= Av_3+Hu_2+Lu_1, & 0 &= Bv_3+Cu_4+Hu_2-Lu_1,\\
0 &= Av_4+Gu_2+Ku_1, & 0 &= Bv_4-Cu_3-Gu_2+Ku_1,\\
0 &= Av_2-Hu_3-Gu_4, & 0 &= Bv_2-Fu_1-Hu_3+Gu_4,\\
0 &= Av_1-Lu_3-Ku_4, & 0 &= Bv_1+Fu_2+Lu_3-Ku_4,\\
0 &= Du_3+Gv_1-Kv_2, & 0 &= Eu_3-Fv_4-Gv_1-Kv_2,\\
0 &= Du_4-Hv_1+Lv_2, & 0 &= Eu_4-Fu_3-Hv_1-Lv_2,\\
0 &= Du_1-Gv_3+Hv_4, & 0 &= Eu_1-Cv_2+Gv_3+Hv_4,\\
0 &= Du_2+Kv_3-Lv_4, & 0 &= Eu_2+Cv_1+Kv_3+Lv_4.
\end{aligned}
\tag{3.3}
$$

In order to extract the matrix elements $A,\ldots,L$ it is convenient to contract the equations with spinors appearing in the equation using the antisymmetric tensor $\varepsilon^{\alpha\beta}$. To avoid clutter we introduce an antisymmetric scalar product for spinors

$$\langle u,v\rangle = \varepsilon^{\alpha\beta}u_\alpha v_\beta. \tag{3.4}$$

In analogy to the scattering amplitudes in four-dimensional Yang–Mills theory using the spinor-helicity formalism we shall call this product a twistor bracket. Moreover, we shall use the short notation

$$\langle kl\rangle := \langle u_k,u_l\rangle,\quad \langle\bar kl\rangle := \langle v_k,u_l\rangle,\quad \langle\bar k\bar l\rangle := \langle v_k,v_l\rangle. \tag{3.5}$$

Somewhat remarkably, all constraints can be solved simultaneously on the ten matrix elements leaving just one overall factor T for the amplitude. The solution reads

$$\begin{aligned}
A &= T\,, & D &= -T\,\frac{\langle\bar{3}\bar{4}\rangle}{\langle 12\rangle}\,, & G &= +T\,\frac{\langle\bar{4}1\rangle}{\langle 12\rangle}\,,\\
\tfrac{1}{2}(A+B) &= -T\,\frac{\langle\bar{3}1\rangle\langle 24\rangle}{\langle 12\rangle\langle\bar{3}\bar{4}\rangle}\,, & \tfrac{1}{2}(D+E) &= +T\,\frac{\langle\bar{3}1\rangle\langle\bar{1}\bar{3}\rangle}{\langle 12\rangle\langle\bar{3}\bar{4}\rangle}\,, & H &= +T\,\frac{\langle\bar{3}1\rangle}{\langle 12\rangle}\,,\\
\tfrac{1}{2}(A-B) &= +T\,\frac{\langle 14\rangle\langle\bar{3}2\rangle}{\langle 12\rangle\langle\bar{3}\bar{4}\rangle}\,, & \tfrac{1}{2}(D-E) &= -T\,\frac{\langle\bar{2}\bar{3}\rangle\langle\bar{3}2\rangle}{\langle 12\rangle\langle\bar{3}\bar{4}\rangle}\,, & K &= -T\,\frac{\langle\bar{4}2\rangle}{\langle 12\rangle}\,,\\
\tfrac{1}{2}C &= -T\,\frac{\langle\bar{3}1\rangle\langle\bar{3}2\rangle}{\langle 12\rangle\langle\bar{3}\bar{4}\rangle}\,, & \tfrac{1}{2}F &= +T\,\frac{\langle\bar{3}1\rangle\langle\bar{1}4\rangle}{\langle 12\rangle\langle\bar{3}\bar{4}\rangle}\,, & L &= -T\,\frac{\langle\bar{3}2\rangle}{\langle 12\rangle}\,.
\end{aligned} \tag{3.6}$$

Clearly there are many equivalent ways of writing the matrix elements which also explains why many of the 32 constraints are degenerate. A useful set of identities for the four scattering particles with $p_1 + p_2 + p_3 + p_4 = 0$ is given by

$$\frac{\langle lm\rangle}{\langle kn\rangle} = \frac{\langle\bar{k}l\rangle}{\langle\bar{m}n\rangle} = \frac{\langle\bar{k}\bar{n}\rangle}{\langle\bar{l}\bar{m}\rangle}\,, \qquad \langle\bar{k}k\rangle = -2im, \qquad \{k,l,m,n\} = \{1,2,3,4\}. \tag{3.7}$$

Additionally there is a cyclic identity which holds for any four two-component spinors

$$0 = \langle a,b\rangle\langle c,d\rangle + \langle b,c\rangle\langle a,d\rangle + \langle c,a\rangle\langle b,d\rangle. \tag{3.8}$$

There are three simple relations among the matrix elements which can be checked using the above identities

$$\begin{aligned}
0 &= AD + GL - HK,\\
0 &= AD - BE + CF,\\
0 &= (A-B)(D-E) - CF + 4GL.
\end{aligned} \tag{3.9}$$

The remaining seven matrix elements are independent: they can be adjusted freely by choosing one overall factor, one fermion phase for each leg ($u_k, v_k \mapsto e^{\pm i\alpha_k} u_k, v_k$) and the two Mandelstam invariants $s = (p_1 + p_2)^2$ and $t = (p_1 + p_4)^2$. Note that by a Lorentz transformation one can change only three of the fermion phases. Thus if one uses a particular choice of spinor polarization as a function of momenta, e.g. (A.9), then only six elements are independent.

3.2 Mixed Scattering

Next we will consider scattering of mixed matter fields. Twisted hypermultiplets transform under supersymmetry equivalently to untwisted multiplets, however with the statistics of the on-shell particles flipped. To obtain the twisted scattering amplitudes we can thus simply replace a $(\phi_a, \psi_{\dot{a}})$ by a $(\tilde{\psi}_a, \tilde{\phi}_{\dot{a}})$ and insert the appropriate signs due to the change of statistics.

For correct overall statistics and parity of the internal symmetry, we can only twist multiplets in pairs. First we twist particles 3 and 4 of the four-particle scattering amplitude (3.2). A suitable ansatz for the mixed scattering amplitude is

$$\begin{aligned}
\langle\mathcal{T}|\phi_a\phi_b\tilde{\psi}_c\tilde{\psi}_d\rangle &= \left(+\tfrac{1}{2}(A+B)\varepsilon_{ad}\varepsilon_{bc}+\tfrac{1}{2}(A-B)\varepsilon_{ac}\varepsilon_{bd}\right)\delta^3(p_1+p_2+p_3+p_4),\\
\langle\mathcal{T}|\psi_{\dot a}\psi_{\dot b}\tilde{\phi}_{\dot c}\tilde{\phi}_{\dot d}\rangle &= \left(-\tfrac{1}{2}(D+E)\varepsilon_{\dot a\dot d}\varepsilon_{\dot b\dot c}-\tfrac{1}{2}(D-E)\varepsilon_{\dot a\dot c}\varepsilon_{\dot b\dot d}\right)\delta^3(p_1+p_2+p_3+p_4),\\
\langle\mathcal{T}|\phi_a\psi_{\dot b}\tilde{\psi}_c\tilde{\phi}_{\dot d}\rangle &= +G\varepsilon_{ac}\varepsilon_{\dot b\dot d}\,\delta^3(p_1+p_2+p_3+p_4),\\
\langle\mathcal{T}|\psi_{\dot a}\phi_b\tilde{\phi}_{\dot c}\tilde{\psi}_d\rangle &= -L\varepsilon_{\dot a\dot c}\varepsilon_{bd}\,\delta^3(p_1+p_2+p_3+p_4),\\
\langle\mathcal{T}|\phi_a\phi_b\tilde{\phi}_{\dot c}\tilde{\phi}_{\dot d}\rangle &= +\tfrac{1}{2}C\varepsilon_{ab}\varepsilon_{\dot c\dot d}\,\delta^3(p_1+p_2+p_3+p_4),\\
\langle\mathcal{T}|\phi_a\psi_{\dot b}\tilde{\phi}_{\dot c}\tilde{\psi}_d\rangle &= -H\varepsilon_{ad}\varepsilon_{\dot b\dot c}\,\delta^3(p_1+p_2+p_3+p_4),\\
\langle\mathcal{T}|\psi_{\dot a}\psi_{\dot b}\tilde{\psi}_c\tilde{\psi}_d\rangle &= -\tfrac{1}{2}F\varepsilon_{\dot a\dot b}\varepsilon_{cd}\,\delta^3(p_1+p_2+p_3+p_4),\\
\langle\mathcal{T}|\psi_{\dot a}\phi_b\tilde{\psi}_c\tilde{\phi}_{\dot d}\rangle &= +K\varepsilon_{\dot a\dot d}\varepsilon_{bc}\,\delta^3(p_1+p_2+p_3+p_4).
\end{aligned}\tag{3.10}$$

The constraints due to supersymmetry turn out to be exactly the same as in (3.3): Due to the change of statistics of particles 3 and 4, we have to flip the signs of all instances of u_4, v_4. Furthermore, the signs of the matrix elements C, D, E, G, K in (3.10) have been flipped with respect to those in (3.2). Altogether the sign flips cancel out and the solution (3.6) applies to the mixed scattering matrix as well. Note that for the amplitudes $A, \ldots, L$ we can use a different prefactor T which will be denoted by $T_{12\tilde{3}\tilde{4}}$. In general a particle index $\tilde{k}$ will indicate a twisted hypermultiplet.

Finally, let us state the result for the scattering matrix of four twisted multiplets

$$\begin{aligned}
\langle\mathcal{T}|\tilde{\psi}_a\tilde{\psi}_b\tilde{\psi}_c\tilde{\psi}_d\rangle &= \left(+\tfrac{1}{2}(A+B)\varepsilon_{ad}\varepsilon_{bc}+\tfrac{1}{2}(A-B)\varepsilon_{ac}\varepsilon_{bd}\right)\delta^3(p_1+p_2+p_3+p_4),\\
\langle\mathcal{T}|\tilde{\phi}_{\dot a}\tilde{\phi}_{\dot b}\tilde{\phi}_{\dot c}\tilde{\phi}_{\dot d}\rangle &= \left(+\tfrac{1}{2}(D+E)\varepsilon_{\dot a\dot d}\varepsilon_{\dot b\dot c}+\tfrac{1}{2}(D-E)\varepsilon_{\dot a\dot c}\varepsilon_{\dot b\dot d}\right)\delta^3(p_1+p_2+p_3+p_4),\\
\langle\mathcal{T}|\tilde{\psi}_a\tilde{\phi}_{\dot b}\tilde{\psi}_c\tilde{\phi}_{\dot d}\rangle &= -G\varepsilon_{ac}\varepsilon_{\dot b\dot d}\,\delta^3(p_1+p_2+p_3+p_4),\\
\langle\mathcal{T}|\tilde{\phi}_{\dot a}\tilde{\psi}_b\tilde{\phi}_{\dot c}\tilde{\psi}_d\rangle &= -L\varepsilon_{\dot a\dot c}\varepsilon_{bd}\,\delta^3(p_1+p_2+p_3+p_4),\\
\langle\mathcal{T}|\tilde{\psi}_a\tilde{\psi}_b\tilde{\phi}_{\dot c}\tilde{\phi}_{\dot d}\rangle &= +\tfrac{1}{2}C\varepsilon_{ab}\varepsilon_{\dot c\dot d}\,\delta^3(p_1+p_2+p_3+p_4),\\
\langle\mathcal{T}|\tilde{\psi}_a\tilde{\phi}_{\dot b}\tilde{\phi}_{\dot c}\tilde{\psi}_d\rangle &= +H\varepsilon_{ad}\varepsilon_{\dot b\dot c}\,\delta^3(p_1+p_2+p_3+p_4),\\
\langle\mathcal{T}|\tilde{\phi}_{\dot a}\tilde{\phi}_{\dot b}\tilde{\psi}_c\tilde{\psi}_d\rangle &= +\tfrac{1}{2}F\varepsilon_{\dot a\dot b}\varepsilon_{cd}\,\delta^3(p_1+p_2+p_3+p_4),\\
\langle\mathcal{T}|\tilde{\phi}_{\dot a}\tilde{\psi}_b\tilde{\psi}_c\tilde{\phi}_{\dot d}\rangle &= +K\varepsilon_{\dot a\dot d}\varepsilon_{bc}\,\delta^3(p_1+p_2+p_3+p_4).
\end{aligned}\tag{3.11}$$

Here the signs of H, K, C, F have been flipped with respect to (3.2). Flipping as well the signs of u_2, v_2, u_4, v_4 results in the same set of constraints (3.3) whose solution is given by (3.6). The prefactor for this scattering process will be denoted by $T_{\tilde{1}\tilde{2}\tilde{3}\tilde{4}}$.

3.3 Scattering with $\mathcal{N} > 4$ Supersymmetry

Let us now consider the additional constraints that follow if we extend the supersymmetry to $\mathcal{N} = 5$. We have new invariance conditions of the type $\langle\mathcal{T}|\tilde{\mathfrak{Q}}_\alpha|\phi_1\phi_1\phi_2\tilde{\psi}_2\rangle$ which in principle give sixteen constraints on the eight independent matrix structures

$$\begin{aligned}
&T_{1234}, && T_{12\tilde{3}\tilde{4}}, && T_{1\tilde{2}3\tilde{4}}, && T_{1\tilde{2}\tilde{3}4},\\
&T_{\tilde{1}\tilde{2}34}, && T_{\tilde{1}2\tilde{3}4}, && T_{\tilde{1}23\tilde{4}}, && T_{\tilde{1}\tilde{2}\tilde{3}\tilde{4}}.
\end{aligned}\tag{3.12}$$

However most of the constraints are redundant and there are only six which are independent. We choose these to be

$$
\begin{aligned}
&+\langle\bar{1}\bar{2}\rangle T_{1\tilde{2}3\tilde{4}}+\langle\bar{1}\bar{3}\rangle T_{12\tilde{3}\tilde{4}}-\langle\bar{1}4\rangle T_{1234}=0,\\
&-\langle\bar{1}\bar{2}\rangle T_{\tilde{1}23\tilde{4}}+\langle\bar{2}\bar{3}\rangle T_{12\tilde{3}\tilde{4}}-\langle\bar{2}4\rangle T_{1234}=0,\\
&+\langle\bar{1}\bar{2}\rangle T_{1\tilde{2}\tilde{3}4}-\langle\bar{1}3\rangle T_{1234}-\langle\bar{1}\bar{4}\rangle T_{12\tilde{3}\tilde{4}}=0,\\
&-\langle\bar{1}2\rangle T_{\tilde{1}\tilde{2}34}+\langle\bar{3}2\rangle T_{1\tilde{2}\tilde{3}4}+\langle\bar{4}2\rangle T_{1\tilde{2}3\tilde{4}}=0,\\
&+\langle\bar{2}1\rangle T_{\tilde{1}\tilde{2}34}+\langle\bar{3}1\rangle T_{\tilde{1}2\tilde{3}4}+\langle\bar{4}1\rangle T_{\tilde{1}23\tilde{4}}=0,\\
&+\langle 12\rangle T_{\tilde{1}\tilde{2}34}-\langle 13\rangle T_{\tilde{1}\tilde{2}\tilde{3}4}+\langle\bar{4}1\rangle T_{\tilde{1}\tilde{2}\tilde{3}\tilde{4}}=0.
\end{aligned}
\tag{3.13}
$$

We can thus express all scattering elements in terms of T_{1234} and $T_{12\tilde{3}\tilde{4}}$

$$
\begin{aligned}
T_{\tilde{1}\tilde{2}\tilde{3}\tilde{4}}&=\frac{\langle 12\rangle}{\langle\bar{4}1\rangle}\left(\frac{\langle 13\rangle}{\langle 34\rangle}+\frac{\langle\bar{2}\bar{4}\rangle}{\langle\bar{1}\bar{2}\rangle}\right)T_{12\tilde{3}\tilde{4}}-\frac{\langle 12\rangle}{\langle\bar{3}\bar{4}\rangle}T_{1234}\,,\\
T_{\tilde{1}\tilde{2}34}&=\frac{\langle 12\rangle}{\langle 34\rangle}\,T_{12\tilde{3}\tilde{4}}\,,\\
T_{1\tilde{2}3\tilde{4}}&=\frac{-\langle\bar{1}\bar{3}\rangle T_{12\tilde{3}\tilde{4}}+\langle\bar{1}4\rangle T_{1234}}{\langle\bar{1}\bar{2}\rangle}\,,\\
T_{1\tilde{2}\tilde{3}4}&=\frac{\langle\bar{1}\bar{4}\rangle T_{12\tilde{3}\tilde{4}}+\langle\bar{1}3\rangle T_{1234}}{\langle\bar{1}\bar{2}\rangle}\,,\\
T_{\tilde{1}2\tilde{3}4}&=\frac{-\langle\bar{2}\bar{4}\rangle T_{12\tilde{3}\tilde{4}}-\langle\bar{2}3\rangle T_{1234}}{\langle\bar{1}\bar{2}\rangle}\,,\\
T_{\tilde{1}23\tilde{4}}&=\frac{\langle\bar{2}\bar{3}\rangle T_{12\tilde{3}\tilde{4}}-\langle\bar{2}4\rangle T_{1234}}{\langle\bar{1}\bar{2}\rangle}\,.
\end{aligned}
\tag{3.14}
$$

There are further, similar, relations if we consider the $\mathcal{N}=6$ algebra, however the multiplet structure is slightly more complicated. In the case of $\mathcal{N}=8$ the algebra is comprised of two copies of $\mathfrak{psu}(2|2)$ and the scattering matrix takes a simple tensor product form. Let us consider in a little more detail the $\mathcal{N}=8$ case. We must decompose the symplectic gauge indices into $\mathfrak{so}(4)_{\rm gauge}\times\mathfrak{su}(2)\times\mathfrak{su}(2)$ indices

$$
\begin{aligned}
\phi^A_a&=\phi^A_{a\hat{a}}, &\qquad \tilde{\phi}^A_{\dot{a}}&=\tilde{\phi}^A_{\dot{a}\tilde{a}}\\
\psi^A_{\dot{a}}&=\psi^A_{\dot{a}\hat{a}}, &\qquad \tilde{\phi}^A_a&=\tilde{\phi}^A_{a\tilde{a}}\ .
\end{aligned}
\tag{3.15}
$$

The $\mathcal{N}=4$ prefactors similarly decompose e.g.

$$
T^{A\hat{a},B\hat{b},C\hat{c},D\hat{d}}_{1234}=\langle\mathcal{T}|\phi^A_{1\hat{a}}\phi^B_{1\hat{b}}\phi^C_{2\hat{c}}\psi^D_{2\hat{d}}\rangle
\tag{3.16}
$$

These are all related by the additional $\mathfrak{su}(2)$'s and so we can pick a single representative in each sector in terms of which we can express all other elements. Let us define

$$
\begin{aligned}
T^{\mathcal{N}=8}_{12\tilde{3}\tilde{4}}&=T^{A\hat{1},B\hat{2},C\tilde{1},D\tilde{2}}_{12\tilde{3}\tilde{4}}\\
T^{\mathcal{N}=8}_{1234}&=T^{A\hat{1},B\hat{2},C\hat{1},D\hat{2}}_{1234}
\end{aligned}
\tag{3.17}
$$

As before we consider the constraints following from invariance conditions such as

$$
\langle\mathcal{T}|\tilde{\mathfrak{Q}}_{\alpha\tilde{e}\hat{f}}|\phi^A_1\phi^B_1\phi^C_2\tilde{\psi}^D_2\rangle=\langle\mathcal{T}|\tilde{\mathfrak{Q}}_{\alpha\tilde{e}\hat{f}}|\phi^A_{1\hat{a}}\phi^B_{1\hat{b}}\phi^C_{2\hat{c}}\tilde{\psi}^D_{2\tilde{d}}\rangle\ .
\tag{3.18}
$$

and in addition to (3.14) we can find a relation between $T_{12\tilde{3}\tilde{4}}$ and T_{1234}

$$
T^{\mathcal{N}=8}_{12\tilde{3}\tilde{4}}=\frac{\langle\bar{1}4\rangle}{\langle\bar{1}\bar{3}\rangle}\,T^{\mathcal{N}=8}_{1234}\ .
\tag{3.19}
$$

3.4 Crossing Symmetry

First consider the exchange of particles $1 \leftrightarrow 2$. In the scattering matrix (3.2,3.6) it corresponds to the following exchange of elements

$$\begin{array}{lllll} A \leftrightarrow +A, & B \leftrightarrow -B, & C \leftrightarrow -C, & G \leftrightarrow -K, & H \leftrightarrow -L, \\ D \leftrightarrow -D, & E \leftrightarrow +E, & F \leftrightarrow +F, & K \leftrightarrow -G, & L \leftrightarrow -H. \end{array} \tag{3.20}$$

It is straight-forward to verify that the whole scattering matrix is symmetric under the interchange of particles if the prefactor obeys $S_{2134} = S_{1234}$. Similarly, the exchange $3 \leftrightarrow 4$ leads to the following map of matrix elements

$$\begin{array}{lllll} A \leftrightarrow +A, & B \leftrightarrow -B, & C \leftrightarrow +C, & G \leftrightarrow +H, & H \leftrightarrow +G, \\ D \leftrightarrow -D, & E \leftrightarrow +E, & F \leftrightarrow -F, & K \leftrightarrow +L, & L \leftrightarrow +K. \end{array} \tag{3.21}$$

Again this leaves the scattering matrix invariant provided that $S_{1243} = S_{1234}$.

As a third type of crossing we consider the cyclic permutation $1 \to 2 \to 3 \to 4 \to 1$. It turns out to reshuffle the matrix elements in a more elaborate fashion

$$\begin{array}{ll} A_{2341} \to -\frac{1}{2}(A_{1234} - B_{1234}), & B_{2341} \to +\frac{1}{2}(3A_{1234} + B_{1234}), \\ D_{2341} \to +\frac{1}{2}(D_{1234} - E_{1234}), & E_{2341} \to -\frac{1}{2}(3D_{1234} + E_{1234}), \\ G_{2341} \to +L_{1234}, & L_{2341} \to -G_{1234}, \\ C_{2341} \to +2K_{1234}, & K_{2341} \to +\frac{1}{2}F_{1234}, \\ F_{2341} \to -2H_{1234}, & H_{2341} \to -\frac{1}{2}C_{1234}. \end{array} \tag{3.22}$$

Cyclic crossing symmetry on all the matrix elements is achieved by demanding

$$1 \to 2 \to 3 \to 4 \to 1: \quad T_{2341} = -T_{1234} \frac{\langle 23\rangle\langle\bar{4}1\rangle}{\langle 12\rangle\langle\bar{4}3\rangle}\,. \tag{3.23}$$

Finally, consider the twisted scattering amplitudes (3.10,3.11). These have the same crossing relations up to some signs due to statistics. A summary of the crossing relations is provided in the following table:

$$\begin{array}{c|c|c} 1 \leftrightarrow 2 & 3 \leftrightarrow 4 & 1 \to 2 \to 3 \to 4 \to 1 \\ \hline T_{2134} = +T_{1234} & T_{1243} = +T_{1234} & T_{2341} = -T_{1234} \dfrac{\langle 23\rangle\langle\bar{4}1\rangle}{\langle 12\rangle\langle\bar{4}3\rangle} \\ T_{21\tilde{3}\tilde{4}} = +T_{12\tilde{3}\tilde{4}} & T_{12\tilde{4}\tilde{3}} = -T_{12\tilde{3}\tilde{4}} & T_{2\tilde{3}\tilde{4}1} = -T_{12\tilde{3}\tilde{4}} \dfrac{\langle 23\rangle\langle\bar{4}1\rangle}{\langle 12\rangle\langle\bar{4}3\rangle} \\ T_{\tilde{2}\tilde{1}\tilde{3}\tilde{4}} = -T_{\tilde{1}\tilde{2}\tilde{3}\tilde{4}} & T_{\tilde{1}\tilde{2}\tilde{4}\tilde{3}} = -T_{\tilde{1}\tilde{2}\tilde{3}\tilde{4}} & T_{\tilde{2}\tilde{3}\tilde{4}\tilde{1}} = +T_{\tilde{1}\tilde{2}\tilde{3}\tilde{4}} \dfrac{\langle 23\rangle\langle\bar{4}1\rangle}{\langle 12\rangle\langle\bar{4}3\rangle} \end{array} \tag{3.24}$$

Note that the cyclic crossing relation for the mixed hypermultiplets maps between scattering matrices with different hypermultiplet assignments.

3.5 Two-to-Two Particle Scattering

The scattering amplitude $\langle \mathcal{T}_{1234}|$ in (3.2) is written such that all four particles are on an equal footing. For various purposes, however, it is convenient to write the scattering amplitude as an operator $\mathcal{T}^{43}_{12}$ acting on two-particle states and returning (a linear combination of) two-particle states.

To convert between the two pictures, let us first introduce a two-particle state $|\mathbf{1}\rangle$ which is invariant under the super-Poincaré algebra. In particular, its total momentum must be zero, $p_2 = -p_1$. This implies the following relation for the polarization spinors

$$u_2 = ie^{+i\alpha}v_1, \qquad v_2 = ie^{-i\alpha}u_1, \tag{3.25}$$

with some free parameter α representing the relative polarization between the spinors. It is then straight-forward to confirm that the following composite state is annihilated by all generators[9]

$$|\mathbf{1}\rangle = \int d^3p\, 2\pi\delta(p^2+m^2)\left(\varepsilon^{ab}|\phi_a\phi_b\rangle + ie^{i\alpha}\varepsilon^{\dot a\dot b}|\psi_{\dot a}\psi_{\dot b}\rangle\right). \tag{3.26}$$

This state is invariant under the full super-Poincaré algebra. Note that without the integration over the mass shell it would only be invariant under supercharges and internal rotations which form an ideal of the algebra. The momenta $p_{1,2}$ of the individual particles break Lorentz invariance. We have inserted a normalization factor of 2π corresponding to the imaginary part of a propagator $2\,\mathrm{Im}(p^2+m^2-i\epsilon)^{-1} = 2\pi\delta(p^2+m^2)$.

Now we can define the two-to-two scattering operator $\mathcal{T}^{43}_{12}$ as

$$\mathcal{T}^{43}_{12}|\mathcal{X}_1\mathcal{X}_2\rangle = \tfrac{1}{2}\langle\mathcal{T}_{12\bar 3\bar 4}|\mathcal{X}_1\mathcal{X}_2\mathbf{1}_{\bar 4 4}\mathbf{1}_{\bar 3 3}\rangle, \tag{3.27}$$

which is invariant by construction. The factor of $1/2$ is a symmetry factor to account for two identical outgoing particle multiplets; it is compensated by the phase space integrals in the S-matrix of Sec. 3.5 which count each state twice modulo permutation. More explicitly, using the action (3.2), the operator takes the form

$$\begin{aligned}
\mathcal{T}|\phi_a\phi_b\rangle &= 2\pi^2\int d^3p\,\delta(p_3^2+m^2)\,\delta(p_4^2+m^2)\big(A|\phi_{(a}\phi_{b)}\rangle + B|\phi_{[a}\phi_{b]}\rangle + \tfrac{1}{2}C\varepsilon_{ab}\varepsilon^{\dot c\dot d}|\psi_{\dot c}\psi_{\dot d}\rangle\big),\\
\mathcal{T}|\psi_{\dot a}\psi_{\dot b}\rangle &= 2\pi^2\int d^3p\,\delta(p_3^2+m^2)\,\delta(p_4^2+m^2)\big(D|\psi_{(\dot a}\psi_{\dot b)}\rangle + E|\psi_{[\dot a}\psi_{\dot b]}\rangle + \tfrac{1}{2}F\varepsilon_{\dot a\dot b}\varepsilon^{cd}|\phi_c\phi_d\rangle\big),\\
\mathcal{T}|\phi_a\psi_{\dot b}\rangle &= 2\pi^2\int d^3p\,\delta(p_3^2+m^2)\,\delta(p_4^2+m^2)\big(G|\psi_{\dot b}\phi_a\rangle + H|\phi_a\psi_{\dot b}\rangle\big),\\
\mathcal{T}|\psi_{\dot a}\phi_b\rangle &= 2\pi^2\int d^3p\,\delta(p_3^2+m^2)\,\delta(p_4^2+m^2)\big(K|\psi_{\dot a}\phi_b\rangle + L|\phi_b\psi_{\dot a}\rangle\big).
\end{aligned} \tag{3.28}$$

Here we have labeled the two outgoing particles as 4 and 3. We have also fixed the above phase to $\alpha = \frac{1}{2}\pi$. In other words, the polarization spinors between particles $3,4$ and their conjugates $\bar 3, \bar 4$ are related by

$$u_{\bar k} = +v_k, \qquad v_{\bar k} = -u_k \tag{3.29}$$

and the matrix elements $A,\ldots,L$ equal those in (3.6) with indices $A_{12\bar 3\bar 4},\ldots L_{12\bar 3\bar 4}$.

[9]This expression implies the use of polarization spinors $u(p), v(p)$ with definite phase for a given momentum p, see the discussion in Sec. 2.3. If we consider $u_{1,2}, v_{1,2}$ as the fundamental degrees of freedom, we should choose the integral $\int d^2u_1\, d^2v_1\, d^2u_2\, d^2v_2\, \delta(\langle v_1, u_1\rangle + 2im)\,\delta^3(p_1+p_2)\ldots$.

3.6 Worldsheet Scattering Matrix for AdS/CFT Integrability

The extended $\mathfrak{psu}(2|2)$ algebra plays an important role in the investigation of integrability in the planar AdS/CFT correspondence between strings on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ super Yang–Mills theory [29, 30, 36]. It also appears analogously in the recently discussed duality between strings on $AdS_4 \times \mathbb{CP}^3$ and $\mathcal{N} = 6$ Chern–Simons theory [20, 21, 37–39]. The algebra serves as the symmetry in a light cone gauge of string theory or equivalently in a ferromagnetic excitation picture of gauge theory spin chains.

For the two body scattering of the $\mathcal{N} = 4$ super Yang–Mills spin chain each of the excitations can be one of 16 flavors and so the scattering is described by a $16^2 \times 16^2$ matrix. In [29] it was shown that the symmetries determine the matrix structure uniquely and so it is determined up to an overall phase. The same result holds for the scattering of worldsheet excitations on the string worldsheet [30, 31].

Hofman and Maldacena, [30] (see also [33]), pointed out that the constraints imposed by the $\mathfrak{psu}(2|2)$ algebra on the spin chain were exactly those that a four particle scattering amplitude in $2 + 1$-dimensions in a theory with the same super-algebra would have. They further pointed out that the "dynamic" nature of the spin chain scattering, whereby the length of the chain changes in certain scattering processes, is related to the non-Lorentz invariance of the $2 + 1$ scattering matrix. Under an overall rotation it picks up a phase due to the fermions spin.

While the matrix structure of the scattering amplitude is identical between the spin chain/worldsheet theory and the $2 + 1$ Lorentz invariant theories the kinematics are quite distinct which can be seen in the difference between the overall two- and three-dimensional momentum delta functions. In two dimensions the scattering momenta can not change in magnitude and at most can be exchanged between particles. In three dimensions the final state phase space is larger and includes the relative angle between the two particles.

The matrix elements $A, \ldots, L$ are related in terms of the spinors u, v from the supersymmetry representation (2.9). In [29, 32] the supersymmetry representation is specified in terms of the parameters a, b, c, d instead. We thus have to relate these sets of parameters first: This is easily achieved by

$$u = \sqrt{2im} \begin{pmatrix} +a \\ -c \end{pmatrix}, \quad v = \sqrt{2im} \begin{pmatrix} -b \\ +d \end{pmatrix}. \tag{3.30}$$

The constraint $\langle v, u \rangle = -2im$ is equivalent to $ad - bc = 1$. This simple choice leads to the following incoming momentum components[10]

$$\begin{aligned} p_0 - p_1 &= -2ig\; m\alpha(1 - x^+/x^-), \\ p_0 + p_1 &= -2ig\; m\alpha^{-1}(1 - x^-/x^+), \\ p_2 &= -2g\; m(x^+ - x^-) + im. \end{aligned} \tag{3.31}$$

[10] Unfortunately, it turns out that p_0 and p_2 are purely imaginary for physical magnons of the worldsheet theory. For magnons of the mirror worldsheet theory [40], however, all momentum components are real. In fact one could choose for incoming particles $u = \sqrt{im}(+a + ic, -c - ia)$, $v = \sqrt{im}(-b - id, +d + ib)$ which leads to exactly the same S-matrix elements. In that case p_0 and p_2 are interchanged and multiplied by i thus making them real. For the alternative choice, the intermediate expressions are somewhat cluttered and hence we shall stick to the above unphysical choice.

Note that the parameters a, b, c, d associated to the magnons depend on the ordering of magnons, see [32] for details. The correct assignment is

$$\begin{aligned}
u_1 &= \sqrt{2igm}\,\gamma_1 \begin{pmatrix} 1 \\ -i\alpha^{-1}/x_1^+ \end{pmatrix}, \\
v_1 &= \sqrt{-2igm}\,\gamma_1^{-1}(x_1^+ - x_1^-) \begin{pmatrix} i\alpha/x_1^- \\ 1 \end{pmatrix}, \\
u_2 &= \sqrt{2igm}\,\gamma_2\xi_1 \begin{pmatrix} 1 \\ -i\alpha^{-1}x_1^-/x_1^+x_2^+ \end{pmatrix}, \\
v_2 &= \sqrt{-2igm}\,\gamma_2^{-1}\xi_1^{-1}(x_2^+ - x_2^-) \begin{pmatrix} i\alpha x_1^+/x_1^-x_2^- \\ 1 \end{pmatrix}, \\
u_3 = -v_{\bar{3}} &= \sqrt{2igm}\,\gamma_1\xi_2 \begin{pmatrix} 1 \\ -i\alpha^{-1}x_2^-/x_1^+x_2^+ \end{pmatrix}, \\
v_3 = +u_{\bar{3}} &= \sqrt{-2igm}\,\gamma_1^{-1}\xi_2^{-1}(x_1^+ - x_1^-) \begin{pmatrix} i\alpha x_2^+/x_1^-x_2^- \\ 1 \end{pmatrix}, \\
u_4 = -v_{\bar{4}} &= \sqrt{2igm}\,\gamma_2 \begin{pmatrix} 1 \\ -i\alpha^{-1}/x_2^+ \end{pmatrix}, \\
v_4 = +u_{\bar{4}} &= \sqrt{-2igm}\,\gamma_2^{-1}(x_2^+ - x_2^-) \begin{pmatrix} i\alpha/x_2^- \\ 1 \end{pmatrix}.
\end{aligned} \tag{3.32}$$

Note that the energies are related as $(E_3, E_4) = (E_1, E_2)$. Thus the prefactor T should contain a factor of $\delta(E_3 - E_1)$. Let us thus compute the contribution from the delta functions[11]

$$2\pi^2 \int d^3p\,\delta(p_3^2 + m^2)\delta(p_4^2 + m^2)\delta(E_3 - E_1) = \frac{\pi^2}{2|p_{1x}p_{2y} - p_{1y}p_{2x}|}\,. \tag{3.33}$$

We will thus need a compensating factor $\varDelta_{12}$ for the prefactor

$$\varDelta_{12} = \frac{2}{\pi^2}\,(p_{1x}p_{2y} - p_{1y}p_{2x}) = \frac{4ig^2m^2}{\pi^2}\left(1 - \frac{x_1^+}{x_1^-}\right)\left(1 - \frac{x_2^+}{x_2^-}\right)\left(1 - \frac{x_1^-x_2^-}{x_1^+x_2^+}\right). \tag{3.34}$$

Substituting the representation spinors (3.32) into the matrix elements (3.6) one recovers the S-matrix presented in [32] provided that the prefactors of the S-matrices are related as follows

$$T_{12\bar{3}\bar{4}} = \delta(E_1 - E_3)\varDelta_{12}S_{12}^0\,\frac{x_2^+ - x_1^-}{x_2^- - x_1^+}\,. \tag{3.35}$$

The matrix elements $A, \ldots, L$ have been normalized such that they can be compared directly to the results of [29, 32]. A priori the magnon S-matrix depends on nine parameters, $S^0, x_1, x_2, g, \alpha, \gamma_1, \gamma_2, \xi_1, \xi_2$. As the matrix elements in (3.6) have only seven degrees of freedom, there must be two directions in the nine-dimensional parameter space along which the S-matrix is invariant. From (3.32) one can easily infer that the parameters $\gamma_1, \gamma_2, \xi_1, \xi_2$ correspond to phases of the fermion spinors, see [30]. In the integrable system these parameters and the phase factor S^0 are usually fixed leaving

[11] We will consistently drop the contribution from diagonal scattering where $(p_1, p_2) = (p_3, p_4)$.

only three degrees of freedom x_1, x_2, g. In that sense, the integrable S-matrix is merely a special case of the most general spacetime S-matrix.

Let us now consider the crossing symmetry of the magnon S-matrix [41]. and compare it to the crossing studied in Sec. 3.4. Crossing of the magnon S-matrix corresponds to interchanging particles 2 and 4. From iterating (3.24) we can derive the $2 \leftrightarrow 4$ crossing relation

$$T_{1432} = \frac{\langle 14\rangle}{\langle 12\rangle}\frac{\langle \bar{3}2\rangle}{\langle \bar{3}4\rangle} T_{1234} \quad \text{or} \quad T_{1\bar{4}\bar{3}2} = \frac{\langle 1\bar{4}\rangle}{\langle 12\rangle}\frac{\langle 32\rangle}{\langle 3\bar{4}\rangle} T_{123\bar{4}}. \tag{3.36}$$

This combination of spinors takes the following form in $x^{\pm}$ notation

$$\frac{\langle 1\bar{4}\rangle}{\langle 12\rangle}\frac{\langle 32\rangle}{\langle 3\bar{4}\rangle} = \frac{x_2^- - x_1^-}{x_2^+ - x_1^-}\,\frac{1/x_2^- - x_1^+}{1/x_2^+ - x_1^+}\,. \tag{3.37}$$

We furthermore have to relate the crossed prefactor $T_{1\bar{4}\bar{3}2}$ to the crossed prefactor $S^0_{1\bar{2}}$ and also fix the parameters ξ_k [12]

$$T_{1\bar{4}\bar{3}2} = \delta(E_1 - E_3)\mathit{\Delta}_{12}\left(S^0_{1\bar{2}}\,\frac{1/x_2^+ - x_1^-}{1/x_2^- - x_1^+}\right)^{-1}, \qquad \xi_k^2 = \frac{x_k^+}{x_k^-}\,. \tag{3.38}$$

We then recover the crossing relation [41, 32]

$$1 = \frac{S^0_{12}S^0_{1\bar{2}}}{\xi_1^2}\,\frac{x_2^+ - x_1^-}{x_2^- - x_1^-}\,\frac{1/x_2^+ - x_1^+}{1/x_2^- - x_1^+}\,. \tag{3.39}$$

Given this formal equivalence, it is natural to ask what role the known integrable structures of the the spin chain might play in the $2+1$ dimensional supersymmetric Chern–Simons theory. However due to the different kinematical structure, and in particular because of the delta-function prefactor, the Chern–Simons scattering matrix does not generically satisfy the Yang–Baxter equation.

3.7 Six-Particle Scattering

Let us briefly comment on scattering of more than four particles. In fact, scattering must always involve an even number of external physical particles due to charge conservation: All particles transform as a doublet of one of the two internal $\mathfrak{su}(2)$ symmetries. In other words they form a doublet of the diagonal $\mathfrak{su}(2)$. A singlet of this $\mathfrak{su}(2)$ can only be composed from an even number of doublets.

The next non-trivial case is thus six external particles. Altogether there are $4^6 = 4096$ components most of which are zero due to charge conservation. Taking into account the $\mathfrak{su}(2) \times \mathfrak{su}(2)$ internal symmetry, there are 70 remaining invariant structures. Finally, supersymmetry relates most of them and there are only two invariant structures leading to two prefactors [32], see also [42]. These can, for instance, be obtained from the purely bosonic and purely fermionic scattering processes

$$\langle \mathcal{T}|\phi_1\phi_1\phi_1\phi_2\phi_2\phi_2\rangle, \qquad \langle \mathcal{T}|\psi_1\psi_1\psi_1\psi_2\psi_2\psi_2\rangle. \tag{3.40}$$

[12] We do not understand the appearance of the inverse in the formula, but it is necessary to make the below crossing relation work.

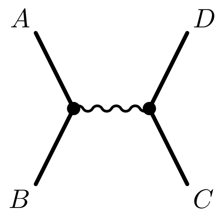

Figure 3: Sample Feynman graph and color structure.

In this work we will not need higher-particle scattering because it contributes to unitarity relations starting from three loops. As there are no physical gluon states all scattering amplitudes must involve an even number of external particles. For two-to-two scattering this implies that in the unitarity relations we must have four internal, cut legs which implies at least three loop momenta. Nevertheless, it would be interesting to see whether one can set up recursion relations similar to those obtained for $\mathcal{N} = 4$ SYM [9, 10] or find related generating functions for amplitudes [43]. The form of the Chern–Simons matter scattering amplitudes is actually quite similar to those of $\mathcal{N} = 4$ SYM but again one must be careful to take into account the significant differences due to the three-dimensional kinematics.

4 Color Structures

Feynman diagrams in a gauge theory consist of two parts, spacetime functions and color structures. Here we shall discuss the color structures relevant to four-point scattering in $\mathcal{N} \geq 4$ Chern–Simons theories at tree level and at one loop. The scattering amplitudes will then be written as linear combinations of these color structures multiplied by space-time functions. In fact, it usually suffices to compute so-called color-ordered amplitudes. However, because the gauge group decomposes into multiple factors potentially of different rank it is worth the effort of analyzing the structures in detail. It will turn out that the numerical factors from the color structure crucially depend on the colors of the external legs.

4.1 Color Graphs

We start by introducing a graphical notation which will be very useful to classify color structures. Consider, for example, the scattering of two particles by exchange of a gluon, cf. the Feynman graph in Fig. 3. The Feynman rules associate a color factor to each vertex (e.g. T_{MAB}) and to each line (e.g. K^{MN}). In gauge theories the color structure of Fig. 3 is typically

$$T_{MAB} K^{MN} T_{NCD}. \tag{4.1}$$

We shall use the same graph Fig. 3 to denote this color structure. Nevertheless, the correspondence between Feynman graphs and color structures is not one-to-one: There can be vertices with a composite color structure, e.g. a single vertex can be of the form (4.1). Therefore several Feynman diagrams will have one and the same color structure. Furthermore, different color structures are often related by some identities.

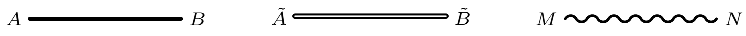

Figure 4: Color lines: L^{AB}, $\tilde{L}^{\tilde{A}\tilde{B}}$, K^{MN}.

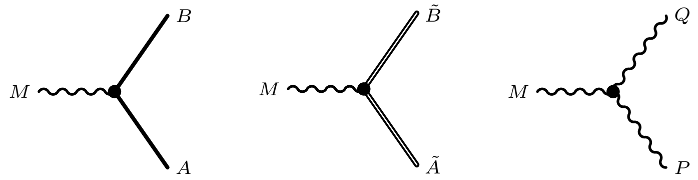

Figure 5: Color vertices: M_{MAB}, $\tilde{M}_{M\tilde{A}\tilde{B}}$, F_{MPQ}.

We now set up the specific rules for a generic $\mathcal{N} = 4$ Chern–Simons model, cf. App. B for a brief summary. There are three types of fields: untwisted matter, twisted matter and gluons. We shall use solid, double and wiggly lines to distinguish between them, see Fig. 4. The associated color factors are L^{AB}, $\tilde{L}^{\tilde{A}\tilde{B}}$ and K^{MN}, respectively. There are also three types of vertices: They connect a gluon to two untwisted fields, two twisted fields or to two further gluons. The vertices are depicted in Fig. 5 and they correspond to the structures $M_{MAB} = K_{MN}M^{N}_{AB}$, $\tilde{M}_{M\tilde{A}\tilde{B}} = K_{MN}\tilde{M}^{N}_{\tilde{A}\tilde{B}}$ and $F_{MPQ} = K_{MN}F^{N}_{PQ}$, respectively. All the terms in the Lagrangian in App. B.2 have a graphical representation using the above lines and 3-vertices.

In a gauge theory the vertices are structure constants of the gauge group. They therefore obey a host of identities, e.g. Jacobi identities. In our case there are five Jacobi identities, see App. B.1. They all have the same form and are summarized graphically in Fig. 6. Note that the Jacobi identity only exists if all involved vertices exist: There is no Jacobi identity for a gluon line joining a twisted with an untwisted vertex! Here our main interest is the enumeration of distinct structures and not their precise prefactors. For instance, we shall not always pay close attention to signs.

4.2 Tree Level

Let us first consider a scattering process of four untwisted matter fields. At tree level we need two 3-vertices to connect the four external lines. There are three ways in which this can be done, see Fig. 7. We shall denote the structures by

$$\Upsilon^{(0)}_{AB,CD} = M_{MAB}K^{MN}M_{NCD}. \tag{4.2}$$

They have the obvious eight-fold symmetries $\Upsilon^{(0)}_{AB,CD} = \Upsilon^{(0)}_{BA,CD} = \Upsilon^{(0)}_{CD,AB}$. Furthermore, the Jacobi identity in Fig. 6 relates these three structures

$$\Upsilon^{(0)}_{AB,CD} + \Upsilon^{(0)}_{AC,DB} + \Upsilon^{(0)}_{AD,BC} = 0. \tag{4.3}$$

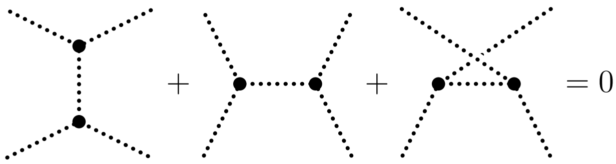

Figure 6: Jacobi identities for color structures.

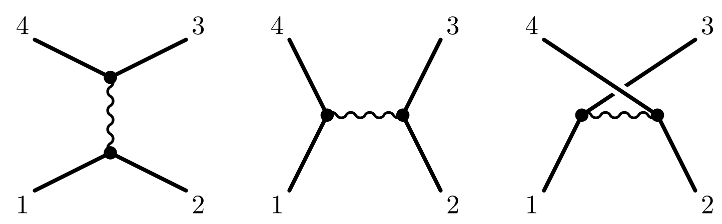

Figure 7: Untwisted tree graphs: $\Upsilon^{(0)}_{12,34}$, $\Upsilon^{(0)}_{13,24}$, $\Upsilon^{(0)}_{14,23}$.

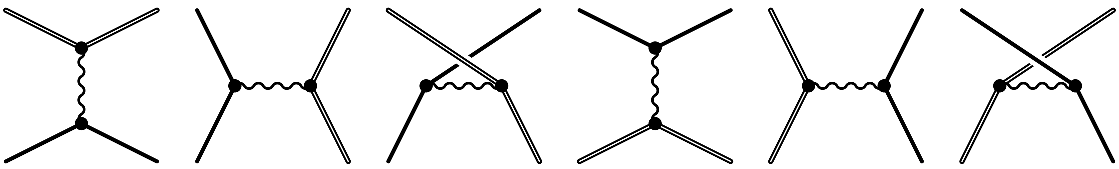

Figure 8: Mixed tree graphs: $\Upsilon^{(0)}_{12,\tilde{3}\tilde{4}}$, $\Upsilon^{(0)}_{12,\tilde{3}\tilde{4}}$, $\Upsilon^{(0)}_{13,\tilde{2}\tilde{4}}$, $\Upsilon^{(0)}_{\tilde{1}\tilde{2},34}$, $\Upsilon^{(0)}_{\tilde{1}\tilde{2},34}$, $\Upsilon^{(0)}_{\tilde{1}\tilde{3},24}$.

Now any scattering amplitude at tree level can be written as

$$T = T_s \Upsilon^{(0)}_{12,34} + T_t \Upsilon^{(0)}_{14,23} + T_u \Upsilon^{(0)}_{13,24}. \tag{4.4}$$

The Jacobi identity (4.3), however, states that the basis $\Upsilon^{(0)}_{12,34}, \Upsilon^{(0)}_{13,24}, \Upsilon^{(0)}_{14,23}$ is over-complete. Thus for any δ the amplitude is equivalent to

$$T = (T_s + \delta)\Upsilon^{(0)}_{12,34} + (T_t + \delta)\Upsilon^{(0)}_{14,23} + (T_u + \delta)\Upsilon^{(0)}_{13,24}. \tag{4.5}$$

We can use this freedom to remove one of the coefficients, for example $\delta = -T_s$ simplifies the amplitude to

$$T = T'_t \Upsilon^{(0)}_{14,23} + T'_u \Upsilon^{(0)}_{13,24}. \tag{4.6}$$

Next we consider scattering amplitudes between two untwisted and two twisted fields. They can be written using the symbol

$$\Upsilon^{(0)}_{AB,\tilde{C}\tilde{D}} = M_{MAB} K^{MN} \tilde{M}_{N\tilde{C}\tilde{D}}. \tag{4.7}$$

There are six permutations for the function: $\Upsilon^{(0)}_{12,\tilde{3}\tilde{4}}$, $\Upsilon^{(0)}_{13,\tilde{2}\tilde{4}}$, $\Upsilon^{(0)}_{14,\tilde{2}\tilde{3}}$, $\Upsilon^{(0)}_{23,\tilde{1}\tilde{4}}$, $\Upsilon^{(0)}_{24,\tilde{1}\tilde{3}}$, $\Upsilon^{(0)}_{34,\tilde{1}\tilde{2}}$, see Fig. 8. In this case there are no Jacobi identities because there is no vertex to connect untwisted and twisted field directly.

Finally, the amplitudes for four twisted fields are analogous to the untwisted fields discussed above. Altogether there are 2 + 6 + 2 color structures for four-particle scattering at tree level.

4.3 One Loop

For four particle scattering at the one-loop level there must be four 3-vertices which can be connected in various ways. It is obvious that the graph has one internal loop which permits a rough classification: The loop can have two sides (bubble), three sides (triangle) or four sides (box). A bubble can be understood to dress a line while a triangle dresses a vertex. Bubbles and vertices are in fact closely related, see Fig. 9

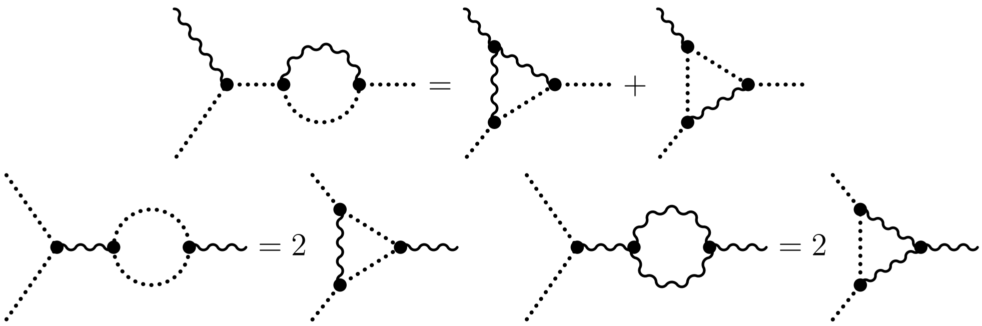

Figure 9: Bubble-triangle relations.

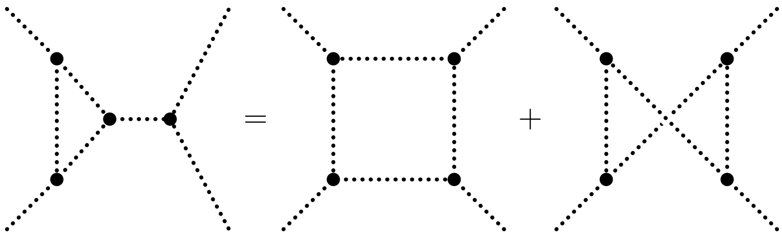

Figure 10: Triangle-box relation.

Consider a bubble connected to a vertex by a line. Applying the Jacobi identity (Fig. 6) to the connecting line will move the loop onto the vertex. Consider instead a triangle with two sides of equal kind (in our model triangles always have this property). Applying the Jacobi identity (Fig. 6) to the third side will move the loop onto the line at the opposite side. We can thus convert freely between bubbles and triangles. The only exception where this is not possible is for configurations of mixed particles which lack a Jacobi identity.

A similar relation holds between triangles and boxes. Consider two adjacent vertices, one of them being dressed by a loop, see Fig. 10. Then apply the Jacobi identity (Fig. 6) to the connecting line. This yields two boxes. The boxes are distinct and thus this conversion is a one-way procedure: Triangles can be converted to boxes (unless they contain mixed particles), but not vice versa.

Our strategy for enumerating independent one-loop structures for four-particle scattering is clear. We should convert bubbles to triangles and triangles to boxes as far as possible.

We start with only untwisted particles. Clearly the color structures can be converted to boxes by the above procedure. A box has the underlying structure

$$\mathit{\Upsilon}^{(1)\,\Box}_{AB,CD} = M_{MAE} M_{NCF} K^{MN} K^{PQ} L^{EG} L^{FH} M_{PGB} M_{QHD}. \tag{4.8}$$

It has a fourfold symmetry $\mathit{\Upsilon}^{(1)\,\Box}_{AB,CD} = \mathit{\Upsilon}^{(1)\,\Box}_{BA,DC} = \mathit{\Upsilon}^{(1)\,\Box}_{CD,AB}$. In total there are six boxes, all of the same structure, but with a permutation of the external legs, see Fig. 11. The Jacobi identity relates all six of these, but only at the expense of triangles which have been eliminated earlier. It turns out that the basis of six boxes is minimal.

For four external untwisted fields there is also the option to have twisted particles run in the internal loop. In this case the loop must be a bubble dressing the central gluon line. Jacobi identities are ineffective here and thus there are three structures denoted by $\mathit{\Upsilon}^{(1)\,\tilde{\circ}}_{12,34}$, $\mathit{\Upsilon}^{(1)\,\tilde{\circ}}_{14,23}$, $\mathit{\Upsilon}^{(1)\,\tilde{\circ}}_{13,24}$ (see Fig. 12). Bubble graphs are defined by

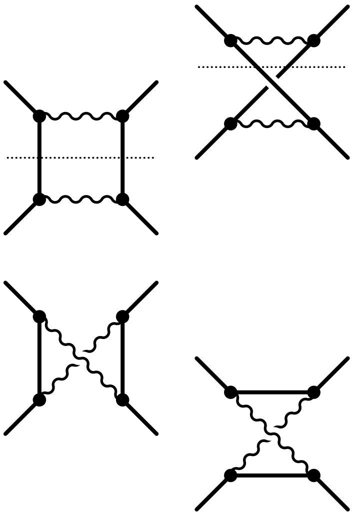

Figure 11: Purely untwisted scattering at one loop. Clockwise from top left: $\Upsilon^{(1)\,\square}_{14,23}$, $\Upsilon^{(1)\,\square}_{13,24}$, $\Upsilon^{(1)\,\square}_{13,42}$, $\Upsilon^{(1)\,\square}_{12,43}$, $\Upsilon^{(1)\,\square}_{12,34}$, $\Upsilon^{(1)\,\square}_{14,32}$. The horizontal and vertical dotted lines indicate possible unitarity cuts in the s- and t-channels, respectively, to be discussed in Sec. 6.4.

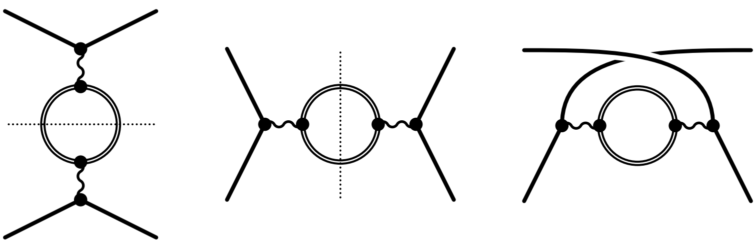

Figure 12: Untwisted scattering with internal twisted loop: $\Upsilon^{(1)\,\tilde{\circ}}_{12,34}$, $\Upsilon^{(1)\,\tilde{\circ}}_{14,23}$, $\Upsilon^{(1)\,\tilde{\circ}}_{13,24}$. The horizontal and vertical dotted lines indicate possible unitarity cuts in the s- and t-channels, respectively, to be discussed in Sec. 6.5.

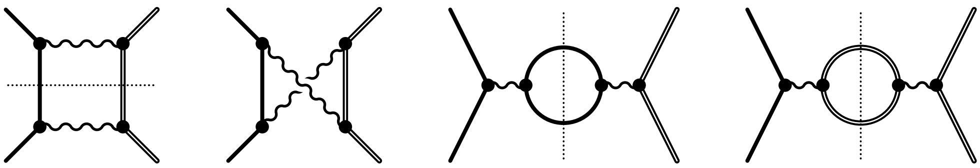

Figure 13: Mixed scattering at one loop with untwisted particles $1,4$ and twisted particles $2,3$: $\Upsilon^{(1)\,\Box}_{14,\tilde{2}\tilde{3}}$, $\Upsilon^{(1)\,\Box}_{14,\tilde{3}\tilde{2}}$, $\Upsilon^{(1)\,\circ}_{14,\tilde{2}\tilde{3}}$, $\Upsilon^{(1)\,\tilde{\circ}}_{14,\tilde{2}\tilde{3}}$.

$$
\begin{aligned}
\Upsilon^{(1)\,\circ}_{AB,CD} &= M_{MAB}K^{MP}M_{PEF}L^{EG}L^{FH}M_{QGH}K^{QN}M_{NCD},\\
\Upsilon^{(1)\,\tilde{\circ}}_{AB,CD} &= M_{MAB}K^{MP}\tilde{M}_{P\tilde{E}\tilde{F}}\tilde{L}^{\tilde{E}\tilde{G}}\tilde{L}^{\tilde{F}\tilde{H}}\tilde{M}_{Q\tilde{G}\tilde{H}}K^{QN}M_{NCD},
\end{aligned}
\tag{4.9}
$$

they have the same eightfold symmetry as tree graphs $\Upsilon^{(0)}_{AB,CD}$, but they do not obey an identity like (4.3). Note that the untwisted bubble can be expanded to a sum of boxes

$$
\Upsilon^{(1)\,\circ}_{AB,CD} = 2\Upsilon^{(1)\,\Box}_{AD,BC} + 2\Upsilon^{(1)\,\Box}_{AC,BD}.
\tag{4.10}
$$

Finally, we must enumerate the structures for mixed four-particle scattering at one loop. It turns out that some graphs can be promoted to a box while others cannot. The latter ones can however be brought to the form of a bubble dressing the central gluon line. There are four diagrams $\Upsilon^{(1)\,\Box}_{14,\tilde{2}\tilde{3}}$, $\Upsilon^{(1)\,\Box}_{14,\tilde{3}\tilde{2}}$, $\Upsilon^{(1)\,\circ}_{14,\tilde{2}\tilde{3}}$, $\Upsilon^{(1)\,\tilde{\circ}}_{14,\tilde{2}\tilde{3}}$ for each assignment of the untwisted and twisted particles to the external legs, see Fig. 13; 24 in total. The structures are analogous to those in (4.8,4.9) but with some structure constants replaced by twisted ones.

Before we close this part, it is useful to mention that the one-loop structures can be understood as squares of tree structures. For example the box can be written as an iterated tree

$$
\Upsilon^{(1)\,\Box}_{AB,CD} = \Upsilon^{(0)}_{AE,CF}L^{EG}L^{FH}\Upsilon^{(0)}_{GB,HD},
\tag{4.11}
$$

or $\Upsilon^{(0)}_{16,25}\Upsilon^{(0)}_{64,53} = \Upsilon^{(1)\,\Box}_{14,23}$ for short. For the three basic tree color structures $\Upsilon^{(0)}_{12,34}$, $\Upsilon^{(0)}_{14,23}$ and $\Upsilon^{(0)}_{13,24}$ we can set up a convenient multiplication table:

$$
\begin{array}{c|ccc}
\cdot & \Upsilon^{(0)}_{56,34} & \Upsilon^{(0)}_{53,64} & \Upsilon^{(0)}_{63,54}\\
\hline
\Upsilon^{(0)}_{12,56} & \Upsilon^{(1)\,\circ}_{12,34} & -\tfrac{1}{2}\Upsilon^{(1)\,\circ}_{12,34} & -\tfrac{1}{2}\Upsilon^{(1)\,\circ}_{12,34}\\
\Upsilon^{(0)}_{16,25} & -\tfrac{1}{2}\Upsilon^{(1)\,\circ}_{12,34} & \Upsilon^{(1)\,\Box}_{14,23} & \Upsilon^{(1)\,\Box}_{13,24}\\
\Upsilon^{(0)}_{15,26} & -\tfrac{1}{2}\Upsilon^{(1)\,\circ}_{12,34} & \Upsilon^{(1)\,\Box}_{13,24} & \Upsilon^{(1)\,\Box}_{14,23}
\end{array}
\tag{4.12}
$$

4.4 Planar Limit and Color Ordering

It is often convenient to consider gauge groups with a very large rank where the class of planar Feynman diagrams contributes dominantly. At the one-loop level it is in fact often sufficient to just compute the planar Feynman diagrams and all the non-planar corrections follow by completion of the color structures. Color ordering for scattering amplitudes is also based intrinsically on the availability of many colors. Here we discuss the large-N behavior of the color structures discussed above.

A prototypical $\mathcal{N}=4$ supersymmetric Chern–Simons model with mixed hypermultiplets is a quiver theory with $\mathrm{U}(N_k)$ gauge groups, see Fig. 1 on page 5. The gauge fields belong to the adjoint of the $\mathrm{U}(N_k)$ while the matter fields are bi-fundamentals

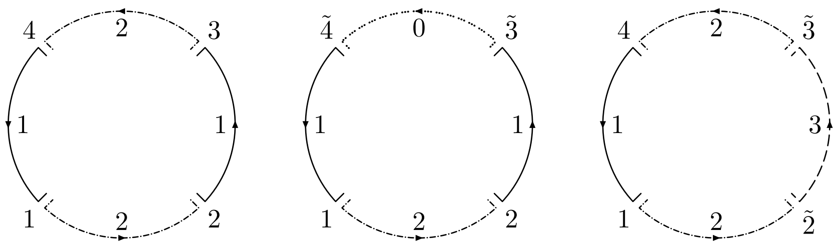

Figure 14: Three color ordering structures $\mathrm{Tr}(\bar{1}2)(\bar{2}1)(\bar{1}2)(\bar{2}1)$ for 1234, $\mathrm{Tr}(\bar{1}2)(\bar{2}1)(\bar{1}0)(\bar{0}1)$ for $12\tilde{3}\tilde{4}$, $\mathrm{Tr}(\bar{1}2)(\bar{2}3)(\bar{3}2)(\bar{2}1)$ for $1\tilde{2}\tilde{3}4$

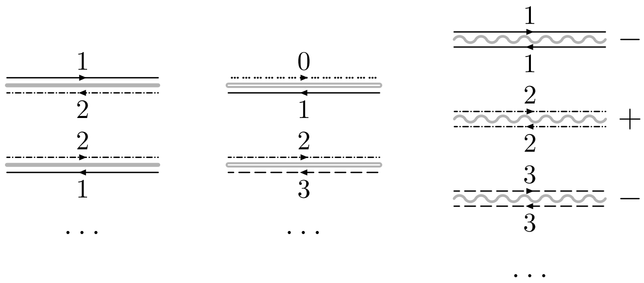

Figure 15: Double line notation for untwisted, twisted and gluon color lines, cf. Fig. 4. Gluon lines have associated sign factors.

connecting two adjacent gauge group factors. Let us for definiteness assume that untwisted matter connects $\mathrm{U}(N_{2k-1})$ to $\mathrm{U}(N_{2k})$ and twisted matter connects $\mathrm{U}(N_{2k})$ to $\mathrm{U}(N_{2k+1})$, cf. Fig. 1. In the planar limit all the N_k are taken to be proportional to some large number N.

Now we shall consider color ordering of the legs in a scattering graph: Each leg is assigned a pair of fundamental color indices $(\bar{k}, k \pm 1)$. Two adjacent legs have a common but mutually conjugate color index $\ldots, k)(\bar{k}, \ldots$. A sample color ordering structure for four untwisted fields is thus $\mathrm{Tr}(\bar{1}2)(\bar{2}1)(\bar{1}2)(\bar{2}1)$. A similar color ordering structure for two untwisted and two twisted fields would be $\mathrm{Tr}(\bar{1}2)(\bar{2}1)(\bar{1}0)(\bar{0}1)$, see Fig. 14.

A color ordering structure can be applied to a color graph in order to yield a polynomial in the ranks N_k. It is straight-forward to evaluate the polynomial when the graph is represented in a double line notation: The lines of a color structure (see Fig. 4) are thickened to a ribbon and the two sides of the ribbon are attributed a

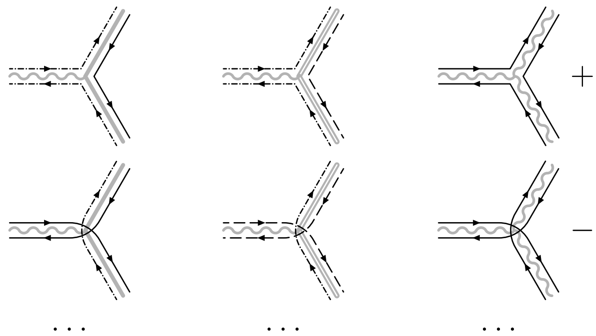

Figure 16: Double line notation for vertices, cf. Fig. 5. Pure gluon vertices have associated sign factors.

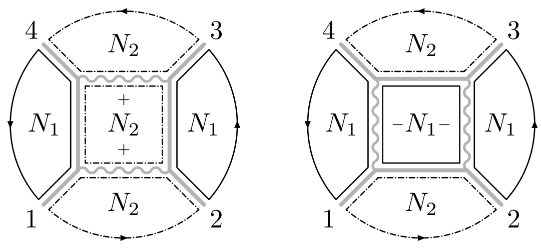

Figure 17: Two box graphs $\varUpsilon^{(1)\square}_{14,23}$ and $\varUpsilon^{(1)\square}_{12,43}$ with different assignment of internal gluons contracted with the color structure $\operatorname{Tr}(\bar{1}2)(\bar{2}1)(\bar{1}2)(\bar{2}1)$ in the planar limit.

certain color N_k, see Fig. 15. The color indices must be adjacent for matter lines and equal for gauge lines as explained above. A vertex connects sides of equal color in two possible ways, see Fig. 16. Each closed loop of color k then contributes one power of N_k to the monomial associated to the ribbon graph. A (possibly incomplete) set of rules to determine the sign of a ribbon graph is as follows: Signs originate from gluon lines with odd color (Fig. 15) as well as from one out of two pure gluon vertices (Fig. 16). Note that we will not be careful about some overall signs of color graphs when they are not related in some way. Sample ribbon graphs are provided in Fig. 17. It is obvious that the large-N asymptotics follows from the planar structure of the graph. However, the precise distribution of the $N_k \sim N$ factors is not as easily recognized. In the example in Fig. 17 the two different orientations of the box lead to two different leading-N contributions, $N_1^2 N_2^3$ vs. $N_1^3 N_2^2$. Let us therefore evaluate the color structures discussed above which appear in the field theory calculation at one loop.

Pure Scattering. For scattering of four untwisted particles we shall always take the standard color ordering of $N_1^{-2} N_2^{-2} \operatorname{Tr}(\bar{1}2)(\bar{2}1)(\bar{1}2)(\bar{2}1)$, cf. Fig. 14, to evaluate color structures. The prefactor cancels the color factors which originate from the color ordering structure itself and they make the large-N expansion more transparent. For the tree graphs in Fig. 7 we obtain the following exact results

$$\varUpsilon^{(0)}_{12,34} \to -1 + \frac{1}{N_3 N_4}\,, \qquad \varUpsilon^{(0)}_{14,23} \to +1 - \frac{1}{N_3 N_4}\,, \qquad \varUpsilon^{(0)}_{13,24} \to 0. \tag{4.13}$$

The large-N asymptotics agrees with the planar structure of the underlying graphs, the first two are planar while the third one is non-planar. Also the Jacobi identity (4.3) is fulfilled.

Next we evaluate the box graphs in Fig. 11, see Fig. 18 for an explicit example,

$$\begin{aligned}
&\varUpsilon^{(1)\square}_{14,23} \to N_2 - \frac{2}{N_1} + \frac{1}{N_2}\,, \qquad \varUpsilon^{(1)\square}_{14,32} \to -\frac{2}{N_1} + \frac{2}{N_2}\,, \qquad \varUpsilon^{(1)\square}_{13,24} \to 0\,,\\
&\varUpsilon^{(1)\square}_{12,43} \to N_1 - \frac{2}{N_2} + \frac{1}{N_1}\,, \qquad \varUpsilon^{(1)\square}_{12,34} \to +\frac{2}{N_1} - \frac{2}{N_2}\,, \qquad \varUpsilon^{(1)\square}_{13,42} \to 0\,.
\end{aligned} \tag{4.14}$$

The bubble graphs in Fig. 12 yield similar expressions

$$\begin{aligned}
&\varUpsilon^{(1)\circ}_{12,34} \to 2N_2 - \frac{4}{N_1} + \frac{2}{N_2}\,, \qquad \varUpsilon^{(1)\circ}_{14,23} \to 2N_1 - \frac{4}{N_2} + \frac{2}{N_1}\,, \qquad \varUpsilon^{(1)\circ}_{13,24} \to 0\,.\\
&\varUpsilon^{(1)\tilde{\circ}}_{12,34} \to 2N_0 + \frac{2N_3}{N_1 N_2}\,, \qquad \varUpsilon^{(1)\tilde{\circ}}_{14,23} \to 2N_3 + \frac{2N_0}{N_1 N_2}\,, \qquad \varUpsilon^{(1)\tilde{\circ}}_{13,24} \to 0\,.
\end{aligned} \tag{4.15}$$

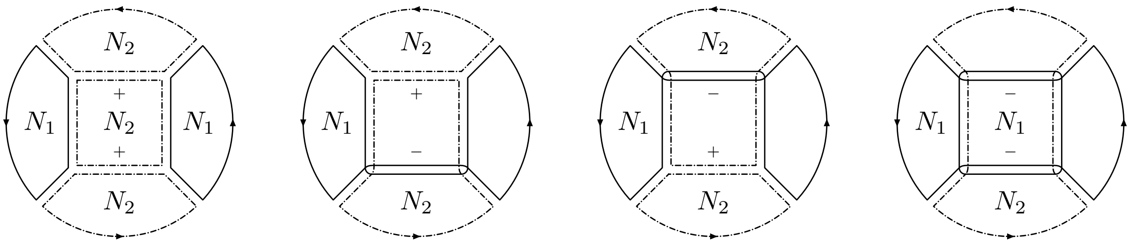

Figure 18: Full evaluation of the box graph $\Upsilon^{(1)\,\square}_{14,23}$ contracted with the color structure $N_1^{-2}N_2^{-2}\,\mathrm{Tr}(\bar{1}2)(\bar{2}1)(\bar{1}2)(\bar{2}1)$: $N_2 - N_1^{-1} - N_1^{-1} + N_2^{-1}$.

The untwisted bubbles are related to the boxes via (4.10) and the above expressions obey the rule. Note that we have evaluated the twisted bubbles under the assumption of many gauge group factors in the $\mathcal{N} = 4$ quiver diagram (Fig. 1). For $\mathcal{N} = 5, 6, 8$ supersymmetric models twisted and untwisted representations are the same and thus the bubbles in (4.15) must be the same as well

$$\Upsilon^{(1)\,\circ} = \Upsilon^{(1)\,\tilde{\circ}}. \tag{4.16}$$

The expressions (4.15) for $N_3 \to N_1$ and $N_0 \to N_2$ do not reflect the equality because the above assumptions for evaluating the twisted bubble $\Upsilon^{(1)\,\tilde{\circ}}$ do not apply in a closed quiver of length two (see Fig. 1 on page 5). Instead we must set $\Upsilon^{(1)\,\tilde{\circ}} \to \Upsilon^{(1)\,\circ}$ for $\mathcal{N} = 5, 6, 8$.

Mixed Scattering. Our standard color ordering for mixed scattering of the type $12\tilde{3}\tilde{4}$ will be $N_0^{-1}N_1^{-2}N_2^{-1}\,\mathrm{Tr}(\bar{1}2)(\bar{2}1)(\bar{1}2)(\bar{2}1)$, cf. Fig. 14. The single tree diagram for this assignment of twisted legs evaluates to

$$\Upsilon^{(0)}_{12,\tilde{3}\tilde{4}} \to -1. \tag{4.17}$$

The one-loop graphs can be found in Fig. 13, they yield

$$\Upsilon^{(1)\,\square}_{12,\tilde{3}\tilde{4}} \to N_1, \qquad \Upsilon^{(1)\,\square}_{12,\tilde{4}\tilde{3}} \to \frac{1}{N_1}, \qquad \Upsilon^{(1)\,\circ}_{12,\tilde{3}\tilde{4}} \to 2N_2 - \frac{2}{N_1}, \qquad \Upsilon^{(1)\,\tilde{\circ}}_{12,\tilde{3}\tilde{4}} \to 2N_0 - \frac{2}{N_1}. \tag{4.18}$$

Finally let us consider another assignment of twisted legs $1\tilde{2}\tilde{3}4$ which will become useful later. The standard color ordering will be $N_1^{-1}N_2^{-2}N_3^{-1}\,\mathrm{Tr}(\bar{1}2)(\bar{2}3)(\bar{3}2)(\bar{2}1)$, cf. Fig. 14. The color ordered tree graph reads

$$\Upsilon^{(0)}_{14,\tilde{2}\tilde{3}} \to +1. \tag{4.19}$$

while the color ordered loop amplitudes in Fig. 13 yield

$$\Upsilon^{(1)\,\square}_{14,\tilde{2}\tilde{3}} \to N_2, \qquad \Upsilon^{(1)\,\square}_{14,\tilde{3}\tilde{2}} \to \frac{1}{N_2}, \qquad \Upsilon^{(1)\,\circ}_{14,\tilde{2}\tilde{3}} \to 2N_1 - \frac{2}{N_2}, \qquad \Upsilon^{(1)\,\tilde{\circ}}_{14,\tilde{2}\tilde{3}} \to 2N_3 - \frac{2}{N_2}. \tag{4.20}$$

For purely twisted scattering we use the ordering $N_2^{-2}N_3^{-2}\,\mathrm{Tr}(\bar{3}2)(\bar{2}3)(\bar{3}2)(\bar{2}3)$, but the results will be analogous to those of purely untwisted scattering discussed above.

5 Four-Particle Scattering in Field Theory

In this section we compare the results for the four particle scattering matrix, obtained in the previous section using the supersymmetry algebra, with the predictions of perturbative field theory computations at the tree and one-loop level.

In what is to follow, we shall start with the $\mathcal{N}=4$ theory without twisted hypermultiplets, we shall then build on that by including twisted hypermultiplets but without imposing any particular conditions on the representations under which the two matter multiplets transform. Thus the solutions we obtain will hold for theories with $\mathcal{N}=4,5,6$ and 8 supersymmetries. We will relegate some of the details regarding the field theory conventions to the appendices where one can find, for example, the explicit expression for the action (B.5) and the oscillator expansion of the fields (B.15).

We can define, as is usual, two particle in and out states for the scalars and fermions in terms of their free oscillator expressions:

$$|B,p_2;A,p_1\rangle_{\mathrm{in}}=\sqrt{2E_1}\sqrt{2E_2}\,C_B^\dagger(p_2)\,C_A^\dagger(p_1)|0,t=-\infty\rangle \tag{5.1}$$

where A,B denote, here collectively, all the particle labels and $C^\dagger_{A,B}$ are the corresponding positive energy creation operators. One can then define the $\mathcal{S}$-matrix elements between in and out states as usual which in turn, in the notation of (Section 3.5), defines the operator $\mathcal{T}_{12}^{43}$ via the usual relation $\mathcal{S}=1+i\mathcal{T}$. Thus,

$$_{\mathrm{out}}\langle p_4,p_3|p_2,p_1\rangle_{\mathrm{in}}=\delta_{1(3}\delta_{4)2}+i\delta\left(p_1+p_2-p_3-p_4\right)\ T_{12\bar{3}\bar{4}}\ . \tag{5.2}$$

However to connect with the four-particle scattering matrix used in (3.2), where all the momenta are on an equal footing and we have negative energy particles, we must rather consider four point correlation functions. As is standard, we identify the one-particle irreducible four point functions with all momenta incoming, $\mathit{\Gamma}(p_1,p_2,p_3,p_4)$ with $-i\ T_{1234}$.

In doing this we must be slightly careful regarding the definition of our asymptotic states. At tree-level we will simply include an addition factor $\sqrt{4\pi/k}$ per external field but at higher loops we must include the non-trivial field renormalization that occurs.

5.1 Pure Amplitudes at Tree Level

Let us initially consider the element of $\mathcal{T}$ given by

$$A=i\big\langle\phi_1^A(p_1)\,\phi_1^B(p_2)\,\phi_2^C(p_3)\,\phi_2^D(p_4)\big\rangle_{\mathrm{1PI}} \tag{5.3}$$

where A denotes the total contribution for untwisted scalar to scalar scattering, transforming in the symmetric representation of $\mathfrak{su}(2)$ and without any implicit requirements of color ordering on the indices A,B,C,D.

Explicit evaluation of the complete, non-color ordered matrix element A follows from the Feynman diagrams (Figure 19) and, in terms of a simplified coupling

$$g=\frac{4\pi}{k}\,, \tag{5.4}$$

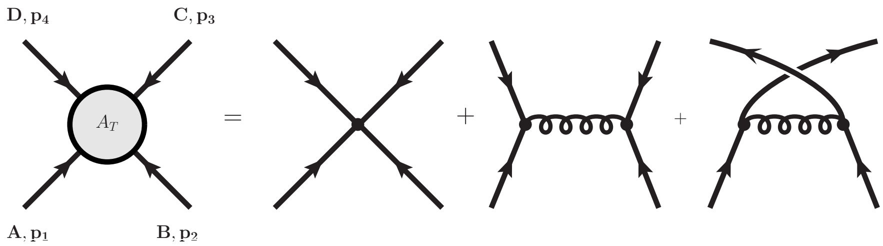

Figure 19: Diagrams for the A_T element at tree level.

gives the expression

$$A = g\ K_{MN}\left(M^M_{AD}M^N_{BC}A_t + M^M_{AC}M^N_{BD}A_u\right), \tag{5.5}$$

with the color stripped amplitudes A_t, A_u

$$A_t = 2i\left(im - \frac{\varepsilon_{\mu\nu\rho}p_1^\mu p_3^\nu p_4^\rho}{(p_1+p_4)^2}\right), \qquad A_u = 2i\left(im - \frac{\varepsilon_{\mu\nu\rho}p_1^\mu p_4^\nu p_3^\rho}{(p_1+p_3)^2}\right). \tag{5.6}$$

There are equivalent expressions for A_t expressed in terms the three dimensional Lorentz invariants s, t, u or the spinors, which themselves are related by,

$$\begin{aligned} s &= (p_1+p_2)^2 = -\langle 1\bar{2}\rangle\langle \bar{1}2\rangle = \langle 1\bar{1}\rangle^2 - \langle 12\rangle\langle \bar{1}\bar{2}\rangle, \\ t &= (p_1+p_4)^2 = -\langle 1\bar{4}\rangle\langle \bar{1}4\rangle = \langle 1\bar{1}\rangle^2 - \langle 14\rangle\langle \bar{1}\bar{4}\rangle, \\ u &= (p_1+p_3)^2 = -\langle 1\bar{3}\rangle\langle \bar{1}3\rangle = \langle 1\bar{1}\rangle^2 - \langle 13\rangle\langle \bar{1}\bar{3}\rangle, \\ \sqrt{-stu} &= \pm\langle 1\bar{2}\rangle\langle 2\bar{3}\rangle\langle 3\bar{1}\rangle. \end{aligned} \tag{5.7}$$

In terms of these variables we can write [13]

$$A_t = i\frac{\langle 12\rangle\langle \bar{2}\bar{4}\rangle}{\langle \bar{4}1\rangle} = i\left(2im - \frac{\sqrt{-stu}}{t}\right), \tag{5.8}$$

The A_t term in the above expression captures the contribution from one ordering of the contact term indices and the t-channel gluon exchange, while the second term A_u is the remaining part of the contact term and the u-channel gluon exchange. Due to the choice of $\mathfrak{su}(2)$ indices there is no s-channel gluon exchange diagram. In obtaining this answer we have used the fact that the Chern–Simons propagator is given by

$$\left\langle A^\mu_M(p)A^\nu_N(q)\right\rangle = g\ K_{MN}\frac{\varepsilon^{\mu\nu\rho}p_\rho}{2p^2}\,\delta^3(p+q). \tag{5.9}$$

As discussed in Section 4.2 using the symmetries of K_{MN} and M^M_{AB} in conjunction with the fundamental identity imply that it is possible to write any tree level amplitude involving only untwisted hypermultiplets as

$$A^{(0)} = A_t\Upsilon^{(0)}_{14,23} + A_u\Upsilon^{(0)}_{13,24}. \tag{5.10}$$

[13] In the left-most expression there is a sign ambiguity due to the square root. In checking such properties as crossing it is therefore always preferable to use the spinor formulation.

Note that (5.5) is already of this form. For amplitudes where this is not automatically the case, we can eliminate the s-channel color structure using the fundamental identity

$$K_{MN}\left(M^M_{AC}M^N_{BD}+M^M_{CB}M^N_{AD}+M^M_{BA}M^N_{CD}\right)=0. \tag{5.11}$$

The color ordered contribution to the amplitude is taken to be simply the first (t-channel) term of the above expression, after we strip away the color factor: A_t. The second (u-channel) term is related to the first term by crossing.

The undetermined prefactor T appearing in the matrix elements (3.6) simply equals the A element. Thus the color ordered normalization factor T_t is nothing but the t-channel amplitude for scalars transforming in the symmetric representation of $\mathfrak{su}(2)$

$$T_t=A_t=i\frac{\langle 12\rangle\langle\bar{2}\bar{4}\rangle}{\langle\bar{4}1\rangle}\,. \tag{5.12}$$

Once this factor has been set, as above, we can check for the other matrix elements all of which will have the same prefactor:

$$g\ K_{MN}M^M_{AD}M^N_{BC}=g\ \varUpsilon^{(0)}_{14,23} \tag{5.13}$$

and for which the tree level perturbative computations yield:

$$\begin{aligned}
A&=+i\frac{\langle 12\rangle\langle\bar{2}\bar{4}\rangle}{\langle\bar{4}1\rangle}\,, & D&=-i\frac{\langle\bar{2}\bar{4}\rangle\langle\bar{3}\bar{4}\rangle}{\langle\bar{4}1\rangle}\,, & G&=+i\langle\bar{2}\bar{4}\rangle\,,\\
\tfrac{1}{2}(A+B)&=-i\frac{\langle\bar{3}1\rangle\langle 24\rangle\langle\bar{2}\bar{4}\rangle}{\langle\bar{4}1\rangle\langle\bar{3}\bar{4}\rangle}\,, & \tfrac{1}{2}(D+E)&=-i\frac{\langle\bar{3}1\rangle\langle\bar{1}\bar{3}\rangle\langle\bar{2}\bar{4}\rangle}{\langle\bar{4}1\rangle\langle\bar{3}\bar{4}\rangle}\,, & H&=+i\frac{\langle\bar{3}1\rangle\langle\bar{2}\bar{4}\rangle}{\langle\bar{4}1\rangle}\,,\\
\tfrac{1}{2}(A-B)&=-i\frac{\langle 14\rangle\langle\bar{1}\bar{3}\rangle}{\langle\bar{3}\bar{4}\rangle}\,, & \tfrac{1}{2}(D-E)&=+i\frac{\langle\bar{1}\bar{3}\rangle\langle\bar{2}\bar{3}\rangle}{\langle\bar{3}\bar{4}\rangle}\,, & K&=-i\frac{\langle\bar{4}2\rangle\langle\bar{2}\bar{4}\rangle}{\langle\bar{4}1\rangle}\,,\\
\tfrac{1}{2}C&=-i\frac{\langle\bar{3}1\rangle\langle\bar{1}\bar{3}\rangle}{\langle\bar{3}\bar{4}\rangle}\,, & \tfrac{1}{2}F&=+i\frac{\langle\bar{2}\bar{4}\rangle\langle\bar{1}\bar{3}\rangle}{\langle\bar{3}\bar{4}\rangle}\,, & L&=+i\langle\bar{1}\bar{3}\rangle\,.
\end{aligned} \tag{5.14}$$

These results are in manifest agreement with the predictions from the supersymmetry algebra in (3.6). We can now write the complete tree-level untwisted-untwisted scattering prefactor

$$T^{(0)}_{1234}=\varUpsilon^{(0)}_{14,23}\left(ig\frac{\langle 12\rangle\langle\bar{2}\bar{4}\rangle}{\langle\bar{4}1\rangle}\right)+\varUpsilon^{(0)}_{13,24}\left(ig\frac{\langle 12\rangle\langle\bar{2}\bar{3}\rangle}{\langle\bar{3}1\rangle}\right). \tag{5.15}$$

Since the twistor brackets satisfy a host of non-linear identities such as (3.7,3.8) there is no canonical way of representing the results of the perturbative computations however the above seems to be particularly simple. We note the following identities have been used to compute the scattering amplitudes involving four fermions i.e. the matrix elements D and E:

$$\begin{aligned}
(u_{\bar{4}}\gamma_\mu v_1)\,(u_{\bar{3}}\gamma_\nu v_2)\,\frac{\varepsilon^{\mu\nu\rho}(p_1+p_4)_\rho}{t}&=2\,\frac{\langle\bar{2}\bar{4}\rangle\langle\bar{3}\bar{4}\rangle}{\langle 1\bar{4}\rangle}\,,\\
(u_{\bar{3}}\gamma_\mu u_{\bar{4}})\,(v_1\gamma_\nu v_2)\,\frac{\varepsilon^{\mu\nu\rho}(p_1+p_2)_\rho}{s}&=2\left(\frac{\langle 12\rangle\langle\bar{2}\bar{4}\rangle}{\langle 1\bar{4}\rangle}\right)\left(\frac{\langle\bar{3}2\rangle\langle\bar{2}\bar{3}\rangle}{\langle 12\rangle\langle\bar{3}\bar{4}\rangle}\right).
\end{aligned} \tag{5.16}$$

On the l.h.s., we have the contributions of gluon exchanges between two fermions as they would usually appear in a (tree level) perturbative computation. The r.h.s. are the relevant twistorial expressions.

Given the explicit expressions for the matrix elements it is straightforward to check that they satisfy the crossing relations described in Section 3.4. For example the invariance under exchange $3 \leftrightarrow 4$ is immediate from (5.15) and using the identities (3.7) one can see that it is invariant under exchange of $1 \leftrightarrow 2$. To further see that the prefactor transforms as

$$T^{(0)}_{2341} = -\frac{\langle 23\rangle\langle\bar{4}1\rangle}{\langle 12\rangle\langle\bar{4}3\rangle}\, T^{(0)}_{1234} \tag{5.17}$$

under $1 \to 2 \to 3 \to 4 \to 1$ we need to use (3.8) and the Jacobi identities relating the tree level color structures.

5.2 Mixed Amplitudes at Tree Level

The 'twisted-untwisted' multiplet scattering is much the same and the overall prefactor T, corresponding to the A element, is

$$A = T = \langle \mathcal{T}|\phi_{(a}\phi_{b)}\tilde{\psi}_{(c}\tilde{\psi}_{d)}\rangle. \tag{5.18}$$

In this case is the color ordered amplitude is defined to be the coefficient of the single mixed color structure at tree level

$$g\ K_{MN}M^M_{BA}\tilde{M}^N_{DC} = g\ \Upsilon^{(0)}_{12,\tilde{3}\tilde{4}}. \tag{5.19}$$

We have the following perturbative tree-level results:

$$\begin{aligned}
A &= -i\langle 34\rangle\,, & D &= +i\langle\bar{1}\bar{2}\rangle\,, & G &= -i\langle 3\bar{2}\rangle\,,\\
\tfrac{1}{2}(A+B) &= +i\frac{\langle\bar{2}4\rangle\langle 24\rangle}{\langle\bar{3}4\rangle}\,, & \tfrac{1}{2}(D+E) &= -i\frac{\langle\bar{2}4\rangle\langle\bar{1}\bar{3}\rangle}{\langle\bar{3}4\rangle}\,, & H &= -i\langle\bar{2}4\rangle\,,\\
\tfrac{1}{2}(A-B) &= +i\frac{\langle\bar{1}4\rangle\langle 14\rangle}{\langle\bar{3}4\rangle}\,, & \tfrac{1}{2}(D-E) &= -i\frac{\langle\bar{2}\bar{3}\rangle\langle\bar{1}4\rangle}{\langle\bar{3}4\rangle}\,, & K &= -i\langle 3\bar{1}\rangle\,,\\
\tfrac{1}{2}C &= +i\frac{\langle\bar{2}4\rangle\langle\bar{3}2\rangle}{\langle\bar{3}4\rangle}\,, & \tfrac{1}{2}F &= -i\frac{\langle\bar{2}4\rangle\langle\bar{1}4\rangle}{\langle\bar{3}4\rangle}\,, & L &= -i\langle\bar{1}4\rangle\,.
\end{aligned} \tag{5.20}$$

Once again, we note that the perturbative tree level results completely agree with the computations based on the supersymmetry algebra (3.6). In this case the complete result for tree level untwisted-twisted scattering is given by

$$T^{(0)}_{12\tilde{3}\tilde{4}} = -ig\langle 34\rangle\ \Upsilon^{(0)}_{12,\tilde{3}\tilde{4}}. \tag{5.21}$$

For the sake of completeness, we list some of the key identities that are useful for converting the results obtained from standard perturbation theory to the twistorial expressions used in the expression for the scattering matrix

$$
\begin{aligned}
(v_{\bar{3}}\varepsilon v_{\bar{4}})\left[\frac{\langle\bar{3}1\rangle\langle 24\rangle}{\langle 12\rangle\langle\bar{3}4\rangle}+\frac{\langle 14\rangle\langle\bar{3}2\rangle}{\langle 12\rangle\langle\bar{3}4\rangle}\right] &= -2\varepsilon^{\mu\nu\sigma}(v_{\bar{4}}\sigma_\mu v_{\bar{3}})\frac{(p_1)_\nu(p_2)_\sigma}{s}\,,\\
(v_{\bar{3}}\varepsilon v_{\bar{4}})\left[\frac{\langle\bar{3}2\rangle\langle 2\bar{3}\rangle}{\langle 12\rangle\langle\bar{3}4\rangle}-\frac{\langle\bar{3}1\rangle\langle\bar{1}\bar{3}\rangle}{\langle 12\rangle\langle\bar{3}4\rangle}\right] &= +2\varepsilon^{\mu\nu\sigma}\,(v_1\sigma_\mu v_2)\,\frac{(p_3)_\nu(p_4)_\sigma}{s}\,,\\
2\,(v_{\bar{3}}\varepsilon v_{\bar{4}})\left[\frac{\langle\bar{3}1\rangle\langle\bar{1}4\rangle}{\langle 12\rangle\langle\bar{3}4\rangle}\right] &= -\frac{\varepsilon^{\mu\nu\sigma}}{s}(p_1+p_2)_\sigma\,(v_{\bar{4}}\sigma_\mu v_{\bar{3}})\,(v_1\sigma_\nu v_2)\;,\\
(v_{\bar{3}}\varepsilon v_{\bar{4}})\left[\frac{\langle\bar{3}1\rangle\langle\bar{3}2\rangle}{\langle 12\rangle\langle\bar{3}4\rangle}\right] &= -\frac{\varepsilon^{\mu\nu\sigma}(p_3-p_4)_\mu(p_1)_\nu(p_2)_\sigma}{s}\,.
\end{aligned}
\tag{5.22}
$$

As it is also useful for later considerations we record the field theory result for scattering when particles 2 and 3 are twisted. In this case the overall prefactor, corresponding to the A element, is

$$
A = T = \langle\mathcal{T}|\phi_{(a}\tilde{\psi}_{b)}\tilde{\psi}_{(c}\phi_{d)}\rangle = i\,\frac{\langle\bar{1}2\rangle\langle 12\rangle}{\langle\bar{3}2\rangle} = -i\,\frac{\langle\bar{4}3\rangle\langle 43\rangle}{\langle\bar{2}3\rangle}\,. \tag{5.23}
$$

Thus we have

$$
T^{(0)}_{1\tilde{2}\tilde{3}4} = -ig\frac{\langle\bar{1}2\rangle\langle 12\rangle}{\langle 2\bar{3}\rangle}\,\Upsilon^{(0)}_{14,\tilde{2}\tilde{3}}. \tag{5.24}
$$

In the case of scattering between the twisted matter content of the theory, we have to consider the fact that the quartic bosonic vertex as well as the fermion mass terms come with opposite signs as compared to the untwisted sector. The explicit formulae for the scattering of the untwisted multiplet can be readily adapted to the case at hand. The matrix elements for any four untwisted fields can be taken over to their twisted counterparts by changing the sign of the mass and replacing $u(p)$ by $v(p)$ and vice versa. Thus the matrix element D is given by

$$
D = i\frac{\langle\bar{1}\bar{2}\rangle\langle 24\rangle}{\langle 4\bar{1}\rangle} = i\left(-2im - \frac{\sqrt{-stu}}{t}\right)\,, \tag{5.25}
$$

where the relevant color prefactor is $g\Upsilon^{(0)}_{\tilde{1}\tilde{4},\tilde{2}\tilde{3}}$. It is a straightforward exercise to see that (3.6) continues to describe the four particle scattering matrix relevant to the twisted sector of the theory. In particular the overall prefactor, $T = A$, for this sector is given by

$$
T = A = -i\frac{\langle 24\rangle\langle 34\rangle}{\langle 4\bar{1}\rangle} \tag{5.26}
$$

and so

$$
T^{(0)}_{\tilde{1}\tilde{2}\tilde{3}\tilde{4}} = \Upsilon^{(0)}_{\tilde{1}\tilde{4},\tilde{2}\tilde{3}}\left(-ig\,\frac{\langle 24\rangle\langle 34\rangle}{\langle 4\bar{1}\rangle}\right) + \Upsilon^{(0)}_{\tilde{1}\tilde{3},\tilde{2}\tilde{4}}\left(-ig\,\frac{\langle 23\rangle\langle 43\rangle}{\langle 3\bar{1}\rangle}\right)\,. \tag{5.27}
$$

Finally one can check that the untwisted-twisted and twisted-twisted scattering matrix elements satisfy the crossing relations in Section 3.4

If one chooses the matter to be in representations such that there is extended $N = 5$ supersymmetry there are additional relations between the pure twisted-twisted, pure untwisted-untwisted and the mixed untwisted-twisted scattering as described in Section 3.3. It is straightforward to check that the perturbative calculations are consistent with those additional relations. In particular, if the untwisted and twisted multiplets transform in the same gauge representation, supersymmetry implies that

$$T_{12\tilde{3}\tilde{4}} = \frac{\langle\bar{3}1\rangle\langle\bar{2}\bar{1}\rangle T_{\tilde{1}\tilde{2}\tilde{3}\tilde{4}} + \langle 12\rangle\langle\bar{2}4\rangle T_{1234}}{\langle 12\rangle\langle\bar{2}\bar{3}\rangle + \langle 14\rangle\langle\bar{4}\bar{3}\rangle} \tag{5.28}$$

which is indeed satisfied by the tree-level expressions above.

To check the $\mathcal{N}=8$ relations one must make use of the simplifications in the color structure that occur for the gauge group SO(4)

$$\begin{aligned}
\Upsilon^{(0)}_{12,34} &= M^M_{AB} K_{MN} M^N_{CD} \propto \epsilon_{ABCD}\epsilon^{\hat{a}\hat{b}}\epsilon^{\hat{c}\hat{d}} \\
\Upsilon^{(0)}_{13,24} &= M^M_{AC} K_{MN} M^N_{CD} \propto -\epsilon_{ABCD}\epsilon^{\hat{a}\hat{c}}\epsilon^{\hat{b}\hat{d}} \\
\Upsilon^{(0)}_{14,23} &= M^M_{AC} K_{MN} M^N_{CD} \propto \epsilon_{ABCD}\epsilon^{\hat{a}\hat{d}}\epsilon^{\hat{b}\hat{c}} \ .
\end{aligned} \tag{5.29}$$

Using these relations one can check that the constraints (3.19) are satisfied.

5.3 One-Loop Amplitudes in Pure $\mathcal{N}=4$ SCS

In this section we shall calculate the one-loop correction to the scattering matrix using standard off-shell methods. As is well known, it is possible using unitarity relations, to reconstruct the imaginary part of the one-loop amplitudes from the phase space integral over products of tree-level amplitudes (which can of course be extended to higher orders). This provides a very efficient method for calculating scattering amplitudes and the results that follow from unitarity for the theories at hand, elaborated upon in detail in the next section, are in perfect agreement with the predictions following from the supersymmetry algebra. However the drawback of these methods is the so called "polynomial ambiguity" whereby the cut construction can miss contributions which are rational functions of the kinematic invariants lacking a cut. In four dimensional super Yang–Mills all one-loop massless amplitudes are cut constructible (for discussion see e.g. [44]) as all rational terms are related to terms with cuts at $\mathcal{O}(\epsilon^0)$ (where ϵ is the dimensional regularisation parameter). This is true for the maximally supersymmetric $\mathcal{N}=4$ case but also for $\mathcal{N}=1$ theories. However it is certainly not true in general and non-supersymmetric Yang–Mills theories are not one-loop cut constructable. Thus while it is reasonable to expect the one-loop amplitudes in the supersymmetric Chern–Simons matter theories to be cut-constructible it is by no means guaranteed. The off-shell methods provide a check that the cuts are indeed capturing all the amplitude and that there no rational piece that is unrelated to a logarithm. In principle one could also fix any rational function by working to sufficiently high order in ϵ however this can be involved and the direct calculation is quite feasible for the two to two scattering.

We will consider the matrix element G in (3.2)

$$G\ \delta^3(p_1+p_2+p_3+p_4) = -\langle\mathcal{T}|\phi_1\psi_{\dot{1}}\psi_{\dot{2}}\phi_2\rangle \tag{5.30}$$

as it involves the fewest number of Feynman diagrams at the one-loop level. We initially consider the contribution with only gluons or untwisted matter running in loops and then separately add the contribution from twisted matter. Let us now make a few remarks about the color structure; following an examination of interactions following from the action (B.5) we can use the Jacobi identities on the vertices to see that, in the notation of Sec. 4.3, only box-like structures occur. The color structures that appear in the s-channel diagrams are:

$$\Upsilon^{(1)\square}_{14,23} \text{ and } \Upsilon^{(1)\square}_{13,24} , \tag{5.31}$$

in the t- and u-channels we have, respectively,

$$\Upsilon^{(1)\square}_{13,42}, \Upsilon^{(1)\square}_{12,43} \qquad \text{and} \qquad \Upsilon^{(1)\square}_{12,34}, \Upsilon^{(1)\square}_{14,32}. \tag{5.32}$$

Thus we see that all box-like color structures enumerated in Sec. 4.3 can appear. However the coefficients of the different structures are all related by crossing and so we need only calculate a single coefficient. We will thus focus on the s-channel contribution to the color-ordered amplitude which occurs with the prefactor $\Upsilon^{(1)\square}_{14,23}$.

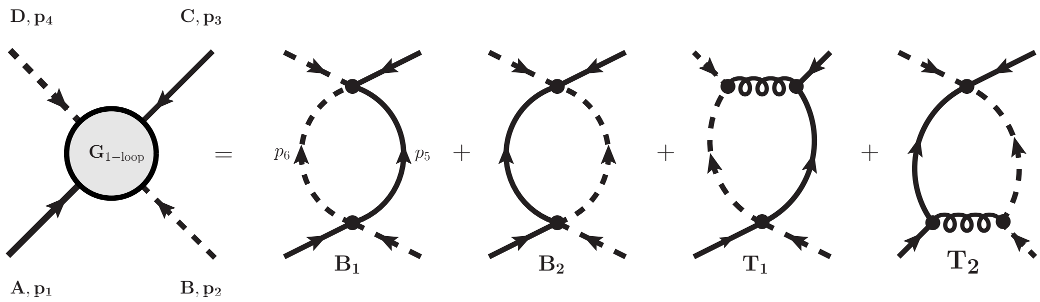

Figure 20: Diagrams for the G element at one-loop.

The one-loop correction to G involves four Feynman diagrams (Figure 20). Two of these diagrams correspond to 'bubbles' with a scalar and a fermionic propagator forming a closed loop and two are 'triangles', where a gluon exchange between the intermediate scalar and fermion fields completes the loop. The complete answer is given by

$$G^{(1)} = i\, \Upsilon^{(1)\square}_{14,23} \int d^3\ell\, (B_1 + B_2 + T_1 + T_2) \tag{5.33}$$

The integrands for the bubble diagrams with all momenta incoming are:

$$\begin{aligned} B_1 &= -i \frac{v_2 \varepsilon (i p_6 + m\varepsilon) \varepsilon v_4}{2(p_6^2 + m^2)(p_5^2 + m^2)} , \\ B_2 &= -i \frac{v_2 \varepsilon (i p_5 + m\varepsilon) \varepsilon v_4}{2(p_6^2 + m^2)(p_5^2 + m^2)} . \end{aligned} \tag{5.34}$$

The integrands for the triangles are:

$$\begin{aligned} T_1 &= \frac{i}{D} \left(v_2 \varepsilon (i p_6 + m\varepsilon) \bar{\sigma}^\rho v_4\, \varepsilon_{\rho\nu\mu} (p_6 + p_4)^\nu (p_5 - p_3)^\mu \right) , \\ T_2 &= \frac{i}{D'} \left(v_2 \bar{\sigma}^\rho (i p_5 + m\varepsilon) \varepsilon v_4\, \varepsilon_{\rho\nu\mu} (p_2 - p_5)^\nu (p_1 + p_6)^\mu \right) . \end{aligned} \tag{5.35}$$

It is to be understood that all the spinors and σ matrices carry lower indices and we use the shorthand notation where $(p_i)_{\alpha\beta} = (p_i)_\mu (\sigma^\mu)_{\alpha\beta}$. We denote σ matrices with raised indices as $\bar{\sigma}$ i.e. $(\bar{\sigma}^\mu)^{\alpha\beta} = \varepsilon^{\alpha\gamma}\varepsilon^{\beta\delta}(\sigma^\mu)_{\gamma\delta}$. We also have introduced

$$\begin{aligned} D &= 2(p_6^2 + m^2)(p_5^2 + m^2)(p_6 + p_4)^2 \quad \text{and} \\ D' &= 2(p_6^2 + m^2)(p_5^2 + m^2)(p_5 - p_2)^2 \end{aligned} \tag{5.36}$$

to denote the triangle denominators. There is of course only one independent loop momentum as p_5 and p_6 are related by the kinematical constraints $p_6 + p_5 = p_1 + p_2 = -(p_3 + p_4)$. We now note the following identities:

$$\begin{aligned} v_2\varepsilon(ip_6 + m\varepsilon)\bar{\sigma}^\rho v_4 &= -iv_2(\varepsilon\eta^{\rho\kappa} + \bar{\sigma}_\chi\varepsilon^{\chi\kappa\rho})v_4 \ (p_6 + p_2)_\kappa, \\ v_2\bar{\sigma}^\rho(ip_5 + m\varepsilon)\varepsilon v_4 &= -iv_2(\eta^{\rho\kappa}\varepsilon + \varepsilon^{\rho\kappa\chi}\bar{\sigma}_\chi)v_4 \ (p_5 - p_4)_\kappa. \end{aligned} \tag{5.37}$$

The use of these identities in T_1 and T_2 respectively, generates four terms for each of the triangle diagrams. Using $\bar{p}_{ij} = p_i - p_j$ and $p_{ij} = p_i + p_j$ for brevity, we have

$$\begin{aligned} T_{1\text{a}} &= -\frac{2}{D}\,(v_2\varepsilon v_4)\ \varepsilon(p_5, p_3, p_4 - p_2), \\ T_{1\text{b}} &= +\frac{1}{D}\,(v_2\bar{\sigma}^\rho\eta_{\rho\lambda}v_4)\ p_{64}^2\bar{p}_{53}^\lambda, \\ T_{1\text{c}} &= +\frac{1}{D}\,(v_2\bar{\sigma}^\rho\eta_{\rho\lambda}v_4)\ (p_5^2 + m^2)p_{64}^\lambda, \\ T_{1\text{d}} &= +\frac{1}{D}\,(v_2\bar{\sigma}^\rho\eta_{\rho\lambda}v_4)\ (\bar{p}_{42}\cdot\bar{p}_{53}p_{64}^\lambda - \bar{p}_{42}\cdot p_{64}\bar{p}_{53}^\lambda). \end{aligned} \tag{5.38}$$

Similarly, we obtain

$$\begin{aligned} T_{2\text{a}} &= -\frac{2}{D'}\,(v_2\varepsilon v_4)\ \varepsilon(p_4 - p_2, p_6, p_1), \\ T_{2\text{b}} &= +\frac{1}{D'}\,(v_2\bar{\sigma}^\rho\eta_{\rho\lambda}v_4)\ \bar{p}_{52}^2 p_{16}^\lambda, \\ T_{2\text{c}} &= +\frac{1}{D'}\,(v_2\bar{\sigma}^\rho\eta_{\rho\lambda}v_4)\ (p_6^2 + m^2)\bar{p}_{52}^\lambda, \\ T_{2\text{d}} &= +\frac{1}{D'}\,(v_2\bar{\sigma}^\rho\eta_{\rho\lambda}v_4)\ (\bar{p}_{42}.\bar{p}_{25}p_{16}^\lambda - \bar{p}_{42}.p_{16}\bar{p}_{25}^\lambda). \end{aligned} \tag{5.39}$$

After introducing Feynman parameters to simplify the denominators, shifting the loop momenta and dropping terms linear in the loop momenta it can be straightforwardly seen that the terms $T_{1\text{a},2\text{a}}$ and $T_{1\text{d},2\text{d}}$ cancel using the relation

$$(v_2\varepsilon v_4)\ \varepsilon(p_4, p_3, p_2) = -v_2\bar{\sigma}^\lambda v_4\Big[p_2\cdot\bar{p}_{24}\ (p_1)_\lambda - p_1\cdot\bar{p}_{24}\ (p_2)_\lambda\Big]. \tag{5.40}$$

Further, the terms $T_{1\text{b}}$, $T_{2\text{b}}$, B_1 and B_2 combine to give

$$T_{1\text{b}} + T_{2\text{b}} + B_1 + B_2 = +i\frac{v_2\varepsilon(ip_1 - m\varepsilon)\varepsilon v_4}{(p_6^2 + m^2)(p_5^2 + m^2)} \tag{5.41}$$

which corresponds to an s-channel massive scalar bubble integral, $I_m(s)$, with a coefficient proportional to $\langle\bar{2}\bar{1}\rangle\langle 1\bar{4}\rangle$. The massive bubble integral can be evaluated

$$\begin{aligned} I_m(s) &= \int d^3\ell\ \frac{1}{(\ell^2 + m^2)\,((\ell - p_{12})^2 + m^2)} \\ &= \frac{i\pi^2}{\sqrt{-s}}\ln\left(\frac{2m + \sqrt{-s}}{2m - \sqrt{-s}}\right). \end{aligned} \tag{5.42}$$

Finally there is the contribution from the terms T_{1c} and T_{2c} which can be written as

$$T_{1c} + T_{2c} = \frac{-im\,(v_2 \varepsilon v_4)}{(\tilde{\ell}^2 + \Delta)^2}(x-1), \tag{5.43}$$

where we have introduced the Feynman parameter x and $\Delta = (1-x)^2 m^2$. In fact the integral over the loop momenta and Feynman parameter can be trivially done and the result is

$$\int dx \int d\ell^3 \frac{1}{(\tilde{\ell}^2 + \Delta)^2}(x-1) = \frac{\pi^2}{m}\,. \tag{5.44}$$

This thus contributes to the amplitude a term proportional to the tree level contribution

$$\pi^2 \langle \bar{2}\bar{4} \rangle \Upsilon^{(1)\,\square}_{14,23}\,, \tag{5.45}$$

however it is cancelled by identical factors coming from the renormalization of the fermionic fields and which contribute to the S-matrix via the LSZ reduction formula.

We first consider the diagrams contributing to the fermionic self-energy (Figure 21), a tadpole with scalars in the loop and a gluon correction. (There is also in principle a contribution from the twisted fields, if they are present, however their contribution vanishes due to color index contractions.) The tadpole diagram gives an integrand

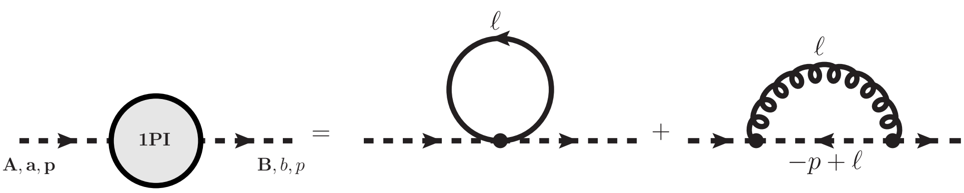

Figure 21: Diagrams for fermionic propagator at one-loop.

$$T(p) = -iK_{MN} M^M_{A\hat{A}} M^N_{\hat{B}B} L^{\hat{B}\hat{A}} \varepsilon^{\dot{a}\dot{b}} \frac{\epsilon}{\ell^2 + m^2} \tag{5.46}$$

and the gluon correction is

$$I(p) = -K_{MN} M^M_{A\hat{A}} M^N_{\hat{B}B} L^{\hat{B}\hat{A}} \varepsilon^{\dot{a}\dot{b}} \frac{\bar{\sigma}^\mu\,(i(\ell - p) + \varepsilon m)\,\bar{\sigma}^\nu \varepsilon_{\mu\rho\nu} \ell^\rho}{2\,[(\ell-p)^2 + m^2]\,\ell^2}. \tag{5.47}$$

For simplicity we strip off the color and flavor indices, then using the relations

$$\begin{aligned} \sigma^\lambda \bar{\sigma}^\nu &= -\eta^{\lambda\nu} + \sigma_\kappa \varepsilon \varepsilon^{\kappa\lambda\nu}, \\ \bar{\sigma}^\lambda \sigma^\nu &= -\eta^{\lambda\nu} + \bar{\sigma}_\kappa \varepsilon \varepsilon^{\kappa\lambda\nu} \end{aligned} \tag{5.48}$$

we can simplify the gluon contribution

$$\begin{aligned} I(p) &= -\frac{-i\ell \cdot (\ell - p)\,\varepsilon - m\,\ell}{[(\ell-p)^2 + m^2]\,\ell^2} \\ &= -\frac{-i\varepsilon}{(\ell-p)^2 - m^2} + \frac{\ell \cdot p\,\varepsilon - m\,\ell}{[(\ell-p)^2 + m^2]\,\ell^2} \end{aligned} \tag{5.49}$$

where the first term can be seen to cancel against the tadpole diagram. We can simplify the remaining term by introducing Feynman parameter x and shifting the loop-momenta. Thus, with $\varDelta = x(1-x)p^2 + xm^2$, we have

$$M^2(p) = im \int dx \int d^3\ell\, x\, \frac{\varepsilon(ip + p^2/m\ \varepsilon)\varepsilon}{[\ell^2 + \varDelta(p)]^2}\,. \tag{5.50}$$

Iterating these 1PI diagrams we find the correction to the propagator

$$\frac{(ip + m\varepsilon)}{p^2 + m^2 + M^2(p)} = Z_{\mathrm{f}}(p)\, \frac{ip + \varepsilon m}{p^2 + m^2} + \text{terms regular as } p_0 \to E(p) \tag{5.51}$$

where we see that the mass remains unchanged (the pole is not shifted as the correction is proportional to the inverse propagator) and that the one-loop shift in the field renormalization is $\delta Z_{\mathrm{f}}(p) = ig^2\pi^2$. The one-loop correction to the bosonic propagator can be easily seen to be zero. The relevant diagrams are given in (Figure 22).

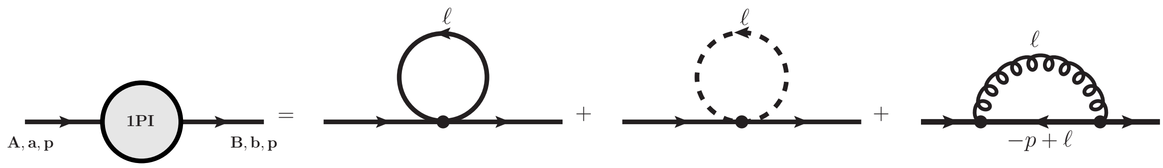

Figure 22: Diagrams for bosonic propagator at one-loop.

In this case the boson and fermion contributions exactly cancel and the gluon contribution is zero due to the ε tensor in the propagator, thus $Z_{\mathrm{b}}(p) = 1$. There are almost identical contributions to the matter in the twisted hypermultiplets.

There is also a non-vanishing one-loop correction to the gluon propagator which is a known effect in supersymmetric Chern–Simons theories (see e.g. [45]) and indeed in this case we find the same result. The gluon self-interaction cancels against the ghost loop while the fermion and bosonic contributions add to give a correction that is similar to the four-dimensional YM propagator. However we should point out that as there are no physical gluon states, there is no point in interpreting this correction as a field renormalization entering into the scattering matrix.

It is worth here pausing to make a comment regarding the color structures that can arise from the corrections to the S-matrix due to the field renormalizations. These have the structure of bubbles on fermionic legs labelled (B, p_2) and (D, p_4) attached to tree-level diagrams $\varUpsilon^{(0)}_{14,23}$ and $\varUpsilon^{(0)}_{13,24}$, for example

$$\left[K_{MN} M^{M}_{DE} M^{NEF}\right]\left[K_{PQ} M^{P}_{AF} M^{Q}_{BC}\right]\,. \tag{5.52}$$

However making use of the identities described in Section 4.3 and in particular Figure 9 we can express these in the basis of one-loop box diagrams to find the relevant term i.e. the coefficient of the structure $\varUpsilon^{(1)\square}_{14,23}$.

The one-loop contribution to the G element from the field renormalization is

$$\varDelta G^{(1)} = \left(\sqrt{Z_{\mathrm{b}}(p_1) Z_{\mathrm{f}}(p_2) Z_{\mathrm{b}}(p_3) Z_{\mathrm{f}}(p_4)} - 1\right) G^{(0)} = -\pi^2 \langle \bar{2}\bar{4}\rangle \tag{5.53}$$

which can be seen to cancel the contribution (5.45).

Thus we find that

$$G^{(1)} = i \langle\bar{2}\bar{1}\rangle\langle 1\bar{4}\rangle\, I_m(s). \tag{5.54}$$

This is the complete s-channel contribution at one-loop, whose overall factor is $\Upsilon^{(1)\,\square}_{14,23}$, for an $\mathcal{N} = 4$ theory without twisted hypermultiplets. There of course remain the other color ordering and the t-channel and u-channel diagrams; the t-channel diagrams are identical to those above after exchanging the external momenta while the u-channel contributions are slightly more complicated. However, as stated above they are all related to the calculated piece, once one accounts for the appropriate color factors. Using this element we can determine the one-loop piece of the overall factor undetermined by the symmetries

$$T^{(1)} = \frac{G^{(1)}}{G^{(0)}} T^{(0)} = i\langle\bar{1}\bar{2}\rangle\langle 12\rangle I_m(s) \tag{5.55}$$

and thus

$$\begin{aligned} T^{(1)}_{1234} = \quad & \left(\Upsilon^{(1)\,\square}_{14,23} + \Upsilon^{(1)\,\square}_{13,24}\right) \left[i\langle\bar{1}\bar{2}\rangle\langle 12\rangle I_m(s)\right] \\ + \quad & \left(\Upsilon^{(1)\,\square}_{13,43} + \Upsilon^{(1)\,\square}_{12,43}\right) \left[i\frac{\langle\bar{2}\bar{3}\rangle\langle\bar{1}2\rangle\langle 34\rangle}{\langle\bar{1}4\rangle}\, I_m(t)\right] \\ + \quad & \left(\Upsilon^{(1)\,\square}_{12,34} + \Upsilon^{(1)\,\square}_{14,32}\right) \left[i\frac{\langle\bar{2}\bar{4}\rangle\langle 12\rangle\langle\bar{3}4\rangle}{\langle\bar{3}1\rangle}\, I_m(u)\right]. \end{aligned} \tag{5.56}$$

5.4 Mixed Amplitudes at One Loop

We now include the contributions from the twisted hypermultiplets which give rise to two massive bubble diagrams (Figure 23). The color structure arises in the s-channel from diagrams with twisted fields in the loop is $\Upsilon^{(1)\,\tilde{\circ}}_{12,34}$. The contribution to the matrix

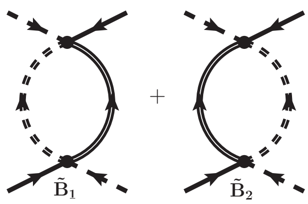

Figure 23: Twisted contribution to the G element at one-loop.

element from these diagrams is thus

$$G^{(1)} = i\Upsilon^{(1)\,\tilde{\circ}}_{12,34} \int d^3\ell \big(\tilde{B}_1 + \tilde{B}_2\big) \tag{5.57}$$

where the integrands for these twisted bubble diagrams are:

$$\begin{aligned} \tilde{B}_1 &= -i\frac{v_2\varepsilon(ip_6 - m\varepsilon)\varepsilon v_4}{2(p_6^2+m^2)(p_5^2+m^2)}\,, \\ \tilde{B}_2 &= -i\frac{v_2\varepsilon(ip_5 - m\varepsilon)\varepsilon v_4}{2(p_6^2+m^2)(p_5^2+m^2)}\,. \end{aligned} \tag{5.58}$$

Combining these we find the same result as in (5.41) but with the opposite sign,

$$\tilde{B}_1 + \tilde{B}_2 = -i \frac{v_2 \varepsilon (i p_1 - m\varepsilon) \varepsilon v_4}{2(p_6^2 + m^2)(p_5^2 + m^2)} \,. \tag{5.59}$$

Including the t- and u-channel contributions the one-loop contribution to the untwisted-untwisted scattering matrix is

$$\begin{aligned} \mathit{\Delta} T^{(1)}_{1234} = \quad & \Upsilon^{(1)\,\tilde{\circ}}_{12,34} \left[-\tfrac{i}{2} \langle \bar{1}\bar{2} \rangle \langle 12 \rangle I_m(s) \right] \\ + \quad & \Upsilon^{(1)\,\tilde{\circ}}_{12,43} \left[-\frac{i}{2} \frac{\langle \bar{2}\bar{3} \rangle \langle \bar{1}2 \rangle \langle 34 \rangle}{\langle \bar{1}4 \rangle} I_m(t) \right] \\ + \quad & \Upsilon^{(1)\,\tilde{\circ}}_{13,24} \left[-\frac{i}{2} \frac{\langle \bar{2}\bar{4} \rangle \langle 12 \rangle \langle \bar{3}4 \rangle}{\langle \bar{3}1 \rangle} I_m(u) \right] . \end{aligned} \tag{5.60}$$

We recall the contribution from the untwisted fields (5.56) and note that we can rewrite the the combination of untwisted color boxes as untwisted bubbles so that the total answer is

$$\begin{aligned} T^{(1)}_{1234} = \quad & \left(\Upsilon^{(1)\,\circ}_{12,34} - \Upsilon^{(1)\,\tilde{\circ}}_{12,34} \right) \left[\tfrac{i}{2} \langle \bar{1}\bar{2} \rangle \langle 12 \rangle I_m(s) \right] \\ + \quad & \left(\Upsilon^{(1)\,\circ}_{12,43} - \Upsilon^{(1)\,\tilde{\circ}}_{12,43} \right) \left[\frac{i}{2} \frac{\langle \bar{2}\bar{3} \rangle \langle \bar{1}2 \rangle \langle 34 \rangle}{\langle \bar{1}4 \rangle} I_m(t) \right] \\ + \quad & \left(\Upsilon^{(1)\,\circ}_{13,24} - \Upsilon^{(1)\,\tilde{\circ}}_{13,24} \right) \left[\frac{i}{2} \frac{\langle \bar{2}\bar{4} \rangle \langle 12 \rangle \langle \bar{3}4 \rangle}{\langle \bar{3}1 \rangle} I_m(u) \right] . \end{aligned} \tag{5.61}$$

In the special case, of $\mathcal{N} > 4$ supersymmetry where both the twisted and untwisted multiplets are in the same representation of the gauge group, the color structures will be equal and so lead to a cancellation. In this case we simply find that

$$T^{(1)}_{1234} = 0. \tag{5.62}$$

For the other sectors there are almost identical diagrams between untwisted-twisted hypermultiplets, namely for the element $H_{12\tilde{3}\tilde{4}}$, which implies

$$T^{(1)}_{12\tilde{3}\tilde{4}} = 0 \tag{5.63}$$

and for twisted-twisted scattering, $L_{\tilde{1}\tilde{2}\tilde{3}\tilde{4}}$, which implies

$$T^{(1)}_{\tilde{1}\tilde{2}\tilde{3}\tilde{4}} = 0. \tag{5.64}$$

However we will leave the more complete treatment to the substantially more efficient unitarity methods of the next section. Note that a vanishing one-loop contribution is obviously in agreement with the $\mathcal{N} > 4$ constraints on scattering amplitudes discussed in Sec. 3.3.

6 Scattering Unitarity

The scattering matrix $\mathcal{S} = 1 + i\mathcal{T}$ in a reasonable quantum field theory is expected to be unitary, $\mathcal{S}^\dagger \mathcal{S} = 1$. For the scattering amplitudes $\mathcal{T}$ it implies the unitarity condition

$$-\, i(\mathcal{T} - \mathcal{T}^\dagger) = \mathcal{T}^\dagger \mathcal{T}. \tag{6.1}$$

In this section we would like to compare the one-loop field theory results of the previous section with scattering unitarity. In particular, we want to see whether the field theory results stand a chance of being cut constructible.

6.1 Adjoint and Multiplication

In order to confirm unitarity for the amplitudes derived above, we should first understand how to take the adjoint and how to multiply two-to-two scattering amplitudes $\mathcal{T}[T]$, where T is the overall factor as defined in (3.6).

The adjoint scattering amplitude has the same structure as the original amplitude but with different matrix elements

$$\begin{aligned}
A^\dagger_{12\bar{3}\bar{4}} &= (A_{43\bar{2}\bar{1}})^*, & D^\dagger_{12\bar{3}\bar{4}} &= (D_{43\bar{2}\bar{1}})^*,\\
B^\dagger_{12\bar{3}\bar{4}} &= (B_{43\bar{2}\bar{1}})^*, & E^\dagger_{12\bar{3}\bar{4}} &= (E_{43\bar{2}\bar{1}})^*,\\
C^\dagger_{12\bar{3}\bar{4}} &= (F_{43\bar{2}\bar{1}})^*, & F^\dagger_{12\bar{3}\bar{4}} &= (C_{43\bar{2}\bar{1}})^*,\\
G^\dagger_{12\bar{3}\bar{4}} &= (L_{43\bar{2}\bar{1}})^*, & L^\dagger_{12\bar{3}\bar{4}} &= (G_{43\bar{2}\bar{1}})^*,\\
H^\dagger_{12\bar{3}\bar{4}} &= (H_{43\bar{2}\bar{1}})^*, & K^\dagger_{12\bar{3}\bar{4}} &= (K_{43\bar{2}\bar{1}})^*.
\end{aligned} \tag{6.2}$$

Here the spinors $\bar{1}, \bar{2}, \bar{3}, \bar{4}$ are conjugate to $1, 2, 3, 4$, respectively, according to (3.29)

$$u_{\bar{k}} = +v_k, \qquad v_{\bar{k}} = -u_k. \tag{6.3}$$

For unitary representations (2.13), or more generally by replacing $u^*_\alpha \to v_\alpha$, $v^*_\alpha \to u_\alpha$, the adjoint matrix elements take the same form as the original matrix elements (3.6). We can thus write the adjoint scattering amplitude as a regular scattering amplitude

$$\mathcal{T}[T]^\dagger = \mathcal{T}[T^\dagger], \tag{6.4}$$

but instead of T with the prefactor $T^\dagger$ defined by

$$T^\dagger_{12\bar{3}\bar{4}} = (T_{43\bar{2}\bar{1}})^*. \tag{6.5}$$

This is because for unitary representations the adjoint scattering matrix obeys the same symmetries as the original one.

Iterative two-to-two particle scattering also satisfies the transformation laws of overall two-to-two scattering, hence

$$\mathcal{T}[T']\,\mathcal{T}[T''] = \mathcal{T}[T]. \tag{6.6}$$

The following relations between the matrix elements ensure that the product takes the expected form

$$\begin{aligned}
\frac{T''_{12\bar{5}\bar{6}}T'_{65\bar{3}\bar{4}}}{T_{12\bar{3}\bar{4}}} &= \frac{A''_{12\bar{5}\bar{6}}A'_{65\bar{3}\bar{4}}}{A_{12\bar{3}\bar{4}}} = \frac{D''_{12\bar{5}\bar{6}}D'_{65\bar{3}\bar{4}}}{D_{12\bar{3}\bar{4}}}\\
&= \frac{B''_{12\bar{5}\bar{6}}B'_{65\bar{3}\bar{4}} + C''_{12\bar{5}\bar{6}}F'_{65\bar{3}\bar{4}}}{B_{12\bar{3}\bar{4}}} = \frac{E''_{12\bar{5}\bar{6}}E'_{65\bar{3}\bar{4}} + F''_{12\bar{5}\bar{6}}C'_{65\bar{3}\bar{4}}}{E_{12\bar{3}\bar{4}}}\\
&= \frac{B''_{12\bar{5}\bar{6}}C'_{65\bar{3}\bar{4}} + C''_{12\bar{5}\bar{6}}E'_{65\bar{3}\bar{4}}}{C_{12\bar{3}\bar{4}}} = \frac{E''_{12\bar{5}\bar{6}}F'_{65\bar{3}\bar{4}} + F''_{12\bar{5}\bar{6}}B'_{65\bar{3}\bar{4}}}{F_{12\bar{3}\bar{4}}}\\
&= \frac{G''_{12\bar{5}\bar{6}}K'_{65\bar{3}\bar{4}} + H''_{12\bar{5}\bar{6}}G'_{65\bar{3}\bar{4}}}{G_{12\bar{3}\bar{4}}} = \frac{K''_{12\bar{5}\bar{6}}K'_{65\bar{3}\bar{4}} + H''_{12\bar{5}\bar{6}}G'_{65\bar{3}\bar{4}}}{K_{12\bar{3}\bar{4}}}\\
&= \frac{G''_{12\bar{5}\bar{6}}L'_{65\bar{3}\bar{4}} + H''_{12\bar{5}\bar{6}}H'_{65\bar{3}\bar{4}}}{H_{12\bar{3}\bar{4}}} = \frac{K''_{12\bar{5}\bar{6}}L'_{65\bar{3}\bar{4}} + L''_{12\bar{5}\bar{6}}H'_{65\bar{3}\bar{4}}}{L_{12\bar{3}\bar{4}}}\,.
\end{aligned} \tag{6.7}$$

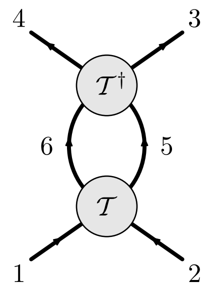

Figure 24: Particle setup for the unitarity relations.

The prefactor of the product is thus simply given by

$$T_{12\bar{3}\bar{4}} = 2\pi^2 \int d^3p\, \delta(p_5^2 + m^2)\, \delta(p_6^2 + m^2)\, T''_{12\bar{5}\bar{6}}\, T'_{65\bar{3}\bar{4}}. \tag{6.8}$$

Similar relations hold for scattering matrices involving twisted hypermultiplets introduced in Sec. 3.2. We will not present these in detail here, but merely apply them where needed.

6.2 Unitarity Relations

Now we are in a position to consider unitarity for two-to-two scattering amplitudes from field theory. We neglect intermediate states with more than two particles. This approximation is exact at the two-loop level because physical particles can only be created or annihilated in pairs. According to the above considerations, unitarity leads to the following relation for the prefactor, see also Fig. 24,

$$-iT_{12\bar{3}\bar{4}} + i(T_{43\bar{2}\bar{1}})^* = 2\pi^2 \int d^3p\, \delta(p_5^2 + m^2)\, \delta(p_6^2 + m^2)\, T_{12\bar{5}\bar{6}}\, (T_{43\bar{5}\bar{6}})^* + \mathcal{O}(g^4). \tag{6.9}$$

It is convenient to go to the center of mass frame and thus make a choice for the momenta of the particles

$$\begin{aligned}
p_1 &= (E, +p, 0), & p_2 &= (E, -p, 0),\\
p_3 &= (E, -p\cos\alpha, -p\sin\alpha), & p_4 &= (E, +p\cos\alpha, +p\sin\alpha),\\
p_5 &= (E, -p\cos\beta, -p\sin\beta), & p_6 &= (E, +p\cos\beta, +p\sin\beta).
\end{aligned} \tag{6.10}$$

In this frame the integral over the delta functions can be evaluated

$$2\pi^2 \int d^3p\, \delta(p_5^2 + m^2)\, \delta(p_6^2 + m^2)\, F_{123456} = \int \frac{\pi^2 d\beta}{4E}\, F_{123456}. \tag{6.11}$$

Substituting the loop expansion of the prefactor

$$T = gT^{(0)} + g^2T^{(1)} + g^3T^{(2)} + \ldots, \qquad g = \frac{4\pi}{k}\,, \tag{6.12}$$

we obtain the unitarity relations up to two loops

$$\begin{aligned}
-iT^{(0)}_{12\bar{3}\bar{4}} + i\big(T^{(0)}_{43\bar{2}\bar{1}}\big)^* &= 0,\\
-iT^{(1)}_{12\bar{3}\bar{4}} + i\big(T^{(1)}_{43\bar{2}\bar{1}}\big)^* &= \int \frac{\pi^2 d\beta}{4E}\, T^{(0)}_{12\bar{5}\bar{6}}\, \big(T^{(0)}_{43\bar{5}\bar{6}}\big)^* = \int \frac{\pi^2 d\beta}{4E}\, T^{(0)}_{12\bar{5}\bar{6}}\, T^{(0)}_{\bar{6}\bar{5}34},\\
-iT^{(2)}_{12\bar{3}\bar{4}} + i\big(T^{(2)}_{43\bar{2}\bar{1}}\big)^* &= \int \frac{\pi^2 d\beta}{4E}\, \Big(T^{(0)}_{12\bar{5}\bar{6}}\, \big(T^{(1)}_{43\bar{5}\bar{6}}\big)^* + T^{(1)}_{12\bar{5}\bar{6}}\, \big(T^{(0)}_{43\bar{5}\bar{6}}\big)^*\Big).
\end{aligned} \tag{6.13}$$

The above relations hold for a $\mathcal{N}=4$ supersymmetric model with only one type of hypermultiplet. If there are both types of hypermultiplets present, the relation (6.9) has to be extended and supplemented by

$$
\begin{aligned}
-iT_{12\bar{3}\bar{4}}+i\big(T_{43\bar{2}\bar{1}}\big)^{*} &= \int \frac{\pi^2 d\beta}{4E}\Big(T_{12\bar{5}\bar{6}}\left(T_{43\bar{5}\bar{6}}\right)^{*}+T_{12\tilde{\bar{5}}\tilde{\bar{6}}}\left(T_{43\tilde{\bar{5}}\tilde{\bar{6}}}\right)^{*}\Big)+\mathcal{O}(g^4),\\
-iT_{12\tilde{\bar{3}}\tilde{\bar{4}}}+i\big(T_{\tilde{4}\tilde{3}\bar{2}\bar{1}}\big)^{*} &= \int \frac{\pi^2 d\beta}{4E}\Big(T_{12\bar{5}\bar{6}}\left(T_{\tilde{4}\tilde{3}\bar{5}\bar{6}}\right)^{*}+T_{12\tilde{\bar{5}}\tilde{\bar{6}}}\left(T_{\tilde{4}\tilde{3}\tilde{\bar{5}}\tilde{\bar{6}}}\right)^{*}\Big)+\mathcal{O}(g^4),\\
-iT_{1\tilde{2}\tilde{\bar{3}}\bar{4}}+i\big(T_{4\tilde{3}\tilde{\bar{2}}\bar{1}}\big)^{*} &= \int \frac{\pi^2 d\beta}{4E}\Big(T_{1\tilde{2}\tilde{\bar{5}}\bar{6}}\left(T_{4\tilde{3}\tilde{\bar{5}}\bar{6}}\right)^{*}+T_{1\tilde{2}\bar{5}\tilde{\bar{6}}}\left(T_{4\tilde{3}\bar{5}\tilde{\bar{6}}}\right)^{*}\Big)+\mathcal{O}(g^4). \qquad (6.14)
\end{aligned}
$$

Relations among the other prefactors can be obtained by applying discrete symmetries. The loop expansion for all of these is analogous to (6.13). As discussed in Sec. 3.3 there are further constraints for amplitudes in models with $\mathcal{N}>4$ supersymmetry. The above relations obey these constraints.

6.3 Tree Level

One can assign untwisted and twisted hypermultiplets to the legs of two-to-two scattering amplitudes in 8 different ways. As discussed in Sec. 3.2 they all have an equivalent matrix structure, but the prefactors are different in general.

The tree-level prefactors with alike hypermultiplets in the in/out channels have the following color structures, cf. Sec. 4.2,

$$
\begin{aligned}
T^{(0)}_{12\bar{3}\bar{4}} &= \Upsilon^{(0)}_{14,23}t^{(0)}_{12\bar{3}\bar{4}}+\Upsilon^{(0)}_{13,24}t^{(0)}_{12\bar{4}\bar{3}}, &\quad T^{(0)}_{12\tilde{\bar{3}}\tilde{\bar{4}}} &= \Upsilon^{(0)}_{12,\tilde{3}\tilde{4}}t^{(0)}_{12\tilde{\bar{3}}\tilde{\bar{4}}},\\
T^{(0)}_{\tilde{1}\tilde{2}\bar{3}\bar{4}} &= \Upsilon^{(0)}_{\tilde{1}\tilde{2},34}t^{(0)}_{\tilde{1}\tilde{2}\bar{3}\bar{4}}, &\quad T^{(0)}_{\tilde{1}\tilde{2}\tilde{\bar{3}}\tilde{\bar{4}}} &= \Upsilon^{(0)}_{\tilde{1}\tilde{4},\tilde{2}\tilde{3}}t^{(0)}_{\tilde{1}\tilde{2}\tilde{\bar{3}}\tilde{\bar{4}}}+\Upsilon^{(0)}_{\tilde{1}\tilde{3},\tilde{2}\tilde{4}}t^{(0)}_{\tilde{1}\tilde{2}\tilde{\bar{4}}\tilde{\bar{3}}}. \qquad (6.15)
\end{aligned}
$$

The coefficient functions t have been evaluated in field theory in Sec. 5.1,5.2

$$
\begin{aligned}
t^{(0)}_{12\bar{3}\bar{4}} &= -i\frac{\langle 12\rangle\langle \bar{2}4\rangle}{\langle 14\rangle}, &\quad t^{(0)}_{12\tilde{\bar{3}}\tilde{\bar{4}}} &= -i\langle \bar{3}\bar{4}\rangle,\\
t^{(0)}_{\tilde{1}\tilde{2}\bar{3}\bar{4}} &= -i\langle 12\rangle, &\quad t^{(0)}_{\tilde{1}\tilde{2}\tilde{\bar{3}}\tilde{\bar{4}}} &= -i\frac{\langle 2\bar{4}\rangle\langle \bar{3}\bar{4}\rangle}{\langle \bar{4}\bar{1}\rangle}. \qquad (6.16)
\end{aligned}
$$

For mixed hypermultiplets in the in/out channels the color structure of the prefactor reads

$$
\begin{aligned}
T^{(0)}_{1\tilde{2}\tilde{\bar{3}}\bar{4}} &= \Upsilon^{(0)}_{14,\tilde{2}\tilde{3}}t^{(0)}_{1\tilde{2}\tilde{\bar{3}}\bar{4}}, &\quad T^{(0)}_{1\tilde{2}\bar{3}\tilde{\bar{4}}} &= \Upsilon^{(0)}_{13,\tilde{2}\tilde{4}}t^{(0)}_{1\tilde{2}\tilde{\bar{4}}\bar{3}},\\
T^{(0)}_{\tilde{1}2\tilde{\bar{3}}\bar{4}} &= \Upsilon^{(0)}_{24,\tilde{1}\tilde{3}}t^{(0)}_{2\tilde{1}\tilde{\bar{3}}\bar{4}}, &\quad T^{(0)}_{\tilde{1}2\bar{3}\tilde{\bar{4}}} &= \Upsilon^{(0)}_{24,\tilde{1}\tilde{3}}t^{(0)}_{2\tilde{1}\bar{4}\tilde{\bar{3}}}, \qquad (6.17)
\end{aligned}
$$

with the single coefficient function

$$
t^{(0)}_{1\tilde{2}\tilde{\bar{3}}\bar{4}} = i\frac{\langle \bar{3}\bar{4}\rangle\langle \bar{3}4\rangle}{\langle \bar{2}\bar{3}\rangle}. \qquad (6.18)
$$

Using the spinor identities (3.7), it is straightforward to confirm the tree-level unitarity conditions

$$\begin{aligned}
\big(t^{(0)}_{43\bar{2}\bar{1}}\big)^* &= i\,\frac{\langle\bar{4}\bar{3}\rangle\langle 3\bar{1}\rangle}{\langle\bar{4}\bar{1}\rangle} = t^{(0)}_{12\bar{3}\bar{4}},\\
\big(t^{(0)}_{43\tilde{\bar{2}}\tilde{\bar{1}}}\big)^* &= i\langle 21\rangle = t^{(0)}_{\tilde{1}\tilde{2}\bar{3}\bar{4}},\\
\big(t^{(0)}_{\tilde{4}\tilde{3}\tilde{\bar{2}}\tilde{\bar{1}}}\big)^* &= i\,\frac{\langle\bar{3}1\rangle\langle 21\rangle}{\langle 14\rangle} = t^{(0)}_{\tilde{1}\tilde{2}\tilde{\bar{3}}\tilde{\bar{4}}},\\
\big(t^{(0)}_{4\tilde{3}\tilde{\bar{2}}\bar{1}}\big)^* &= -i\frac{\langle 21\rangle\langle 2\bar{1}\rangle}{\langle 32\rangle} = t^{(0)}_{1\tilde{2}\tilde{\bar{3}}\bar{4}}.
\end{aligned} \tag{6.19}$$

6.4 Pure Matter at One Loop

Next we consider one-loop unitarity for a model with only one type of hypermultiplet. First the color structure of the integrand in (6.13) is investigated

$$T^{(0)}_{12\bar{5}\bar{6}}T^{(0)}_{65\bar{3}\bar{4}} = 2\varUpsilon^{(1)\,\square}_{14,23}t^{(0)}_{12\bar{5}\bar{6}}t^{(0)}_{65\bar{3}\bar{4}} + 2\varUpsilon^{(1)\,\square}_{13,24}t^{(0)}_{12\bar{5}\bar{6}}t^{(0)}_{65\bar{4}\bar{3}} \tag{6.20}$$

We have used the crossing property $t^{(0)}_{12\bar{3}\bar{4}} = t^{(0)}_{21\bar{4}\bar{3}}$ and identified the color structure as a box $\varUpsilon^{(0)}_{16,25}\varUpsilon^{(0)}_{64,53} = \varUpsilon^{(1)\,\square}_{14,23}$, cf. (4.12). For convenience we have indicated the unitarity cuts in Fig. 11 on page 25. We evaluate and simplify the product of coefficient functions

$$\frac{\pi^2}{4E}\,t^{(0)}_{12\bar{5}\bar{6}}t^{(0)}_{65\bar{3}\bar{4}} = -i\pi^2\left(\frac{ie^{i\beta}}{e^{i\beta}-1} - \frac{ie^{i\beta}}{e^{i\beta}-e^{i\alpha}}\right)t^{(0)}_{12\bar{3}\bar{4}} - \frac{\pi^2p^2}{E}\,. \tag{6.21}$$

Note that the integrand has single poles at the angles $\beta = 0$ and $\beta = \alpha$. At these points the momenta of the intermediate particles $6, 5$ agree precisely with the ones of the ingoing particles $1, 2$ ($\beta = 0$) or outgoing particles $4, 3$ ($\beta = \alpha$). They originate from a gluon exchange with zero momentum. Fortunately the poles have exactly opposite residues and thus we can ignore their contribution altogether. The unitarity condition leads to an imaginary part which is independent of the overall scattering angle α

$$\begin{aligned}
-iT^{(1)}_{12\bar{3}\bar{4}} + i\big(T^{(1)}_{43\bar{2}\bar{1}}\big)^* &= \int\frac{\pi^2 d\beta}{4E}\,T^{(0)}_{12\bar{5}\bar{6}}T^{(0)}_{65\bar{3}\bar{4}} = -\frac{4\pi^3p^2}{E}\left(\varUpsilon^{(1)\,\square}_{14,23} + \varUpsilon^{(1)\,\square}_{13,24}\right)\\
&= -\frac{2\pi^3p^2}{E}\,\varUpsilon^{(1)\,\circ}_{12,34} = -\frac{\pi^3\langle 12\rangle\langle\bar{1}\bar{2}\rangle}{\sqrt{\langle 1\bar{2}\rangle\langle\bar{1}2\rangle}}\,\varUpsilon^{(1)\,\circ}_{12,34}.
\end{aligned} \tag{6.22}$$

The conversion between different color structures is due to the identity (4.10) and the final transformation $\sqrt{\langle 1\bar{2}\rangle\langle\bar{1}2\rangle} = 2E$ and $\langle 12\rangle\langle\bar{1}\bar{2}\rangle = 4p^2$ makes the result independent of a specific frame. This expression is in agreement with the field theory calculation (5.56) in Sec. 5.3: The expression (5.56) obeys $T^{(1)}_{12\bar{3}\bar{4}} = \big(T^{(1)}_{43\bar{2}\bar{1}}\big)^*$ except for branch cut discontinuities in the loop integrals. The integral $I_m(s)$ in (5.42) has a branch cut with discontinuity ($\sqrt{-s} = 2E$)

$$I_m(s-i\epsilon) - I_m(s+i\epsilon) = -\frac{2\pi^3}{\sqrt{-s}}\,\theta(\sqrt{-s}-2m) = -\frac{\pi^3}{E}\,\theta(E-m). \tag{6.23}$$

The bubble integrals $I_m(t), I_m(u)$ in the t- and u-channels clearly have no cuts in the physical region. For the one-loop expression (5.56)

$$T^{(1)}_{12\bar{3}\bar{4}} = \tfrac{i}{2}\langle 12\rangle\langle\bar{1}\bar{2}\rangle\,I_m(s-i\epsilon)\,\varUpsilon^{(1)\,\circ}_{12,34} + \ldots = 2ip^2\,I_m(s-i\epsilon)\,\varUpsilon^{(1)\,\circ}_{12,34} + \ldots \tag{6.24}$$

we thus get full agreement with unitarity (6.22). We can in fact do even better and compare the results before integrating over the loop momenta phase space. Taking the integrand of the Feynman diagram calculation (Section 5.3) and putting all momenta on-shell, one finds agreement with (6.21). It can be seen that the pole terms indeed come from the 'triangle' diagrams which involve gluon exchange whereas the finite piece gets contributions from both the 'triangle' and 'bubble' diagrams.

It is a curious fact that the full one-loop amplitude $T^{(1)}_{12\bar{3}\bar{4}}$ from field theory is a linear combination of massive scalar bubbles I_m without further rational parts. The coefficients of the bubbles can be reconstructed from unitarity in all channels. Expanding cuts using the inverse of (6.23), i.e. the minimal replacement, therefore yields the full one-loop amplitude from unitarity. It gives a hint that amplitudes in $\mathcal{N} = 4$ Chern–Simons theories may be cut constructible.

Finally we would like to consider the total cross section of the scattering process of two hypermultiplets. For that purpose we shall set $\alpha = 0$ so that the in and out states are the same. The cross section is proportional to

$$2\,\mathrm{Im}\,T^{(1)}_{12\bar{2}\bar{1}} = \int \frac{\pi^2 d\beta}{4E}\,\bigl|T^{(0)}_{12\bar{5}\bar{6}}\bigr|^2 = -\frac{2\pi^3 p^2}{E}\,\Upsilon^{(1)\,\circ}_{12,34}. \tag{6.25}$$

Curiously it appears that it is negative although the integrand itself is manifestly positive. Let us thus have a closer look at the integrand

$$\frac{\pi^2}{4E}\,\bigl|t^{(0)}_{12\bar{5}\bar{6}}\bigr|^2 = \frac{\pi^2}{4E}\,t^{(0)}_{12\bar{5}\bar{6}}t^{(0)}_{65\bar{3}\bar{4}} = \frac{\pi^2 E}{\sin^2(\frac{1}{2}\beta)} - \frac{\pi^2 p^2}{E} = \pi^2 E\,\cot^2(\tfrac{1}{2}\beta) + \frac{\pi^2 m^2}{E}\,. \tag{6.26}$$

In the last form it is manifestly positive. Due to a double pole the integral is infinite and needs to be regularized. A principal value prescription (or any other contour in the complex plane) will show that the $1/\sin^2$ term does not contribute. The finite remainder is however negative.

Essentially we have dropped a contribution from forward scattering where a gluon with zero momentum and zero energy is exchanged. Thus the peculiarity can be associated to an infrared divergence. It is in fact very similar to the collinear divergences encountered in Yang–Mills theories, but it is milder: It can only appear for gluons with zero momentum whereas for Yang–Mills it appears for all light-like gluons. The effects are nevertheless similar. The reason why the IR singularity cannot directly be seen in the result is related to the fact that in odd spacetime dimensions there are no divergences at one loop.

6.5 Mixed Matter at One Loop

We now consider one-loop scattering unitarity in a theory with both types of hypermultiplets using the relations (6.14).

The integrands of the first integral in (6.14) have the color structures

$$T^{(0)}_{12\bar{5}\bar{6}}T^{(0)}_{65\bar{3}\bar{4}} = 2\Upsilon^{(1)\,\square}_{14,23}t^{(0)}_{12\bar{5}\bar{6}}t^{(0)}_{65\bar{3}\bar{4}} + 2\Upsilon^{(1)\,\square}_{13,24}t^{(0)}_{12\bar{5}\bar{6}}t^{(0)}_{65\bar{4}\bar{3}}, \qquad T^{(0)}_{12\bar{\tilde{5}}\bar{\tilde{6}}}T^{(0)}_{\tilde{6}\tilde{5}\bar{3}\bar{4}} = \Upsilon^{(1)\,\tilde{\circ}}_{12,34}t^{(0)}_{12\bar{\tilde{5}}\bar{\tilde{6}}}t^{(0)}_{\tilde{6}\tilde{5}\bar{3}\bar{4}}, \tag{6.27}$$

where we have used the composition of trees to loop in (4.12), see also Fig. 11,12. The remaining coefficients evaluate to

$$\frac{\pi^2}{4E}\,t^{(0)}_{12\bar{5}\bar{6}}t^{(0)}_{65\bar{3}\bar{4}} = -i\pi^2\left(\frac{ie^{i\beta}}{e^{i\beta}-1} - \frac{ie^{i\beta}}{e^{i\beta}-e^{i\alpha}}\right)t^{(0)}_{12\bar{3}\bar{4}} - \frac{\pi^2 p^2}{E}\,, \qquad \frac{\pi^2}{4E}\,t^{(0)}_{12\bar{\tilde{5}}\bar{\tilde{6}}}t^{(0)}_{\tilde{6}\tilde{5}\bar{3}\bar{4}} = \frac{\pi^2 p^2}{E}\,. \tag{6.28}$$

In the integral over β the residues cancel and only the constant pieces remain

$$- iT^{(1)}_{12\bar{3}\bar{4}} + i\big(T^{(1)}_{43\bar{2}\bar{1}}\big)^* = -\frac{2\pi^3 p^2}{E} \left(\Upsilon^{(1)\,\circ}_{12,34} - \Upsilon^{(1)\,\tilde{\circ}}_{12,34}\right). \tag{6.29}$$

Again this result agrees with the field theory computation (5.61). Furthermore the field theory result is again a linear combination of massive scalar bubbles I_m hinting at cut constructibility.

The two integrands of the second integral in (6.14) have the following color structures, cf. Fig. 13

$$T^{(0)}_{12\bar{5}\bar{6}} T^{(0)}_{65\tilde{\bar{3}}\tilde{\bar{4}}} = -\Upsilon^{(1)\,\circ}_{12,\tilde{3}\tilde{4}} t^{(0)}_{12\bar{5}\bar{6}} t^{(0)}_{65\tilde{\bar{3}}\tilde{\bar{4}}}, \qquad T^{(0)}_{12\tilde{\bar{5}}\tilde{\bar{6}}} T^{(0)}_{\tilde{6}\tilde{5}\tilde{\bar{3}}\tilde{\bar{4}}} = -\Upsilon^{(1)\,\tilde{\circ}}_{12,\tilde{3}\tilde{4}} t^{(0)}_{12\tilde{\bar{5}}\tilde{\bar{6}}} t^{(0)}_{\tilde{6}\tilde{5}\tilde{\bar{3}}\tilde{\bar{4}}}. \tag{6.30}$$

The coefficient functions evaluate to

$$\begin{aligned} \frac{\pi^2}{4E}\, t^{(0)}_{12\bar{5}\bar{6}} t^{(0)}_{65\tilde{\bar{3}}\tilde{\bar{4}}} &= -i\pi^2 \left(\frac{ie^{i\beta}}{e^{i\beta}-1} - \frac{i(E+m)}{2E}\right) t^{(0)}_{12\tilde{\bar{3}}\tilde{\bar{4}}}\,, \\ \frac{\pi^2}{4E}\, t^{(0)}_{12\tilde{\bar{5}}\tilde{\bar{6}}} t^{(0)}_{\tilde{6}\tilde{5}\tilde{\bar{3}}\tilde{\bar{4}}} &= +i\pi^2 \left(\frac{ie^{i\beta}}{e^{i\beta}-e^{i\alpha}} - \frac{i(E+m)}{2E}\right) t^{(0)}_{12\tilde{\bar{3}}\tilde{\bar{4}}}. \end{aligned} \tag{6.31}$$

Note that again there are two poles with residues proportional to the tree-level amplitude. Here the poles originate from the two different terms in the integrand. It is not entirely clear how to perform the integral over the poles. For practical purposes, let us assume a principal value prescription. The unitarity integral then evaluates to

$$- iT^{(1)}_{12\tilde{\bar{3}}\tilde{\bar{4}}} + i\big(T^{(1)}_{\tilde{4}\tilde{3}\bar{2}\bar{1}}\big)^* = -\frac{i\pi^2(E+m)\langle\bar{3}\bar{4}\rangle}{2E} \left(\Upsilon^{(1)\,\circ}_{12,\tilde{3}\tilde{4}} - \Upsilon^{(1)\,\tilde{\circ}}_{12,\tilde{3}\tilde{4}}\right), \tag{6.32}$$

In the third integrand of (6.14) we find a single color structure

$$T^{(0)}_{1\tilde{2}\tilde{\bar{5}}\bar{6}} T^{(0)}_{6\tilde{5}\tilde{\bar{3}}\bar{4}} = T_{1\tilde{2}\bar{5}\tilde{\bar{6}}} T_{\tilde{6}5\tilde{\bar{3}}\bar{4}} = \Upsilon^{(1)\,\Box}_{14,\tilde{2}\tilde{3}} t^{(0)}_{1\tilde{2}\tilde{\bar{5}}\bar{6}} t^{(0)}_{6\tilde{5}\tilde{\bar{3}}\bar{4}}. \tag{6.33}$$

The coefficient function yields

$$\frac{\pi^2}{4E}\, t^{(0)}_{1\tilde{2}\tilde{\bar{5}}\bar{6}} t^{(0)}_{6\tilde{5}\tilde{\bar{3}}\bar{4}} = i\pi^2 \left(\frac{ie^{i\beta}}{e^{i\beta}-1} - \frac{ie^{i\beta}}{e^{i\beta}-e^{i\alpha}}\right) t^{(0)}_{1\tilde{2}\tilde{\bar{3}}\bar{4}}\,. \tag{6.34}$$

Here both poles are present and there is no constant piece. The integral thus vanishes exactly

$$- iT^{(1)}_{12\tilde{\bar{3}}\tilde{\bar{4}}} + i\big(T^{(1)}_{\tilde{4}\tilde{3}\bar{2}\bar{1}}\big)^* = 0, \tag{6.35}$$

Finally we would like to mention the curious fact that all three integrals vanish for model with $\mathcal{N} = 5, 6, 8$ extended supersymmetry where untwisted and twisted fields are equivalent $\Upsilon^{(1)\,\circ} = \Upsilon^{(1)\,\tilde{\circ}}$

$$- iT^{(1)}_{12\bar{3}\bar{4}} + i\big(T^{(1)}_{43\bar{2}\bar{1}}\big)^* = -iT^{(1)}_{12\tilde{\bar{3}}\tilde{\bar{4}}} + i\big(T^{(1)}_{\tilde{4}\tilde{3}\bar{2}\bar{1}}\big)^* = -iT^{(1)}_{1\tilde{2}\tilde{\bar{3}}\bar{4}} + i\big(T^{(1)}_{4\tilde{3}\tilde{\bar{2}}\bar{1}}\big)^* = 0. \tag{6.36}$$

It implies a remarkable feature that the scattering amplitudes at one loop are free from unitarity cuts. Moreover the field theory calculations in Sec. 5.3,5.4 suggest that the one-loop amplitudes vanish altogether $T^{(1)} = 0$. We shall discuss the further implications below.

6.6 Two-Loop Puzzle

The result of vanishing one-loop unitarity cuts (6.36) in $\mathcal{N} = 5, 6, 8$ supersymmetric models leads to a puzzle. The point is that the one-loop amplitudes must be rational functions of the momenta which in (highly) supersymmetric theories often implies that the amplitudes $T^{(1)}$ vanish altogether. Our field theory computations in Sec. 5.4 confirm this result for our model. In this case, however, unitarity (6.13) implies that the two-loop amplitudes are merely rational functions. Blindly following the argument leads to no loop corrections at all which is hard to believe.

There are good reasons to believe that the two-loop amplitudes from field theory are neither zero nor merely rational functions (cf. the discussion in the conclusions). This also leads to a much more realistic pattern of non-trivial corrections at higher loop orders. However, how does this match with our observation of vanishing one-loop contributions? The point is perhaps that our model does suffer from IR divergences in spite of having only massive physical particles. The zero mode of the Chern–Simons gauge field appears to cause the IR problems and in the above discussions we have seen several instances of such singularities. The singularities effectively require to regularize the model before computing quantum corrections.

The most reliable regulator arguably is dimensional regularization/reduction where loop integrals are performed in a spacetime of dimension $D = 3 - 2\epsilon$. Our results then imply merely that $T^{(1)} = 0 + \mathcal{O}(\epsilon)$. The integrand of the two-loop unitarity relation must be suppressed likewise $T^{(0)}T^{(1)} = 0 + \mathcal{O}(\epsilon)$. However, the integral can very well produce $1/\epsilon$ divergent terms such that the two-loop unitarity integral is finite $\int T^{(0)}T^{(1)} = \mathcal{O}(\epsilon^0)$ (or even divergent).

It would be very desirable to perform a two-loop computation in dimensional regularization based on both field theory and unitarity, and consequently compare the two results.

7 Conclusions

In this paper we have considered the spacetime S-matrices of various supersymmetric Chern–Simons matter theories focussing on the mass deformed $\mathcal{N} \geq 4$ theories whose super-Poincaré group contains the supergroup PSU(2|2). We have presented the tree-level and one-loop four particle amplitudes derived using both symmetry arguments and explicit perturbative calculations. This extended PSU(2|2) symmetry group is almost the same as that which occurs in the light-cone gauge fixed worldsheet theory of strings in $AdS_5 \times S^5$ or, equivalently, as the group of symmetries preserved by the ferromagnetic vacuum in the spin chain picture of maximally supersymmetric four dimensional Yang–Mills. As in that context, the superalgebra greatly constrains the two-to-two S-matrix as it interrelates every element and determines the entire matrix structure up to an overall factor. This leads to the intriguing observation that the two-body spacetime S-matrix of these three-dimensional Chern–Simons matter theories is the same as the two-dimensional spin-chain/worldsheet S-matrix that plays such a central role in the AdS/CFT correspondence. Furthermore this S-matrix is known to be equivalent to Shastry's R-matrix for the Hubbard model [32]. In many respects this similarity is purely formal and the kinematics are obviously quite different. For example, due to the different kinematical structure the Chern–Simons spacetime S-matrix, unlike the two-dimensional integrable S-matrices, does not satisfy the Yang–Baxter equation. Nonetheless it is certainly tempting to ask how far this analogy may be extended and whether there are structures in common with the integrable systems if only for certain kinematical regimes.

That all four particle scattering amplitudes should be related is perhaps not surprising for a theory with extended supersymmetry, indeed in four-dimensional $\mathcal{N} = 4$ Yang–Mills super-Ward identities imply similar relations. For the three-dimensional Chern–Simons theories with additional twisted matter there are of course additional unrelated amplitudes though in the cases where the supersymmetry is extended to $\mathcal{N} = 5, 6$ or 8 there are further relations between amplitudes. For the general $\mathcal{N} \geq 4$ case we have fixed, by explicit calculation, the tree level contribution to the overall factor undetermined by the global symmetries. We have discussed in detail the color structures that occur in the perturbative calculations as, especially beyond tree level, these calculations are substantially simplified by separately treating the color and kinematical contributions. Having the color structure in hand one can focus on the color ordered amplitudes, which in turn can be calculated efficiently using unitarity methods (whose validity we explicitly verified at one-loop).

For generic $\mathcal{N} = 4$ theories we find a one-loop contribution to the overall prefactor corresponding to a massive bubble diagram and, interestingly, when we include additional twisted matter we find an identical contribution but with the opposite sign. Thus when the twisted and untwisted matter are in the same gauge group representations, such as in the $\mathcal{N} = 5, 6, 8$ theories, we find that all the one-loop amplitudes vanish. It is possible that in these cases there is an additional symmetry related to the exchange of twisted and untwisted matter that explains this seeming coincidence; if this is so it is not unreasonable to ask whether this continues at higher odd orders in the perturbative expansion. However before going to three loops there remains the question of how to find non-trivial two-loop scattering; naive application of the unitarity method implies the vanishing of the two-loop amplitudes as a consequence of the vanishing of all one-loop amplitudes. In order to solve this puzzle it would be worthwhile to carry out an explicit two-loop calculation either using unitarity methods, but being careful to keep higher orders in the dimensional regularization parameter ϵ, or using off-shell methods.

Taking the mass deformation to zero, a limit that appears to be smooth, the theories we consider are in one-to-one correspondence with $\mathcal{N} \geq 4$ superconformal Chern–Simons theories which connects our results to the $\mathrm{AdS}_4/\mathrm{CFT}_3$ correspondence. In this context it is obvious to ask whether one can find, for the planar $\mathcal{N} = 6$ theory, a similar relationship between scattering amplitudes and Wilson loops as was found in the $\mathcal{N} = 4$ SYM/$AdS_5 \times S^5$ case. On general grounds [46, 47] we certainly expect that the IR asymptotics of the scattering amplitudes should be related to the behavior of light-like Wilson loops with a cusp which for conformal theories is related to the anomalous dimensions of specific twist operators [48] (see also [49–51,4]). Of course for $\mathcal{N} = 4$ Yang–Mills the relationship between amplitudes and Wilson loops goes beyond the IR divergent piece to include the finite contributions. The four particle amplitudes display an iterative structure in perturbation theory which can be combined into an all order exponential form, the finite piece of which is also governed by the cusp anomalous dimension [47, 52]. It would be interesting to see if the same is true for the Chern–Simons theory or indeed whether the even more general relation between Wilson loops and MHV amplitudes, see for example [2], can be generalized. A moderate indication is that the NLO scattering amplitudes vanish identically for $\mathcal{N} > 4$ SCS and the same is true for NLO Wilson loops [53], see also [54]. Relatedly it may be worthwhile to look for a version of the dual conformal symmetry found in four-dimensional amplitudes [6]. Given this possible relation between scattering amplitudes/Wilson loops/twist operators and previous results [20, 21, 38] on the anomalous dimensions of twist operators in the planar limit we might expect that the two-loop scattering amplitudes in $\mathcal{N} = 6$ CS are related to one-loop amplitudes in four-dimensional $\mathcal{N} = 4$ YM. Another hint that this may indeed be the case is the similarity of the one-loop correction to the CS gluon propagator to the YM propagator. Furthermore, at strong coupling the fact that the relevant, non-compact, part of the geometry dual to the $\mathcal{N} = 6$ theory, [20], is similar leads one to believe that such a relationship between the four-dimensional YM and three dimensional CS is plausible. Certainly in the analysis of the spectrum of spinning strings/twist operators marked similarity to the $AdS_5 \times S^5$ case is apparent and the proof of the relationship between open strings dual to Wilson loops and spinning strings dual to twist operators via analytic continuation [50] goes through exactly as in the AdS_5 geometry. However while classical string solution dual to four particle scattering amplitudes is almost identical [1, 5] it should be mentioned that the full geometry felt by the string, particularly for the fermions, is different and it is not certain that the arguments relating Wilson loops to scattering amplitudes at strong coupling via (fermionic) T-duality will be valid in this theory [1, 23, 22].

The particle representations that are used in the present paper are merely the simplest representations of the symmetry algebra. It is conceivable that the mass-deformed CS model has bound states which transform in larger representations. It might be interesting to determine the spectrum of such composite particles and also compute their scattering matrices (by means of unitarity).

Finally, in attempting to answer questions involving higher loops, amplitudes involving larger numbers of particles or bound states the global symmetries will naturally be less restrictive and there will be more independent elements. It may therefore be useful to see whether one can adapt the methods of recursion relations [10], generalized unitarity [12] and the use of complex momenta [8, 13] to the current context.

The analysis carried out in this paper raises several interesting questions about SCS theories, their relationship to Yang–Mills theories and the $\mathrm{AdS}_4/\mathrm{CFT}_3$ correspondence. There are of course, several issues pertaining specifically to a better understanding scattering processes in $\mathcal{N} \geq 4$ SCS theories that require further studies. As explained in the preceding sections, the resolution of the puzzle regarding a two loop contribution to the scattering matrix that is neither zero nor purely rational, and an understanding of the relationship between scattering amplitudes and Wilson loops, would yield vital insights into the gauge theories in question as well as the $\mathrm{AdS}_4/\mathrm{CFT}_3$ correspondence.

Looking beyond the immediate problems and puzzles posed by this paper, we would like to point out that possible connections between mass deformations of Chern–Simons models and Yang–Mills theories in three spacetime dimensions, are worth investigating in greater detail. In the case of massless Chern–Simons models, both the maximally supersymmetric $\mathcal{N} = 8$ BLG theory, as well as the non-supersymmetric pure Chern–Simons theory are expected describe the strongly coupled dynamics of $\mathcal{N} = 8$ and $\mathcal{N} = 0$ Yang–Mills theories respectively. In the later case, the vacuum wave functionals of the Chern–Simons theory, namely the Wess–Zumino–Witten model, and that of the strongly coupled gluonic theory are known to be the same: a fact that can be established using a gauge invariant Hamiltonian formulation of pure Yang–Mills theory [55]. In the maximally supersymmetric case, the strongly coupled Yang–Mills theory is expected to flow to the SCS theory at strong coupling due to the standard dualities between D2 and M2 brane dynamics [56]. Direct evidence relating the Lagrangians of the two theories via a Higgs mechanism has also been uncovered in [57]. However, the relationship between mass-deformed SCS theories and Yang–Mills theories, if any, remains unclear. In this context, it is worth noting that, in the special case of three spacetime dimensions, it is possible to carry out mass-deformations of super Yang–Mills theories on $\mathbb{R}^{1,2}$ by using a non-local, gauge invariant mass-term for the gluons, and ordinary quadratic mass-terms for the matter fields. Appropriately mass-deformed super Yang–Mills theories can also be shown to be related to matrix models in plane wave type backgrounds by the methods of dimensional reduction [58]. It is thus worth investigating if the interrelationships between massless SCS and super Yang–Mills theories has a parallel in connections between mass deformed Chern–Simons models and massive Yang–Mills theories of the type investigated in [58].

On a related note, it might be interesting to investigate the role of mass-deformed algebras in constraining the spacetime physics of other gauge theories, such as $\mathcal{N} = 8$ supersymmetric Yang–Mills theory on $\mathbb{R} \times S^2$ and other related Yang–Mills Chern–Simons theories constructed in [59, 33].

Acknowledgements. We would like to thank L. Dixon, J. Henn, J. Plefka, S.-J. Rey and R. Roiban for many useful and enlightening discussions.

A Conventions

A.1 Spacetime

We give a brief summary of spacetime index conventions used in this paper following the conventions of HLLLP [16] to a large extent.

Vectors. For vector indices we choose the signature of spacetime and the antisymmetric tensor according to

$$\eta^{\mu\nu} = \mathrm{diag}(-,+,+), \qquad \eta_{\mu\nu} = \mathrm{diag}(-,+,+), \qquad \varepsilon^{012} = +1. \qquad \varepsilon_{012} = -1. \quad \text{(A.1)}$$

Spinors. We start by defining a basis of real symmetric and antisymmetric 2×2 matrices:

$$[\sigma^\mu]^{\alpha\beta} = \begin{pmatrix} - & 0 \\ 0 & - \end{pmatrix}, \begin{pmatrix} + & 0 \\ 0 & - \end{pmatrix}, \begin{pmatrix} 0 & - \\ - & 0 \end{pmatrix}, \qquad \varepsilon^{\alpha\beta} = \begin{pmatrix} 0 & + \\ - & 0 \end{pmatrix}. \quad \text{(A.2)}$$

The conjugate basis with lower indices is defined by $\sigma^\mu_{\alpha\beta} = \varepsilon_{\alpha\gamma}\varepsilon_{\beta\delta}\sigma^{\mu,\gamma\delta}$ and $\varepsilon_{\alpha\beta} = \varepsilon_{\alpha\gamma}\varepsilon_{\beta\delta}\varepsilon^{\gamma\delta}$

$$[\sigma^\mu]_{\alpha\beta} = \begin{pmatrix} - & 0 \\ 0 & - \end{pmatrix}, \begin{pmatrix} - & 0 \\ 0 & + \end{pmatrix}, \begin{pmatrix} 0 & + \\ + & 0 \end{pmatrix}, \qquad \varepsilon_{\alpha\beta} = \begin{pmatrix} 0 & + \\ - & 0 \end{pmatrix}. \quad \text{(A.3)}$$

If we lower only one spinor index $[\gamma^\mu]_\alpha{}^\beta = -\varepsilon_{\alpha\gamma}[\sigma^\mu]^{\gamma\beta} = \varepsilon^{\beta\gamma}[\sigma^\mu]_{\alpha\gamma}$ we obtain

$$[\gamma^\mu]_\alpha{}^\beta = \begin{pmatrix} 0 & + \\ - & 0 \end{pmatrix}, \begin{pmatrix} 0 & + \\ + & 0 \end{pmatrix}, \begin{pmatrix} + & 0 \\ 0 & - \end{pmatrix} = i\sigma^2, \sigma^1, \sigma^3, \tag{A.4}$$

where the latter three σ^k refer to the standard Pauli matrices. The gamma matrices obey the algebra

$$\gamma^\mu\gamma^\nu = \eta^{\mu\nu} + \varepsilon^{\mu\nu\rho}\gamma_\rho. \tag{A.5}$$

Spinors ψ will usually carry a lower spinor index ψ_α so that one can conveniently multiply gamma matrices to their left, $\gamma_\mu\psi$. To close off a sequence of gamma matrices from the left one can use a spinor ψ followed by ε to raise the index $\psi_\alpha\varepsilon^{\alpha\beta}$. Barred spinors $\bar\psi = \psi^*\varepsilon$ have an upper index $\bar\psi^\alpha = \psi^*_\beta\varepsilon^{\beta\alpha}$ and one can multiply gamma matrices to their right, $\bar\psi\gamma_\mu$.

To convert between vectors and bi-spinors we use the map

$$p_{\alpha\beta} = p_\mu\sigma^\mu_{\alpha\beta} = \begin{pmatrix} -p_0 - p_1 & p_2 \\ p_2 & -p_0 + p_1 \end{pmatrix}, \qquad p_\mu = -\tfrac{1}{2}\sigma_\mu^{\alpha\beta}p_{\alpha\beta}. \tag{A.6}$$

Conversion from HLLLP notation. For compatibility reasons we adopt the spinor conventions used in HLLLP [16] with the only exception of the $\varepsilon_{\alpha\beta}$ symbol with lower indices. The reason is that lowering both indices of $\varepsilon^{\alpha\beta}$ with two $\varepsilon^{\mathrm{HLLLP}}_{\alpha\beta}$ according to the prescription in [16] leads to $-\varepsilon^{\mathrm{HLLLP}}_{\alpha\beta}$. We thus choose the opposite sign for $\varepsilon_{\alpha\beta}$. This is consistent with the fact that $\det\varepsilon = +1$: The relative normalization of totally antisymmetric tensors ε with upper and lower indices should be determined by the determinant of the matrix to raise or lower indices (here: the same ε).

In order to avoid sign confusions we will always raise or lower spinor indices explicitly by means of ε tensors. All the symbols will have a definite position of spinor indices; we commonly use lower indices which are contracted by $\varepsilon^{\alpha\beta}$.

Thus, the conversion from HLLLP [16] to our notation consists of the following two replacements

$$\psi^\alpha_{\mathrm{HLLLP}} = \varepsilon^{\alpha\beta}\psi_\beta, \qquad \varepsilon^{\mathrm{HLLLP}}_{\alpha\beta} = -\varepsilon_{\alpha\beta}. \tag{A.7}$$

A.2 Polarization Spinors

Consider now the Dirac equation

$$(\gamma^\mu\partial_\mu - m)\psi = 0. \tag{A.8}$$

It is solved by $\psi = \exp(+ip_\mu x^\mu)u(+p)$ and $\psi = \exp(-ip_\mu x^\mu)u(-p)$ with the polarization spinors

$$u(p) = \frac{1}{\sqrt{p_0 - p_1}}\begin{pmatrix} p_2 - im \\ p_1 - p_0 \end{pmatrix}, \qquad v(p) = \frac{1}{\sqrt{p_0 - p_1}}\begin{pmatrix} p_2 + im \\ p_1 - p_0 \end{pmatrix}. \tag{A.9}$$

These are normalized such that

$$v_\alpha(p)\,u_\beta(p) = -p_{\alpha\beta} - im\varepsilon_{\alpha\beta}. \tag{A.10}$$

Obviously, the Dirac equation with opposite mass

$$(\gamma^\mu \partial_\mu + m)\tilde{\psi} = 0 \tag{A.11}$$

has the solutions $\tilde{\psi} = \exp(+ip_\mu x^\mu)v(+p)$ and $\tilde{\psi} = \exp(-ip_\mu x^\mu)v(-p)$ with u replaced by v interchanged. By construction, it is also clear that for inverted momentum one obtains

$$u(-p) = i\,\mathrm{sign}(p_0)\,v(p), \qquad v(-p) = i\,\mathrm{sign}(p_0)\,u(p). \tag{A.12}$$

Finally, let us note that the two polarization spinors are related complex conjugation

$$u(p)^* = -i\,u(-p^*), \qquad v(p)^* = -i\,v(-p^*). \tag{A.13}$$

A.3 Completeness Relations and Conversion

We list two completeness relations for symmetrized bispinors

$$\sigma_\mu^{\alpha\beta}\sigma^\mu_{\gamma\delta} = -\delta^\alpha_\gamma \delta^\beta_\delta - \delta^\alpha_\delta \delta^\beta_\gamma, \qquad \varepsilon^{\alpha\beta}\varepsilon_{\gamma\delta} = \delta^\alpha_\gamma \delta^\beta_\delta - \delta^\alpha_\delta \delta^\beta_\gamma. \tag{A.14}$$

These can be used to convert between vectors and symmetric bispinors

$$a_{\alpha\beta} = \sigma^\mu_{\alpha\beta} a_\mu, \qquad a_\mu = -\tfrac{1}{2}\sigma_\mu^{\alpha\beta} a_{\alpha\beta}. \tag{A.15}$$

Furthermore in three dimensions vectors and two-forms are equivalent

$$a_\rho = \tfrac{1}{2}\varepsilon_{\mu\nu\rho} a^{\mu\nu}, \qquad a_{\mu\nu} = -\varepsilon_{\mu\nu\rho} a^\rho. \tag{A.16}$$

Thus we can also convert directly between symmetric bispinors and two-forms

$$a_{\alpha\beta} = -\tfrac{1}{2}\sigma^\mu_{\alpha\gamma}\varepsilon^{\gamma\delta}\sigma^\nu_{\delta\beta} a_{\mu\nu}, \qquad a_{\mu\nu} = -\tfrac{1}{2}\sigma_\mu^{\alpha\gamma}\varepsilon_{\gamma\delta}\sigma_\nu^{\delta\beta} a_{\alpha\beta}. \tag{A.17}$$

B The $\mathcal{N} = 4$ Chern–Simons Model

In this appendix we define the $\mathcal{N} = 4$ supersymmetric Chern–Simons model and give a summary of its symmetries.

B.1 Definitions

We start by listing the basic fields, symbols and indices that appear in the model.

Types of Indices.

- $M, N, P, \ldots$: gauge adjoint indices,
- $A, B, C, \ldots$: gauge untwisted representation indices,
- $\tilde{A}, \tilde{B}, \tilde{C}, \ldots$: gauge twisted representation indices,
- $\alpha, \beta, \gamma, \ldots$: spacetime spinor indices (cf. App. A.1),
- $\mu, \nu, \rho, \ldots$: spacetime vector indices (cf. App. A.1),

- $a, b, c, \ldots$: flavor indices of first $\mathfrak{su}(2)$,
- $\dot{a}, \dot{b}, \dot{c}, \ldots$: flavor indices of second $\mathfrak{su}(2)$,
- $\tilde{a}, \tilde{b}, \tilde{c}, \ldots$: flavor indices of third $\mathfrak{su}(2)$,
- $\hat{a}, \hat{b}, \hat{c}, \ldots$: flavor indices of fourth $\mathfrak{su}(2)$.

Gauge Invariant Symbols.

- gauge algebra structure constants $F^M_{NP} = -F^M_{PN}$,
- gauge algebra (Cartan–Killing) metric $K_{MN} = K_{NM}$,
- untwisted (twisted) representation T^A_{MB} ($\tilde{T}^{\tilde{A}}_{M\tilde{B}}$),
- untwisted (twisted) moments $M^M_{AB} = M^M_{BA}$ ($\tilde{M}^M_{\tilde{A}\tilde{B}} = \tilde{M}^M_{\tilde{B}\tilde{A}}$),
- untwisted (twisted) metric $L_{AB} = -L_{BA}$ ($\tilde{L}_{\tilde{A}\tilde{B}} = -\tilde{L}_{\tilde{B}\tilde{A}}$).

The untwisted structure constants $F^M_{NP}, T^A_{MB}, M^M_{AB}$ obey the Jacobi identities of a Lie superalgebra

$$\begin{aligned}
0 &= F^M_{RN} F^N_{PQ} + F^M_{RP} F^R_{QN} + F^M_{RQ} F^R_{NP}, \\
0 &= T^A_{CM} T^C_{NB} - T^A_{CN} T^C_{MB} + T^A_{PB} F^P_{MN}, \\
0 &= F^M_{NP} M^P_{AB} + M^M_{AC} T^C_{NB} + M^M_{BC} T^C_{NA}, \\
0 &= T^A_{MB} M^M_{CD} + T^A_{MC} M^M_{DB} + T^A_{MD} M^M_{BC}.
\end{aligned} \tag{B.1}$$

Furthermore the compatibility of structure constants and metric implies the relation

$$L_{AC} T^C_{MB} = K_{MN} M^N_{AB}. \tag{B.2}$$

The twisted constants $F^M_{NP}, \tilde{T}^{\tilde{A}}_{M\tilde{B}}, \tilde{M}^M_{\tilde{A}\tilde{B}}, \tilde{L}_{\tilde{A}\tilde{B}}$ obey the same Lie superalgebra relations. The Lie superalgebra for the twisted sector need not be isomorphic to the one for the "untwisted" sector; the even parts defined through F^M_{NP} must be isomorphic, but the odd parts $\tilde{T}^{\tilde{A}}_{M\tilde{B}}, \tilde{M}^M_{\tilde{A}\tilde{B}}, \tilde{L}_{\tilde{A}\tilde{B}}$ can differ.

Fields and Combinations. The most general non-abelian model is based upon the following five types of fields:

- gauge field $\mathcal{A}^M_{\alpha\beta} = \sigma^\mu_{\alpha\beta} A^M_\mu$,
- untwisted scalars ϕ^A_a and fermions $\psi^A_{\alpha\dot{b}}$,
- twisted scalars $\tilde{\phi}^{\tilde{A}}_{\dot{a}}$ and fermions $\tilde{\psi}^{\tilde{A}}_{\alpha b}$.

The following combinations of fields (field strength, covariant derivatives, moments, currents) have proven useful

$$
\begin{aligned}
\mathcal{F}^M_{\alpha\beta} &= -\tfrac{1}{2}\varepsilon^{\gamma\delta}\partial_{\alpha\gamma}\mathcal{A}^M_{\beta\delta} - \tfrac{1}{2}\varepsilon^{\gamma\delta}\partial_{\beta\gamma}\mathcal{A}^M_{\alpha\delta} - \tfrac{1}{2}\varepsilon^{\gamma\delta}F^M_{NP}\mathcal{A}^N_{\alpha\gamma}\mathcal{A}^P_{\beta\delta}, \\
\mathcal{D}_{\alpha\beta}\mathcal{X}^A &= \partial_{\alpha\beta}\mathcal{X}^A + T^K_{MB}\mathcal{A}^M_{\alpha\beta}\mathcal{X}^B, \qquad \mathcal{D}_{\alpha\beta}\mathcal{X}^{\tilde{A}} = \partial_{\alpha\beta}\mathcal{X}^{\tilde{A}} + \tilde{T}^{\tilde{A}}_{M\tilde{B}}\mathcal{A}^M_{\alpha\beta}\mathcal{X}^{\tilde{B}}, \\
\mathcal{M}^M_{ab} &= M^M_{AB}\phi^A_a\phi^B_b, \qquad \tilde{\mathcal{M}}^M_{\dot{a}\dot{b}} = \tilde{M}^M_{\tilde{A}\tilde{B}}\tilde{\phi}^{\tilde{A}}_{\dot{a}}\tilde{\phi}^{\tilde{B}}_{\dot{b}}, \\
\mathcal{J}^M_{\alpha b\dot{c}} &= M^M_{AB}\phi^A_b\psi^B_{\alpha\dot{c}}, \qquad \tilde{\mathcal{J}}^M_{\alpha\dot{b}c} = \tilde{M}^M_{\tilde{A}\tilde{B}}\tilde{\phi}^{\tilde{A}}_{\dot{b}}\tilde{\psi}^{\tilde{B}}_{\alpha c}.
\end{aligned} \tag{B.3}
$$

With respect to the conventions in [16] we have rescaled the fields for our convenience as follows:

$$
\begin{aligned}
\phi^A_a &\to +\sqrt{4\pi}\, q^A_a, \qquad & \tilde{\phi}^{\tilde{A}}_{\dot{a}} &\to -\sqrt{4\pi}\, \tilde{q}^{\tilde{A}}_{\dot{a}}, \\
\psi^A_{\alpha\dot{b}} &\to +\sqrt{4\pi}\, \psi^A_{\alpha\dot{b}}, \qquad & \tilde{\psi}^{\tilde{A}}_{\alpha b} &\to +\sqrt{4\pi}\, \tilde{\psi}^{\tilde{A}}_{\alpha b}.
\end{aligned} \tag{B.4}
$$

B.2 Action

The action which appears in the path integral as e^{iS} is defined as $S = \int d^3x\, \mathcal{L}$ with the Lagrangian:

$$
\begin{aligned}
(4\pi/k)\mathcal{L} = &-\tfrac{1}{2}K_{MN}\varepsilon^{\beta\gamma}\varepsilon^{\delta\kappa}\varepsilon^{\lambda\alpha}\mathcal{A}^M_{\alpha\beta}\partial_{\gamma\delta}\mathcal{A}^N_{\kappa\lambda} \\
&- \tfrac{1}{6}K_{MN}F^N_{PQ}\varepsilon^{\beta\gamma}\varepsilon^{\delta\kappa}\varepsilon^{\lambda\alpha}\mathcal{A}^M_{\alpha\beta}\mathcal{A}^P_{\gamma\delta}\mathcal{A}^Q_{\kappa\lambda} \\
&+ \tfrac{1}{4}L_{AB}\varepsilon^{ab}\varepsilon^{\gamma\kappa}\varepsilon^{\delta\lambda}\mathcal{D}_{\gamma\delta}\phi^A_a\mathcal{D}_{\kappa\lambda}\phi^B_b \\
&- \tfrac{1}{2}m^2L_{AB}\varepsilon^{ab}\phi^A_a\phi^B_b \\
&+ \tfrac{i}{2}L_{AB}\varepsilon^{\dot{a}\dot{b}}\varepsilon^{\gamma\kappa}\varepsilon^{\lambda\delta}\psi^A_{\gamma\dot{a}}\mathcal{D}_{\kappa\lambda}\psi^B_{\delta\dot{b}} \\
&+ \tfrac{i}{2}mL_{AB}\varepsilon^{\dot{a}\dot{b}}\varepsilon^{\gamma\delta}\psi^A_{\gamma\dot{a}}\psi^B_{\delta\dot{b}} \\
&+ \tfrac{1}{4}\tilde{L}_{\tilde{A}\tilde{B}}\varepsilon^{\dot{a}\dot{b}}\varepsilon^{\gamma\kappa}\varepsilon^{\delta\lambda}\mathcal{D}_{\gamma\delta}\tilde{\phi}^{\tilde{A}}_{\dot{a}}\mathcal{D}_{\kappa\lambda}\tilde{\phi}^{\tilde{B}}_{\dot{b}} \\
&- \tfrac{1}{2}m^2\tilde{L}_{\tilde{A}\tilde{B}}\varepsilon^{\dot{a}\dot{b}}\tilde{\phi}^{\tilde{A}}_{\dot{a}}\tilde{\phi}^{\tilde{B}}_{\dot{b}} \\
&+ \tfrac{i}{2}\tilde{L}_{\tilde{A}\tilde{B}}\varepsilon^{ab}\varepsilon^{\gamma\kappa}\varepsilon^{\lambda\delta}\tilde{\psi}^{\tilde{A}}_{\gamma a}\mathcal{D}_{\kappa\lambda}\tilde{\psi}^{\tilde{B}}_{\delta b} \\
&- \tfrac{i}{2}m\tilde{L}_{\tilde{A}\tilde{B}}\varepsilon^{ab}\varepsilon^{\gamma\delta}\tilde{\psi}^{\tilde{A}}_{\gamma a}\tilde{\psi}^{\tilde{B}}_{\delta b} \\
&+ \tfrac{i}{4}K_{MN}\varepsilon^{ab}\varepsilon^{\dot{c}\dot{d}}\varepsilon^{\kappa\lambda}\mathcal{J}^M_{\kappa a\dot{c}}\mathcal{J}^N_{\lambda b\dot{d}} \\
&+ \tfrac{i}{4}K_{MN}\varepsilon^{\dot{a}\dot{b}}\varepsilon^{cd}\varepsilon^{\kappa\lambda}\tilde{\mathcal{J}}^M_{\kappa\dot{a}c}\tilde{\mathcal{J}}^N_{\lambda\dot{b}d} \\
&+ iK_{MN}\varepsilon^{ad}\varepsilon^{\dot{b}\dot{c}}\varepsilon^{\kappa\lambda}\mathcal{J}^M_{\kappa a\dot{b}}\tilde{\mathcal{J}}^N_{\lambda\dot{c}d} \\
&- \tfrac{i}{4}L_{AC}T^C_{MB}\varepsilon^{\dot{a}\dot{c}}\varepsilon^{\dot{b}\dot{d}}\varepsilon^{\kappa\lambda}\tilde{\mathcal{M}}^M_{\dot{a}\dot{b}}\psi^A_{\kappa\dot{c}}\psi^B_{\lambda\dot{d}} \\
&- \tfrac{i}{4}\tilde{L}_{\tilde{A}\tilde{C}}\tilde{T}^{\tilde{C}}_{M\tilde{B}}\varepsilon^{ac}\varepsilon^{bd}\varepsilon^{\kappa\lambda}\mathcal{M}^M_{ab}\tilde{\psi}^{\tilde{A}}_{\kappa c}\tilde{\psi}^{\tilde{B}}_{\lambda d} \\
&+ \tfrac{1}{6}mK_{MN}\varepsilon^{bc}\varepsilon^{da}\mathcal{M}^M_{ab}\mathcal{M}^N_{cd} \\
&- \tfrac{1}{6}mK_{MN}\varepsilon^{\dot{b}\dot{c}}\varepsilon^{\dot{d}\dot{a}}\tilde{\mathcal{M}}^M_{\dot{a}\dot{b}}\tilde{\mathcal{M}}^N_{\dot{c}\dot{d}} \\
&+ \tfrac{1}{96}K_{MN}F^N_{PQ}\varepsilon^{bc}\varepsilon^{de}\varepsilon^{fa}\mathcal{M}^M_{ab}\mathcal{M}^P_{cd}\mathcal{M}^Q_{ef} \\
&+ \tfrac{1}{96}K_{MN}F^N_{PQ}\varepsilon^{\dot{b}\dot{c}}\varepsilon^{\dot{d}\dot{e}}\varepsilon^{\dot{f}\dot{a}}\tilde{\mathcal{M}}^M_{\dot{a}\dot{b}}\tilde{\mathcal{M}}^P_{\dot{c}\dot{d}}\tilde{\mathcal{M}}^Q_{\dot{e}\dot{f}} \\
&+ \tfrac{1}{16}\tilde{L}_{\tilde{C}\tilde{D}}\tilde{T}^{\tilde{C}}_{M\tilde{A}}\tilde{T}^{\tilde{D}}_{N\tilde{B}}\varepsilon^{\dot{a}\dot{b}}\varepsilon^{ce}\varepsilon^{df}\mathcal{M}^M_{cd}\mathcal{M}^N_{ef}\tilde{\phi}^{\tilde{A}}_{\dot{a}}\tilde{\phi}^{\tilde{B}}_{\dot{b}} \\
&+ \tfrac{1}{16}L_{CD}T^C_{MA}T^D_{NB}\varepsilon^{ab}\varepsilon^{\dot{c}\dot{e}}\varepsilon^{\dot{d}\dot{f}}\tilde{\mathcal{M}}^M_{\dot{c}\dot{d}}\tilde{\mathcal{M}}^N_{\dot{e}\dot{f}}\phi^A_a\phi^B_b.
\end{aligned} \tag{B.5}
$$

This is equivalent to the action presented in [16] after using the dictionary in (B.4).

B.3 Symmetries

Here we collect the global symmetries of the model.

Rotations. The $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ flavor and $\mathfrak{sl}(2)$ Lorentz rotations act on the corresponding indices of some field $\mathcal{X}$ as follows

$$\begin{aligned}
\mathfrak{R}_{ab}\mathcal{X}_c &= \tfrac{i}{2}\varepsilon_{bc}\mathcal{X}_a + \tfrac{i}{2}\varepsilon_{ac}\mathcal{X}_b, \\
\dot{\mathfrak{R}}_{\dot{a}\dot{b}}\mathcal{X}_{\dot{c}} &= \tfrac{i}{2}\varepsilon_{\dot{b}\dot{c}}\mathcal{X}_{\dot{a}} + \tfrac{i}{2}\varepsilon_{\dot{a}\dot{c}}\mathcal{X}_{\dot{b}}, \\
\mathfrak{L}_{\alpha\beta}\mathcal{X}_\gamma &= \tfrac{1}{2}\varepsilon_{\beta\gamma}\mathcal{X}_\alpha + \tfrac{1}{2}\varepsilon_{\alpha\gamma}\mathcal{X}_\beta.
\end{aligned} \tag{B.6}$$

Translations. The momentum generators act by covariant derivatives

$$\begin{aligned}
\mathfrak{P}_{\alpha\beta}\mathcal{X}^A &= -i\mathcal{D}_{\alpha\beta}\mathcal{X}^A, \\
\mathfrak{P}_{\alpha\beta}\mathcal{X}^{\tilde{A}} &= -i\mathcal{D}_{\alpha\beta}\mathcal{X}^{\tilde{A}}, \\
\mathfrak{P}_{\alpha\beta}\mathcal{A}^M_{\gamma\delta} &= \tfrac{i}{2}\varepsilon_{\beta\delta}\mathcal{F}^M_{\alpha\gamma} + \tfrac{i}{2}\varepsilon_{\beta\gamma}\mathcal{F}^M_{\alpha\delta} + \tfrac{i}{2}\varepsilon_{\alpha\delta}\mathcal{F}^M_{\beta\gamma} + \tfrac{i}{2}\varepsilon_{\alpha\gamma}\mathcal{F}^M_{\beta\delta}.
\end{aligned} \tag{B.7}$$

Supersymmetry. Supersymmetry generators act on the fields according to the rules

$$\begin{aligned}
\mathfrak{Q}_{\alpha b\dot{c}}\phi^A_d &= \varepsilon_{bd}\psi^A_{\alpha\dot{c}}, \\
\mathfrak{Q}_{\alpha b\dot{c}}\tilde{\phi}^{\tilde{A}}_{\dot{d}} &= \varepsilon_{\dot{c}\dot{d}}\tilde{\psi}^{\tilde{A}}_{\alpha b}, \\
\mathfrak{Q}_{\alpha b\dot{c}}\psi^A_{\delta\dot{e}} &= i\varepsilon_{\dot{c}\dot{e}}\big(\mathcal{D}_{\alpha\delta} - m\varepsilon_{\alpha\delta}\big)\phi^A_b + i\varepsilon_{\alpha\delta}\big(\tfrac{1}{6}\mathcal{M}^M_{bf}\varepsilon^{fg}\varepsilon_{\dot{c}\dot{e}} - \tfrac{1}{2}\delta^g_b\tilde{\mathcal{M}}^M_{\dot{c}\dot{e}}\big)T^A_{MB}\phi^B_g, \\
\mathfrak{Q}_{\alpha b\dot{c}}\tilde{\psi}^{\tilde{A}}_{\delta e} &= i\varepsilon_{be}\big(\mathcal{D}_{\alpha\delta} + m\varepsilon_{\alpha\delta}\big)\tilde{\phi}^{\tilde{A}}_{\dot{c}} + i\varepsilon_{\alpha\delta}\big(\tfrac{1}{6}\varepsilon_{be}\tilde{\mathcal{M}}^M_{\dot{c}\dot{f}}\varepsilon^{\dot{f}\dot{g}} - \tfrac{1}{2}\mathcal{M}^M_{be}\delta^{\dot{g}}_{\dot{c}}\big)\tilde{T}^{\tilde{A}}_{M\tilde{B}}\tilde{\phi}^{\tilde{B}}_{\dot{g}}, \\
\mathfrak{Q}_{\alpha b\dot{c}}\mathcal{A}^A_{\delta\epsilon} &= \tfrac{1}{2}\varepsilon_{\alpha\delta}\mathcal{J}^M_{\epsilon b\dot{c}} + \tfrac{1}{2}\varepsilon_{\alpha\epsilon}\mathcal{J}^M_{\delta b\dot{c}} + \tfrac{1}{2}\varepsilon_{\alpha\delta}\tilde{\mathcal{J}}^M_{\epsilon\dot{c}b} + \tfrac{1}{2}\varepsilon_{\alpha\epsilon}\tilde{\mathcal{J}}^M_{\delta\dot{c}b}.
\end{aligned} \tag{B.8}$$

The supersymmetry variation δ of [16] corresponds to the action of $\delta = i\eta^{\alpha b\dot{c}}\mathfrak{Q}_{\alpha b\dot{c}}$ with a fermionic field η. Lowering some of the indices of this field yields

$$\eta^\alpha{}_d{}^{\dot{c}} = \eta^{\alpha b\dot{c}}\varepsilon_{bd}, \quad \eta^{\alpha b}{}_{\dot{d}} = \eta^{\alpha b\dot{c}}\varepsilon_{\dot{c}\dot{d}}, \quad \eta_\delta{}^b{}_{\dot{e}} = \eta^{\alpha b\dot{c}}\varepsilon_{\alpha\delta}\varepsilon_{\dot{c}\dot{e}}, \quad \eta_{\delta e}{}^{\dot{c}} = \eta^{\alpha b\dot{c}}\varepsilon_{\alpha\delta}\varepsilon_{be}. \tag{B.9}$$

B.4 Interacting Symmetry Algebra

The symmetry algebra takes the form described in Sec. 2.2. However, it is well known that the symmetry algebra in an interacting gauge theory closes only on shell and modulo (field-dependent) gauge transformations. The additional terms form an ideal of the algebra and thus can be factored out consistently by acting only on on-shell, gauge-invariant states.

Commutators. The additional terms in the commutators can be written explicitly as

$$
\begin{aligned}
\{\mathfrak{Q}_{\alpha b\dot{c}}, \mathfrak{Q}_{\delta e\dot{f}}\} &= \varepsilon_{be}\varepsilon_{\dot{c}\dot{f}}(+\mathfrak{P}_{\alpha\delta} + \mathfrak{E}_{\alpha\delta}) \\
&\quad + \varepsilon_{\alpha\delta}\varepsilon_{\dot{c}\dot{f}}(-2m\mathfrak{R}_{be} - \tfrac{1}{2}\mathfrak{G}[\mathcal{M}_{be}] + \mathfrak{E}_{be}) \\
&\quad + \varepsilon_{\alpha\delta}\varepsilon_{be}(+2m\dot{\mathfrak{R}}_{\dot{c}\dot{f}} - \tfrac{1}{2}\tilde{\mathfrak{G}}[\tilde{\mathcal{M}}_{\dot{c}\dot{f}}] + \mathfrak{E}_{\dot{c}\dot{f}}), \\
[\mathfrak{P}_{\alpha\beta}, \mathfrak{Q}_{\gamma d\dot{e}}] &= \tfrac{1}{2}\varepsilon_{\beta\gamma}\mathfrak{G}[\mathcal{J}_{\alpha d\dot{e}} + \tilde{\mathcal{J}}_{\alpha\dot{e}d}] + \tfrac{1}{2}\varepsilon_{\alpha\gamma}\mathfrak{G}[\mathcal{J}_{\beta d\dot{e}} + \tilde{\mathcal{J}}_{\beta\dot{e}d}], \\
[\mathfrak{P}_{\alpha\beta}, \mathfrak{P}_{\gamma\delta}] &= \tfrac{i}{2}\varepsilon_{\beta\delta}\mathfrak{G}[\mathcal{F}_{\alpha\gamma}] + \tfrac{i}{2}\varepsilon_{\beta\gamma}\mathfrak{G}[\mathcal{F}_{\alpha\delta}] + \tfrac{i}{2}\varepsilon_{\alpha\delta}\mathfrak{G}[\mathcal{F}_{\beta\gamma}] + \tfrac{i}{2}\varepsilon_{\alpha\gamma}\mathfrak{G}[\mathcal{F}_{\beta\delta}].
\end{aligned}
\tag{B.10}
$$

The generator $\mathfrak{E}$ annihilates on-shell fields and the generators $\mathfrak{G}[\mathcal{X}]$ are field-dependent gauge transformations.

Gauge Transformations. The generators $\mathfrak{G}[\mathcal{X}]$ are gauge variations with variation parameter $\mathcal{X}^M$

$$
\begin{aligned}
\mathfrak{G}[\mathcal{X}]\mathcal{Y}^A &= -iT^A_{MB}\mathcal{X}^M\mathcal{Y}^B, \\
\mathfrak{G}[\mathcal{X}]\mathcal{Y}^{\tilde{A}} &= -i\tilde{T}^{\tilde{A}}_{M\tilde{B}}\mathcal{X}^M\mathcal{Y}^{\tilde{B}}, \\
\mathfrak{G}[\mathcal{X}]\mathcal{A}^M_{\alpha\beta} &= i\mathcal{D}_{\alpha\beta}\mathcal{X}^M.
\end{aligned}
\tag{B.11}
$$

The gauge transformation generate an ideal of the full symmetry algebra: One can show that commutators of gauge transformations close onto gauge transformations, explicitly

$$
\big[\mathfrak{J}, \mathfrak{G}[\mathcal{X}]\big] = \mathfrak{G}[\mathfrak{J}\mathcal{X}]. \tag{B.12}
$$

Equation of Motion Generators. The action of the generators $\mathfrak{E}[\mathcal{X}]$ is defined as

$$
\begin{aligned}
\mathfrak{E}_{\alpha\beta}\psi^A_{\gamma\dot{d}} &= \tfrac{i}{2}\varepsilon_{\beta\gamma}\mathrm{E}[\psi^A_{\alpha\dot{d}}] + \tfrac{i}{2}\varepsilon_{\alpha\gamma}\mathrm{E}[\psi^A_{\beta\dot{d}}], \\
\mathfrak{E}_{\alpha\beta}\tilde{\psi}^{\tilde{A}}_{\gamma d} &= \tfrac{i}{2}\varepsilon_{\beta\gamma}\mathrm{E}[\tilde{\psi}^A_{\alpha d}] + \tfrac{i}{2}\varepsilon_{\alpha\gamma}\mathrm{E}[\tilde{\psi}^A_{\beta d}], \\
\mathfrak{E}_{\alpha\beta}\mathcal{A}^M_{\gamma\delta} &= -\tfrac{i}{2}\varepsilon_{\beta\gamma}\mathrm{E}[\mathcal{A}^M_{\alpha\delta}] - \tfrac{i}{2}\varepsilon_{\beta\delta}\mathrm{E}[\mathcal{A}^M_{\alpha\gamma}] - \tfrac{i}{2}\varepsilon_{\alpha\gamma}\mathrm{E}[\mathcal{A}^M_{\beta\delta}] - \tfrac{i}{2}\varepsilon_{\alpha\delta}\mathrm{E}[\mathcal{A}^M_{\beta\gamma}], \\
\mathfrak{E}_{ab}\tilde{\psi}^{\tilde{A}}_{\gamma d} &= -\tfrac{i}{2}\varepsilon_{bd}\mathrm{E}[\tilde{\psi}^{\tilde{A}}_{\gamma a}] - \tfrac{i}{2}\varepsilon_{ad}\mathrm{E}[\tilde{\psi}^{\tilde{A}}_{\gamma b}], \\
\mathfrak{E}_{\dot{a}\dot{b}}\psi^A_{\gamma\dot{d}} &= -\tfrac{i}{2}\varepsilon_{\dot{b}\dot{d}}\mathrm{E}[\psi^A_{\gamma\dot{a}}] - \tfrac{i}{2}\varepsilon_{\dot{a}\dot{d}}\mathrm{E}[\psi^A_{\gamma\dot{b}}].
\end{aligned}
\tag{B.13}
$$

They annihilate on-shell fields because $\mathrm{E}[\mathcal{X}] = 0$ is defined as the equation of motion for the field $\mathcal{X}$

$$
\begin{aligned}
\mathrm{E}[\phi^A_a] &= \tfrac{1}{2}\varepsilon^{\gamma\kappa}\varepsilon^{\delta\lambda}\mathcal{D}_{\gamma\delta}\mathcal{D}_{\kappa\lambda}\phi^A_a - m^2\phi^A_a \\
&\quad + \tfrac{i}{2}\varepsilon^{\dot{c}\dot{d}}\varepsilon^{\kappa\lambda}T^A_{N\,B}\psi^B_{\kappa\dot{c}}\mathcal{J}^N_{\lambda a\dot{d}} + i\varepsilon^{\dot{b}\dot{c}}\varepsilon^{\kappa\lambda}T^A_{N\,B}\psi^B_{\kappa\dot{b}}\tilde{\mathcal{J}}^N_{\lambda\dot{c}a} \\
&\quad - \tfrac{i}{2}\tilde{M}^M_{\bar{C}\bar{D}}\varepsilon^{bd}\varepsilon^{\kappa\lambda}T^A_{M\,B}\phi^B_b\tilde{\psi}^{\bar{C}}_{\kappa a}\tilde{\psi}^{\bar{D}}_{\lambda d} + \tfrac{2}{3}m\varepsilon^{bd}T^A_{N\,B}\phi^B_b\mathcal{M}^N_{ad} \\
&\quad + \tfrac{1}{16}F^N_{PQ}T^A_{N\,B}\varepsilon^{bc}\varepsilon^{de}\mathcal{M}^P_{ab}\mathcal{M}_{cd}\phi^B_e + \tfrac{1}{4}\tilde{M}^M_{\bar{A}\bar{D}}\tilde{T}^{\bar{D}}_{N\,\bar{B}}\varepsilon^{bf}T^A_{M\,B}\phi^B_b\mathcal{M}^M_{af}\tilde{\phi}^{\bar{A}}_{\dot{c}}\tilde{\phi}^{\bar{B}}_{\dot{d}} \\
&\quad + \tfrac{1}{8}L_{CD}T^{CA}_M T^D_{N\,B}\varepsilon^{\dot{c}\dot{e}}\varepsilon^{\dot{d}\dot{f}}\tilde{\mathcal{M}}^M_{\dot{c}\dot{d}}\tilde{\mathcal{M}}^N_{\dot{e}\dot{f}}\phi^B_a, \\
\mathrm{E}[\tilde{\phi}^{\tilde{A}}_{\dot{a}}] &= \tfrac{1}{2}\varepsilon^{\gamma\kappa}\varepsilon^{\delta\lambda}\mathcal{D}_{\gamma\delta}\mathcal{D}_{\kappa\lambda}\phi^A_{\dot{a}} - m^2\phi^A_{\dot{a}} \\
&\quad + \tfrac{i}{2}\varepsilon^{cd}\varepsilon^{\kappa\lambda}\tilde{T}^{\bar{A}}_{N\,\bar{B}}\tilde{\psi}^{\bar{B}}_{\kappa c}\tilde{\mathcal{J}}^N_{\lambda\dot{a}d} + i\varepsilon^{bc}\varepsilon^{\kappa\lambda}\tilde{T}^{\bar{A}}_{N\,\bar{B}}\tilde{\psi}^{\bar{B}}_{\kappa b}\mathcal{J}^N_{\lambda c\dot{a}}
\end{aligned}
$$

$$
\begin{aligned}
&- \tfrac{i}{2} M^M_{CD} \varepsilon^{\dot b\dot d} \varepsilon^{\kappa\lambda} \tilde T^{\bar A}_{M\bar B} \tilde\phi^{\bar B}_{\dot b} \psi^C_{\kappa\dot a} \psi^D_{\lambda\dot d} - \tfrac{2}{3} m \varepsilon^{\dot b\dot d} \tilde T^{\bar A}_{N\bar B} \tilde\phi^{\bar B}_{\dot b} \tilde{\mathcal{M}}^N_{\dot a\dot d} \\
&+ \tfrac{1}{16} F^N_{PQ} \tilde T^{\bar A}_{N\bar B} \varepsilon^{\dot b\dot c} \varepsilon^{\dot d\dot e} \tilde{\mathcal{M}}^P_{\dot a\dot b} \tilde{\mathcal{M}}_{\dot c\dot d} \tilde\phi^{\bar B}_{\dot e} + \tfrac{1}{4} M^M_{AD} T^D_{N\,B} \varepsilon^{\dot b\dot f} \tilde T^{\bar A}_{M\bar B} \tilde\phi^{\bar B}_{\dot b} \tilde{\mathcal{M}}^M_{\dot a\dot f} \phi^A_c \phi^B_d \\
&+ \tfrac{1}{8} \tilde L_{\bar C\bar D} \tilde T^{\bar C\bar A}_M \tilde T^{\bar D}_{N\,\bar B} \varepsilon^{ce} \varepsilon^{df} \mathcal{M}^M_{cd} \mathcal{M}^N_{ef} \tilde\phi^{\bar B}_{\dot a}\,,
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{E}[\psi^A_{\alpha\dot b}] &= \varepsilon^{\gamma\delta} \mathcal{D}_{\alpha\gamma} \psi^A_{\delta\dot b} + m\psi^A_{\alpha\dot b} + T^A_{MB} (\mathcal{J}^M_{\alpha c\dot b} + \tilde{\mathcal{J}}^M_{\alpha\dot b c}) \varepsilon^{cd} \phi^B_d\,, \\
\mathrm{E}[\tilde\psi^{\tilde A}_{\alpha b}] &= \varepsilon^{\gamma\delta} \mathcal{D}_{\alpha\gamma} \tilde\psi^{\tilde A}_{\delta b} - m\tilde\psi^{\tilde A}_{\alpha b} + \tilde T^{\tilde A}_{M\tilde B} (\mathcal{J}^M_{\alpha b\dot c} + \tilde{\mathcal{J}}^M_{\alpha\dot c b}) \varepsilon^{\dot c\dot d} \tilde\phi^{\tilde B}_{\dot d}\,, \\
\mathrm{E}[\mathcal{A}^M_{\alpha\beta}] &= \mathcal{F}^M_{\alpha\beta} + \tfrac{1}{2} M^M_{AB} \varepsilon^{cd} \phi^A_c \mathcal{D}_{\alpha\beta} \phi^B_d + \tfrac{1}{2} \tilde M^M_{\tilde A\tilde B} \varepsilon^{\dot c\dot d} \tilde\phi^{\tilde A}_{\dot c} \mathcal{D}_{\alpha\beta} \tilde\phi^{\tilde B}_{\dot d} \\
&\quad + \tfrac{i}{2} M^M_{AB} \varepsilon^{\dot c\dot d} \psi^A_{\alpha\dot c} \psi^B_{\beta\dot d} + \tfrac{i}{2} \tilde M^M_{\tilde A\tilde B} \varepsilon^{cd} \tilde\psi^{\tilde A}_{\alpha c} \tilde\psi^{\tilde B}_{\beta d}.
\end{aligned} \tag{B.14}
$$

The commutators of the generators $\mathfrak{E}$ close onto further generators $\mathfrak{E}$ which annihilate all on-shell fields. Thus they also form an ideal of the symmetry algebra.

Oscillator Expansion. We use the following oscillator expansions to the fields, using the basis of solutions found in (A.2):

$$
\begin{aligned}
\phi^A_a(x) &= \int \frac{d^2p}{\sqrt{2E(p)}} \left(e^{-ip\cdot x} a^A_a(p) + e^{ip\cdot x} a^{A\dagger}_a(p)\right) \\
\tilde\phi^A_{\dot a}(x) &= \int \frac{d^2p}{\sqrt{2E(p)}} \left(e^{-ip\cdot x} \tilde a^A_{\dot a}(p) + e^{ip\cdot x} \tilde a^{A\dagger}_{\dot a}(p)\right) \\
\psi^A_{\dot a}(x) &= \int \frac{d^2p}{\sqrt{2E(p)}} \left(u(p) e^{ip\cdot x} b^{\dagger A}_{\dot a}(p) + v(p) e^{-ip\cdot x} b^A_{\dot a}(p)\right) \\
\tilde\psi^A_a(x) &= \int \frac{d^2p}{\sqrt{2E(p)}} \left(v(p) e^{ip\cdot x} \tilde b^{\dagger A}_a(p) + u(p) e^{-ip\cdot x} \tilde b^A_a(p)\right).
\end{aligned} \tag{B.15}
$$

$\phi(x)$ and $\psi(x)$ are the bosonic and fermionic fields in the action found in Appendix B and the action (B.5). Requiring that the linearized $\mathcal{N}=4$ algebra be realized on bosonic/fermionic oscillator states, leads to (2.9). We note that the positive and negative energy frequency parts for the twisted fermions are related to the conjugates of those for the untwisted ones. This has to do with the fact that the twisted fermions have a negative mass compared to the untwisted ones.

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