Published by IOP Publishing for SISSA

Received: *March 15, 2009*
Accepted: *April 4, 2009*
Published: *April 29, 2009*

On integrable backgrounds self-dual under fermionic T-duality

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Abstract: We study the fermionic T-duality symmetry of integrable Green-Schwarz sigma-models on AdS backgrounds with Ramond-Ramond fluxes in various dimensions. We show that sigma-models based on supercosets of PSU supergroups, such as $AdS_2 \times S^2$ and $AdS_3 \times S^3$ are self-dual under fermionic T-duality, while supercosets of OSp supergroups such as non-critical AdS_2 and AdS_4 models, and the critical $AdS_4 \times \mathbb{C}P^3$ background are not. We present a general algebraic argument to when a supercoset is expected to have a fermionic T-duality symmetry, and when it will fail to have one.

Keywords: String Duality, Duality in Gauge Field Theories

ArXiv ePrint: 0902.3805

doi:10.1088/1126-6708/2009/04/120

JHEP04(2009)120

Contents

1 Introduction and summary

Recently, Alday and Maldacena proposed [1] that planar $\mathcal{N} = 4$ SU(N) SYM MHV gluon scattering amplitudes at leading order in the strong 't Hooft coupling expansion can be calculated using the dual gravity (string) description. A crucial step in the calculation procedure is an application of an ordinary bosonic T-duality transformation to the four CFT coordinates of $AdS_5 \times S^5$. This suggestion implies that such amplitudes possess a dual conformal symmetry at strong coupling originating from the fact that AdS_5 is self-dual under the T-duality. This dual symmetry has been observed also in gluon scattering amplitudes calculated in the weakly-coupled gauge theory description [2, 3].

Using the AdS/CFT duality it was shown that such a symmetry is expected to be valid at all values of the 't Hooft coupling [4, 5]. This was done by proving that the aforementioned T-duality together with a novel T-duality of Grassmann-odd coordinates of the target-superspace form an exact quantum duality under which the full $AdS_5 \times S^5$ superstring is self-dual. Since both the original background and its dual possess superconformal symmetry, it means that each of them has both the manifest superconformal symmetry and a dual one. Furthermore, [4, 5] have linked the duals of the superconformal Noether currents to the non-local currents implied by the integrability of the superstring on $AdS_5 \times S^5$ [6].

In view of these results it is natural to inquire how ubiquitous this dual superconformal symmetry is. The aim of this paper is to consider this question by analyzing the fermionic T-duality symmetry of integrable Green-Schwarz sigma-models on AdS backgrounds with Ramond-Ramond fluxes in various dimensions. Some of these sigma-models have been constructed in [7–10].

We show that sigma-models based on supercosets of PSU supergroups, such as $AdS_2 \times S^2$ and $AdS_3 \times S^3$ are self-dual under fermionic T-duality. Supercosets of OSp supergroups such as non-critical AdS_2 and AdS_4 models, and the critical $AdS_4 \times \mathbb{C}P^3$ background (whose coset model was constructed and explored in [11–15] and the full model was constructed in [22]) are not self-dual. In the OSp models we find that the Buscher procedure [16, 17] fails due to a lack of appropriate quadratic terms. This is because the Cartan-Killing bilinear form of the ortho-symplectic group is non-zero only for products of different Grassmann-odd generators. Thus, one may expect this to imply that in those cases in which a dual theory exists, the theory does not have a dual superconformal symmetry.

The paper is organized as follows. In section 2 we show that the $AdS_p \times S^p$ ($p = 2, 3$) target-spaces based on PSU cosets are self-dual under a combination of bosonic and fermionic T-duality. In section 3 we consider models based on supercosets of the ortho-symplectic supergroup, for which the Buscher procedure [16, 17] of gauging an isometry of the target-space in order to obtain the T-dual sigma-model fails. These include the non-critical superstring on AdS_2 with four supersymmetries and AdS_4 with eight supersymmetries, the supercoset construction of $AdS_4 \times \mathbb{C}\mathrm{P}^3$, and a model of AdS_2 with eight supersymmetries. In section 4 we present a general algebraic argument to when a supercoset is expected to have a fermionic T-duality symmetry, and when it will fail to have one. In the appendices we provide details on the relevant superalgebras that are used in the paper.

2 $AdS_p \times S^p$ target-spaces

In this section we show that the $AdS_p \times S^p$ ($p = 2, 3$) target-spaces based on PSU supercosets are self-dual under a combination of bosonic and fermionic T-duality.

2.1 The $AdS_2 \times S^2$ target-space

The target superspace whose bosonic part is $AdS_2 \times S^2$ can be realized as the coset space $\mathrm{PSU}(1, 1|2)/(\mathrm{U}(1) \times \mathrm{U}(1))$.[1] The Green-Schwarz sigma-model for supercosets with a $\mathbb{Z}_4$

[1] For this to be a superstring, this superspace has to be supplemented by an additional internal CFT with the appropriate central charge. It is not clear that such a description exists using such a partial Green-Schwarz action (the hybrid model discussed in [18] might be a better option for this kind of superstring construction).

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automorphism is given by the action [7]

$$S = \frac{R^2}{4\pi\alpha'}\int d^2z \mathrm{Str}\left(J_2\bar{J}_2 + \frac{1}{2}J_1\bar{J}_3 - \frac{1}{2}\bar{J}_1 J_3\right), \tag{2.1}$$

where $J = g^{-1}\partial g$ for $g \in G$ and J_i is the current J restricted to the invariant subspace $\mathcal{H}_i$ of the $\mathbb{Z}_4$ automorphism of the algebra of the group G. Using the psu$(1,1|2)$ algebra given in appendix D, the sigma-model (2.1) takes the form

$$\begin{aligned} S = \frac{R^2}{4\pi\alpha'}\int d^2z \Big[&\frac{1}{2}(J_P - J_K)(\bar{J}_P - \bar{J}_K) + \frac{1}{2}J_D\bar{J}_D + \frac{1}{2}J_{R_1}\bar{J}_{R_1} + \frac{1}{2}J_{R_2}\bar{J}_{R_2} - \\ &- \frac{i}{2}\eta_{\alpha\beta}(J_{Q_\alpha}\bar{J}_{Q_\beta} - J_{\hat{Q}_\alpha}\bar{J}_{\hat{Q}_\beta} + J_{S_\alpha}\bar{J}_{S_\beta} - J_{\hat{S}_\alpha}\bar{J}_{\hat{S}_\beta})\Big], \end{aligned} \tag{2.2}$$

where $\eta_{12} = \eta_{21} = 1$ and zero otherwise. The analysis here will follow that of [4]. A general group element $g \in \mathrm{PSU}(1,1|2)$ can be parameterized as

$$g = e^{xP + x'K + \theta^\alpha Q_\alpha + \xi^\alpha S_\alpha} e^B, \quad e^B \equiv e^{\hat{\theta}^\alpha \hat{Q}_\alpha + \hat{\xi}^\alpha \hat{S}_\alpha} y^D e^{\sum y^i R_i / y}. \tag{2.3}$$

We partially fix the κ-symmetry such that $\xi^\alpha = 0$, and we use the $\mathrm{U}(1)\times\mathrm{U}(1)$ gauge symmetry generated by $P + K$ and R_3 to set $x' = 0$, thus the coset representative is

$$g = e^{xP + \theta^\alpha Q_\alpha} e^B. \tag{2.4}$$

Using the fact that the group generated by $\{\hat{Q}, \hat{S}, D, R_i\}$ transforms P and Q_α among themselves, the components of the Maurer-Cartan 1-form are

$$\begin{aligned} &J_P = [e^{-B}(dxP + d\theta^\alpha Q_\alpha)e^B]_P, && J_{Q_\alpha} = [e^{-B}(dxP + d\theta^\beta Q_\beta)e^B]_{Q_\alpha}, \\ &J_K = 0, \quad J_{\hat{Q}_\alpha} = [e^{-B}de^B]_{\hat{Q}_\alpha}, && J_{S_\alpha} = 0, \quad J_{\hat{S}_\alpha} = [e^{-B}de^B]_{\hat{S}_\alpha}, \\ &J_D = [e^{-B}de^B]_D, && J_{R_i} = [e^{-B}de^B]_{R_i}. \end{aligned} \tag{2.5}$$

This sigma-model is T-dualized in the directions of the Abelian sub-algebra formed by the generators P and Q_α according to the procedure of [16, 17], by introducing the gauge fields A, $\bar{A}$ for the translation P and A^α, $\bar{A}^\alpha$ for the supercharges Q_α and the corresponding Lagrange multipliers $\widetilde{x}$ and $\widetilde{\theta}_\alpha$:

$$\begin{aligned} S = \frac{R^2}{4\pi\alpha'}\int d^2z \Big[&\frac{1}{2}[e^{-B}(AP + A^\alpha Q_\alpha)e^B]_P[e^{-B}(\bar{A}P + \bar{A}^\alpha Q_\alpha)e^B]_P - \\ &- \frac{i}{2}\eta_{\alpha\beta}[e^{-B}(AP + A^\gamma Q_\gamma)e^B]_{Q_\alpha}[e^{-B}(\bar{A}P + \bar{A}^\gamma Q_\gamma)e^B]_{Q_\beta} + \\ &+ \frac{1}{2}J_D\bar{J}_D + \frac{1}{2}J_{R_1}\bar{J}_{R_1} + \frac{1}{2}J_{R_2}\bar{J}_{R_2} + \frac{i}{2}\eta_{\alpha\beta}(J_{\hat{Q}_\alpha}\bar{J}_{\hat{Q}_\beta} + J_{\hat{S}_\alpha}\bar{J}_{\hat{S}_\beta}) + \\ &+ \widetilde{x}(\bar{\partial}A - \partial\bar{A}) + \widetilde{\theta}_\alpha(\bar{\partial}A^\alpha - \partial\bar{A}^\alpha)\Big]. \end{aligned} \tag{2.6}$$

It is convenient to change variables to

$$A' = [e^{-B}(AP + A^\alpha Q_\alpha)e^B]_P, \quad A'^\alpha = [e^{-B}(AP + A^\beta Q_\beta)e^B]_{Q_\alpha} \tag{2.7}$$

and similarly for the right-moving gauge fields. Using the inverted relations

$$A = [e^B(A'P + A'^\alpha Q_\alpha)e^{-B}]_P, \quad A^\alpha = [e^B(A'P + A'^\beta Q_\beta)e^{-B}]_{Q_\alpha}, \tag{2.8}$$

the action in terms of the new variables reads

$$\begin{aligned} S = \frac{R^2}{4\pi\alpha'} \int d^2z \Big[& \frac{1}{2} A'\bar{A}' - \frac{i}{2}\eta_{\alpha\beta} A'^\alpha \bar{A}'^\beta + \cdots - \\ & - \bar{\partial}\widetilde{x}(A'[e^B P e^{-B}]_P + A'^\alpha [e^B Q_\alpha e^{-B}]_P) + \\ & + \partial\widetilde{x}\,(\bar{A}'[e^B P e^{-B}]_P + \bar{A}'^\alpha [e^B Q_\alpha e^{-B}]_P) - \\ & - \bar{\partial}\widetilde{\theta}_\alpha \left(A'[e^B P e^{-B}]_{Q_\alpha} + A'^\beta [e^B Q_\beta e^{-B}]_{Q_\alpha} \right) + \\ & + \partial\widetilde{\theta}_\alpha \left(\bar{A}'[e^B P e^{-B}]_{Q_\alpha} + \bar{A}'^\beta [e^B Q_\beta e^{-B}]_{Q_\alpha} \right) \Big], \end{aligned} \tag{2.9}$$

where $\ldots$ denotes the spectator terms. Since the gauge fields appear quadratically in the action, one can integrate them out by substituting their equations of motion

$$\begin{aligned} A' &= -2[e^B \partial\widetilde{x} P e^{-B}]_P - 2[e^B \partial\widetilde{\theta}_\alpha P e^{-B}]_{Q_\alpha} = -2[e^{-B}(\partial\widetilde{x}K + i\partial\widetilde{\theta}_\alpha S^\alpha)e^B]_K, \\ \bar{A}' &= 2[e^B \bar{\partial}\widetilde{x} P e^{-B}]_P + 2[e^B \bar{\partial}\widetilde{\theta}_\alpha P e^{-B}]_{Q_\alpha} = 2[e^{-B}(\bar{\partial}\widetilde{x}K + i\bar{\partial}\widetilde{\theta}_\alpha S^\alpha)e^B]_K, \\ A'^\alpha &= 2i\eta^{\alpha\beta}([e^B \partial\widetilde{x} Q_\beta e^{-B}]_P - [e^B \partial\widetilde{\theta}_\gamma Q_\beta e^{-B}]_{Q_\gamma}) = -2\eta^{\alpha\beta}[e^{-B}(\partial\widetilde{x}K + i\partial\widetilde{\theta}_\gamma S^\gamma)e^B]_{S^\beta}, \\ \bar{A}'^\alpha &= 2i\eta^{\alpha\beta}([e^B \bar{\partial}\widetilde{x} Q_\beta e^{-B}]_P - [e^B \bar{\partial}\widetilde{\theta}_\gamma Q_\beta e^{-B}]_{Q_\gamma}) = -2\eta^{\alpha\beta}[e^{-B}(\bar{\partial}\widetilde{x}K + i\bar{\partial}\widetilde{\theta}_\gamma S^\gamma)e^B]_{S^\beta}. \end{aligned} \tag{2.10}$$

and obtain after rescaling $\widetilde{x} \to \frac{1}{2}\widetilde{x}$ and $\widetilde{\theta}_\alpha \to \frac{1}{2}\widetilde{\theta}_\alpha$

$$\begin{aligned} S = \frac{R^2}{4\pi\alpha'} \int d^2z \Big[& \frac{1}{2}[e^{-B}(\partial\widetilde{x}K + i\partial\widetilde{\theta}_\alpha S^\alpha)e^B]_K [e^{-B}(\bar{\partial}\widetilde{x}K + i\bar{\partial}\widetilde{\theta}_\alpha S^\alpha)e^B]_K - \\ & - \frac{i}{2}\eta_{\alpha\beta}[e^{-B}(\partial\widetilde{x}K + i\partial\widetilde{\theta}_\gamma S^\gamma)e^B]_{S_\alpha}[e^{-B}(\bar{\partial}\widetilde{x}K + i\bar{\partial}\widetilde{\theta}_\gamma S^\gamma)e^B]_{S_\beta} + \ldots \Big], \end{aligned} \tag{2.11}$$

where we used $\epsilon_{\alpha\beta}$ to lower the spinor indices of $\eta^{\alpha\beta}$.

In order to show the self-duality of this background under the above T-duality, the original action has to be brought to the same form as (2.11). One can easily check by using the $\epsilon_{\beta\gamma}\sigma_\alpha^{ij\gamma} = \epsilon_{\alpha\gamma}\sigma_\beta^{ij\gamma}$ that the psu$(1,1|2)$ algebra admits the automorphism

$$P \leftrightarrow K, \quad D \to -D, \quad Q_\alpha \leftrightarrow S^\alpha, \quad \hat{Q}_\alpha \leftrightarrow \hat{S}_\alpha, \tag{2.12}$$

with the rest of the generators unchanged. Applying this automorphism combined with the change of variables

$$x \to \widetilde{x}, \quad \theta^\alpha \to i\widetilde{\theta}_\alpha, \quad \hat{\theta}^\alpha \leftrightarrow \hat{\xi}^\alpha, \quad y_i \to \frac{y_i}{y^2}, \tag{2.13}$$

to (2.5) one obtains (2.11).

In order to complete the proof of quantum mechanical equivalence we also have to show that the Jacobian functional determinant from the change of variables (2.7) is the identity. The transformation of variables was done using e^B. Since it is in a unitary subgroup of the PSU$(1,1|2)$ group, its super-determinant is equal to one and hence the Jacobian of the transformation is trivial and does not induce any shift of the dilaton. In order to see this explicitly, one can treat (A, A^α) as a vector and write the adjoint action of the generators D, R_{ij}, $\hat{Q}_\alpha$ and $\hat{S}_\alpha$ as matrices acting on this vector

$$\begin{aligned}
[\mathrm{ad}_D] &= \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2}\delta_\alpha{}^\beta \end{pmatrix}, & [\mathrm{ad}_{R_{ij}}] &= \begin{pmatrix} 0 & 0 \\ 0 & -\frac{1}{2}\sigma_{ij\alpha}{}^\beta \end{pmatrix}, \\
[\mathrm{ad}_{\hat{Q}_\alpha}] &= \begin{pmatrix} 0 & -\epsilon_{\alpha\beta} \\ 0 & 0 \end{pmatrix}, & [\mathrm{ad}_{\hat{S}_\alpha}] &= \begin{pmatrix} 0 & 0 \\ -i\delta_\alpha{}^\beta & 0 \end{pmatrix},
\end{aligned} \tag{2.14}$$

which evidently have a vanishing supertrace. Hence, the super-determinant of similarity transformations with elements of the group generated by these generators is the identity.

2.2 The $AdS_3 \times S^3$ target-space

We construct the Green-Schwarz sigma-model on $AdS_3 \times S^3$ using the supercoset manifold $(\mathrm{PSU}(1,1|2) \times \mathrm{PSU}(1,1|2))/(\mathrm{SU}(1,1) \times \mathrm{SU}(2))$ with 16 supersymmetry generators. Using the $\mathbb{Z}_4$ structure (E.4) we have

$$\begin{aligned}
J_2 &= J_a^-(P_a - K_a) + J_D D + J_\mu R_\mu = \frac{1}{2}(J_{P_a} - J_{K_a})(P_a - K_a) + J_D D + J_\mu R_\mu \quad (2.15) \\
J_1 &= J_{S^I_{\alpha\hat{\alpha}} + a^{IJ} Q^J_{\alpha\hat{\alpha}}}(S^I_{\alpha\hat{\alpha}} + a^{IK} Q^K_{\alpha\hat{\alpha}}) = \frac{1}{2}(J_{S^I_{\alpha\hat{\alpha}}} + a^{IJ} J_{Q^J_{\alpha\hat{\alpha}}})(S^I_{\alpha\hat{\alpha}} + a^{IK} Q^K_{\alpha\hat{\alpha}}) \\
J_3 &= J_{S^I_{\alpha\hat{\alpha}} - a^{IJ} Q^J_{\alpha\hat{\alpha}}}(S^I_{\alpha\hat{\alpha}} - a^{IK} Q^K_{\alpha\hat{\alpha}}) = \frac{1}{2}(J_{S^I_{\alpha\hat{\alpha}}} - a^{IJ} J_{Q^J_{\alpha\hat{\alpha}}})(S^I_{\alpha\hat{\alpha}} - a^{IK} Q^K_{\alpha\hat{\alpha}}).
\end{aligned}$$

Thus,

$$\mathrm{Str}(J_2 \bar{J}_2) = -\frac{1}{2}(J_{P_a} - J_{K_a})(\bar{J}_{P_b} - \bar{J}_{K_b})\eta^{ab} + J_D \bar{J}_D + J_{R_\mu} \bar{J}_{R_\mu}, \tag{2.16}$$

$$\mathrm{Str}(J_1 \bar{J}_3) = \frac{1}{2}(\sigma^3)^{IK} \epsilon_{\alpha\beta} \epsilon_{\hat{\alpha}\hat{\beta}} (J_{S^I_{\alpha\hat{\alpha}}} + a^{IJ} J_{Q^J_{\alpha\hat{\alpha}}})(\bar{J}_{S^K_{\beta\hat{\beta}}} - a^{KL} \bar{J}_{Q^L_{\beta\hat{\beta}}}), \tag{2.17}$$

and the action reads

$$\begin{aligned}
S = \frac{R^2}{4\pi\alpha'} \int d^2 z \Big[&-\frac{1}{2}(J_{P_a} - J_{K_a})(\bar{J}_{P_b} - \bar{J}_{K_b})\eta_{ab} + J_D \bar{J}_D + J_{R_\mu} \bar{J}_{R_\mu} \\
&+\frac{1}{2}(\sigma^3)^{IK} \epsilon_{\hat{\alpha}\hat{\beta}} \epsilon_{\alpha\beta} \Big(J_{S^I_{\alpha\hat{\alpha}}} \bar{J}_{S^K_{\beta\hat{\beta}}} + J_{Q^I_{\alpha\hat{\alpha}}} \bar{J}_{Q^K_{\beta\hat{\beta}}} \Big) \Big].
\end{aligned} \tag{2.18}$$

Next we parameterize the supergroup element

$$\begin{aligned}
g &= \exp(x^a P_a + \theta^{\alpha\hat{\alpha}1} Q^1_{\alpha\hat{\alpha}}) \exp(\theta^{\alpha\hat{\alpha}2} Q^2_{\alpha\hat{\alpha}} + \xi^{\alpha\hat{\alpha}1} S^1_{\alpha\hat{\alpha}}) y^D \exp(y_\mu R_\mu / y) \\
&\equiv \exp(x^a P_a + \theta^{\alpha\hat{\alpha}1} Q^1_{\alpha\hat{\alpha}}) e^B.
\end{aligned} \tag{2.19}$$

Define the currents

$$J = g^{-1} dg = j + \mathrm{j}, \qquad j = e^{-B}(dx^a P_a + d\theta^{\alpha\hat{\alpha}1} Q^1_{\alpha\hat{\alpha}}) e^B, \qquad \mathrm{j} = e^{-B} de^B. \tag{2.20}$$

Using these definitions and the algebra (E.1) we get the currents

$$
\begin{aligned}
&J_{P_a} = j_{P_a}, \qquad J_{Q^1_{\alpha\hat\alpha}} = j_{Q^1_{\alpha\hat\alpha}}, \qquad J_{Q^2_{\alpha\hat\alpha}} = \mathrm{j}_{Q^2_{\alpha\hat\alpha}}, \qquad J_{S^1_{\alpha\hat\alpha}} = \mathrm{j}_{S^1_{\alpha\hat\alpha}}, \qquad J_{S^2_{\alpha\hat\alpha}} = 0 \\
&J_{K_a} = 0, \qquad J_D = \mathrm{j}_D, \qquad J_{R_\mu} = \mathrm{j}_{R_\mu},
\end{aligned} \tag{2.21}
$$

the action reads

$$
\begin{aligned}
S = \frac{R^2}{4\pi\alpha'}\int d^2z\Big[&-\frac{1}{2} j_{P_a}\bar{j}_{P_b}\eta_{ab} + \mathrm{j}_D\bar{\mathrm{j}}_D + \mathrm{j}_{R_\mu}\bar{\mathrm{j}}_{R_\mu} \\
&+\frac{1}{2}\epsilon_{\alpha\beta}\epsilon_{\hat\alpha\hat\beta}\Big(\mathrm{j}_{S^1_{\alpha\hat\alpha}}\bar{\mathrm{j}}_{S^1_{\beta\hat\beta}} + j_{Q^1_{\alpha\hat\alpha}}\bar{j}_{Q^1_{\beta\hat\beta}} - \mathrm{j}_{Q^2_{\alpha\hat\alpha}}\bar{\mathrm{j}}_{Q^2_{\beta\hat\beta}}\Big)\Big].
\end{aligned} \tag{2.22}
$$

Introducing gauge fields and Lagrange multipliers we get the action

$$
\begin{aligned}
S = \frac{R^2}{4\pi\alpha'}\int d^2z\Big[&-\frac{1}{2} A'_{P_a}\bar{A}'_{P_b}\eta_{ab} + \mathrm{j}_D\bar{\mathrm{j}}_D + \mathrm{j}_{R_\mu}\bar{\mathrm{j}}_{R_\mu} \\
&+\frac{1}{2}\epsilon_{\alpha\beta}\epsilon_{\hat\alpha\hat\beta}\Big(\mathrm{j}_{S^1_{\alpha\hat\alpha}}\bar{\mathrm{j}}_{S^1_{\beta\hat\beta}} + A'_{Q^1_{\alpha\hat\alpha}}\bar{A}'_{Q^1_{\beta\hat\beta}} - \mathrm{j}_{Q^2_{\alpha\hat\alpha}}\bar{\mathrm{j}}_{Q^2_{\beta\hat\beta}}\Big) \\
&+\widetilde{x}_a(\bar\partial A^a - \partial\bar{A}^a) + \widetilde{\theta}^1_{\alpha\hat\alpha}(\bar\partial A_{Q^1_{\alpha\hat\alpha}} - \partial\bar{A}_{Q^1_{\alpha\hat\alpha}})\Big],
\end{aligned}
$$

where

$$
\begin{aligned}
&A' = e^{-B}(A^a P_a + A^{1\alpha\hat\alpha}Q^1_{\alpha\hat\alpha})e^B, \qquad A^{1\alpha\hat\alpha} \equiv A_{Q^1_{\alpha\hat\alpha}}, \qquad A_{P_a} \equiv A^a, \\
&A = e^B(A'^a P_a + A'^{1\alpha\hat\alpha}Q^1_{\alpha\hat\alpha})e^{-B}.
\end{aligned} \tag{2.23}
$$

Solving for A' we get

$$
\begin{aligned}
A'_a &= 2(\partial\widetilde{x}_b[e^B P_a e^{-B}]_{P_b} + \partial\widetilde{\theta}^1_{\alpha\hat\alpha}[e^B P_a e^{-B}]_{Q^1_{\alpha\hat\alpha}}) && (2.24)\\
&= 2\eta_{ac}[e^{-B}(\partial\widetilde{x}^b K_b + \partial\widetilde{\theta}^{1\alpha\hat\alpha}S^2_{\alpha\hat\alpha})e^B]_{K_c} \\
\bar{A}'_a &= -2(\bar\partial\widetilde{x}_b[e^B P_a e^{-B}]_{P_b} + \bar\partial\widetilde{\theta}^1_{\alpha\hat\alpha}[e^B P_a e^{-B}]_{Q^1_{\alpha\hat\alpha}}) && (2.25)\\
&= -2\eta_{ac}[e^{-B}(\bar\partial\widetilde{x}^b K_b + \bar\partial\widetilde{\theta}^{1\alpha\hat\alpha}S^2_{\alpha\hat\alpha})e^B]_{K_c} \\
\frac{1}{2}\epsilon_{\alpha\beta}\epsilon_{\hat\alpha\hat\beta}A'_{Q^1_{\beta\hat\beta}} &= -(\partial\widetilde{x}_b[e^B Q^1_{\alpha\hat\alpha}e^{-B}]_{P_b} - \partial\widetilde{\theta}^1_{\beta\hat\beta}[e^B Q^1_{\alpha\hat\alpha}e^{-B}]_{Q^1_{\beta\hat\beta}}) && (2.26)\\
&= -\epsilon_{\alpha\beta}\epsilon_{\hat\alpha\hat\beta}[e^{-B}(\partial\widetilde{x}^b K_b + \partial\widetilde{\theta}^{1\gamma\hat\gamma}S^2_{\gamma\hat\gamma})e^B]_{S^2_{\beta\hat\beta}} \\
\frac{1}{2}\epsilon_{\alpha\beta}\epsilon_{\hat\alpha\hat\beta}\bar{A}'_{Q^1_{\beta\hat\beta}} &= -(\bar\partial\widetilde{x}_b[e^B Q^1_{\alpha\hat\alpha}e^{-B}]_{P_b} - \bar\partial\widetilde{\theta}^1_{\beta\hat\beta}[e^B Q^1_{\alpha\hat\alpha}e^{-B}]_{Q^1_{\beta\hat\beta}}) && (2.27)\\
&= -\epsilon_{\alpha\beta}\epsilon_{\hat\alpha\hat\beta}[e^{-B}(\bar\partial\widetilde{x}^b K_b + \bar\partial\widetilde{\theta}^{1\gamma\hat\gamma}S^2_{\gamma\hat\gamma})e^B]_{S^2_{\beta\hat\beta}}. && (2.28)
\end{aligned}
$$

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Thus,

$$
\begin{aligned}
S &= \frac{R^2}{4\pi\alpha'}\int d^2z\bigg[\frac{1}{2}A'_{P_a}\bar{A}'_{P_b}\eta_{ab}+\mathrm{j}_D\bar{\mathrm{j}}_D+\mathrm{j}_{R_\mu}\bar{\mathrm{j}}_{R_\mu} \\
&\qquad +\frac{1}{2}\epsilon_{\alpha\beta}\epsilon_{\hat\alpha\hat\beta}\Big(\mathrm{j}_{S^1_{\alpha\hat\alpha}}\bar{\mathrm{j}}_{S^1_{\beta\hat\beta}}-A'_{Q^1_{\alpha\hat\alpha}}\bar{A}'_{Q^1_{\beta\hat\beta}}-\mathrm{j}_{Q^2_{\alpha\hat\alpha}}\bar{\mathrm{j}}_{Q^2_{\beta\hat\beta}}\Big)\bigg] \qquad (2.29)\\
&= \frac{R^2}{4\pi\alpha'}\int d^2z\bigg[-\frac{1}{2}4[e^{-B}(\partial\widetilde{x}^bK_b+\partial\widetilde{\theta}^{1\alpha\hat\alpha}S^2_{\alpha\hat\alpha})e^B]_{K_a}[e^{-B}(\bar\partial\widetilde{x}^bK_b+\bar\partial\widetilde{\theta}^{1\alpha\hat\alpha}S^2_{\alpha\hat\alpha})e^B]_{K_b}\eta^{ab} \\
&\qquad +\mathrm{j}_D\bar{\mathrm{j}}_D+\mathrm{j}_{R_\mu}\bar{\mathrm{j}}_{R_\mu}+\frac{1}{2}\epsilon_{\alpha\beta}\epsilon_{\hat\alpha\hat\beta}\Big(\mathrm{j}_{S^1_{\alpha\hat\alpha}}\bar{\mathrm{j}}_{S^1_{\beta\hat\beta}}-\mathrm{j}_{Q^2_{\alpha\hat\alpha}}\bar{\mathrm{j}}_{Q^2_{\beta\hat\beta}} \\
&\qquad -4[e^{-B}(\partial\widetilde{x}^bK_b+\partial\widetilde{\theta}^{1\gamma\hat\gamma}S^2_{\gamma\hat\gamma})e^B]_{S^2_{\alpha\hat\alpha}}[e^{-B}(\bar\partial\widetilde{x}^bK_b+\bar\partial\widetilde{\theta}^{1\gamma\hat\gamma}S^2_{\gamma\hat\gamma})e^B]_{S^2_{\beta\hat\beta}}\Big)\bigg]. \qquad (2.30)
\end{aligned}
$$

We can define J''s

$$J' = e^{-B}(2\partial\widetilde{x}^bK_b+2\partial\widetilde{\theta}^{1\gamma\hat\gamma}S^2_{\gamma\hat\gamma})e^B, \qquad (2.31)$$

and the action reads

$$
\begin{aligned}
S &= \frac{R^2}{4\pi\alpha'}\int d^2z\bigg[-\frac{1}{2}J'_{P_a}\bar{J}'_{P_b}\eta_{ab}+\mathrm{j}_D\bar{\mathrm{j}}_D+\mathrm{j}_{R_\mu}\bar{\mathrm{j}}_{R_\mu} \qquad (2.32)\\
&\qquad +\frac{1}{2}\epsilon_{\alpha\beta}\epsilon_{\hat\alpha\hat\beta}\Big(\mathrm{j}_{S^1_{\alpha\hat\alpha}}\bar{\mathrm{j}}_{S^1_{\beta\hat\beta}}-J'_{S^2_{\alpha\hat\alpha}}\bar{J}'_{S^2_{\beta\hat\beta}}-\mathrm{j}_{Q^2_{\alpha\hat\alpha}}\bar{\mathrm{j}}_{QS^2_{\beta\hat\beta}}\Big)\bigg].
\end{aligned}
$$

Applying the following automorphism of the algebra

$$
\begin{array}{lll}
D\to -D, & K_a\leftrightarrow -\sigma^3_{ab}P_b, & J_{ab}\to -J_{ab}, \\
R_1\to R_1, & R_2\to -R_2, & R_3\to -R_3, \\
N_{12}\to -N_{12}, & N_{23}\to N_{23}, & N_{31}\to -N_{31}, \\
Q^I_{\alpha\hat\alpha}\leftrightarrow \sigma^1_{IJ}\sigma^1_{\hat\alpha\hat\beta}S^J_{\alpha\hat\beta}, & &
\end{array} \qquad (2.33)
$$

followed by the change of variables

$$
\begin{array}{llll}
x_a\to -2\sigma^3_{ab}\widetilde{x}_b, & \theta^1_{\alpha\hat\alpha}\to 2\sigma^1_{\hat\alpha\hat\beta}\widetilde{\theta}^2_{\alpha\hat\beta}, & & \\
\theta^2_{\alpha\hat\alpha}\leftrightarrow \sigma^1_{\hat\alpha\hat\beta}\xi^1_{\alpha\hat\beta}, & y_1\to y_1/y^2, & y_2\to -y_2/y^2, & y_3\to -y_3/y^2,
\end{array} \qquad (2.34)
$$

the action (2.32) is mapped to the original one (2.22).

3 Models based on the ortho-symplectic supergroup

In this section we consider models based on supercosets of the ortho-symplectic supergroup, for which the Buscher procedure [16, 17] of gauging an isometry of the target-space in order to obtain the T-dual sigma-model fails. These include the non-critical superstring on AdS_2 with four supersymmetries and AdS_4 with eight supersymmetries, the supercoset construction of $AdS_4\times\mathbb{C}\mathrm{P}^3$ which is conjectured to be dual to superconformal Chern-Simons theory in three dimensions [19], and a model of AdS_2 with eight supersymmetries.

3.1 The Green-Schwarz sigma-model on AdS_2 with four supersymmetries

The non-critical AdS_2 background with RR-flux can be realized as the supercoset $\mathrm{OSp}(2|2)/(\mathrm{SO}(1,1)\times\mathrm{SO}(2))$. The Green-Schwarz action is of the form (2.1) (see appendix A for the details of the algebra and the conventions that are used).

Fixing the $\mathrm{SO}(1,1)\times\mathrm{SO}(2)$ gauge symmetry and κ-symmetry one can choose the coset representative

$$g = e^{xP+\theta Q} e^{\bar\theta\bar Q+\bar\xi\bar S} y^D. \tag{3.1}$$

The generators P and Q form an Abelian subalgebra, which we will attempt to T-dualize. In this parameterization the Maurer-Cartan current J takes the form

$$\begin{aligned} J =& \frac{1}{y}\left(\partial x - 2\partial\theta\bar\theta\right)P + \frac{1}{y^{1/2}}\partial\theta Q + \frac{1}{y^{1/2}}\left[\left(2\partial\theta\bar\theta - \partial x\right)\bar\xi + \partial\bar\theta\right]\bar Q + y^{1/2}\partial\bar\xi\bar S + \\ &+ \left(\frac{\partial y}{y} - 2\partial\theta\bar\xi\right)D + i\partial\theta\bar\xi R. \end{aligned} \tag{3.2}$$

The sigma-model can be written explicitly as

$$\begin{aligned} S =& \frac{R^2}{4\pi\alpha'}\int d^2z\Big[-\frac{1}{2}(J_P+J_K)(\bar J_P+\bar J_K) + \frac{1}{2}J_D\bar J_D + J_Q\bar J_{\bar Q} + J_{\bar Q}\bar J_Q - \\ &- J_S\bar J_{\bar S} - J_{\bar S}\bar J_S\Big]. \end{aligned} \tag{3.3}$$

The action in terms of the coordinates is then given by

$$\begin{aligned} S =& \frac{R^2}{4\pi\alpha'}\int d^2z\Big[\frac{-\partial x\bar\partial x + \partial y\bar\partial y}{2y^2} - \frac{1}{y^2}\bar\theta(\partial x\bar\partial\theta + \partial\theta\bar\partial x) + \frac{1}{y}\bar\xi(\partial y\bar\partial\theta + \partial\theta\bar\partial y) + \\ &+ \frac{4}{y}\bar\theta\bar\xi\partial\theta\bar\partial\theta + \frac{1}{y}\bar\xi(\partial\theta\bar\partial x - \partial x\bar\partial\theta) + \frac{1}{y}(\partial\theta\bar\partial\bar\theta + \partial\bar\theta\bar\partial\theta)\Big]. \end{aligned} \tag{3.4}$$

Note that the action is indeed quadratic in θ so naively one should be able to T-dualize it along that coordinate.

An equivalent action can be written using two gauge fields [16, 17]

$$\begin{aligned} S =& \frac{R^2}{4\pi\alpha'}\int d^2z\Big[\frac{-A_x\bar A_x + \partial y\bar\partial y}{2y^2} - \frac{1}{y^2}\bar\theta(A_x\bar A_\theta + A_\theta\bar A_x) + \frac{1}{y}\bar\xi(\partial y\bar A_\theta + A_\theta\bar\partial y) + \\ &+ \frac{4}{y}\bar\theta\bar\xi A_\theta\bar A_\theta + \frac{1}{y}\bar\xi(A_\theta\bar A_x - A_x\bar A_\theta) + \frac{1}{y}(A_\theta\bar\partial\bar\theta + \partial\bar\theta\bar A_\theta) + \widetilde{x}(\bar\partial A_x - \partial\bar A_x) + \\ &+ \widetilde{\theta}(\bar\partial A_\theta - \partial\bar A_\theta)\Big]. \end{aligned} \tag{3.5}$$

The classical equations of motion for the gauge fields are

$$\frac{1}{2y^2}A_x + \frac{1}{y^2}\bar\theta A_\theta - \frac{1}{y}\bar\xi A_\theta - \partial\widetilde{x} = 0, \tag{3.6}$$

$$\frac{1}{2y^2}\bar A_x + \frac{1}{y^2}\bar\theta\bar A_\theta + \frac{1}{y}\bar\xi\bar A_\theta + \bar\partial\widetilde{x} = 0, \tag{3.7}$$

$$\frac{1}{y^2}\bar\theta A_x - \frac{1}{y}\bar\xi\partial y - \frac{4}{y}\bar\theta\bar\xi A_\theta + \frac{1}{y}\bar\xi A_x - \frac{1}{y}\partial\bar\theta - \partial\widetilde{\theta} = 0, \tag{3.8}$$

$$\frac{1}{y^2}\bar\theta\bar A_x - \frac{1}{y}\bar\xi\bar\partial y + \frac{4}{y}\bar\theta\bar\xi\bar A_\theta - \frac{1}{y}\bar\xi\bar A_x + \frac{1}{y}\bar\partial\bar\theta + \bar\partial\widetilde{\theta} = 0. \tag{3.9}$$

Since Grassmann variables such as $\bar{\theta}$ and $\bar{\xi}$ have non-trivial kernels, one cannot solve for A_θ and for $\bar{A}_\theta$. Solving for A_x and $\bar{A}_x$ one gets

$$A_x = -2\bar{\theta}A_\theta + 2y\bar{\xi}A_\theta + 2y^2\partial\widetilde{x}, \tag{3.10}$$
$$\bar{A}_x = -2\bar{\theta}\bar{A}_\theta - 2y\bar{\xi}\bar{A}_\theta - 2y^2\bar{\partial}\widetilde{x}. \tag{3.11}$$

Substituting (3.10) and (3.11) in the action (this can be done since they appear only quadratically in the action, but the classical procedure should be supplemented by a functional determinant coming from the Gaussian path integration of A_x and $\bar{A}_x$) yields

$$S = \frac{R^2}{4\pi\alpha'}\int d^2z\Big[-2y^2\partial\widetilde{x}\bar{\partial}\widetilde{x} - 2y\bar{\xi}\partial\widetilde{x}\bar{A}_\theta - 2y\bar{\xi}A_\theta\bar{\partial}\widetilde{x} - 2\bar{\theta}\partial\widetilde{x}\bar{A}_\theta + 2\bar{\theta}A_\theta\bar{\partial}\widetilde{x} + \\ + \frac{\partial y\bar{\partial}y}{2y^2} + \frac{1}{y}\bar{\xi}(\partial y\bar{A}_\theta + A_\theta\bar{\partial}y) + \frac{1}{y}(A_\theta\bar{\partial}\bar{\theta} + \partial\bar{\theta}\bar{A}_\theta) - \bar{\partial}\widetilde{\theta}A_\theta + \partial\widetilde{\theta}\bar{A}_\theta\Big]. \tag{3.12}$$

Thus, after integrating out A_x and $\bar{A}_x$ the remaining fermionic gauge fields A_θ and $\bar{A}_\theta$ serve as Lagrange multipliers forcing in the path integration over $\widetilde{\theta}$ that

$$2\bar{\theta}\partial\widetilde{x} - \frac{1}{y}\bar{\xi}\partial y + 2y\bar{\xi}\partial\widetilde{x} - \frac{1}{y}\partial\bar{\theta} - \partial\widetilde{\theta} = 0,$$
$$2\bar{\theta}\bar{\partial}\widetilde{x} + \frac{1}{y}\bar{\xi}\bar{\partial}y - 2y\bar{\xi}\bar{\partial}\widetilde{x} - \frac{1}{y}\bar{\partial}\bar{\theta} - \bar{\partial}\widetilde{\theta} = 0, \tag{3.13}$$

which express the non-zero modes of $\widetilde{\theta}$ in terms of the other fields, effectively reducing the path integration over $\widetilde{\theta}$ only to its zero-modes.

This appears rather strange as the original action did include a term quadratic in θ. A possible explanation is that the quadratic term is κ-symmetry-exact, and hence does not influence the equations of motion.

In order to support the above claim we do the same computation with a different gauge choice for the κ-symmetry. The coset representative in this gauge can be parameterized as

$$g = e^{xP+\theta Q}e^{\bar{\theta}\bar{Q}+\xi S}y^D, \tag{3.14}$$

and the Maurer-Cartan current is

$$J = \frac{1}{y}(\partial x - 2\partial\theta\bar{\theta} - \partial x\bar{\theta}\xi)P + \frac{1}{y^{1/2}}(\partial\theta - \partial x\xi + \partial\theta\bar{\theta}\xi)Q + \frac{1}{y^{1/2}}\partial\bar{\theta}\bar{Q} + y^{1/2}\partial\xi S + \\ + \left(\frac{\partial y}{y} - \partial\bar{\theta}\xi - \partial\xi\bar{\theta}\right)D - \frac{i}{2}(\partial\bar{\theta}\xi + \partial\xi\bar{\theta})R. \tag{3.15}$$

In this gauge the action is

$$S = \frac{R^2}{4\pi\alpha'}\int d^2z\Bigg[\frac{(-1+\bar{\theta}\xi)\partial x\bar{\partial}x + \partial y\bar{\partial}y}{2y^2} - \frac{1}{y^2}\bar{\theta}(\partial x\bar{\partial}\theta + \partial\theta\bar{\partial}x) - \\ - \frac{1}{y}\xi(\partial x\bar{\partial}\bar{\theta} - \partial\bar{\theta}\bar{\partial}x) + \frac{1}{2y}\xi(\partial y\bar{\partial}\bar{\theta} + \partial\bar{\theta}\bar{\partial}y) + \frac{1}{2y}\bar{\theta}(\partial y\bar{\partial}\xi + \partial\xi\bar{\partial}y) + \\ + \frac{1}{2}\bar{\theta}\xi(\partial\bar{\theta}\bar{\partial}\xi - \partial\xi\bar{\partial}\bar{\theta}) + \frac{1}{y}(1+\bar{\theta}\xi)(\partial\theta\bar{\partial}\bar{\theta} + \partial\bar{\theta}\bar{\partial}\theta)\Bigg]. \tag{3.16}$$

This action has no quadratic term in the Grassmann coordinate θ so the usual procedure introduced by Buscher [16, 17] cannot be applied. This lends credence to the explanation that in the different gauge above the procedure failed because the quadratic term is κ-symmetry-exact.

3.2 The AdS_4 target-space

The non-critical superstring on AdS_4 with eight supersymmetries can be constructed as the supercoset $\mathrm{OSp}(2|4)/(\mathrm{SO}(3,1)\times\mathrm{SO}(2))$. Taking $g\in\mathrm{OSp}(2|4)$ in (2.1) one obtains the sigma-model

$$S=-\frac{R^2}{4\pi\alpha'}\int d^2z\Big[\eta_{mn}(J_{P_m}+J_{K_m})(\bar{J}_{P_n}+\bar{J}_{K_n})+J_D\bar{J}_D+ \\ +4i\epsilon_{\alpha\beta}\left(J_{Q_\alpha}\bar{J}_{\hat{Q}_\beta}-J_{\hat{Q}_\alpha}\bar{J}_{Q_\beta}+J_{S_\alpha}\bar{J}_{\hat{S}_\beta}-J_{\hat{S}_\alpha}\bar{J}_{S_\beta}\right), \tag{3.17}$$

where the conventions for the algebra are given in appendix B.

The general group element can be parameterized as

$$g=e^{x^mP_m+w^mK_m+\theta^\alpha Q_\alpha}e^B,\quad e^B=e^{\hat{\theta}^\alpha\hat{Q}_\alpha+\xi^\alpha S_\alpha+\hat{\xi}^\alpha\hat{S}_\alpha}y^De^{\phi R+\omega^{mn}M_{mn}}. \tag{3.18}$$

We assume that we can partially gauge-fix the κ-symmetry such that $\hat{\xi}^\alpha=0$ and we will also fix the $\mathrm{SO}(3,1)\times\mathrm{SO}(2)$ gauge symmetry by setting $w^m=0$, $\phi=0$ and $\omega^{mn}=0$ essentially picking the specific coset representative

$$g=e^{x^mP_m+\theta^\alpha Q_\alpha}e^B,\quad e^B=e^{\hat{\theta}^\alpha\hat{Q}_\alpha+\xi^\alpha S_\alpha}y^D. \tag{3.19}$$

One can check that the needed components of the Maurer-Cartan 1-form are

$$\begin{aligned}
J_{P_m}&=\left[e^{-B}(dx^nP_n+d\theta^\alpha Q_\alpha)e^B\right]_{P_m}, && J_{K_m}=0,\\
J_{Q_\alpha}&=\left[e^{-B}(dx^mP_m+d\theta^\beta Q_\beta)e^B\right]_{Q_\alpha}, && J_{S_\alpha}=\left[e^{-B}de^B\right]_{S_\alpha},\\
J_{\hat{Q}_\alpha}&=\left[e^{-B}de^B\right]_{\hat{Q}_\alpha},\quad J_{\hat{S}_\alpha}=0, && J_D=\left[e^{-B}de^B\right]_D.
\end{aligned} \tag{3.20}$$

The action then becomes

$$\begin{aligned}
S=-\frac{R^2}{4\pi\alpha'}\int d^2z\Big[&\eta_{mn}\left[e^{-B}(\partial x^lP_l+\partial\theta^\alpha Q_\alpha)e^B\right]_{P_m}\left[e^{-B}(\bar\partial x^pP_p+\bar\partial\theta^\beta Q_\beta)e^B\right]_{P_n}+\\
&+\left[e^{-B}\partial e^B\right]_D\left[e^{-B}\bar\partial e^B\right]_D+4i\epsilon_{\alpha\beta}\Big(\left[e^{-B}(\partial x^mP_m+\partial\theta^\gamma Q_\gamma)e^B\right]_{Q_\alpha}\left[e^{-B}\bar\partial e^B\right]_{\hat{Q}_\alpha}\\
&-\left[e^{-B}\partial e^B\right]_{\hat{Q}_\alpha}\left[e^{-B}(\bar\partial x^mP_m+\bar\partial\theta^\gamma Q_\gamma)e^B\right]_{Q_\beta}\Big)\Big].
\end{aligned} \tag{3.21}$$

Replacing the partial derivatives of x^m and of θ^α by the gauge fields A^m, $\bar{A}^m$, A^α and $\bar{A}^\alpha$ and then performing the field redefinition

$$\begin{aligned}
A'^m&=\left[e^{-B}(A^nP_n+A^\alpha Q_\alpha)e^B\right]_{P_m},\\
A'^\alpha&=\left[e^{-B}(A^mP_m+A^\beta Q_\beta)e^B\right]_{Q_\alpha}
\end{aligned} \tag{3.22}$$

with similar expressions for the right-moving sector and introducing the Lagrange multiplier terms forcing the gauge fields to be flat the new equivalent action takes the form

$$S = -\frac{R^2}{4\pi\alpha'}\int d^2z\Big[\eta_{mn}A'^m\bar{A}'^n + J_D\bar{J}_D + 4i\epsilon_{\alpha\beta}\Big(A'^\alpha\left[e^{-B}\bar{\partial}e^B\right]_{\hat{Q}_\beta} - \\ - \left[e^{-B}\partial e^B\right]_{\hat{Q}_\alpha}\bar{A}'^\beta\Big) + \widetilde{x}_m(\bar{\partial}A^m - \partial\bar{A}^m) + \widetilde{\theta}_\alpha(\bar{\partial}A^\alpha - \partial\bar{A}^\alpha)\Big]. \tag{3.23}$$

As can be immediately seen, just as for the case of the AdS_2 space discussed in section 3.1 for the (physically trivial) AdS_2 the fermionic gauge fields A'^α and $\bar{A}'^\alpha$ appear only linearly in the action and hence integration over them will yield constraints rather than equations of motion.

3.3 The $AdS_4 \times \mathbb{C}P^3$ target-space

Using the same procedure as for the $AdS_n \times S^n$ models we construct the Green-Schwarz sigma model action (2.1) using the bilinear forms of (C.21),

$$S = \frac{R^2}{4\pi\alpha'}\int d^2z\Big[-2\eta_{ab}(J_{P_a}+J_{K_a})(\bar{J}_{P_b}+\bar{J}_{K_b}) \tag{3.24}\\ -J_D\bar{J}_D - 2(J_{R_{kl}}\bar{J}_{R_{\dot{p}\dot{q}}} + \bar{J}_{R_{kl}}J_{R_{\dot{p}\dot{q}}})(\delta^{k\dot{p}}\delta^{l\dot{q}} - \delta^{k\dot{q}}\delta^{l\dot{p}}) \\ -2i\delta^{\dot{k}l}C_{\alpha\beta}\Big(J_{Q^l_\alpha}\bar{J}_{Q^{\dot{k}}_\beta} - \bar{J}_{Q^l_\alpha}J_{Q^{\dot{k}}_\beta} - J_{S^l_\alpha}\bar{J}_{S^{\dot{k}}_\beta} + \bar{J}_{S^l_\alpha}J_{S^{\dot{k}}_\beta}\Big)\Big].$$

We proceed by taking the gauge-fixed coset representative

$$g = e^{x^aP_a + \theta^\alpha_l Q^l_\alpha}e^B, \quad e^B \equiv e^{\theta^\alpha_{\dot{l}}Q^{\dot{l}}_\alpha + \xi^\alpha_l S^l_\alpha}y^D e^{(\Sigma y^{kl}R_{kl} + \Sigma y^{\dot{k}\dot{l}}R_{\dot{k}\dot{l}})/y}, \tag{3.25}$$

where we also fixed six out of the eight κ-symmetry degrees of freedom by setting $\xi^\alpha_l = 0$. P_a and $\theta^\alpha_l Q^l_\alpha$ form an Abelian subalgebra.

Next, we would like to write the currents in terms of this parameterization, but in this case the commutations relations $[Q,R] \sim \bar{Q}$ and $\{Q,S\} \sim R$ prevent us from getting two kinds of currents as in (2.20), so

$$J_T = [e^{-B}(dx^aP_a + d\theta^\alpha_l Q^l_\alpha)e^B]_T + [e^{-B}de^B]_T \equiv j_T + \mathrm{j}_T. \tag{3.26}$$

We have

$$j = j_{P_a}P_a + j_{Q^\alpha_l}Q^\alpha_l + j_{Q^\alpha_{\dot{l}}}Q^\alpha_{\dot{l}} + j_{R_{kl}}R_{kl} + j_{R_{\dot{k}\dot{l}}}R_{\dot{k}\dot{l}} + j_{\lambda_{\kappa\dot{l}}}\lambda_{\kappa\dot{l}}, \tag{3.27}$$

and

$$\mathrm{j} = \mathrm{j}_{Q^\alpha_l}Q^\alpha_l + \mathrm{j}_{Q^\alpha_{\dot{l}}}Q^\alpha_{\dot{l}} + \mathrm{j}_{S^\alpha_l}S^\alpha_l + \mathrm{j}_{S^\alpha_{\dot{l}}}S^\alpha_{\dot{l}} + \mathrm{j}_{R_{kl}}R_{kl} + \mathrm{j}_{R_{\dot{k}\dot{l}}}R_{\dot{k}\dot{l}} + \mathrm{j}_{\lambda_{\kappa\dot{l}}}\lambda_{\kappa\dot{l}} + \mathrm{j}_D D + \mathrm{j}_{M_{mn}}M_{mn}. \tag{3.28}$$

The action in terms of these currents reads

$$S = \frac{R^2}{4\pi\alpha'}\int d^2z\Big[-2\eta^{ab}j_{P_a}\bar{j}_{P_b} - \mathrm{j}_D\bar{\mathrm{j}}_D \tag{3.29}\\ -2((j+\mathrm{j})_{R_{kl}}(\bar{j}+\bar{\mathrm{j}})_{R_{\dot{p}\dot{q}}} + (\bar{j}+\bar{\mathrm{j}})_{R_{kl}}(j+\mathrm{j})_{R_{\dot{p}\dot{q}}})(\delta^{k\dot{p}}\delta^{l\dot{q}} - \delta^{k\dot{q}}\delta^{l\dot{p}}) \\ -2i\delta^{\dot{k}l}C_{\alpha\beta}\Big((j+\mathrm{j})_{Q^l_\alpha}(\bar{j}+\bar{\mathrm{j}})_{Q^{\dot{k}}_\beta} - (\bar{j}+\bar{\mathrm{j}})_{Q^l_\alpha}(j+\mathrm{j})_{Q^{\dot{k}}_\beta} - \mathrm{j}_{S^l_\alpha}\mathrm{j}_{S^{\dot{k}}_\beta} + \mathrm{j}_{S^l_\alpha}\mathrm{j}_{S^{\dot{k}}_\beta}\Big)\Big].$$

We add gauge fields instead of the x_a and θ_l^α derivatives and a suitable Lagrange multiplier term. We define

$$A' = e^{-B}(A^a P_a + A_l^\alpha Q_\alpha^l)e^B, \tag{3.30}$$

(where $A^a \equiv A_{P_a}$ and $A_l^\alpha \equiv A_{Q_\alpha^l}$) and the action becomes

$$\begin{aligned} S = \frac{R^2}{4\pi\alpha'} \int d^2z \Big[& -2\eta_{ab} A'_{P_a} \bar{A}'_{P_b} - \mathrm{j}_D \bar{\mathrm{j}}_D \\ & -2((A'+\mathrm{j})_{R_{kl}}(\bar{A}'+\bar{\mathrm{j}})_{R_{\dot{p}\dot{q}}} + (\bar{A}'+\bar{\mathrm{j}})_{R_{kl}}(A'+\mathrm{j})_{R_{\dot{p}\dot{q}}})(\delta^{k\dot{p}}\delta^{l\dot{q}} - \delta^{k\dot{q}}\delta^{l\dot{p}}) \\ & -2i\delta^{\dot{k}l} C_{\alpha\beta}\Big((A'+\mathrm{j})_{Q_\alpha^l}(\bar{A}'+\bar{\mathrm{j}})_{Q_\beta^{\dot{k}}} - (\bar{A}'+\bar{\mathrm{j}})_{Q_\alpha^l}(A'+\mathrm{j})_{Q_\beta^{\dot{k}}} - \mathrm{j}_{S_\alpha^l}\mathrm{j}_{S_\beta^{\dot{k}}} + \mathrm{j}_{S_\alpha^l}\mathrm{j}_{S_\beta^{\dot{k}}}\Big) \\ & +\widetilde{x}^a(\partial\bar{A}_a - \bar{\partial}A_a) + \widetilde{\theta}_l^\alpha(\partial\bar{A}_\alpha^l - \bar{\partial}A_\alpha^l)\Big]. \end{aligned} \tag{3.31}$$

3.3.1 Current expansion in fermions

To zeroth order in the fermions $(\theta_l^\alpha, \theta_{\dot{l}}^\alpha, \xi_l^\alpha)$ we get a bosonic sigma model with no fermions, which we can T-dualize,

$$\begin{aligned} S_0 = \frac{R^2}{4\pi\alpha'} \int d^2z \Bigg[& -2y^2\eta_{ab} A_{P_a} \bar{A}_{P_b} - j_D \bar{j}_D \\ & -2\Big\{ \mathrm{j}_{R_{kl}} \bar{\mathrm{j}}_{R_{\dot{p}\dot{q}}} + \bar{\mathrm{j}}_{R_{kl}} \mathrm{j}_{R_{\dot{p}\dot{q}}} \Big\}(\delta^{k\dot{p}}\delta^{l\dot{q}} - \delta^{k\dot{q}}\delta^{l\dot{p}}) + \widetilde{x}^a(\partial\bar{A}_a - \bar{\partial}A_a)\Bigg], \end{aligned} \tag{3.32}$$

where $\mathrm{j} = e^{-B}de^B$, $e^B = y^D e^{(\Sigma y^{kl}R_{kl} + \Sigma y^{\dot{k}\dot{l}}R_{\dot{k}\dot{l}})/y}$.

Next, leaving only θ_l^α terms we find

$$e^{-B}(A^a P_a + A_\lambda^\alpha Q_\alpha^l)e^B = yA^a P_a + y^{1/2} A_k^\alpha \left(f_l^k \left(\frac{y^{mn}}{y}, \frac{y^{\dot{m}\dot{n}}}{y} \right) Q_\alpha^l + g_{\dot{l}}^k \left(\frac{y^{mn}}{y}, \frac{y^{\dot{m}\dot{n}}}{y} \right) Q_\alpha^{\dot{l}} \right), \tag{3.33}$$

thus the A''s in terms of the A's are

$$A'^a = yA^a, \quad A_l'^\alpha = y^{1/2} A_k^\alpha f_l^k \left(\frac{y^{mn}}{y}, \frac{y^{\dot{m}\dot{n}}}{y} \right), \quad A_{\dot{l}}'^\alpha = y^{1/2} A_k^\alpha g_{\dot{l}}^k \left(\frac{y^{mn}}{y}, \frac{y^{\dot{m}\dot{n}}}{y} \right). \tag{3.34}$$

Plugging these in the action we have

$$\begin{aligned} S_1 &= \frac{R^2}{4\pi\alpha'} \int d^2z \Big[-2y^2\eta_{ab} A_{P_a} \bar{A}_{P_b} - \mathrm{j}_D \bar{\mathrm{j}}_D - 2\Big\{ \mathrm{j}_{R_{kl}} \bar{\mathrm{j}}_{R_{\dot{p}\dot{q}}} + \bar{\mathrm{j}}_{R_{kl}} \mathrm{j}_{R_{\dot{p}\dot{q}}} \Big\}(\delta^{k\dot{p}}\delta^{l\dot{q}} - \delta^{k\dot{q}}\delta^{l\dot{p}}) \\ & \quad -2iy\delta^{\dot{k}l} C_{\alpha\beta}\Big(A_n^\alpha f_l^n \bar{A}_m^\beta g_{\dot{k}}^m - \bar{A}_n^\alpha f_l^n A_m^\beta g_{\dot{k}}^m \Big) + \widetilde{x}^a(\partial\bar{A}_a - \bar{\partial}A_a) + \widetilde{\theta}_l^\alpha(\partial\bar{A}_\alpha^l - \bar{\partial}A_\alpha^l)\Big] \\ &= \frac{R^2}{4\pi\alpha'} \int d^2z \Big[-2y^2\eta_{ab} A_{P_a} \bar{A}_{P_b} - \mathrm{j}_D \bar{\mathrm{j}}_D - 2\Big\{ \mathrm{j}_{R_{kl}} \bar{\mathrm{j}}_{R_{\dot{p}\dot{q}}} + \bar{\mathrm{j}}_{R_{kl}} \mathrm{j}_{R_{\dot{p}\dot{q}}} \Big\}(\delta^{k\dot{p}}\delta^{l\dot{q}} - \delta^{k\dot{q}}\delta^{l\dot{p}}) \\ & \quad -2iy\delta^{\dot{k}l} C_{\alpha\beta} A_n^\alpha \bar{A}_m^\beta \Big(f_l^n g_{\dot{k}}^m - f_l^m g_{\dot{k}}^n \Big) + \widetilde{x}^a(\partial\bar{A}_a - \bar{\partial}A_a) + \widetilde{\theta}_l^\alpha(\partial\bar{A}_\alpha^l - \bar{\partial}A_\alpha^l)\Big]. \end{aligned} \tag{3.35}$$

The bosonic T-duality works as before, but now we also have a quadratic term for the fermions. The equation of motion for the fermions is

$$A_m^\alpha \delta^{l\dot{k}} \left(f_l^n g_{\dot{k}}^m - f_l^m g_{\dot{k}}^n \right) = -\frac{i}{2y} C^{\alpha\beta} \partial\theta_\beta^n. \tag{3.36}$$

The matrix $M^{mn} \equiv \delta^{l\dot{k}}\left(f_l^n g_{\dot{k}}^m - f_l^m g_{\dot{k}}^n\right) = -4iy^{\dot{n}\dot{m}}/y + O(y^{\dot{n}\dot{m}}y^{kl}/y^2)$, is an antisymmetric three-dimensional matrix and hence has a vanishing determinant, so we cannot solve for A_l^α.

Next, to first order in ξ_l^α we have nontrivial $\mathrm{j}_{S_\alpha^l}$ and $\mathrm{j}_{S_\alpha^{\dot{l}}}$, but these terms do not mix with the A's so we will ignore them. The A's change as follows,

$$\begin{aligned} &A_k^\beta \to A_k^\beta - iA^a(\gamma_a)_\alpha{}^\beta \xi_k^\alpha \\ &A^a \to A^a \\ &A_{kl} \to A_k^\beta C_{\beta\alpha}\xi_l^\alpha, \qquad\qquad A'^{rs} \to h_{kl}^{rs}(y^{ij}/y)A_k^\beta C_{\beta\alpha}\xi_l^\alpha. \end{aligned} \tag{3.37}$$

Thus the change in (3.36) goes like

$$(A_m^\alpha - iA^a(\gamma_a)_\beta{}^\alpha\xi_m^\beta)\delta^{l\dot{k}}\left(f_l^n g_{\dot{k}}^m - f_l^m g_{\dot{k}}^n\right) = -\frac{i}{2y}C^{\alpha\beta}\partial\theta_\beta^n + k^{\alpha n}(\xi, \mathrm{j}_{R_{kl}}, y, \ldots), \tag{3.38}$$

and again, we have the same singular matrix M^{mn} multiplying the gauge fields A. The second equation, which we can think of as the first order correction to the bosonic T-duality equation, is

$$2y^2A^a + 2\delta^{\dot{k}l}\xi_l^\gamma(\gamma^a C)_{\gamma\beta}(\sqrt{y}A_m^\beta g_{\dot{k}}^m + \mathrm{j}_{\dot{k}}^\beta) = -\partial\tilde{x}^a. \tag{3.39}$$

We can plug the solution for A^m in (3.38), but we will just get an expression of the form

$$A_l^\beta(\delta_\beta^\alpha\delta_m^l + \phi_\beta^\alpha F_m^l)M^{mn} = -\frac{i}{2y}C^{\alpha\beta}\partial\theta_\beta^n + k^{\alpha n}(\xi, \mathrm{j}_{R_{kl}}, y, \ldots) + \cdots \tag{3.40}$$

where ϕ, F, k and $\ldots$ are functions of the coordinates but not of A, so again we cannot solve for A_l^β.

Going next to first order in $\theta_{\dot{l}}^\alpha$ (but leaving out terms of order $\theta_{\dot{l}}^\alpha\xi_k^\beta$) we have nontrivial $\mathrm{j}_{Q_\alpha^l}$ and $\mathrm{j}_{Q_\alpha^{\dot{l}}}$, and also

$$A^a \to A^a + A_k^\beta\delta^{\dot{l}k}(\gamma^a C)_{\beta\alpha}\theta_{\dot{l}}^\alpha \tag{3.41}$$

with respect to (3.35), so the action we get is

$$\begin{aligned} S_2 = \frac{R^2}{4\pi\alpha'}\int d^2z\Big[&-2y^2\eta_{ab}(A^a + A_k^\beta\delta^{\dot{l}k}(\gamma^a C)_{\beta\alpha}\theta_{\dot{l}}^\alpha)(\bar{A}^b + \bar{A}_m^\gamma\delta^{\dot{n}m}(\gamma^b C)_{\gamma\delta}\theta_{\dot{n}}^\delta) &(3.42)\\ &-\mathrm{j}_D\bar{\mathrm{j}}_D - 2\Big\{(h_{rs}^{kl}A_r^\beta C_{\beta\alpha}\xi_s^\alpha + \mathrm{j}_{kl})\bar{\mathrm{j}}_{\dot{p}\dot{q}} + (h_{mn}^{kl}\bar{A}_m^\beta C_{\beta\alpha}\xi_n^\alpha + \bar{\mathrm{j}}_{kl})\mathrm{j}_{\dot{p}\dot{q}}\Big\}(\delta^{k\dot{p}}\delta^{l\dot{q}} - \delta^{k\dot{q}}\delta^{l\dot{p}}) \\ &-2i\delta^{\dot{k}l}C_{\alpha\beta}\Big(y(f_l^m(A_m^\alpha - iA^a(\gamma_a)_\delta{}^\alpha\xi_m^\delta) + \mathrm{j}_l^\alpha)(g_{\dot{k}}^n(\bar{A}_n^\beta - i\bar{A}^b(\gamma_b)_\gamma{}^\beta\xi_n^\gamma) + \bar{\mathrm{j}}_{\dot{k}}^\beta) \\ &-y(f_l^m(\bar{A}_m^\alpha - i\bar{A}^a(\gamma_a)_\delta{}^\alpha\xi_m^\delta) + \bar{\mathrm{j}}_l^\alpha)(g_{\dot{k}}^n(A_n^\beta - iA^b(\gamma_b)_\gamma{}^\beta\xi_n^\gamma) + \mathrm{j}_{\dot{k}}^\beta) - \mathrm{j}_{S_\alpha^l}\mathrm{j}_{S_\beta^{\dot{k}}} + \mathrm{j}_{S_\alpha^l}\mathrm{j}_{S_\beta^{\dot{k}}}\Big) \\ &+\tilde{x}^a(\partial\bar{A}_a - \bar{\partial}A_a) + \tilde{\theta}_l^\alpha(\partial\bar{A}_\alpha^l - \bar{\partial}A_\alpha^l)\Big]. &(3.43) \end{aligned}$$

The equations of motion (to first order in ξ or θ) are,

$$\begin{aligned} -2yA_a + A_k^\beta\delta^{\dot{l}k}(\gamma_a C)_{\beta\alpha}\theta_{\dot{l}}^\alpha + \qquad &(3.44)\\ -2\delta^{\dot{k}l}yC_{\alpha\beta}\Big[g_{\dot{k}}^n(f_l^m A_m^\alpha + \mathrm{j}_l^\alpha)(\gamma_a)_\gamma{}^\beta\xi_n^\gamma - f_l^m(\gamma_a)_\delta{}^\alpha\xi_m^\delta(g_{\dot{k}}^n A_n^\beta + \mathrm{j}_{\dot{k}}^\beta)\Big] = \partial\tilde{x}_a \\ 2i\delta^{\dot{k}l}C_{\alpha\beta}y\Big\{(f_l^m(A_m^\alpha - iA^a(\gamma_a)_\delta{}^\alpha\xi_m^\delta) + \mathrm{j}_l^\alpha)g_{\dot{k}}^n - f_l^n(g_{\dot{k}}^m(A_m^\alpha - iA^a(\gamma_a)_\gamma{}^\alpha\xi_m^\gamma) + \mathrm{j}_{\dot{k}}^\alpha)\Big\} + \qquad &(3.45)\\ -2yA^a\delta^{n\dot{m}}(\gamma_a C)_{\beta\delta}\theta_{\dot{m}}^\delta + F_\beta^n = 0. \end{aligned}$$

Plugging (3.44) in (3.45) we find

$$2i\delta^{\dot{k}l}C_{\alpha\beta}y\left\{\left(f_l^m\left(A_m^\alpha+i\frac{\partial\widetilde{x}^a}{2y}(\gamma_a)_\delta{}^\alpha\xi_m^\delta\right)+\mathfrak{j}_l^\alpha\right)g_{\dot{k}}^n-f_l^n\left(g_{\dot{k}}^m\left(A_m^\alpha+i\frac{\partial\widetilde{x}^a}{2y}(\gamma_a)_\gamma{}^\alpha\xi_m^\gamma\right)+\mathfrak{j}_{\dot{k}}^\alpha\right)\right\}+ \\ +\partial\widetilde{x}^a\delta^{n\dot{m}}(\gamma_a C)_{\beta\delta}\theta_{\dot{m}}^\delta+F_\beta^n=0. \tag{3.46}$$

Thus again the A_l^α will multiply the same singular matrix.

More generally, the bosonic singular matrix M will get corrections with even powers of the fermionic variables,

$$M_{mn}=M_{mn}^s+M_{mn}^2+M_{mn}^4+\cdots, \tag{3.47}$$

where M_{mn}^{2n} is a matrix function with a power of $2n$ of the fermionic variables $\theta_{\dot{k}}^\alpha$ and/or ξ_k^α, and M_{mn}^s is the purely bosonic singular matrix. In order to solve the equations of motion we will need the inverse of M_{mn}, but such an inverse does not exists, because all the matrices involving fermions should cancel each other order by order and we should get,

$$\delta_{mn}=M_{mk}M_{kn}^{-1}=M_{mk}^s(M^{-1})_{kn}^s+\cdots=M_{mk}^s(M^{-1})_{kn}^s, \tag{3.48}$$

but M_{mn}^s is singular.

3.4 The AdS_2 target-space with eight supersymmetries

Similarly to section 2.1 we construct a Green-Schwarz sigma-model action on AdS_2, but using a different supercoset, namely PSU$(1,1|2)/(\mathrm{U}(1)\times\mathrm{SU}(2))$, so this time we will have eight supersymmetries rather then four. Using the $\mathbb{Z}_4$ structure of this super-algebra given in (D.7) we construct the Green-Schwarz action

$$S=\frac{R^2}{4\pi\alpha'}\int d^2z\left(-J_P\bar{J}_K-J_K\bar{J}_P+\frac{i}{2}\epsilon_{\alpha\beta}(J_{Q_\alpha}\bar{J}_{S_\beta}-J_{\hat{Q}_\alpha}\bar{J}_{\hat{S}_\beta}-J_{S_\alpha}\bar{J}_{Q_\beta}+J_{\hat{S}_\alpha}\bar{J}_{\hat{Q}_\beta}\right). \tag{3.49}$$

We parameterize the coset representatives such that

$$g=e^{xP+\theta^\alpha Q_\alpha}e^B,\qquad e^B=e^{\hat{\theta}^\alpha\hat{Q}_\alpha+\xi^\alpha S_\alpha+\hat{\xi}^\alpha\hat{S}_\alpha}e^{yK}. \tag{3.50}$$

If we define the following currents

$$J=g^{-1}dg=j+\mathfrak{j},\quad j=e^{-B}(dxP+d\theta^\alpha Q_\alpha)e^B,\qquad \mathfrak{j}=e^{-B}de^B, \tag{3.51}$$

Expanding j to zeroth order in the fermions in B we get,

$$j_{(0)}=dxP+s\theta^\alpha Q_\alpha+2ydxD+id\theta^\alpha\hat{S}_\alpha-y^2dxK. \tag{3.52}$$

Thus by using the Buscher procedure of introducing gauge fields as in the examples above, to this order the action will not have quadratic terms in the fermions, so all the quadratic $d\theta$ (or more precisely A^α) terms in the action, coming from higher orders in the fermions in B, will multiply some fermions, and we will not be able to solve the equations of motion for the fermions. The reason again is that we will have to multiply the equations of motion by a singular matrix.

4 A general analysis

In this section we present a general algebraic argument to when a supercoset is expected to have a fermionic T-duality symmetry, and when it will fail to have one.

Let us assume a superconformal algebra $\mathcal{G}$ with a $\mathbb{Z}_4$ automorphism structure with the zero grading subalgebra $\mathcal{H}$, with the usual conformal bosonic generators P_μ, K_μ, D, $J_{\mu\nu}$, supercharges Q, $\bar{Q}$, superconformal generators S, $\bar{S}$ and R-symmetry generators R_a. We can find an Abelian subalgebra $\mathcal{A}$ composed of a subset of the P's and Q's if we can find an anti-commuting combination of supercharges. We denote by $A, B, ..$ the indices of $\mathcal{A}$. Uppercase letters denote both bosonic and fermionic indices, lowercase letters bosonic ones and Greek letters fermionic ones, $A = \{a, \alpha\}$. We gauge fix the $\mathcal{H}$ symmetry and parameterize the coset element as follows,

$$g = e^{x_A T^A} e^B, \quad x_A T^A \in \mathcal{A}, \quad B \in \mathcal{G} \ominus \mathcal{A}. \tag{4.1}$$

More specifically we parameterize

$$e^B = \exp(\bar{\theta}\bar{Q} + \xi S + \bar{\xi}\bar{S}) y^D \exp(y^\mu R_\mu / y), \tag{4.2}$$

where contraction of the fermions is understood (R_μ includes only the generators in the supercoset, $R_\mu \in \mathcal{G}/\mathcal{H}$). The left-invariant one-form current will split into two pieces,

$$J = e^{-B} dx_a T^a e^B + e^{-B} de^B \equiv j + \mathrm{j}, \tag{4.3}$$

where j is independent of the coordinates associated with the Abelian subalgebra x_a. Generally, j and j can take any value in $\mathcal{G}$. We give special indices to these generators as follows,

$$j = j_I T^I, \quad \mathrm{j} = \mathrm{j}_W T^W \tag{4.4}$$

(in general a is both in I and W and $T^I \cap T^W \neq \varnothing$), so we can write the general Green-Schwarz sigma model action as,

$$\begin{aligned} S = \int d^2 z \Big[& j_I^{(2)} \bar{j}_J^{(2)} \eta^{IJ} + \mathrm{j}_W^{(2)} \bar{j}_J^{(2)} \eta^{WJ} + j_I^{(2)} \bar{\mathrm{j}}_X^{(2)} \eta^{IX} + \mathrm{j}_W^{(2)} \bar{\mathrm{j}}_X^{(2)} \eta^{WX} \\ & + \frac{1}{2} \Big(j_I^{(1)} \bar{j}_J^{(3)} \eta^{IJ} + \mathrm{j}_W^{(1)} \bar{j}_J^{(3)} \eta^{WJ} + j_I^{(1)} \bar{\mathrm{j}}_X^{(3)} \eta^{IX} + \mathrm{j}_W^{(1)} \bar{\mathrm{j}}_X^{(3)} \eta^{WX} \Big) \\ & - \frac{1}{2} \Big(\bar{j}_I^{(1)} j_J^{(3)} \eta^{IJ} + \bar{\mathrm{j}}_W^{(1)} j_J^{(3)} \eta^{WJ} + \bar{j}_I^{(1)} \mathrm{j}_X^{(3)} \eta^{IX} + \bar{\mathrm{j}}_W^{(1)} \mathrm{j}_X^{(3)} \eta^{WX} \Big) \Big] \\ = \int d^2 z \Big[& A_I'^{(2)} \bar{A}_J'^{(2)} \eta^{IJ} + \mathrm{j}_W^{(2)} \bar{A}_J'^{(2)} \eta^{WJ} + A_I'^{(2)} \bar{\mathrm{j}}_X^{(2)} \eta^{IX} + \mathrm{j}_W^{(2)} \bar{\mathrm{j}}_X^{(2)} \eta^{WX} \\ & + \frac{1}{2} \Big(A_I'^{(1)} \bar{A}_J'^{(3)} \eta^{IJ} + \mathrm{j}_W^{(1)} \bar{A}_J'^{(3)} \eta^{WJ} + A_I'^{(1)} \bar{\mathrm{j}}_X^{(3)} \eta^{IX} + \mathrm{j}_W^{(1)} \bar{\mathrm{j}}_X^{(3)} \eta^{WX} \Big) \\ & - \frac{1}{2} \Big(\bar{A}_I'^{(1)} A_J'^{(3)} \eta^{IJ} + \bar{\mathrm{j}}_W^{(1)} A_J'^{(3)} \eta^{WJ} + \bar{A}_I'^{(1)} \mathrm{j}_X^{(3)} \eta^{IX} + \bar{\mathrm{j}}_W^{(1)} \mathrm{j}_X^{(3)} \eta^{WX} \Big) \\ & + \widetilde{x}^A (\partial \bar{A}_A - \bar{\partial} A_A) \Big], \end{aligned} \tag{4.5}$$

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where A' equals j when replacing the coordinate derivatives dx_A with the gauge field A_A, and $\widetilde{x}_A$ is the Lagrange multiplier. By the structure of j, (4.3), we see that A' depends linearly on A, so quadratic terms will rise only from $A'\bar{A}'$ interactions in the action. We also know that

$$A'^{(1)} = Q^I A'^{(1)}_{Q^I} + \bar{Q}^I A'^{(1)}_{\bar{Q}^I} + S^I A'^{(1)}_{S^I} + \bar{S}^I A'^{(1)}_{\bar{S}^I}, \tag{4.6}$$

and similarly for $A'^{(3)}$. Generally, we encountered two cases where J_{Q^I} is coupled to $\bar{J}_{Q^I}$ or to $\bar{J}_{\bar{Q}^I}$, namely we have terms in the action of the form

$$c\eta^{IJ}(J_{Q^I}\bar{J}_{Q^J} - J_{\bar{Q}^I}\bar{J}_{\bar{Q}^J}), \quad \text{or} \quad c\eta^{IJ}(J_{Q^I}\bar{J}_{\bar{Q}^J} - \bar{J}_{Q^I}J_{\bar{Q}^J}), \tag{4.7}$$

where c is some constant. Let us call these two cases, case I and case II respectively. Case I appeared usually when considering PSU based model and case II when considering ortho-symplectic based model. To zeroth order in the fermions of B we have

$$A'_i = f(y)A_i, \quad A'_{Q^\iota} = g(y)h^\kappa_\iota(y^\mu/y)A_{Q^\kappa}, \quad A'_{\bar{Q}^\iota} = g(y)\widetilde{h}^\kappa_\iota(y^\mu/y)A_{\bar{Q}^\kappa}, \tag{4.8}$$

where the ι, κ indices are the R-symmetry indices of $Q, \bar{Q}$, so Q might have more indices of transformation under the bosonic conformal group, which are the same on both sides of the equations (4.8). To this order, the equations of motion for the bosons are

$$A^{(2)}_I \eta^{IJ} f^2 = F^J(\mathfrak{j}, \widetilde{x}^A), \tag{4.9}$$

while for the fermions we have for the two cases

$$c\eta^{\iota\varsigma}g^2 h^\kappa_\varsigma h^\lambda_\iota A^{(1)}_\lambda = F^\kappa(\mathfrak{j}, \widetilde{x}^A), \quad \text{and} \quad c\eta^{\iota\varsigma}g^2(\widetilde{h}^\kappa_\varsigma h^\lambda_\iota - \widetilde{h}^\lambda_\varsigma h^\kappa_\iota)A^{(1)}_\lambda = G^\kappa(\mathfrak{j}, \widetilde{x}^A), \tag{4.10}$$

respectively. The matrix $m^{\kappa\lambda} \equiv \eta^{\iota\varsigma}\widetilde{h}^{[\kappa}_\varsigma h^{\lambda]}_\iota$ in the second equation is singular when the dimension of the representation of Q under R-symmetry is odd. That is the case for the $AdS_4 \times CP^3$ action. The matrix m will get corrections at higher orders in the fermions of B, but these will contribute additively with even number of fermions to keep m bosonic. Let us call the full matrix M so,

$$M = m + \mathcal{O}(\chi^2), \tag{4.11}$$

where $\chi = \{\bar{\theta}, \xi, \bar{\xi}\}$. The inverse matrix should have the form $M^{-1} = m^{-1} + \mathcal{O}(\chi^2)$, since all the fermions should cancel, so when m is singular so is M.

In case II when the R-symmetry does not mix Q with $\bar{Q}$ (as in the AdS_4 case), we don't get a quadratic term to this order. Actually in this case we will get quadratic terms in A_Q when going to higher orders, so we will get equations of motion for A_Q, but these will always multiply terms of order $O(\chi^2)$ or higher in the fermions, so again we will have a singular matrix multiplying A_Q, and we will not be able to solve for them. The argument above does not rely on fixing any fermionic degrees of freedom (specifically we don't use κ-symmetry).

We saw that there can also be a case III where the WZ term gives an interaction of the form

$$J_Q\bar{J}_S + \cdots, \tag{4.12}$$

and this is the case of AdS_2 with eight supersymmetries realized by the supercoset $\mathrm{PSU}(1,1|2)/(\mathrm{U}(1)\times\mathrm{SU}(2))$. In this case, like for AdS_4, A_S does not get corrections to zeroth order in the fermions, so the quadratic term $\partial\theta\bar{\partial}\theta$ again will be multiplied by a singular fermionic matrix. This case has a different structure since the R-symmetry does not play any role (its generators are in $\mathcal{H}$), instead we have K (special conformal transformations) that mixes Q with $\bar{S}$ and $\bar{Q}$ with S.

Acknowledgments

We would like to thank N. Berkovits, P. A. Grassi, M. Magro, L. Mazzucato, V. Schomerus, D. Sorokin and S. Theisen for valuable discussions. I.A. would like to thank the KITP, UCSB for hospitality during the Fundamental Aspects of String Theory workshop. This research was supported by the German-Israeli Project cooperation (DIP H.52), the German-Israeli Fund (GIF) and supported in part by the National Science Foundation under Grant No. NSF PHY05-51164.

A The osp(2|2) algebra

The osp(2|2) algebra is

$$
\begin{aligned}
&[P,K]=-2D, \quad [D,P]=P, \quad [D,K]=-K, \quad [R,D]=0, \quad [R,P]=0,\\
&[R,K]=0,\\
&[P,Q]=0, \quad [P,\bar{Q}]=0, \quad [K,Q]=S, \quad [K,\bar{Q}]=\bar{S}, \quad [D,Q]=\frac{1}{2}Q,\\
&[D,\bar{Q}]=\frac{1}{2}\bar{Q}, \quad [R,Q]=-iQ, \quad [R,\bar{Q}]=i\bar{Q},\\
&[P,S]=-Q, \quad [P,\bar{S}]=-\bar{Q}, \quad [K,S]=0, \quad [K,\bar{S}]=0, \quad [D,S]=-\frac{1}{2}S,\\
&[D,\bar{S}]=-\frac{1}{2}\bar{S}, \quad [R,S]=-iS, \quad [R,\bar{S}]=i\bar{S},\\
&\{Q,Q\}=0, \quad \{\bar{Q},\bar{Q}\}=0, \quad \{Q,\bar{Q}\}=2P, \quad \{S,S\}=0, \quad \{\bar{S},\bar{S}\}=0,\\
&\{S,\bar{S}\}=2K, \quad \{Q,S\}=0, \quad \{\bar{Q},\bar{S}\}=0,\\
&\{Q,\bar{S}\}=2D-iR, \quad \{\bar{Q},S\}=2D+iR.
\end{aligned} \tag{A.1}
$$

This algebra admits a $\mathbb{Z}_4$ automorphism. The automorphism relevant to our needs is the one implemented by $\Omega X\Omega^{-1}$ for $X\in\mathrm{osp}(2|2)$, where

$$
\Omega=\begin{pmatrix} -i\sigma_1 & 0\\ 0 & \mathbf{1}\end{pmatrix}. \tag{A.2}
$$

Using this automorphism the algebra can be decomposed into $\mathbb{Z}_4$-invariant subspaces $\mathcal{H}_k$ $(k=0\ldots,3)$ such that

$$
\mathcal{H}_k=\{X\in\mathrm{osp}(2|2)|\Omega X\Omega^{-1}=i^kX\}.
$$

These subspaces are

$$\begin{aligned}
\mathcal{H}_0 &= \{P-K,\ R\}, && \text{(A.3)}\\
\mathcal{H}_1 &= \{Q+S,\ \bar{Q}+\bar{S}\}, && \text{(A.4)}\\
\mathcal{H}_2 &= \{P+K,\ D\}, && \text{(A.5)}\\
\mathcal{H}_3 &= \{Q-S,\ \bar{Q}-\bar{S}\}. && \text{(A.6)}
\end{aligned}$$

The Cartan-Killing bilinear form is defined by $g_{XY} = \mathrm{Str}(XY)$. Its non-trivial elements are

$$\begin{aligned}
\mathrm{Str}(PK) &= \mathrm{Str}(KP) = -1, \quad \mathrm{Str}(DD) = \frac{1}{2}, \quad \mathrm{Str}(RR) = 2,\\
\mathrm{Str}(Q\bar{S}) &= \mathrm{Str}(\bar{Q}S) = 2. \qquad \text{(A.7)}
\end{aligned}$$

B The osp(2|4) algebra

The definition of the OSp(2|4) group is that of [20]. The non-trivial brackets of the osp(2|4) algebra are

$$\begin{aligned}
[M_{mn}, M_{pq}] &= \eta_{mp}M_{nq} + \eta_{nq}M_{mp} - \eta_{mq}M_{np} - \eta_{np}M_{mq},\\
[M_{mn}, P_p] &= \eta_{mp}P_n - \eta_{np}P_m, \quad [M_{mn}, K_p] = \eta_{mp}K_n - \eta_{np}K_m,\\
[D, P_m] &= P_m, \quad [D, K_m] = -K_m, \quad [P_m, K_n] = 2\eta_{mn}D - 2M_{mn},\\
[D, Q_\alpha] &= \frac{1}{2}Q_\alpha, \quad [D, \hat{Q}_\alpha] = \frac{1}{2}\hat{Q}_\alpha, \quad [D, S_\alpha] = -\frac{1}{2}S_\alpha, \quad [D, \hat{S}_\alpha] = -\frac{1}{2}\hat{S}_\alpha,\\
[M_{mn}, Q_\alpha] &= \frac{1}{2}(C\gamma_{mn}C^{-1})_\alpha{}^\beta Q_\beta, \quad [M_{mn}, \hat{Q}_\alpha] = \frac{1}{2}(C\gamma_{mn}C^{-1})_\alpha{}^\beta \hat{Q}_\beta,\\
[M_{mn}, S_\alpha] &= \frac{1}{2}(C\gamma_{mn}C^{-1})_\alpha{}^\beta S_\beta, \quad [M_{mn}, \hat{S}_\alpha] = \frac{1}{2}(C\gamma_{mn}C^{-1})_\alpha{}^\beta \hat{S}_\beta,\\
[P_m, S_\alpha] &= i(C\gamma_m C^{-1})_\alpha{}^\beta Q_\beta, \quad [P_m, \hat{S}_\alpha] = i(C\gamma_m C^{-1})_\alpha{}^\beta \hat{Q}_\beta,\\
[K_m, Q_\alpha] &= -i(C\gamma_m C^{-1})_\alpha{}^\beta S_\beta, \quad [K_m, \hat{Q}_\alpha] = -i(C\gamma_m C^{-1})_\alpha{}^\beta \hat{S}_\beta,\\
[R, Q_\alpha] &= iQ_\alpha, \quad [R, \hat{Q}_\alpha] = -i\hat{Q}_\alpha, \quad [R, S_\alpha] = iS_\alpha, \quad [R, \hat{S}_\alpha] = -i\hat{S}_\alpha\\
\{Q_\alpha, \hat{Q}_\beta\} &= 4(C\gamma^m)_{\alpha\beta}P_m, \quad \{S_\alpha, \hat{S}_\beta\} = -4(C\gamma^m)_{\alpha\beta}K_m,\\
\{Q_\alpha, \hat{S}_\beta\} &= -2i(C\gamma^{mn})_{\alpha\beta}M_{mn} + 4i\epsilon_{\alpha\beta}D - 4\epsilon_{\alpha\beta}R,\\
\{\hat{Q}_\alpha, S_\beta\} &= -2i(C\gamma^{mn})_{\alpha\beta}M_{mn} + 4i\epsilon_{\alpha\beta}D + 4\epsilon_{\alpha\beta}R, \qquad \text{(B.1)}
\end{aligned}$$

where $\epsilon_{12} = -\epsilon_{21} = 1$, $\epsilon^{21} = -\epsilon^{12} = 1$, $C_{\alpha\beta} = \epsilon_{\alpha\beta}$ and the Dirac matrices have the index structure $\gamma^{m\alpha}{}_\beta$ and indices are lowered and raised using C. The antisymmetrized Dirac-matrices are $\gamma^{mn} = \frac{1}{2}[\gamma^m, \gamma^n]$. The metric is $\eta = \mathrm{diag}(-1,1,1)$, $m,n,p,q = 0,\ldots,2$ are space-time indices, $\alpha, \beta = 1,2$ are spinor indices and $I, J = 1,2$ are SO(2) R-symmetry indices.

The non-trivial elements of the Cartan-Killing bilinear form are

$$\begin{aligned}
&\mathrm{Str}(M_{mn}M_{pq}) = \eta_{mp}\eta_{nq} - \eta_{mq}\eta_{np}, \quad \mathrm{Str}(DD) = -1, \quad \mathrm{Str}(RR) = -2,\\
&\mathrm{Str}(P_m K_n) = -2\eta_{mn}, \quad \mathrm{Str}(Q_\alpha \hat{S}_\beta) = -8i\epsilon_{\alpha\beta}, \quad \mathrm{Str}(\hat{Q}_\alpha S_\beta) = -8i\epsilon_{\alpha\beta}. \qquad \text{(B.2)}
\end{aligned}$$

The $\mathbb{Z}_4$ automorphism invariant subspaces are

$$\begin{aligned}\mathcal{H}_0 &= \{M_{mn},\ P_m - K_m,\ R\},\\ \mathcal{H}_1 &= \{Q_\alpha - S_\alpha,\ \hat{Q}_\alpha + \hat{S}_\alpha\},\\ \mathcal{H}_2 &= \{P_m + K_m,\ D\},\\ \mathcal{H}_3 &= \{Q_\alpha + S_\alpha,\ \hat{Q}_\alpha - \hat{S}_\alpha\}.\end{aligned} \tag{B.3}$$

C The osp(6|4) algebra in so(1, 2) ⊕ u(3) basis

The osp(6|4) algebra's commutation relations are given by

$$[\lambda_{k\dot{l}}, \lambda_{m\dot{n}}] = 2i(\delta_{m\dot{l}}\lambda_{k\dot{n}} - \delta_{k\dot{n}}\lambda_{m\dot{l}}) \tag{C.1}$$

$$[\lambda_{k\dot{l}}, R_{mn}] = 2i(\delta_{m\dot{l}}R_{kn} - \delta_{n\dot{l}}R_{km}) \tag{C.2}$$

$$[R_{mn}, R_{kl}] = 0, \qquad [R_{mn}, R_{\dot{k}\dot{l}}] = \frac{i}{2}(\delta_{m\dot{k}}\lambda_{n\dot{l}} - \delta_{m\dot{l}}\lambda_{n\dot{k}} - \delta_{n\dot{k}}\lambda_{m\dot{l}} + \delta_{n\dot{l}}\lambda_{m\dot{k}}) \tag{C.3}$$

$$[P_a, P_b] = 0, \qquad [K_a, K_b] = 0, \qquad [P_a, K_b] = 2\eta_{ab}D - 2M_{ab} \tag{C.4}$$

$$[M_{ab}, M_{cd}] = \eta_{ac}M_{bd} + \eta_{bd}M_{ac} - \eta_{ad}M_{bc} - \eta_{bc}M_{ad} \tag{C.5}$$

$$[M_{ab}, P_c] = \eta_{ac}P_b - \eta_{bc}P_a, \qquad [M_{ab}, K_c] = \eta_{ac}K_b - \eta_{bc}K_a \tag{C.6}$$

$$[D, P_a] = P_a, \qquad [D, K_a] = -K_a, \qquad [D, M_{ab}] = 0 \tag{C.7}$$

$$[D, Q^l_\alpha] = \frac{1}{2}Q^l_\alpha, \qquad [D, S^l_\alpha] = -\frac{1}{2}S^l_\alpha \tag{C.8}$$

$$[P_a, Q^l_\alpha] = 0, \qquad [K_a, S^l_\alpha] = 0 \tag{C.9}$$

$$[P_a, S^l_\alpha] = -i(\gamma_a)_\alpha{}^\beta Q^l_\beta, \qquad [K_a, Q^l_\alpha] = i(\gamma_a)_\alpha{}^\beta S^l_\beta \tag{C.10}$$

$$[M_{ab}, Q^l_\alpha] = -\frac{i}{2}(\gamma_{ab})_\alpha{}^\beta Q^l_\beta, \qquad [M_{ab}, S^l_\alpha] = -\frac{i}{2}(\gamma_{ab})_\alpha{}^\beta S^l_\beta \tag{C.11}$$

$$[R_{kl}, Q^{\dot{p}}_\alpha] = i(\delta^{\dot{p}l}Q^k_\alpha - \delta^{\dot{p}k}Q^l_\alpha), \qquad [R_{kl}, S^{\dot{p}}_\alpha] = -i(\delta^{\dot{p}l}S^k_\alpha - \delta^{\dot{p}k}S^l_\alpha) \tag{C.12}$$

$$[R_{\dot{k}\dot{l}}, Q^p_\alpha] = -i(\delta^{p\dot{l}}Q^{\dot{k}}_\alpha - \delta^{p\dot{k}}Q^{\dot{l}}_\alpha), \qquad [R_{\dot{k}\dot{l}}, S^p_\alpha] = i(\delta^{p\dot{l}}S^{\dot{k}}_\alpha - \delta^{p\dot{k}}S^{\dot{l}}_\alpha) \tag{C.13}$$

$$[\lambda_{k\dot{l}}, Q^p_\alpha] = 2i\delta^{p\dot{l}}Q^k_\alpha, \qquad [\lambda_{k\dot{l}}, S^p_\alpha] = 2i\delta^{p\dot{l}}S^k_\alpha \tag{C.14}$$

$$[\lambda_{k\dot{l}}, Q^{\dot{p}}_\alpha] = -2i\delta^{\dot{p}k}Q^{\dot{l}}_\alpha, \qquad [\lambda_{k\dot{l}}, S^{\dot{p}}_\alpha] = -2i\delta^{\dot{p}k}S^{\dot{l}}_\alpha \tag{C.15}$$

$$\{Q^l_\alpha, Q^k_\beta\} = 0, \qquad \{Q^l_\alpha, Q^{\dot{k}}_\beta\} = -\delta^{l\dot{k}}(\gamma^a C)_{\alpha\beta}P_a \tag{C.16}$$

$$\{S^l_\alpha, S^k_\beta\} = 0, \qquad \{S^l_\alpha, S^{\dot{k}}_\beta\} = -\delta^{l\dot{k}}(\gamma^a C)_{\alpha\beta}K_a \tag{C.17}$$

$$\{Q^l_\alpha, S^k_\beta\} = -C_{\alpha\beta}R_{lk}, \qquad \{Q^{\dot{l}}_\alpha, S^{\dot{k}}_\beta\} = -C_{\alpha\beta}R_{\dot{l}\dot{k}} \tag{C.18}$$

$$\{Q^l_\alpha, S^{\dot{k}}_\beta\} = -i\delta^{l\dot{k}}\left(C_{\alpha\beta}D + i\frac{1}{2}(\gamma^{ab}C)_{\alpha\beta}M_{ab}\right) + \frac{1}{2}C_{\alpha\beta}\lambda_{l\dot{k}} \tag{C.19}$$

$$\{Q^{\dot{l}}_\alpha, S^k_\beta\} = i\delta^{\dot{l}k}\left(C_{\alpha\beta}D - i\frac{1}{2}(\gamma^{ab}C)_{\alpha\beta}M_{ab}\right) + \frac{1}{2}C_{\alpha\beta}\lambda_{k\dot{l}} \tag{C.20}$$

The indices take the values $k, l = 1, 2, 3$ and the same for the dotted ones — the **3** and $\bar{\mathbf{3}}$ of u(3), $a, b = 0, 1, 2$ are the **3** of so(1, 2) and $\alpha, \beta, .. = 1, 2$ are the so(2, 1) spinors, and $\eta = \mathrm{diag}(-, +, +)$. The generators satisfy $R^*_{kl} = R_{\dot{k}\dot{l}}$ and $\lambda_{k\dot{l}} = \lambda^*_{l\dot{k}}$ and $Q^{\dot{l}}_\alpha = (Q^l_\alpha)^*$. The

$(\gamma_a)_\alpha{}^\beta$ are the Dirac matrices of so(1,2), and $\gamma_{ab} = \frac{i}{2}[\gamma_a, \gamma_b]$. We raise and lower spinor indices as explained in appendix E.

The bilinear forms are given by

$$\begin{aligned}
\mathrm{Str}(R_{kl}R_{\dot{p}\dot{q}}) &= -2(\delta^{k\dot{p}}\delta^{l\dot{q}} - \delta^{k\dot{q}}\delta^{l\dot{p}}) && \text{(C.21)}\\
\mathrm{Str}(Q^l_\alpha S^{\dot{k}}_\beta) &= 2i\delta^{l\dot{k}}C_{\alpha\beta} && \text{(C.22)}\\
\mathrm{Str}(P_a K_b) &= -2\eta_{ab} && \text{(C.23)}\\
\mathrm{Str}(DD) &= -1 && \text{(C.24)}\\
\mathrm{Str}(M_{ab}M_{cd}) &= \eta_{ac}\eta_{bd} - \eta_{ad}\eta_{bc} && \text{(C.25)}
\end{aligned}$$

The $\mathbb{Z}_4$ automorphism matrix is given by

$$\Omega = \left(\begin{array}{ccc|cc} i\sigma_2 & 0 & 0 & 0 & 0 \\ 0 & i\sigma_2 & 0 & 0 & 0 \\ 0 & 0 & i\sigma_2 & 0 & 0 \\ \hline 0 & 0 & 0 & \sigma_2 & 0 \\ 0 & 0 & 0 & 0 & -\sigma_2 \end{array}\right). \tag{C.26}$$

The $\mathbb{Z}_4$ invariant subspaces of the algebra are

$$\begin{aligned}
\mathcal{H}_0 &= \{P_a - K_a, M_{ab}, \lambda_{l\dot{k}}\},\\
\mathcal{H}_1 &= \{Q^l_\alpha - S^l_\alpha, Q^{\dot{l}}_\alpha - S^{\dot{l}}_\alpha\},\\
\mathcal{H}_2 &= \{P_a + K_a, D, R_{kl}, R_{\dot{k}\dot{l}}\},\\
\mathcal{H}_3 &= \{Q^l_\alpha + S^l_\alpha, Q^{\dot{l}}_\alpha + S^{\dot{l}}_\alpha\}.
\end{aligned} \tag{C.27}$$

D The psu(1,1|2) algebra

The algebra of the PSU(1,1|2) group was developed by using the definition given in [18] (which follows the definition given in [21] applied to matrices whose elements are all Grassmann-even).

The su(1,1|2) algebra in a basis oriented towards the SU(2) R-symmetry is

$$\begin{aligned}
&[D,P] = P, && [D,K] = -K, \quad [P,K] = -2D, \quad [R_i,R_j] = -\epsilon_{ijk}R_k,\\
&[D,Q_\alpha] = \frac{1}{2}Q_\alpha, && [D,\hat{Q}_\alpha] = \frac{1}{2}\hat{Q}_\alpha,\ [D,S_\alpha] = -\frac{1}{2}S_\alpha, \quad [D,\hat{S}_\alpha] = -\frac{1}{2}\hat{S}_\alpha,\\
&[P,Q_\alpha] = [P,\hat{Q}_\alpha] = 0, && [P,S_\alpha] = i\hat{Q}_\alpha, \quad [P,\hat{S}_\alpha] = iQ_\alpha,\\
&[K,Q_\alpha] = i\hat{S}_\alpha, && [K,\hat{Q}_\alpha] = iS_\alpha, \quad [K,S_\alpha] = [K,\hat{S}_\alpha] = 0,\\
&[R_{ij},Q_\alpha] = -\frac{1}{2}\sigma_{ij\alpha}{}^\beta Q_\beta, && [R_{ij},\hat{Q}_\alpha] = -\frac{1}{2}\sigma_{ij\alpha}{}^\beta \hat{Q}_\beta,\\
&[R_{ij},S_\alpha] = -\frac{1}{2}\sigma_{ij\alpha}{}^\beta S_\beta, && [R_{ij},\hat{S}_\alpha] = -\frac{1}{2}\sigma_{ij\alpha}{}^\beta \hat{S}_\beta,\\
&\{Q_\alpha,\hat{Q}_\beta\} = \epsilon_{\alpha\beta}P, && \{S_\alpha,\hat{S}_\beta\} = \epsilon_{\alpha\beta}K, \quad \{Q_\alpha,S_\beta\} = -\frac{i}{2}\epsilon_{\beta\gamma}\sigma^{ij\gamma}_\alpha R_{ij} - i\epsilon_{\alpha\beta}D,\\
&\{Q_\alpha,\hat{S}_\beta\} = \frac{i}{2}\epsilon_{\alpha\beta}\mathbf{1}, && \{\hat{Q}_\alpha,S_\beta\} = -\frac{i}{2}\epsilon_{\alpha\beta}\mathbf{1},\ \{\hat{Q}_\alpha,\hat{S}_\beta\} = \frac{i}{2}\epsilon_{\beta\gamma}\sigma^{ij\gamma}_\alpha R_{ij} + i\epsilon_{\alpha\beta}D,
\end{aligned} \tag{D.1}$$

where $a, b = 1, 2$ are AdS$_2$ indices, $\alpha, \beta, \gamma = 1, 2$ are SU(2) R-symmetry indices, R_i ($i = 1, \ldots, 3$) are the generators of the SU(2) R-symmetry and $R_{ij} = \epsilon_{ijk} R_k$ are defined for convenience. The antisymmetric tensors are defined such that $\epsilon_{12} = 1$, $\epsilon_{123} = 1$ and the generators of SU(2) are $\sigma_{ij\alpha}{}^{\beta} = \frac{1}{2}\left(\sigma_{i\alpha\gamma}\sigma_j^{\gamma\beta} - \sigma_{j\alpha\gamma}\sigma_i^{\gamma\beta}\right)$ with σ_i being the Pauli matrices. In order to get the psu$(1, 1|2)$ algebra, one has to divide by the U(1) generator $\mathbf{1}$. The Grassmann-odd generators in the algebra above are not Hermitian but are formed as linear combinations of the original Hermitian matrices with complex coefficients. These combinations correspond to the multiplying the original Hermitian generators by the complex Killing spinors found by the requiring the supercharges and superconformal transformations to form Abelian subalgebras.

The non-trivial elements of the Cartan-Killing bilinear form are

$$\begin{aligned}
&\mathrm{Str}(PK) = -1, &&\mathrm{Str}(DD) = \frac{1}{2}, &&\mathrm{Str}(R_i R_j) = \frac{1}{2}\delta_{ij},\\
&\mathrm{Str}(Q_\alpha S_\beta) = -i\epsilon_{\alpha\beta}, &&\mathrm{Str}(\hat{Q}_\alpha \hat{S}_\beta) = i\epsilon_{\alpha\beta}
\end{aligned} \tag{D.2}$$

The $\mathbb{Z}_4$ automorphism is the same one as given in [18],

$$\Omega = \begin{pmatrix} \sigma_3 & 0 \\ 0 & i\sigma_3 \end{pmatrix}. \tag{D.3}$$

The $\mathbb{Z}_4$ invariant subspaces of the algebra defined as

$$\mathcal{H}_k = \left\{ X \in \mathrm{psu}(1, 1|2) | \Omega X \Omega^{-1} = i^k X \right\} \tag{D.4}$$

are

$$\begin{aligned}
\mathcal{H}_0 &= \{P + K, R_3\},\\
\mathcal{H}_1 &= \{Q_2 + S_2, \hat{Q}_2 + \hat{S}_2, Q_1 - S_1, \hat{Q}_1 - \hat{S}_1\},\\
\mathcal{H}_2 &= \{P - K, D, R_1, R_2\},\\
\mathcal{H}_3 &= \{Q_1 + S_1, \hat{Q}_1 + \hat{S}_1, Q_2 - S_2, \hat{Q}_2 - \hat{S}_2\}.
\end{aligned} \tag{D.5}$$

Another $\mathbb{Z}_4$ automorphism is given by

$$\Omega = \begin{pmatrix} \sigma_1 & 0 \\ 0 & i\mathbf{I} \end{pmatrix} \tag{D.6}$$

yields the following invariant subspaces of the algebra

$$\begin{aligned}
\mathcal{H}_0 &= \{D, R_i\},\\
\mathcal{H}_1 &= \{Q_\alpha, \hat{Q}_\alpha\},\\
\mathcal{H}_2 &= \{P, K\},\\
\mathcal{H}_3 &= \{S_\alpha, \hat{S}_\alpha\}.
\end{aligned} \tag{D.7}$$

This will give us the supercoset space PSU$(1, 1|2)/(\mathrm{U}(1) \times \mathrm{SU}(2))$ which is isomorphic to AdS_2 with eight supersymmetries.

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E The $\mathbf{psu(1,1|2) \oplus psu(1,1|2)}$ algebra

We build the super-algebra by taking the direct sum of two psu$(1,1|2)$ algebras (where we take complex combinations of the odd generators), so the bosonic part is su$(2) \oplus$ su$(2) \simeq$ so(4) and su$(1,1)\oplus$su$(1,1) \simeq$ so$(2,2)$. The non-vanishing part of the psu$(1,1|2)\oplus$psu$(1,1|2)$ algebra, involving odd generators, is given by

$$
\begin{aligned}
[D,P_a] &= P_a, \quad [D,K_a] = -K_a, \quad [D,J_{ab}] = 0, \\
[P_a,K_a] &= 2(\eta_{ab}D + J_{ab}), \quad [P_c,J_{ab}] = \eta_{ac}P_b - \eta_{bc}P_a, \quad [K_c,J_{ab}] = \eta_{ac}P_b - \eta_{bc}K_a, \\
[R_\mu,R_\nu] &= -N_{\mu\nu}, \quad [R_\rho,N_{\mu\nu}] = \delta_{\rho\mu}R_\nu - \delta_{\rho\nu}R_\mu, \\
[N_{\rho\sigma},N_{\mu\nu}] &= \delta_{\rho\mu}N_{\sigma\nu} + \delta_{\sigma\nu}N_{\rho\mu} - \delta_{\rho\nu}N_{\sigma\mu} - \delta_{\sigma\mu}N_{\rho\nu} \\
[P_a,Q^I_{\alpha\hat\alpha}] &= 0, \qquad [K_a,S^I_{\alpha\hat\alpha}] = 0, \qquad [D,Q^I_{\alpha\hat\alpha}] = \frac{1}{2}Q^I_{\alpha\hat\alpha}, \qquad [D,S^I_{\alpha\hat\alpha}] = -\frac{1}{2}S^I_{\alpha\hat\alpha}, \\
[K_a,Q^I_{\alpha\hat\alpha}] &= i(\hat\gamma_a)_{\hat\alpha}{}^{\hat\beta}S^I_{\alpha\hat\beta}, \qquad [P_a,S^I_{\alpha\hat\alpha}] = i(\hat\gamma_a)_{\hat\alpha}{}^{\hat\beta}Q^I_{\alpha\hat\beta}, \\
[J_{01},Q^I_{\alpha\hat\alpha}] &= -\frac{i}{2}(\hat\gamma_{01})_{\hat\alpha}{}^{\hat\beta}Q^I_{\alpha\hat\beta}, \qquad [J_{01},S^I_{\alpha\hat\alpha}] = -\frac{i}{2}(\hat\gamma_{01})_{\hat\alpha}{}^{\hat\beta}S^I_{\alpha\hat\beta}, \\
[R_\mu,S^I_{\alpha\hat\alpha}] &= \frac{i}{2}(i\hat\gamma_{01})_{\hat\alpha}{}^{\hat\beta}(\gamma_\mu)_\alpha{}^\beta S^I_{\beta\hat\beta}, \qquad [R_\mu,Q^I_{\alpha\hat\alpha}] = -\frac{i}{2}(i\hat\gamma_{01})_{\hat\alpha}{}^{\hat\beta}(\gamma_\mu)_\alpha{}^\beta Q^I_{\beta\hat\beta}, \\
[N_{\mu\nu},S^I_{\alpha\hat\alpha}] &= \frac{i}{2}\hat C_{\hat\alpha}{}^{\hat\beta}(\gamma_{\mu\nu})_\alpha{}^\beta S^I_{\beta\hat\beta}, \qquad [N_{\mu\nu},Q^I_{\alpha\hat\alpha}] = \frac{i}{2}\hat C_{\hat\alpha}{}^{\hat\beta}(\gamma_{\mu\nu})_\alpha{}^\beta Q^I_{\beta\hat\beta}, \\
\{S^I_{\alpha\hat\alpha},S^J_{\beta\hat\beta}\} &= \frac{i}{2}\epsilon^{IJ}C_{\alpha\beta}(\hat\gamma^a\hat C)_{\hat\alpha\hat\beta}K_a, \qquad \{Q^I_{\alpha\hat\alpha},Q^J_{\beta\hat\beta}\} = \frac{i}{2}\epsilon^{IJ}C_{\alpha\beta}(\hat\gamma^a\hat C)_{\hat\alpha\hat\beta}P_a, \\
\{S^I_{\alpha\hat\alpha},Q^J_{\beta\hat\beta}\} &= \frac{i}{2}\epsilon^{IJ}(C_{\alpha\beta}(i\hat C_{\hat\alpha\hat\beta}D - \frac{1}{2}(\hat C\hat\gamma^{ab})_{\hat\alpha\hat\beta}J_{ab}) \\
&\quad +\frac{1}{2}\hat C_{\hat\alpha\hat\beta}N_{\mu\nu}(\gamma^{\mu\nu}C)_{\alpha\beta} - i(\hat\gamma_{01}\hat C)_{\hat\alpha\hat\beta}R_\mu(\gamma^\mu C)_{\alpha\beta})
\end{aligned}
\tag{E.1}
$$

where the indices go as $I = 1,2$, $\hat\alpha = 1,2$, $\alpha = 1,2$,$a = 1,2$, $\mu = 1,2,3$ and $C_{\alpha\beta} = \epsilon_{\alpha\beta}$, $\hat C_{\hat\alpha\hat\beta} = \epsilon_{\hat\alpha\hat\beta}$. The gamma matrices are defined by $(\hat\gamma_a)_{\hat\alpha}{}^{\hat\beta} = \{i\sigma_2, -\sigma^1\}$, $(\hat\gamma_{01})_{\hat\alpha}{}^{\hat\beta} = \frac{i}{2}[\hat\gamma_0,\hat\gamma_1]_{\hat\alpha}{}^{\hat\beta} = \{-i\sigma^3\}$, $(\gamma_\mu)_\alpha{}^\beta = \{\sigma_1,\sigma_3,\sigma_2\}$, $(\gamma_{\mu\nu})_\alpha{}^\beta = \frac{i}{2}[\gamma_\mu,\gamma_\nu]_\alpha{}^\beta = \{\sigma_2,\sigma_1,\sigma_3\}$, where σ are the Pauli matrices, and $\eta_{ab} = \mathrm{diag}(+,-)$. We raise and lower spinor indices using $\psi_\alpha = \psi^\beta\epsilon_{\beta\alpha}$, $\psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta$, where $\epsilon_{12} = -\epsilon_{21} = \epsilon^{12} = -\epsilon^{21} = 1$ and the same for hatted objects. The bilinear Cartan-Killing forms can be rescaled to give

$$
\begin{aligned}
\mathrm{Str}(P_aK_b) &= \eta_{ab}, \qquad \mathrm{Str}(DD) = 1, \quad \mathrm{Str}(J_{ab}J_{cd}) = \delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc} \\
\mathrm{Str}(R_\mu R_\nu) &= \delta_{\mu\nu}, \qquad \mathrm{Str}(N_{\mu\nu}N_{\rho\sigma}) = \delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho} \\
\mathrm{Str}(Q^I_{\alpha\hat\alpha}S^J_{\beta\hat\beta}) &= \epsilon^{IJ}\epsilon_{\alpha\beta}\epsilon_{\hat\alpha\hat\beta}
\end{aligned}
\tag{E.2}
$$

The super-algebra has $\mathbb{Z}_4$ grading structure using the automorphism matrix

$$
\Omega = \begin{pmatrix} 0 & 0 & \sigma_3 & 0 \\ 0 & 0 & 0 & 1_{2\times 2} \\ -i\sigma_3 & 0 & 0 & 0 \\ 0 & i1_{2\times 2} & 0 & 0 \end{pmatrix}
\tag{E.3}
$$

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so

$$\mathcal{H}_0 = \{J_{01}, N_{\mu\nu}, P_0 - K_0, P_1 - K_1\} \tag{E.4}$$
$$\mathcal{H}_1 = \{S^I_{\alpha\hat{\alpha}} + a^{IJ} Q^J_{\alpha\hat{\alpha}}\} \tag{E.5}$$
$$\mathcal{H}_2 = \{R_\mu, D, P_0 + K_0, P_1 + K_1\} \tag{E.6}$$
$$\mathcal{H}_3 = \{S^I_{\alpha\hat{\alpha}} - a^{IJ} Q^J_{\alpha\hat{\alpha}}\} \tag{E.7}$$

where $a^{IJ} = \sigma_1$. So we can get the supercoset space $\mathrm{PSU}(1,1|2)^2/(\mathrm{SU}(1,1) \times \mathrm{SU}(2))$, whose bosonic part is $AdS_3 \times S^3$. We also have another $\mathbb{Z}_4$ grading structure with the same number of supersymmetry generators

$$\mathcal{H}_0 = \{D, J_{01}, N_{\mu\nu}, R_\mu\} \tag{E.8}$$
$$\mathcal{H}_1 = \{Q^I_{\alpha\hat{\alpha}}\} \tag{E.9}$$
$$\mathcal{H}_2 = \{P_a, K_a\} \tag{E.10}$$
$$\mathcal{H}_3 = \{S^I_{\alpha\hat{\alpha}}\} \tag{E.11}$$

which is the four-dimensional space $BDI(2,2) \simeq AdS_2 \times AdS_2$, given by the supercoset $\mathrm{PSU}(1,1|2)^2/(\mathrm{SO}(2)^2 \times \mathrm{SO}(4))$.

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