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T-Duality, dual conformal symmetry and integrability for strings on $AdS_5 \times S^5$

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In recent years two intriguing observations have been made for $\mathcal{N} = 4$ super Yang–Mills theory and for superstrings on $AdS_5 \times S^5$: In the planar limit the computation of the spectrum is vastly simplified by the apparent integrability of the models. Furthermore, planar scattering amplitudes of the gauge theory display remarkable features which have been attributed to the appearance of a dual superconformal symmetry. Here we review the connection of these two developments from the point of view of the classical symmetry by means of a super-T-self-duality. In particular, we show explicitly how the charges of conformal symmetry and of the integrable structure are related to the dual ones.

1 Introduction

Two observations of recent years have led to remarkable progress in $\mathcal{N} = 4$ supersymmetric gauge theory and in IIB string theory on $AdS_5 \times S^5$ as well as in their conjectured duality, the AdS/CFT string/gauge correspondence. One development is the appearance of integrable structures helping dramatically in determining the AdS/CFT spectrum (we refer the reader to reviews on this subject [1–3]). The other observation is that scattering amplitudes in the gauge theory have a much simpler structure than expected (see the reviews [4, 5]). Both phenomena have in common that they hold only in the strict large-N_c alias the planar limit. For a long time the coinciding requirements have led to speculations that both features may be related. Recent works are starting to confirm this idea and to make the connection more rigorous and concrete.

Perhaps the first indication of extended symmetries for scattering amplitudes was found in [6] where it was argued for the existence of a *dual* conformal symmetry in addition to the original conformal symmetry. This observation helped the four-loop unitarity construction of four-gluon scattering [7] producing a result for the cusp anomalous dimension which is in perfect agreement with the prediction based on integrability [8]. Later in [9] scattering amplitudes of the gauge theory were related to certain Wilson loops in the string theory. The key step was the proposal of a T-duality transformation on the worldsheet coordinates of the string which leaves the (bosonic) action invariant. The relationship between scattering amplitudes and Wilson loops was shown to hold also purely within the perturbative gauge theory setup [10–12] even at higher loops [13, 14]. The conformal symmetry of Wilson loops turns into the dual conformal symmetry of scattering amplitudes [10, 15]. Moreover, the dual symmetry also combines with supersymmetry into *dual superconformal* symmetry [16, 17]. On the string theory side the above mentioned T-duality transformation maps between the original and the dual symmetries. For the full supersymmetric string action the standard

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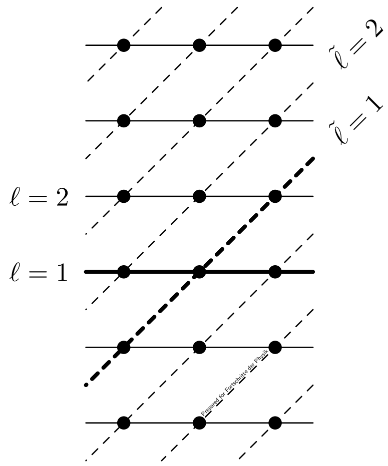

Fig. 1 Embedding of the original and dual superconformal symmetries $Q = Y^{(1)}$, $\tilde{Q} = \tilde{Y}^{(1)}$ into the integrable structure, $Y^{(\ell)}$ or $\tilde{Y}^{(\tilde{\ell})}$.

bosonic T-duality has to be supplemented by a fermionic T-duality in order to map the worldsheet action back to itself [18] (see also some more recent work [19, 20]).

One may wonder what is the closure of the algebra generated by the original and the dual symmetries. Furthermore, what is the image of the integrable structure under T-duality? It turns out that both questions have a common answer [18, 21, 22] (see also [23]). The closure of the algebra is the integrable structure, and the latter is mapped to itself but in a non-trivial manner. This is conveniently illustrated in Fig. 1: The integrable structure consists of the loop algebra of superconformal symmetry, i.e. infinitely many copies of the superconformal generators indexed by an integer label. There are many ways of choosing a closed Lie algebra within its loop algebra, and the original and dual symmetries are two such instances. Very recently, the integrable structure was shown to apply directly to tree-level scattering amplitudes in gauge theory [24]. The loop algebra is quantised to Yangian symmetry. This symmetry is almost identical to the Yangian symmetry for one-loop anomalous dimensions [25]. Thus the simplicity of planar scattering amplitudes is indeed closely related to the integrability of planar AdS/CFT.

This note is a review of the works [9, 18, 21] outlining T-duality for superstrings on $AdS_5 \times S^5$ and how the integrable structure transforms under it. We extend the previous works slightly by making the mapping of the integrable charges more explicit. We start by reviewing the AdS_n coset space sigma model, its integrable structure as well as its T-duality transformations. Then we show how T-duality maps the symmetries and the integrable structure and finally we sketch the extension of the above to the supersymmetric model on $AdS_5 \times S^5$.

2 The AdS_{n+1} sigma model and integrability

The anti-de Sitter spacetime AdS_{n+1} is most conveniently viewed as the symmetric coset space $\mathrm{SO}(n,2)/\mathrm{SO}(n,1)$. Let the algebra $\mathfrak{so}(n,2)$ be spanned by the standard conformal generators: $\mathfrak{L}_{\mu\nu}$ (Lorentz), $\mathfrak{D}$ (dilatation), $\mathfrak{P}_\mu$ (momentum) and $\mathfrak{K}_\mu$ (special conformal). These obey the algebra relations[1]

$$[\mathfrak{L}_{\mu\nu}, \mathfrak{L}_{\rho\sigma}] = \eta_{\nu\rho}\mathfrak{L}_{\mu\sigma} \mp 3 \text{ terms}, \quad [\mathfrak{L}_{\mu\nu}, \mathfrak{P}_\rho] = \eta_{\nu\rho}\mathfrak{P}_\mu - \eta_{\mu\rho}\mathfrak{P}_\nu, \quad [\mathfrak{D}, \mathfrak{P}_\mu] = +\mathfrak{P}_\mu,$$
$$[\mathfrak{P}_\mu, \mathfrak{K}_\nu] = 2\mathfrak{L}_{\mu\nu} + 2\eta_{\mu\nu}\mathfrak{D}, \quad [\mathfrak{L}_{\mu\nu}, \mathfrak{K}_\rho] = \eta_{\nu\rho}\mathfrak{K}_\mu - \eta_{\mu\rho}\mathfrak{K}_\nu, \quad [\mathfrak{D}, \mathfrak{K}_\mu] = -\mathfrak{K}_\mu. \quad (1)$$

[1] We disregard reality conditions of the algebra and use this freedom to remove factors of i into the definition of the generators.

We embed the denominator algebra $\mathfrak{so}(n,1)$ as the invariant space of the $\mathbb{Z}_2$ automorphism Ω of $\mathfrak{so}(n,2)$ defined by

$$\Omega(\mathfrak{L}_{\mu\nu}) = \mathfrak{L}_{\mu\nu}, \qquad \Omega(\mathfrak{D}) = -\mathfrak{D}, \qquad \Omega(\mathfrak{P}_\mu) = \mathfrak{K}_\mu, \qquad \Omega(\mathfrak{K}_\mu) = \mathfrak{P}_\mu. \tag{2}$$

We can formulate the non-linear sigma model on AdS_{n+1} using the $\mathrm{SO}(n,2)$-valued field g and its associated Maurer–Cartan form $J = g^{-1}dg$. The coset space is implemented through the gauge symmetry $g \mapsto gh$ with a $\mathrm{SO}(n,1)$-valued field h for which J acts as a gauge connection. Dynamics of the model is governed by a set of invariant equations for J and $K = J - \Omega(J)$

$$dJ + J \wedge J = 0, \qquad d{*K} + J \wedge {*K} + {*K} \wedge J = 0. \tag{3}$$

The first expression is the Maurer–Cartan equation following from the definition of J and the second is the equation of motion which follows from the standard non-linear sigma model Lagrangian.

This two-dimensional field theory model turns out to be integrable: The gauge connection J can be deformed into the Lax connection

$$A(x) = J + \frac{1}{x^2-1}\,K + \frac{x}{x^2-1}\,{*K}. \tag{4}$$

Provided that the above equations of motion hold, the Lax connection is flat, $dA(x) + A(x) \wedge A(x) = 0$, for all values of the spectral parameter x. Note that the Maurer–Cartan equation for the combination K reads $dK + J \wedge K + K \wedge J - K \wedge K = 0$.

Now the forms J and K are covariant under gauge transformations which makes the identification of conserved quantities cumbersome. We therefore go to a gauge-invariant frame by conjugating with the group element g. The field $k = gKg^{-1}$ obeys the equation $dk - k \wedge k = 0$ as well as $d{*k} = 0$. In other words, k is the Noether current leading to the conserved Noether charges

$$Q = \int {*k}. \tag{5}$$

The Lax connection in the invariant frame is obtained by conjugation of the associated covariant derivative $d + a(x) = g(d + A(x))g^{-1}$, it reads

$$a(x) = \frac{1}{x^2-1}\,k + \frac{x}{x^2-1}\,{*k} = \frac{x^{-2}}{1-x^{-2}}\,k + \frac{x^{-1}}{1-x^{-2}}\,{*k}. \tag{6}$$

It is used to construct the higher charges of the integrable model through its parallel transport, the so-called monodromy $M(x)$, along a curve γ on the worldsheet. Due to the vanishing of $a(x)$ near $x = \infty$, the expansion of $M(x)$ around this point

$$M(x) = \overrightarrow{\mathrm{P}\exp} \int_\gamma a(x) = \exp\left(\sum_{n=1}^{\infty} x^{-n} Y^{(n)}\right) \tag{7}$$

leads to a tower of n-local charges $Y^{(n)}$

$$Q = Y^{(1)} = \int {*k},$$

$$Y^{(2)} = \tfrac{1}{2} \iint_{\sigma_1<\sigma_2} [{*k_1}, {*k_2}] + \int k,$$

$$Y^{(3)} = -\tfrac{1}{6} \iiint_{\sigma_1<\sigma_2<\sigma_3} \big([{*k_1}, [{*k_3}, {*k_2}]] + [{*k_3}, [{*k_1}, {*k_2}]]\big)$$

$$+\frac{1}{2}\iint_{\sigma_1<\sigma_2}\big([k_1,*k_2]+[*k_1,k_2]\big)+\int *k, \quad \ldots \tag{8}$$

The first of these multi-local charges $Y^{(1)}$ is precisely the Noether charge Q and thus the integrable structure enhances the Lie algebra $\mathfrak{so}(n,2)$ to an infinite-dimensional algebra. The higher charges $Y^{(n)}$, $n>1$, are not strictly conserved: Shifting the end-points of the curve γ in $Y^{(n)}$ leads to commutators involving the lower $Y^{(k)}$.

3 Poincaré coordinates and T-self-duality

The above formulation in terms of a coset still leaves a large amount of gauge freedom. For AdS_{n+1} there is a convenient (local) chart of coordinates which fixes the gauge: It is sufficient to specify the coordinates along the $\mathfrak{P}$ and $\mathfrak{D}$ directions of $\mathrm{SO}(n,2)$ because the $\mathrm{SO}(n,1)$-directions $\mathfrak{L}$ and $\mathfrak{P}+\mathfrak{K}$ are unphysical. We can thus choose the group element to be

$$g=\exp(X^\mu\mathfrak{P}_\mu)\exp(\Phi\mathfrak{D}). \tag{9}$$

Here Φ measures the distance to the boundary and a slice of constant Φ is a Minkowski space. Consequently we call (X^μ,Φ) Poincaré coordinates for AdS_{n+1}. The advantage of this chart is that the algebra generated by $\mathfrak{P},\mathfrak{D}$ is triangular and the Maurer–Cartan form takes a simple form

$$J=g^{-1}dg=J_\mathfrak{P}+J_\mathfrak{D},\qquad J_\mathfrak{P}=e^{-\Phi}dX^\mu\,\mathfrak{P}_\mu,\quad J_\mathfrak{D}=d\Phi\,\mathfrak{D}. \tag{10}$$

The Maurer–Cartan equations and the equations of motion read

$$\begin{aligned} 0 &= dJ_\mathfrak{D}, & 0 &= dJ_\mathfrak{P}+J_\mathfrak{D}\wedge J_\mathfrak{P}+J_\mathfrak{P}\wedge J_\mathfrak{D},\\ 0 &= d{*}J_\mathfrak{D}-\tfrac{1}{2}J_\mathfrak{P}\wedge{*}\Omega(J_\mathfrak{P})-\tfrac{1}{2}{*}\Omega(J_\mathfrak{P})\wedge J_\mathfrak{P}, & 0 &= d{*}J_\mathfrak{P}-J_\mathfrak{D}\wedge{*}J_\mathfrak{P}-{*}J_\mathfrak{P}\wedge J_\mathfrak{D}. \end{aligned} \tag{11}$$

In fact, the equations of motion for the coordinates X^μ,Φ are even simpler

$$d(e^{-2\Phi}{*}dX^\mu)=0,\qquad d{*}d\Phi-e^{-2\Phi}dX^\mu\wedge{*}dX_\mu=0. \tag{12}$$

Interestingly, the field X appears in all places only through its derivative dX. This allows to introduce a set of dual fields $(\tilde X^\mu,\tilde\Phi)$ through the relation

$$d\tilde X^\mu=e^{-2\Phi}{*}dX^\mu,\qquad \tilde\Phi=-\Phi. \tag{13}$$

In fact, the transformation is a combination of a formal T-duality on the coordinates X^μ and a flip of the sign of Φ [9]. Note that the transformation between X and $\tilde X$ is non-local because their relation is defined only via their derivatives. It was chosen such that the above equation of motion (12) for X is automatically satisfied, $dd\tilde X=0$. Conversely, closedness of dX leads to an equation of motion for $\tilde X$. Incidentally, it takes precisely the same form as (12). Likewise the equation of motion for $\tilde\Phi$ matches the one of Φ with all fields replaced by their duals

$$d(e^{-2\tilde\Phi}{*}d\tilde X^\mu)=0,\qquad d{*}d\tilde\Phi-e^{-2\tilde\Phi}d\tilde X^\mu\wedge{*}d\tilde X_\mu=0. \tag{14}$$

In the first-order formalism involving the Maurer–Cartan form, the transformation is given through the map

$$\tilde J_\mathfrak{P}={*}J_\mathfrak{P},\qquad \tilde J_\mathfrak{D}=-J_\mathfrak{D}. \tag{15}$$

The set of first-order equations (11) is again mapped to itself, however, the role of Maurer–Cartan equation and equation of motion for $J_\mathfrak{P}$ are interchanged.

4 T-self-duality and symmetries

We are thus in the curious situation that the T-duality transforms the model to itself, namely it is a T-self-duality [9]. The model can be expressed through two sets of variables which incidentally obey the same set of equations. For all quantities expressed through the original variables there must therefore exist a quantity expressed through the dual variables enjoying the same properties. For example, in addition to the Noether charge Q there exists a dual Noether charge $\tilde{Q}$. The associated symmetry is the so-called dual conformal symmetry. This symmetry is a $\mathrm{SO}(n,2)$ group which is not equivalent to the original conformal $\mathrm{SO}(n,2)$ symmetry, although, for example, the Lorentz subgroup of both symmetries coincides. One also comes to the conclusion that the higher integrable charges $Y^{(n)}$ lead to dual charges $\tilde{Y}^{(n)}$. An important question is whether these charges are independent of the original ones and thus whether the model has two coexisting integrable structures. Alternatively, there could be a relation between the two towers of charges, and if so, what is it precisely?

Let us therefore compare the Lax connection and its dual version expressed through the original variables via (15)

$$\begin{aligned} A(x) &= \frac{1}{x^2-1}\big(+x^2 J_{\mathfrak{P}} - \Omega(J_{\mathfrak{P}}) + x{*}J_{\mathfrak{P}} - x{*}\Omega(J_{\mathfrak{P}}) + (x^2+1)J_{\mathfrak{D}} - x{*}J_{\mathfrak{D}}\big),\\ \tilde{A}(x) &= \frac{1}{x^2-1}\big(-x\Omega(J_{\mathfrak{P}}) + xJ_{\mathfrak{P}} - {*}\Omega(J_{\mathfrak{P}}) + x^2{*}J_{\mathfrak{P}} - (x^2+1)J_{\mathfrak{D}} + x{*}J_{\mathfrak{D}}\big). \end{aligned} \tag{16}$$

Looking at the $\mathfrak{D}$-components, it becomes clear that an algebraic transformation to relate $A(x)$ and $\tilde{A}(x')$ cannot involve the Hodge dual or a change of spectral parameter, $x \mapsto x'$, but it must act by merely flipping the sign $J_{\mathfrak{D}} \mapsto -J_{\mathfrak{D}}$. On the other hand the $\mathfrak{P}$-components imply that the spectral parameter must be involved in the transformation. The first two terms suggest two options for the transformation, $J_{\mathfrak{P}} \mapsto x^{-1}J_{\mathfrak{P}}$, $\Omega(J_{\mathfrak{P}}) \mapsto x\Omega(J_{\mathfrak{P}})$ or $J_{\mathfrak{P}} \mapsto -x^{-1}\Omega(J_{\mathfrak{P}})$, $\Omega(J_{\mathfrak{P}}) \mapsto -xJ_{\mathfrak{P}}$. The former does however not lead to the desired result for the Hodge dual terms, while the latter one does. Altogether the transformation can be formulated as an x-dependent automorphism Ω_x [18, 21]

$$\tilde{A}(x) = \Omega_x(A(x)), \qquad \Omega_x(\mathfrak{X}) = (-x)^{\mathfrak{D}}\,\Omega(\mathfrak{X})\,(-x)^{-\mathfrak{D}}. \tag{17}$$

Similarly, the automorphism maps between the parallel transports of the Lax connections A and $\tilde{A}$

$$\Omega_x : \overrightarrow{\mathrm{P}\exp}\int_\gamma A(x) \mapsto \overrightarrow{\mathrm{P}\exp}\int_\gamma \tilde{A}(x). \tag{18}$$

Since the higher integrable charges $Y^{(n)}$ are defined in the invariant frame we have to convert this statement by conjugation with g

$$\overrightarrow{\mathrm{P}\exp}\int_\gamma A(x) = g_-^{-1}\left(\overrightarrow{\mathrm{P}\exp}\int_\gamma a(x)\right)g_+ = g_-^{-1}M(x)g_+. \tag{19}$$

Here $g_\mp$ denote the values of g at the endpoints of the curve γ. The statement is thus

$$\Omega_x\big(g_-^{-1}M(x)g_+\big) = \tilde{g}_-^{-1}\tilde{M}(x)\tilde{g}_+. \tag{20}$$

Using the definition (9) of g, the T-duality transformation (13) and the $\mathfrak{K}$-components of the Noether charges (see below) $\tilde{Q}_{\mathfrak{K}} = (X^\mu_- - X^\mu_+)\mathfrak{K}_\mu$, $Q_{\mathfrak{K}} = (\tilde{X}^\mu_- - \tilde{X}^\mu_+)\mathfrak{K}_\mu$, we obtain a useful expression for the relation between the monodromy matrices

$$\begin{aligned} &\exp(-x^{-1}\tilde{Q}_{\mathfrak{K}})\exp(-\tilde{X}^\mu_-\mathfrak{P}_\mu)\tilde{M}(x)\exp(+\tilde{X}^\mu_-\mathfrak{P}_\mu)\\ &= \Omega_x\big(\exp(-X^\mu_+\mathfrak{P}_\mu)M(x)\exp(+X^\mu_+\mathfrak{P}_\mu)\exp(-x^{-1}Q_{\mathfrak{K}})\big). \end{aligned} \tag{21}$$

Conveniently the conjugations by $\exp(X_+^\mu \mathfrak{P}_\mu)$ and $\exp(\tilde{X}_-^\mu \mathfrak{P}_\mu)$ lead to only finitely many terms due to the nilpotency of $\mathfrak{P}$. As a first step, let us expand all exponents to linear order in the exponent

$$\sum_{n=1}^{\infty} x^{-n} \tilde{Y}^{(n)} \simeq x^{-1} \tilde{Q}_{\mathfrak{K}} + \sum_{n=1}^{\infty} x^{-n} \Omega_x(Y^{(n)}) - x^{-1} \Omega_x(Q_{\mathfrak{K}}). \tag{22}$$

These expressions are exact up to commutators involving the lower charges as well as $X_+^\mu \mathfrak{P}_\mu$ or $\tilde{X}_-^\mu \mathfrak{P}_\mu$. When we split this into components we find the relations ($n \geq 1$)

$$\tilde{Y}_{\mathfrak{K}}^{(n+1)} \simeq -\Omega(Y_{\mathfrak{P}}^{(n)}), \quad \tilde{Y}_{\mathfrak{P}}^{(n)} \simeq -\Omega(Y_{\mathfrak{K}}^{(n+1)}), \quad \tilde{Y}_{\mathfrak{D}}^{(n)} \simeq -Y_{\mathfrak{D}}^{(n)}, \quad \tilde{Y}_{\mathfrak{L}}^{(n)} \simeq Y_{\mathfrak{L}}^{(n)}. \tag{23}$$

We can also write down the first few relations with the omitted commutators explicitly:

$$\tilde{Q}_{\mathfrak{L}} + \tilde{Q}_{\mathfrak{D}} - [\tilde{X}_-^\mu \mathfrak{P}_\mu, \tilde{Q}_{\mathfrak{K}}] = \Omega\big(Q_{\mathfrak{L}} + Q_{\mathfrak{D}} - [X_+^\mu \mathfrak{P}_\mu, Q_{\mathfrak{K}}]\big), \tag{24}$$

$$\tilde{Q}_{\mathfrak{P}} - \big[\tilde{X}_-^\mu \mathfrak{P}_\mu, \tilde{Q}_{\mathfrak{L}} + \tilde{Q}_{\mathfrak{D}} - \tfrac{1}{2}[\tilde{X}_-^\mu \mathfrak{P}_\mu, \tilde{Q}_{\mathfrak{K}}]\big] = \Omega\big(-Y_{\mathfrak{K}}^{(2)} + \tfrac{1}{2}\big[Q_{\mathfrak{L}} + Q_{\mathfrak{D}} - [X_+^\mu \mathfrak{P}_\mu, Q_{\mathfrak{K}}], Q_{\mathfrak{K}}\big]\big),$$

$$\tilde{Y}_{\mathfrak{K}}^{(2)} + \tfrac{1}{2}\big[\tilde{Q}_{\mathfrak{L}} + \tilde{Q}_{\mathfrak{D}} - [\tilde{X}_-^\mu \mathfrak{P}_\mu, \tilde{Q}_{\mathfrak{K}}], \tilde{Q}_{\mathfrak{K}}\big] = \Omega\big(-Q_{\mathfrak{P}} + \big[X_+^\mu \mathfrak{P}_\mu, Q_{\mathfrak{L}} + Q_{\mathfrak{D}} - \tfrac{1}{2}[X_+^\mu \mathfrak{P}_\mu, Q_{\mathfrak{K}}]\big]\big).$$

Let us see in practice, how the duality of charges works. First we work out the components of the Noether current $k = gKg^{-1}$ from (9,10)

$$\begin{aligned} k_{\mathfrak{K}} &= -e^{-2\Phi} dX^\mu \mathfrak{K}_\mu, \\ k_{\mathfrak{D}} &= 2(d\Phi - e^{-2\Phi} X_\mu dX^\mu)\mathfrak{D}, \\ k_{\mathfrak{L}} &= -2e^{-2\Phi} X^\mu dX^\nu \mathfrak{L}_{\mu\nu}, \\ k_{\mathfrak{P}} &= (dX^\mu - 2X^\mu d\Phi - e^{-2\Phi} X^2 dX^\mu + 2e^{-2\Phi} X^\mu X_\nu dX^\nu)\mathfrak{P}_\mu. \end{aligned} \tag{25}$$

Their Hodge duals will be written using the dual coordinates as far as possible

$$\begin{aligned} *k_{\mathfrak{K}} &= -d\tilde{X}^\mu \mathfrak{K}_\mu, \\ *k_{\mathfrak{D}} &= 2(*d\Phi - X_\mu d\tilde{X}^\mu)\mathfrak{D}, \\ *k_{\mathfrak{L}} &= -2X^\mu d\tilde{X}^\nu \mathfrak{L}_{\mu\nu}, \\ *k_{\mathfrak{P}} &= (e^{2\Phi} d\tilde{X}^\mu - 2X^\mu *d\Phi - X^2 d\tilde{X}^\mu + 2X^\mu X_\nu d\tilde{X}^\nu)\mathfrak{P}_\mu. \end{aligned} \tag{26}$$

The $\mathfrak{K}$-components of the Noether charges (5) read

$$Q_{\mathfrak{K}} = \int *k_{\mathfrak{K}} = -\int d\tilde{X}^\mu \mathfrak{K}_\mu = (\tilde{X}_-^\mu - \tilde{X}_+^\mu)\mathfrak{K}_\mu, \qquad \tilde{Q}_{\mathfrak{K}} = \ldots = (X_-^\mu - X_+^\mu)\mathfrak{K}_\mu. \tag{27}$$

These two quantities are independent, and the relationship between the monodromies respects this. Next we consider their $\mathfrak{L}$-components

$$Q_{\mathfrak{L}} = -2\int X^\mu d\tilde{X}^\nu \mathfrak{L}_{\mu\nu}, \qquad \tilde{Q}_{\mathfrak{L}} = -2\int \tilde{X}^\mu dX^\nu \mathfrak{L}_{\mu\nu}. \tag{28}$$

Upon partial integration we recover the expression for $Q_{\mathfrak{L}}$ in $*Q_{\mathfrak{L}}$ up to some boundary terms

$$\tilde{Q}_{\mathfrak{L}} = Q_{\mathfrak{L}} + 2(\tilde{X}_-^\mu X_-^\nu - \tilde{X}_+^\mu X_+^\nu)\mathfrak{L}_{\mu\nu} = Q_{\mathfrak{L}} + 2\big((\tilde{X}_-^\mu - \tilde{X}_+^\mu)X_+^\nu + \tilde{X}_-^\mu(X_-^\nu - X_+^\nu)\big)\mathfrak{L}_{\mu\nu}. \tag{29}$$

This is precisely the $\mathfrak{L}$-component of the first equation in (24), similarly for $\mathfrak{D}$. It is also interesting to consider the bi-local charge $Y_{\mathfrak{K}}^{(2)}$ [18]

$$Y_{\mathfrak{K}}^{(2)} = \tfrac{1}{2} \iint_{\sigma_1 < \sigma_2} [*k_{1,\mathfrak{K}}, *k_{2,\mathfrak{L}+\mathfrak{D}}] + \tfrac{1}{2} \iint_{\sigma_1 < \sigma_2} [*k_{1,\mathfrak{L}+\mathfrak{D}}, *k_{2,\mathfrak{K}}] + \int k_{\mathfrak{K}}. \tag{30}$$

Now because $*k_{\mathfrak{K}}$ is a total derivative the double integral collapses to a single one

$$Y_{\mathfrak{K}}^{(2)} = \tfrac{1}{2}\big[(\tilde{X}_-^\mu + \tilde{X}_+^\mu)\mathfrak{K}_\mu, Q_{\mathfrak{L}} + Q_{\mathfrak{D}}\big] - \int [\tilde{X}^\mu \mathfrak{K}_\mu, *k_{\mathfrak{L}+\mathfrak{D}}] + \int k_{\mathfrak{K}}. \tag{31}$$

After substituting the remaining expressions and some partial integrations one finds the expression for $\Omega(\tilde{Q}_{\mathfrak{P}})$ plus some boundary terms

$$\begin{aligned} Y_{\mathfrak{K}}^{(2)} &= -\Omega(\tilde{Q}_{\mathfrak{P}}) + \tfrac{1}{2}\big[(\tilde{X}_-^\mu + \tilde{X}_+^\mu)\mathfrak{K}_\mu, Q_{\mathfrak{L}} + Q_{\mathfrak{D}}\big] \\ &\quad + 2(X_+ \cdot \tilde{X}_+)\tilde{X}_+^\mu \mathfrak{K}_\mu + 2(X_- \cdot \tilde{X}_-)\tilde{X}_-^\mu \mathfrak{K}_\mu - \tilde{X}_+^2 X_+^\mu \mathfrak{K}_\mu + \tilde{X}_-^2 X_-^\mu \mathfrak{K}_\mu. \end{aligned} \tag{32}$$

Some elementary operations later one recovers precisely the second relation in (24).

5 Super-T-self-duality and integrability

An analog construction exists for the superstring on $AdS_5 \times S^5$ or, equivalently, the sigma model on the coset space $\mathrm{PSU}(2,2|4)/\mathrm{Sp}(1,1) \times \mathrm{Sp}(2)$ coupled to worldsheet gravity. The coset construction is based on a $\mathbb{Z}_4$ grading which allows to split the Maurer–Cartan into four components

$$J = J_0 + J_2 + J_1 + J_{-1}, \qquad \Omega(J_n) = i^n J_n, \quad J_n = \frac{1}{4}\sum_{k=0}^{3} i^{-nk}\Omega^{\circ k}(J). \tag{33}$$

By introducing the combination

$$*K = 2{*}J_2 - J_1 + J_{-1} \tag{34}$$

we can write the Maurer–Cartan equations and the equations of motion in precisely the same way as in (3). Note, however, that splitting these equations into their $\mathbb{Z}_4$ components yields a more complicated set of equations than before. These equations can again be cast into the form of a flatness condition for a Lax connection [26]

$$A(z) = J_0 + \tfrac{1}{2}(z^2 + z^{-2})J_2 + \tfrac{1}{2}(-z^2 + z^{-2}){*}J_2 + zJ_1 + z^{-1}J_{-1}. \tag{35}$$

Note that the bosonic part consisting of the first three terms is the same as above if the spectral parameters are identified as [27]

$$z^2 = \frac{x-1}{x+1}, \qquad x = \frac{1+z^2}{1-z^2}. \tag{36}$$

The $\mathfrak{psu}(2,2|4)$ algebra is spanned by the above $\mathfrak{so}(4,2)$ conformal generators $\mathfrak{L}$, $\mathfrak{D}$, $\mathfrak{P}$, $\mathfrak{K}$, by the internal $\mathfrak{so}(6)$ generators $\mathfrak{R}$ as well as the supercharges $\mathfrak{Q}$, $\bar{\mathfrak{Q}}$, $\mathfrak{S}$, $\bar{\mathfrak{S}}$. The structure of the algebraic relations can conveniently be sketched as in Fig. 2.

$$\Omega(\mathfrak{Q}) \sim i\mathfrak{S}, \qquad \Omega(\mathfrak{S}) \sim i\mathfrak{Q}, \qquad \Omega(\bar{\mathfrak{Q}}) \sim i\bar{\mathfrak{S}}, \qquad \Omega(\bar{\mathfrak{S}}) \sim i\bar{\mathfrak{Q}}. \tag{37}$$

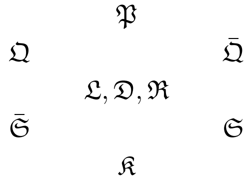

Fig. 2 Structure of the superconformal algebra $\mathfrak{psu}(2,2|4)$. The generators are arranged in the plane according to their charges under two Cartan generators $(\mathfrak{B}, \mathfrak{D})$ which are conserved in commutators.

In addition to the bosonic gauge group $\mathrm{Sp}(1,1) \times \mathrm{Sp}(2)$ with grading 0, the model has a local fermionic kappa symmetry affecting predominantly the components with grading ± 1. These local symmetries can be gauge fixed in many different useful ways. For our purposes it is again advisable to choose them such that as many components of the Maurer–Cartan form as possible become trivial. Again we would like to eliminate the components corresponding to conformal boosts, $J_{\mathfrak{K}} = 0$. In addition we can eliminate half of the fermionic components. One option is to eliminate the components corresponding to all superconformal boosts, $J_{\mathfrak{S}} = J_{\bar{\mathfrak{S}}} = 0$. This is achieved by choosing the group element to be generated by $\mathfrak{P}$, $\mathfrak{Q}$, $\bar{\mathfrak{Q}}$, $\mathfrak{D}$, $\mathfrak{R}$ [28–30]. Fig. 2 shows that commutators of the generators close onto the subset, and we can understand this choice as upper triangular matrices. By the same logic we can instead eliminate $J_{\mathfrak{S}} = J_{\bar{\mathfrak{Q}}} = 0$ by the alternative choice of group element generated by $\mathfrak{P}$, $\mathfrak{Q}$, $\bar{\mathfrak{S}}$, $\mathfrak{D}$, $\mathfrak{R}$ [31].[2]

Both gauges have in common that they make 4 bosonic and 8 fermionic coordinate fields appear only through their derivatives. Therefore one can apply a T-duality transformation to all of these fields. It is a combination of a bosonic T-duality similar to (13) and a so-called fermionic T-duality acting on the fermionic fields [18].[3] In contradistinction to the purely bosonic case, the T-duality transformation does not map the equations of motion into themselves. It rather maps between the equations in the two gauges discussed above.

The formulation of the T-duality transformation for the fields is somewhat complicated and it depends on the precise choice of gauge. In fact, it is much simpler to state the resulting transformation in terms of the Maurer–Cartan form analogously to (15)

$$\tilde{J}_{\mathfrak{P}} = *J_{\mathfrak{P}}, \quad \tilde{J}_{\mathfrak{D}} = -J_{\mathfrak{D}}, \quad \tilde{J}_{\mathfrak{Q}} = iJ_{\mathfrak{Q}}, \quad \tilde{J}_{\bar{\mathfrak{S}}} = \Omega(J_{\bar{\mathfrak{Q}}}), \quad \tilde{J}_{\mathfrak{R}} = \Omega(J_{\mathfrak{R}}). \tag{38}$$

The statement is that when taking the full set of equations for J and restricting them to the above two gauge choices, T-duality will map between the two sets. Instead of proving the statement, we will show that the Lax connections are related by an automorphism as in the bosonic case, which also proves the equivalence of the integrable structure and its dual. We first split the Lax connection into bosonic and fermionic components, $A(z) = A_{\mathrm{B}}(z) + A_{\mathrm{F}}(z)$ which must transform separately. The transformation for the bosonic components works as in the purely bosonic case (17) after identifying x and z according to (36). We are left with comparing the fermionic components of the Lax connection in one gauge and its dual in a different gauge but expressed through the first set of variables via (38)

$$A_{\mathrm{F}}(z) = \tfrac{1}{2}(z + z^{-1})\big(J_{\mathfrak{Q}} + J_{\bar{\mathfrak{Q}}}\big) + \tfrac{1}{2}(z - z^{-1})\big(-i\Omega(J_{\mathfrak{Q}}) - i\Omega(J_{\bar{\mathfrak{Q}}})\big), \tag{39}$$

$$\tilde{A}_{\mathrm{F}}(z) = \tfrac{1}{2}(z + z^{-1})\big(+iJ_{\mathfrak{Q}} + \Omega(J_{\bar{\mathfrak{Q}}})\big) + \tfrac{1}{2}(z - z^{-1})\big(\Omega(J_{\mathfrak{Q}}) + iJ_{\bar{\mathfrak{Q}}}\big). \tag{40}$$

From the bosonic part we know that the automorphism involves the $\mathbb{Z}_4$ transformation Ω. Consequently, here we are forced to interchange the $J_{\mathfrak{Q}}$ terms which implies a z-dependent factor for $J_{\mathfrak{Q}}$. Conversely the $J_{\bar{\mathfrak{Q}}}$ terms must stay in place and the automorphism should be independent of z. Altogether this is achieved by the following transformation [21]

$$\tilde{A}(z) = \Omega_z(A(z)), \qquad \Omega_z(\mathfrak{X}) = \left(-\frac{1+z^2}{1-z^2}\right)^{\mathfrak{D}+\mathfrak{B}} \Omega(\mathfrak{X}) \left(-\frac{1-z^2}{1+z^2}\right)^{\mathfrak{D}+\mathfrak{B}}. \tag{41}$$

Here $\mathfrak{B}$ generates the $\mathfrak{u}(1)$ automorphism of $\mathfrak{psu}(2,2|4)$ which acts exclusively on the fermionic generators $\mathfrak{Q}$, $\bar{\mathfrak{Q}}$, $\mathfrak{S}$, $\bar{\mathfrak{S}}$. More concretely $\mathfrak{B}$ is defined such that,

$$[\mathfrak{D} + \mathfrak{B}, (\mathfrak{P}, \mathfrak{Q})] = +(\mathfrak{P}, \mathfrak{Q}),$$
$$[\mathfrak{D} + \mathfrak{B}, (\mathfrak{D}, \mathfrak{L}, \mathfrak{R}, \bar{\mathfrak{Q}}, \bar{\mathfrak{S}})] = 0,$$

[2] This gauge necessarily requires complexification of the coordinates.

[3] Note that T-duality in $n = 4$ bosonic variables induces a shift of the dilaton which is cancelled precisely by T-duality in $2n = 8$ fermionic variables leading to a quantum mechanically exact self-duality.

$$[\mathfrak{D} + \mathfrak{B}, (\mathfrak{K}, \mathfrak{S})] = -(\mathfrak{K}, \mathfrak{S}). \tag{42}$$

Clearly the automorphism Ω_z is compatible with Ω_x for bosonic generators. The resulting mapping of non-local charges is analogous to the bosonic case; up to commutator terms, it is depicted in Fig. 3.

$$Y^{(r)}_{\mathfrak{P}} \simeq -\tilde{Y}^{(r-1)}_{\mathfrak{K}}$$

$$Y^{(r)}_{\mathfrak{Q}} \simeq -\tilde{Y}^{(r-1)}_{\mathfrak{S}} \qquad Y^{(r)}_{\bar{\mathfrak{Q}}} \simeq \tilde{Y}^{(r\pm 0)}_{\bar{\mathfrak{S}}}$$

$$Y^{(r)}_{\mathfrak{L},\mathfrak{D},\mathfrak{R}} \simeq \pm\tilde{Y}^{(r\pm 0)}_{\mathfrak{L},\mathfrak{D},\mathfrak{R}}$$

$$Y^{(r)}_{\bar{\mathfrak{S}}} \simeq \tilde{Y}^{(r\pm 0)}_{\bar{\mathfrak{Q}}} \qquad Y^{(r)}_{\mathfrak{S}} \simeq -\tilde{Y}^{(r+1)}_{\mathfrak{Q}}$$

$$Y^{(r)}_{\mathfrak{K}} \simeq -\tilde{Y}^{(r+1)}_{\mathfrak{P}}$$

Fig. 3 Mapping between non-local charges for classical superstrings on $AdS_5 \times S^5$ and their duals (up to commutators).

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