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Maxwell-like Lagrangians for higher spins

ANDREA CAMPOLEONI [a,1], DARIO FRANCIA [b,c]

[a]Max-Planck-Institut für Gravitationsphysik,
Albert-Einstein-Institut
Am Mühlenberg 1, D-14476 Golm, Germany
e-mail: andrea.campoleoni@aei.mpg.de

[b]Centro Studi e Ricerche E. Fermi
Piazza del Viminale 1, I-00184 Roma, Italy
[c]Scuola Normale Superiore and INFN
Piazza dei Cavalieri 7, I-56126 Pisa, Italy
e-mail: dario.francia@sns.it

Abstract

We show how implementing invariance under divergence-free gauge transformations leads to a remarkably simple Lagrangian description of massless bosons of any spin. Our construction covers both flat and (A)dS backgrounds and extends to tensors of arbitrary mixed-symmetry type. Irreducible and traceless fields produce single-particle actions, while whenever trace constraints can be dispensed with the resulting Lagrangians display the same reducible, multi-particle spectra as those emerging from the tensionless limit of free open-string field theory. For all explored options the corresponding kinetic operators take essentially the same form as in the spin-one, Maxwell case.

[1]Address after October 1, 2012: Université Libre de Bruxelles, ULB-Campus Plaine CP231, B-1050 Brussels, Belgium.

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1 Introduction

The main purpose of this work is to investigate the conditions under which higher-spin free Lagrangians take their simplest possible forms, exploring the case of massless bosons of any spin and symmetry in Minkowski as well as in (Anti)-de Sitter backgrounds. We find that in all these cases it is possible to keep the corresponding kinetic operators essentially as simple as their spin-one, Maxwell counterpart[2]. For instance, the Lagrangian equations of motion for rank-s symmetric tensors in flat space-time resulting from our approach read

$$(M\,\varphi)_{\,\mu_1\,\cdots\,\mu_s} \,\equiv\, \Box\,\varphi_{\,\mu_1\,\cdots\,\mu_s} \,-\, (\,\partial_{\,\mu_1}\,\partial^{\,\alpha}\,\varphi_{\,\alpha\,\mu_2\,\cdots\,\mu_s} \,+\,\cdots\,) \,=\, 0\,, \tag{1.1}$$

where the dots stand for symmetrisation of indices while the operator M builds the higher-spin extension of the Maxwell field equations,

$$(M\,A)_{\,\mu} \,\equiv\, \Box\,A_{\,\mu} \,-\, \partial_{\,\mu}\,\partial^{\,\alpha}\,A_{\,\alpha} \,=\, 0\,. \tag{1.2}$$

Indeed, considering the on-shell conditions for massless, spin-s propagation [3],

$$\begin{aligned} &\Box\,\varphi_{\,\mu_1\,\cdots\,\mu_s} \,=\, 0\,, && \Box\,\Lambda_{\,\mu_1\,\cdots\,\mu_{s-1}} \,=\, 0\,,\\ &\partial^{\,\alpha}\varphi_{\,\alpha\,\mu_2\,\cdots\,\mu_s} \,=\, 0\,, && \partial^{\,\alpha}\Lambda_{\,\alpha\,\mu_2\,\cdots\,\mu_{s-1}} \,=\, 0\,,\\ &\varphi^{\,\alpha}{}_{\,\alpha\,\mu_3\,\cdots\,\mu_s} \,=\, 0\,, && \Lambda^{\,\alpha}{}_{\,\alpha\,\mu_3\,\cdots\,\mu_{s-1}} \,=\, 0\,, \end{aligned} \tag{1.3}$$

where $\varphi_{\,\mu_1\,\cdots\,\mu_s}$ is a rank-s symmetric tensor subject to the abelian gauge transformation

$$\delta\varphi_{\,\mu_1\,\cdots\,\mu_s} \,=\, \partial_{\,\mu_1}\,\Lambda_{\,\mu_2\,\cdots\,\mu_s} \,+\,\cdots\,, \tag{1.4}$$

it is already possible to notice that if $\varphi_{\,\mu_1\,\cdots\,\mu_s}$ and $\Lambda_{\,\mu_1\,\cdots\,\mu_{s-1}}$ no longer satisfy the first equations in (1.3), as required for the system to be off-shell, then compensating the gauge variation of $\Box\,\varphi_{\,\mu_1\,\cdots\,\mu_s}$ leads to forego the condition on the vanishing of its divergence as well and to construct the combination displayed in (1.1). In this sense one can interpret the Maxwell-like tensor $(M\,\varphi)_{\,\mu_1\,\cdots\,\mu_s}$ as providing the minimal building block necessary for any off-shell extension of (1.3), and our goal in the present paper is to show that the same operator actually also suffices for the same purpose. Differently, the trace conditions in (1.3) appear at this level as optional possibilities and, as we shall see, keeping or discarding them in the off-shell formulation can affect the spectrum of the resulting theories but not the form of the corresponding Lagrangians.

The key idea underlying the whole construction is to allow for a restricted form of gauge symmetry with parameters subject to a suitable set of transversality conditions. For instance, as we show in section 2.1.1, in order to enforce invariance of $(M\,\varphi)_{\,\mu_1\,\cdots\,\mu_s}$ under (1.4) the simplest option is indeed to require that the gauge parameter $\Lambda_{\,\mu_1\,\cdots\,\mu_{s-1}}$ be *divergence-free*,

$$\partial^{\,\alpha}\,\Lambda_{\,\alpha\,\mu_2\,\cdots\,\mu_{s-1}} \,=\, 0\,, \tag{1.5}$$

[2]See [1] for a general motivating discussion and [2] for reviews on the subject of higher spins.

allowing to dispense with the introduction of additional terms involving traces of the field, like those appearing for the same class of tensors in Fronsdal's equation [4]

$$(\mathcal{F}\,\varphi)_{\,\mu_1\,\cdots\,\mu_s}\,=\,(M\,\varphi)_{\,\mu_1\,\cdots\,\mu_s}\,+\,(\,\partial_{\,\mu_1}\,\partial_{\,\mu_2}\,\varphi^{\,\alpha}{}_{\,\alpha\,\mu_3\,\cdots\,\mu_s}\,+\,\cdots)\,=\,0\,. \tag{1.6}$$

A further distinction concerns the analysis of the spectra: while (1.6), supplemented with the condition that the gauge parameter be traceless, describes the propagation of a single massless particle of spin s, the spectrum associated to (1.1) comprises a whole set of particles of spin s, $s-2$, $s-4$, $\cdots$ and so on, down to spin $s=1$ or $s=0$, thus providing a reducible description of higher-spin dynamics. However, without altering the form of the corresponding Lagrangian, easily seen to be given by

$$\mathcal{L}\,=\,\frac{1}{2}\,\varphi_{\,\mu_1\,\cdots\,\mu_s}\,(M\,\varphi)^{\,\mu_1\,\cdots\,\mu_s}\,, \tag{1.7}$$

it is also possible to truncate the particle content of divergence-free theories to the single irreducible representation of highest spin s by further restricting both the field $\varphi_{\,\mu_1\,\cdots\,\mu_s}$ and the gauge parameter $\Lambda_{\,\mu_1\,\cdots\,\mu_{s-1}}$ to be traceless, as originally shown in [5]. Similar considerations apply to symmetric tensors in (A)dS backgrounds, to which our construction extends with no special difficulties both for reducible and irreducible theories, as discussed in section 2.2.1.

Our approach proves especially effective in simplifying the Lagrangian formulation of theories involving tensors with mixed symmetry. In section 2.1.2 we study the general case of multi-symmetric tensors with N families of indices,

$$\varphi_{\,\mu_1\,\cdots\,\mu_{s_1},\,\nu_1\,\cdots\,\nu_{s_2},\,\cdots}\ , \tag{1.8}$$

defining $GL(D)$-reducible representations, showing that a consistent Lagrangian for their massless particle content in flat space-time is simply

$$\begin{aligned}\mathcal{L}=\frac{1}{2}\,\varphi_{\,\mu_1\,\cdots\,\mu_{s_1},\,\nu_1\,\cdots\,\nu_{s_2},\,\cdots}\Big\{\Box\,\varphi^{\,\mu_1\,\cdots\,\mu_{s_1},\,\nu_1\,\cdots\,\nu_{s_2},\,\cdots}\,&-\,(\partial^{\,\mu_1}\,\partial_{\,\alpha}\,\varphi^{\,\alpha\,\mu_2\,\cdots\,\mu_{s_1},\,\nu_1\,\cdots\,\nu_{s_2},\,\cdots}\,+\,\cdots)\\ &-\,(\partial^{\,\nu_1}\,\partial_{\,\alpha}\,\varphi^{\,\mu_1\,\cdots\,\mu_{s_1},\,\alpha\,\nu_2\,\cdots\,\nu_{s_2},\,\cdots}\,+\,\cdots)\\ &-\,\cdots\Big\},\end{aligned} \tag{1.9}$$

where within parentheses symmetrisations over indices belonging to a single family are understood. Gauge invariance of (1.9) under

$$\begin{aligned}\delta\,\varphi_{\,\mu_1\,\cdots\,\mu_{s_1},\,\nu_1\,\cdots\,\nu_{s_2},\,\cdots}\,&=\,(\partial_{\,\mu_1}\,\Lambda_{\,\mu_2\,\cdots\,\mu_{s_1},\,\nu_1\,\cdots\,\nu_{s_2},\,\cdots}\,+\,\cdots)\\ &+\,(\partial_{\,\nu_1}\,\lambda_{\,\mu_1\,\cdots\,\mu_{s_1},\,\nu_2\,\cdots\,\nu_{s_2},\,\cdots}\,+\,\cdots)\\ &+\,\cdots\,,\end{aligned} \tag{1.10}$$

is guaranteed provided the N parameters $\Lambda_{\,\mu_1\,\cdots\,\mu_{s_1-1},\,\nu_1\,\cdots\,\nu_{s_2},\,\cdots}\,,\ \lambda_{\,\mu_1\,\cdots\,\mu_{s_1},\,\nu_1\,\cdots\,\nu_{s_2-1},\,\cdots}\,,\ \cdots,$ each missing one index in a given group of symmetric indices, satisfy a set of transversality conditions where the divergence is computed "symmetrically" with respect to each family:

$$\begin{aligned}
&\partial^{\alpha}\,\Lambda_{\,\alpha\,\mu_2\,\cdots\,\mu_{s_1-1},\,\nu_1\,\cdots\,\nu_{s_2},\,\cdots}\;=\;0\,,\\
&\partial^{\alpha}\,\{\Lambda_{\,\mu_1\,\cdots\,\mu_{s_1-1},\,\alpha\,\nu_2\,\cdots\,\nu_{s_2},\,\cdots}\;+\;\lambda_{\,\alpha\,\mu_2\,\cdots\,\mu_{s_1},\,\nu_1\,\cdots\,\nu_{s_2-1},\,\cdots}\}\;=\;0\,,\\
&\cdots\,.
\end{aligned}\tag{1.11}$$

Thus, together with diagonal constraints forcing the divergence of the $i-$th parameter computed in the $i-$th family to vanish, there are off-diagonal constraints as well, combining the divergence of the $i-$th parameter computed in the $j-$th family with the divergence of the $j-$th parameter computed in the $i-$th family, as synthetically expressed by eq. (2.29) in a more appropriate notation.

Similarly to the symmetric case, this constrained local symmetry is indeed sufficient to ensure that the propagating degrees of freedom eventually sit in the totally transverse reducible tensor of $GL(D-2)$

$$\varphi_{\,i_1\,\cdots\,i_{s_1},\,j_1\,\cdots\,j_{s_2},\,\cdots}\,,\tag{1.12}$$

whose branching in $O(D-2)-$irreps describes the full particle content associated to (1.9). Comparing (1.9) with the constrained Lagrangian of Labastida [6] or with its minimal unconstrained versions given in [7, 8] (see also [9, 10] for reviews) allows to appreciate the advantages of our present approach[3]: while Lagrangians (1.9) always maintain the same form irrespective of the number N of index-families of the tensor φ, in the approach of [6, 7] the need to implement the more conventional kind of gauge invariance calls for the introduction of a number of traces of the basic kinetic tensor increasing with N. Moreover, the same actions (1.9) extend with no modification to the case of tensors in irreducible representations of $GL(D)$ or $O(D)$, so that even in the mixed-symmetry case transverse invariance can accommodate spectra of various degrees of complexity.

Remarkably, proceeding along the same lines it turns out to be possible to extend the Lagrangian formulation for tensors of any symmetry to the case of backgrounds with non-vanishing cosmological constant. Whereas a considerable body of knowledge is by now available for field theories involving symmetric tensors of arbitrary rank, both in flat and in particular in (A)dS backgrounds, where interactions among massless higher-spin particles seem to find a most natural arena [14], when it comes to tensors with mixed-symmetry the situation is vastly different. Indeed, in cosmological spaces not only are interactions for these types of particles so far little explored [15], but even free Lagrangians are available only for special classes of tensors, both in the metric-like approach that we pursue in this paper [16] and in the frame-like approach where the higher-spin degrees of freedom are encoded in sets of generalised vielbeins [17, 18, 19, 20] (see also [21, 22]). This essential gap in our knowledge is especially acute since mixed-symmetry states account for the

[3]Fields of mixed symmetry in Minkowski backgrounds have been subject to intense study since the mid-eighties, following the early progress of string field theory [11]. Here we discuss higher-spin fields as generalisations of the metric tensor for gravity (metric-like approach), while alternative forms of Lagrangians for tensors of mixed-symmetry in flat space have been obtained in a frame-like formulation [12] and in various other approaches,including some BRST-inspired ones [13].

vast majority of the string excitations. In this sense, it seems reasonable to expect that a satisfactory comparison between massless higher-spins and strings may benefit from a more complete understanding of the general types of massless particles allowed in a given space-time dimension.

In this work we propose an action for arbitrary massless fields in (A)dS spaces of any dimension, eq. (2.80) or (2.83), and in the remainder of this introduction we would like to provide a few additional details on the peculiarities of massless particles in (A)dS spaces, in order to better frame the main lines of our procedure.

The investigation of theories involving fields in arbitrary representations of the AdS or dS groups in D dimensions, $O(D-1,2)$ or $O(D,1)$ respectively, besides the technical complications already present in the flat-space analysis, is fraught with additional subtleties that are absent for the more customary symmetric representations. In particular, as first shown by Metsaev in [23, 24], the very notion of single, massless particle does not admit in general a continuous deformation from flat to (A)dS backgrounds and vice-versa, on account of the impossibility of preserving all the gauge symmetries of the flat theory. The analysis of [23, 24] elucidates the on-shell conditions to be satisifed in order for the wave operator to retain the maximal possible amount of gauge symmetry in (A)dS backgrounds, while also providing the further specifications needed to grant unitarity in Anti-de Sitter space. The general result is that, out of the p gauge parameters in principle available for tensors described by Young diagrams possessing p rectangular blocks of different horizontal lengths, *only one* can be kept in (A)dS. Moreover, while gauge invariance alone does not distinguish among the p options available in principle, for the case of Anti-de Sitter spaces unitarity dictates to preserve the parameter represented by the diagram missing one box in the upper rectangular block.

As a general consequence, (A)dS massless "particles" associated with a given diagram describe the propagation of more degrees of freedom than their flat-space peers. The exact branching of these irreducible (A)dS representations in terms of $O(D-2)$ ones (i.e. the structure of the flat-space multiplet corresponding to a single (A)dS particle) was first conjectured in [25] by Brink, Metsaev and Vasiliev and was recently subject to a detailed group-theoretical analysis in [26]; for instance, the unitary BMV multiplet associated with the massless AdS particle $\varphi_{\mu\nu,\rho}$ with the symmetries of the hook tableau $\{2,1\}$ comprises the degrees of freedom of the flat-space particle described by the same hook diagram together with those of a "graviton". However, as already mentioned, while the pattern of flat massless particles branching single (A)dS massless irreps is indeed known in the general case, so far its off-shell realisation has been provided only for special classes of Young diagrams.

In eq. (2.80) we propose Lagrangians for general N-family, $O(D)-$tensor fields in (A)dS, uniquely determined requiring that they preserve the amount of gauge-symmetry dictated by Metsaev's analysis; in particular for the unitary choice identified in [23, 24] our Lagrangian reads

$$\mathcal{L} = \frac{1}{2}\varphi_{\,\mu_1\cdots\mu_{s_1},\,\nu_1\cdots\nu_{s_2},\cdots}\Big\{\Box\,\varphi^{\,\mu_1\cdots\mu_{s_1},\,\nu_1\cdots\nu_{s_2},\,\cdots} - (\nabla^{\,\mu_1}\,\nabla_\alpha\,\varphi^{\,\alpha\,\mu_2\cdots\mu_{s_1},\,\nu_1\cdots\nu_{s_2},\,\cdots} \,+\,\cdots)$$
$$- \,(\nabla^{\,\nu_1}\,\nabla_\alpha\,\varphi^{\mu_1\cdots\mu_{s_1},\,\alpha\,\nu_2\cdots\nu_{s_2},\,\cdots} \,+\,\cdots) \,-\,\cdots$$
$$- \,\frac{1}{L^2}\left[(s_1-t_1-1)(D+s_1-t_1-2)\,-\,\sum_{k\,=\,1}^{p}s_k\,t_k\right]\varphi^{\,\mu_1\cdots\mu_{s_1},\,\nu_1\cdots\nu_{s_2},\cdots}\Big\}\,, \tag{1.13}$$

where s_1 and t_1 identify length and height of the first rectangular block, while the sum runs over the products of lengths and heights of all different blocks. Let us mention that, in this irreducible context, the gauge invariance of (1.13) is meant under transformations involving a fully divergenceless parameter. This result provides a relatively simple generalisation of the corresponding Lagrangian for symmetric tensors in (A)dS (2.55), to which it reduces for $p = 1$ and $t_1 = 1$. However, differently from that example and from the flat-space case, where our construction applies both to reducible and to irreducible tensors, the Lagrangians (1.13) involve traceless tensors in irreducible representations of $GL(D)$, and as such define candidate single-particle theories.

A convenient way to get some intuition about the unfamiliar BMV phenomenon is to observe that, in (A)dS backgrounds, the pattern of *reducible* gauge transformations associated in general to a mixed-symmetry gauge potential *gets unavoidably broken by terms proportional to the (A)dS curvature*. With reference to the example of the (traceless) hook tensor $\{2,\,1\}$ with covariantised gauge variation

$$\delta\,\varphi_{\,\mu\nu,\,\rho} \,=\, \nabla_\mu\,\Lambda_{\,\nu,\,\rho} \,+\, \nabla_\nu\,\Lambda_{\,\mu,\,\rho} \,+\, \nabla_\rho\,\lambda_{\,\mu\nu} \,-\, \frac{1}{2}\left(\nabla_\mu\,\lambda_{\,\nu\rho} \,+\, \nabla_\nu\,\lambda_{\,\mu\rho}\right), \tag{1.14}$$

where $\Lambda_{\,\mu,\,\nu}$ is a two-form while $\lambda_{\,\mu\nu}$ is a symmetric tensor, it is not hard to verify that, under the "gauge-for-gauge" transformations

$$\delta\,\Lambda_{\,\mu,\,\rho} \,=\, \nabla_\rho\,\theta_\mu \,-\, \nabla_\mu\,\theta_\rho, \qquad \delta\,\lambda_{\,\mu\,\nu} \,=\, -\,2\left(\nabla_\mu\,\theta_\nu \,+\, \nabla_\mu\,\theta_\nu\right), \tag{1.15}$$

that would leave the gauge potential invariant in flat space-time, now $\delta\,\varphi_{\,\mu\nu,\,\rho}$ acquires a contribution proportional to the (A)dS curvature:

$$\delta\,\varphi_{\,\mu\nu,\,\rho} \,=\, -\,2\left([\nabla_\rho,\,\nabla_\mu]\,\theta_\nu \,+\, [\nabla_\rho,\,\nabla_\nu]\,\theta_\mu\right). \tag{1.16}$$

This observation implies that even if one were able to find a kinetic tensor for $\varphi_{\,\mu\nu,\,\rho}$ invariant under the *full* transformation (1.14) the corresponding theory would possess too much gauge invariance with respect to the flat case, and thus would not describe anymore the degrees of freedom of the $O(D-2)$ hook. As an alternative to the elimination of one of the parameters of the flat theory, suggested by the result of [23, 24], one can instead "neutralise" the effect of the broken gauge-for-gauge vector θ_μ "promoting" it to play the role of gauge parameter for a new $O(D)-$field, that at this level could be either a symmetric rank-two tensor or a two-form. One could then proceed to construct the full Stueckelberg Lagrangian for both options, taking care of the fact that not only the "standard" gauge invariance but also overall gauge-for-gauge invariance be simultaneously preserved, in such a way that the number of independent gauge components of the resulting (A)dS theory be the same as for the corresponding flat system. The important difference between the two possible options is that, now referring specifically to Anti-de Sitter backgrounds, only for the rank-two symmetric tensor would the corresponding kinetic operator emerge with the correct sign required by a unitary theory, in accordance with the group-theoretical analysis of [23, 24]. Proceeding in this fashion for more general cases one can obtain an independent justification of the full BMV-pattern, and construct the corresponding Stueckelberg Lagrangians smoothly deforming the sum of flat-space Lagrangians appropriate for the description of the corresponding multiplet.

We perform this kind of construction explicitly for the case of tableaux with two rows in section 3.2.2, deforming the corresponding flat-space transverse-invariant Lagrangians presented in section 2.1.2. In particular we show that, starting from the Stueckelberg Lagrangian obtained in this way and performing appropriate off-shell gauge-fixings one reaches indeed our simplified Lagrangians (2.80) together with the corresponding transversality constraints, thus providing a complete proof, at least for these classes of tensors, of the validity of our result. Let us mention that the Stueckelberg construction, also discussed in [25] for the AdS particle with the symmetries of the hook tableau, has been explored in particular by Zinoviev in [18, 19], where frame-like Lagrangians for massive two-family tensors in (A)dS were obtained and their massless and partially massless limits were also discussed.

Whenever a system is found to be invariant under constrained gauge transformations it is natural in our opinion to try and interpret it as resulting from the partial gauge-fixing of a more general theory whose gauge symmetry is not constrained. Indeed, for the irreducible case involving traceless fields and transverse-traceless parameters investigated in [5], the Lagrangian (1.7) can be seen to arise from a partial gauge fixing of Fronsdal's Lagrangian itself, whose formulation requires traceless parameters (and doubly traceless fields). In its turn, the Fronsdal-Labastida theory admits minimal unconstrained extensions given in [27, 28, 7, 8], building on previous formulations [29, 30, 31] where the removal of constraints was linked to the possibility of assigning a dynamical role to the higher-spin curvatures of [32][4]. For the transverse-invariant Lagrangians that we propose in this work the most natural unconstrained extensions should be identified with the "triplets" associated to the tensionless limit of free open string field theory [34, 30, 35, 36] (see also [37, 38, 39]), whenever the corresponding actions are available. However, for the case of (A)dS tensors with mixed symmetry the corresponding unconstrained Lagrangians are not yet known, and exploring the possibility of constructing them and their possible relation with string systems in (A)dS is an interesting question that we leave for future investigation.

[4]For alternative formulations of the free theory of massless higher spins see e.g. [33].

For the case of spin 2 the idea of considering transverse-diffeomorphism invariance is indeed quite old and was explored from a number of perspectives, mainly in connection with so-called unimodular gravity and its relation to the cosmological constant problem [40, 41, 42]. The observation is that, while the variation of the Einstein-Hilbert action performed keeping the determinant of the metric fixed provides only the traceless part of Einstein's equations, the (contracted) Bianchi identity allows to recover the relation between the Ricci scalar and the trace of the stress-energy tensor, up to an arbitrary integration constant appearing in the resulting equation as a cosmological term. The connection with our approach is established observing that, at the linearised level, demanding that the determinant of the metric be gauge invariant requires *transverse* vector parameters, thus providing the first non-trivial example of (1.5). Let us also mention that conditions of transversality on gauge parameters were recently considered in the context of quantum-mechanical models on Kähler manifolds in [43].

We present the main results of this work in section 2, where we perform the construction of transverse-invariant Lagragians in increasing degree of generality, from symmetric tensors in Minkowski space to mixed-symmetry fields in (A)dS. The spectrum of particles propagating in the corresponding equations of motion is then analysed in section 3 exploiting various approaches. As already mentioned, when built out of traceful fields our Lagrangians propagate reducible spectra of free particles. This is the situation where the highest simplification is obtained (taking into account the structure of the Lagrangians, the form of the equations of motion and the analysis of the spectrum) and the closest contact with the tensionless open string is achieved. In section 4 we perform an additional step for the case of symmetric tensors, providing the solution to the problem of dissecting the field φ so as to explicitly identify its lower-spin components; as a result the action gets decomposed into a sum of decoupled terms, one for each particle present in the spectrum of the theory. We conclude summarising our findings while also putting them in a more general perspective, while in the appendices we collect our notations and conventions together with a number of additional technical results.

2 Lagrangians

2.1 Flat backgrounds

2.1.1 Symmetric tensors

Let us consider the Lagrangian[5]

$$\mathcal{L} \,=\, \frac{1}{2}\,\varphi\,M\,\varphi\,, \tag{2.1}$$

where M is the Maxwell operator

$$M \,=\, \Box \,-\, \partial\,\partial\cdot \;\;. \tag{2.2}$$

Up to total derivatives, its gauge variation under $\delta\,\varphi \,=\, \partial\,\Lambda$ is

$$\delta\,\mathcal{L} \,=\, -\,2\binom{s}{2}\,\partial\cdot\partial\cdot\varphi\,\partial\cdot\Lambda\,, \tag{2.3}$$

and thus vanishes assuming the condition of transversality for the gauge parameter:

$$\partial\cdot\Lambda \,=\, 0\,. \tag{2.4}$$

One could alternatively impose a differential constraint on the gauge field of the form $\partial\cdot\partial\cdot\varphi = 0$, that would also guarantee gauge invariance of (2.1). However, in order for this latter condition to be itself gauge invariant, the parameter should satisfy in principle a more involved transversality condition of the form $\partial\cdot\{\Box\,\partial\cdot\Lambda \,+\, \partial\,\partial\cdot\Lambda\} = 0$, so that it does not seem especially convenient to proceed in this direction.

Our interest in these kind of systems has several motivations, the first clearly being the appeal of simplicity. As we will show in this work, they provide an alternative route to the description of massless higher spins in their full generality, finding their original inspiration in the so-called TDiff-invariant spin-2 theories originally considered in [40, 41] and more recently in [42] in connection with unimodular gravity and with the cosmological constant problem. Moreover, in a number of cases one can relate transverse-invariant theories to the triplet Lagrangians emerging from the tensionless limit of the free open string [34, 30, 35], of which they effectively provide a simplified version retaining the same particle content.

A detailed analysis of the spectrum of Lagrangian (2.1) and of its generalisations is presented in section 3. In the specific case of interest in this section one can also connect the corresponding equations of motion,

$$(\Box \,-\, \partial\,\partial\cdot)\,\varphi \,=\, 0\,, \tag{2.5}$$

[5] Our notation and conventions are spelled out in appendix A. Symmetrised indices are always implicit and symmetrisation is understood with no weight factors. In particular "∂" stands for a gradient, while the symbol "$\partial\cdot$" denotes a divergence. Thus, in the Maxwell-like equations for a rank-s tensor, $(\Box - \partial\,\partial\cdot)\varphi = 0$, the second term actually contains s contributions: $\partial\,\partial\cdot\varphi = \partial_{\mu_1}\,\partial^{\alpha}\varphi_{\alpha\mu_2\cdots\mu_s} \,+\, \partial_{\mu_2}\,\partial^{\alpha}\varphi_{\alpha\mu_1\cdots\mu_s} \,+\, \cdots$.

to the reduced Fierz system [3],

$$\begin{aligned} \Box\, \varphi \,&=\, 0\,, \\ \partial \cdot \varphi \,&=\, 0\,. \end{aligned} \tag{2.6}$$

Indeed computing a divergence of (2.5) gives $\partial\, \partial \cdot \partial \cdot \varphi \,=\, 0$, and thereby effectively

$$\partial \cdot \partial \cdot \varphi \,=\, 0\,, \tag{2.7}$$

up to discrete degrees of freedom that we will systematically neglect, since they do not affect the counting of local degrees of freedom which is our main object of interest in the present framework. The remaining, transverse part of the divergence of φ can be gauged away using the divergence-free parameter Λ on account of

$$\delta\, \partial \cdot \varphi \,=\, \Box\, \Lambda\,, \tag{2.8}$$

thus showing the equivalence of (2.5) with (2.6) supplemented by the appropriate residual gauge invariance with parameter satisfying

$$\begin{aligned} \Box\, \Lambda \,&=\, 0\,, \\ \partial \cdot \Lambda \,&=\, 0\,. \end{aligned} \tag{2.9}$$

A standard analysis of (2.6) and (2.9) shows that the propagating polarisations are those associated with the components $\varphi_{\,i_1 \cdots\, i_s}$, where the indices i_k refer to directions transverse to the light-cone. Thus, together with the spin-s degrees of freedom contained in the traceless part of $\varphi_{\,i_1 \cdots\, i_s}$ in $D-2$ Euclidean dimensions, lower-spin representations of spin $s-2k$, with $k = 1, \ldots, [\frac{s}{2}]$, also propagate and sit in the traces of $\varphi_{\,i_1 \cdots\, i_s}$. For the irreducible case describing a single particle of spin s, already studied in [5], it will suffice to observe that, up to a restriction of the space of fields to traceless tensors subject to gauge variations with transverse and traceless parameters, one does not need to modify the form (2.1) of the Lagrangian, that in this sense applies to both reducible and irreducible descriptions. The corresponding equations of motion obtain taking the traceless projection of (2.5) and read

$$\left(\Box \,-\, \partial\, \partial \cdot \,+\, \frac{1}{D\,+\,2\,s\,-\,4}\, \eta\, \partial \cdot \partial \cdot \right) \varphi \,=\, 0\,. \tag{2.10}$$

Let us also mention that in our reducible context, with unconstrained fields subject to transverse and *traceful* gauge variations, multiple divergences of φ of order higher than one and, for even spins, the highest order trace $\varphi^{\,[\frac{s}{2}]}$, provide independent gauge-invariant quantities that could possibly enter modified forms of the Lagrangian. This is in particular true for the spin-2 case, where $\varphi^{\,\prime}$ and $\partial \cdot \partial \cdot \varphi$ could be combined in various forms providing gauge-invariant modifications of (2.1). For arbitrary spins, even limiting the attention to kinetic operators containing not more than two derivatives, we observe that uniqueness of (2.1) is always meant up to the scalar sector of the even-spin case admitting indeed possible deformations, both in the forms of mass terms

$$\Delta\, \mathcal{L}_m \,=\, \frac{1}{2}\, m^{\,2}\, (\varphi^{\,[\frac{s}{2}]})^{\,2}\,, \tag{2.11}$$

or as additional kinetic operators for the scalar member of the multiplet,

$$\Delta\,\mathcal{L}_K \,=\, \frac{a}{2}\,\varphi^{\,[\frac{s}{2}]}\,\Box\,\varphi^{\,[\frac{s}{2}]}\,, \tag{2.12}$$

amenable in principle to change the propagating nature of the latter, and possibly to eliminate it altogether from the spectrum by a suitable choice of the coefficient a[6].

The reducible particle content associated to eq. (2.5) in the absence of trace constraints corresponds to that of the Lagrangians obtained from the BRST action for the free open string, after taking the tensionless limit $\alpha' \to \infty$ in the sense described in [34, 30, 35]. The spin-s block-diagonal term obtained in that approach (after solving for an additional auxiliary field with algebraic equations of motion) reads in fact

$$\mathcal{L} \,=\, \frac{1}{2}\,\varphi\,M\,\varphi \,+\, 2\,\varphi\,\partial^{\,2}\,\mathcal{D} \,-\, 2\binom{s}{2}\,\mathcal{D}\hat{M}\,\mathcal{D}\,, \tag{2.13}$$

where φ and $\mathcal{D}$ are symmetric tensors of ranks s and $s-2$ respectively, subject to the unconstrained gauge transformations $\delta\,\varphi \,=\, \partial\,\Lambda$ and $\delta\,\mathcal{D} \,=\, \partial\cdot\Lambda$, while

$$\hat{M} \,=\, \Box \,+\, \frac{1}{2}\,\partial\,\partial\cdot\,, \tag{2.14}$$

is a sort of deformed Maxwell operator for the field $\mathcal{D}$. In this respect our analysis shows that performing the off-shell gauge-fixing $\mathcal{D} = 0$ does not alter the spectrum of the theory. Conversely, one can generate the Lagrangian (2.13) from (2.1) in two steps: first introducing a Stueckelberg field θ via the redefinition $\varphi \,\to\, \varphi - \partial\,\theta$, with $\delta\,\varphi = \partial\,\Lambda$ and $\delta\,\theta = \Lambda$, and then identifying the divergence of θ with the field $\mathcal{D}$ of (2.13). In this sense the relation between the transverse-invariant Lagrangian (2.1) and the unconstrained triplet Lagrangian (2.13) is analogous to the relative role played by Fronsdal's constrained theory, with traceless gauge parameter and doubly-traceless field, and its minimal unconstrained extension proposed in [27] for the description of irreducible massless particles of spin s.

Modifying the constraints on the field and on the gauge parameter in various ways (which include the option of relaxing them altogether) several possible completions of (2.1) can be found, local and non-local (see also our comments in the Discussion). Concerning the latter, a rationale for the introduction of non-local terms in unconstrained Lagrangians is found whenever it is possible to interpret them as the result of the integration over unphysical fields [44]. The corresponding analysis for the triplets was performed in [36] where it was shown that the elimination of the field $\mathcal{D}$ in (2.13) produces indeed a gauge-invariant completion of (2.1) given by the Lagrangian

$$\begin{aligned}\mathcal{L} \,&=\, \frac{1}{2}\,\varphi\,M\,\varphi \,+\, \sum_{m\,=\,2}^{s}\binom{s}{m}\,\partial\,\cdot^{\,m}\,\varphi\,\frac{1}{\Box^{\,m-1}}\,\partial\,\cdot^{\,m}\,\varphi \\ &=\, \frac{1}{2}\,\sum_{m\,=\,0}^{s}\binom{s}{m}\,\partial\,\cdot^{\,m}\,\varphi\,\frac{1}{\Box^{\,m-1}}\,\partial\,\cdot^{\,m}\,\varphi\,,\end{aligned} \tag{2.15}$$

[6]See [39] for a discussion of reducible triplets in the frame-like approach, where in particular a similar arbitrariness at the level of the scalar component of the multiplet was also noticed.

where $\partial \cdot^{\,m}$ denotes the $m-$th power of the divergence. In the same context it was also shown how to combine the various terms in (2.15) in order to reproduce the square of the corresponding higher-spin curvatures $\mathcal{R}^{(s)}_{\mu_1\cdots\,\mu_s,\,\nu_1\cdots\,\nu_s}$ [32], leading to the compact expression[7]

$$\mathcal{L} \,=\, \frac{(-1)^{\,s}}{2\,(s+1)}\,\mathcal{R}^{(s)}_{\mu_1\cdots\,\mu_s,\,\nu_1\cdots\,\nu_s}\,\frac{1}{\Box^{s-1}}\,\mathcal{R}^{(s)\,\mu_1\cdots\,\mu_s,\,\nu_1\cdots\,\nu_s}\,. \tag{2.16}$$

In (2.16) the spin-1 case corresponds to the only local option, while the non-local Lagrangians obtained for spin $s \geq 2$ build a metric-like generalisation of Maxwell's Lagrangian in its geometric form bearing the same particle content as the triplet system (2.13).

Let us finally observe that, in analogy with the spin-1 case, where the divergence of the field strength defines the equations of motion, in our present setting the kinetic tensor M can be easily related to the first connexion in the de Wit-Freedman hierarchy[8] [32]

$$\Gamma^{(1)}{}_{\rho,\,\mu_s} \,=\, \partial_{\,\rho}\,\varphi_{\,\mu_s} \,-\, \partial_{\,\mu}\,\varphi_{\,\mu_{s-1}\rho}\,, \tag{2.17}$$

which clearly reduces to Maxwell's field strength for $s=1$. Indeed, it is simple to check that the divergence of $\Gamma_{\,\rho,\,\mu_s}$ in the ρ index builds the Maxwell operator (2.2),

$$\partial^{\,\rho}\,\Gamma^{(1)}{}_{\rho,\,\mu_s} \,=\, (M\,\varphi)_{\,\mu_s}\,, \tag{2.18}$$

while the Lagrangian (2.1) can be written as a square of those connexions as follows:

$$\mathcal{L} \,=\, \frac{1}{4\,(s-1)}\,\left\{\,\Gamma_{\,\mu,\,\rho\,\mu_{s-1}} \,-\, (s-2)\,\Gamma_{\,\rho,\,\mu_s}\,\right\}\Gamma^{\,\rho,\,\mu_s}\,. \tag{2.19}$$

This observation suggests a clear parallel with the Fronsdal formulation, where the basic kinetic tensor $\mathcal{F}$ given in (1.6) obtains from the trace of the second connexion in the hierarchy of [32],

$$\Gamma^{(2)}{}_{\rho\rho,\,\mu_s} \,=\, \partial_{\,\rho}^{\,2}\,\varphi_{\,\mu_s} \,-\, \frac{1}{2}\,\partial_{\,\rho}\,\partial_{\,\mu}\,\varphi_{\,\mu_{s-1}\rho} \,+\, \partial_{\,\mu}^{\,2}\,\varphi_{\,\mu_{s-2}\rho\rho}, \tag{2.20}$$

to be computed in the ρ-indices, providing a spin$-s$ generalization of the linearised spin-2 Ricci tensor.

[7] Lagrangians for symmetric bosons and fermions of arbitrary spin were first formulated in terms of metric-like curvatures in [29] while an approach similar in spirit was also proposed for mixed-symmetry bosons in [45, 46]. Out of the infinitely many options available in principle, the unique Lagrangians leading to the correct propagators were given for symmetric tensors in [28], while their massive deformations were discussed in [31], together with a more detailed analysis of the role of curvatures for fermionic theories. Their connection with the minimal local Lagrangians of [27] was given in [44]. More recently, first-order non-linear deformation of the curvatures were also found in [47].

[8] To manipulate the generalised connexions of [32] we resort to a notation [31], where symmetrised indices are denoted with the same symbol, while the subscript denotes the number of indices in a given group. For instance $\partial_{\,\mu}\,\varphi_{\,\mu_{s-1}\rho}$ is a shortcut for $\partial_{\,\mu_1}\,\varphi_{\,\mu_2\,\cdots\,\mu_s\rho} \,+\, \partial_{\,\mu_2}\,\varphi_{\,\mu_1\,\mu_3\,\cdots\,\mu_s\rho} \,+\, \cdots$, with the index ρ excluded from the symmetrization. For the manipulations required in this section the rules of symmetric calculus given in Appendix A.1 apply separately for each group of indices.

2.1.2 Mixed-symmetry tensors

In this section we would like to extend our analysis to tensor fields of mixed symmetry. The basic objects under scrutiny will be *multi-symmetric* tensors,

$$\varphi_{\mu_1\cdots\,\mu_{s_1},\,\nu_1\cdots\,\nu_{s_2},\,\cdots} \,\equiv\, \varphi\,, \tag{2.21}$$

possessing an arbitrary number N of independent sets ("families") of symmetrised indices and thus defining *reducible* $GL(D)$ tensors, of the kind appearing as coefficients in the expansion of the bosonic string field. While this choice guarantees the highest degree of overall simplification, our results can also be adapted to the case where φ defines an irreducible representation of $GL(D)$, or even of $O(D)$, with minor complications from the perspective of the construction of a gauge-invariant Lagrangian.

The proper generalisation of the index-free notation used in the symmetric case was introduced in [7, 8, 9, 10] and is reviewed in appendix A.2. The basic idea is to employ "family" labels to denote operations adding or subtracting space-time indices belonging to a given group. More specifically upper family indices are reserved for operators, like gradients, which add space-time indices, while lower family indices are used for operators, like divergences, which remove them. As a result gradients and divergences acting on the $i-$th family are denoted concisely by $\partial^{\,i}\,\varphi$ and $\partial_{\,i}\,\varphi$ respectively, while $T_{ij}\,\varphi$ refers to a trace contracting one index in the $i-$th family with one index in the $j-$th family. It is also useful to introduce operators, denoted by $S^{\,i}{}_{j}$, whose effect is to displace indices from one family to another while also implementing the corresponding symmetrization; namely

$$S^{\,i}{}_{j}\,\varphi \,\equiv\, \varphi_{\cdots,\,(\mu^i{}_1\cdots\,\mu^i{}_{s_i}|,\,\cdots,\,|\mu^i{}_{s_i+1})\mu^j{}_1\cdots\,\mu^j{}_{s_j-1},\,\cdots}\,, \qquad \text{for}\ \ i\neq j\,, \tag{2.22}$$

while their diagonal members $S^{\,i}{}_{i}$ essentially count the number of indices in the $i-$th family:

$$S^{\,i}{}_{i}\,\varphi \,\equiv\, s_i\ \varphi_{\cdots,\,\mu^i{}_1\cdots\,\mu^i{}_{s_i},\,\cdots}\,. \tag{2.23}$$

Thus, given the multi-symmetric tensor φ in (2.21), the corresponding $GL(D)$-diagram with the same index structure is characterised by the additional condition

$$S^{\,i}{}_{j}\,\varphi \,=\, 0\,, \qquad \text{for}\ i<j\,, \tag{2.24}$$

while supplementing (2.24) with the tracelessness constraint

$$T_{ij}\,\varphi \,=\, 0\,, \qquad \forall\ i,j\,, \tag{2.25}$$

allows to deal directly with irreducible tensors of $O(D)$.

We first consider the case of multi-symmetric tensors (2.21) and postulate Maxwell-like equations of motion

$$M\,\varphi \,\equiv\, (\Box\,-\,\partial^{\,i}\,\partial_{\,i})\,\varphi \,=\, 0\,, \tag{2.26}$$

together with the gauge transformations

$$\delta\,\varphi \,=\, \partial^{\,i}\,\Lambda_{\,i} \tag{2.27}$$

involving a set of multi-symmetric gauge parameters each of them lacking one space-time index in the appropriate group, and thus denoted by Λ_i. Under (2.27) the equations of motion transform according to

$$\delta\,(\Box\,-\,\partial^{\,i}\,\partial_i)\,\varphi\;=\;-\,\frac{1}{2}\,\partial^{\,i}\,\partial^{\,j}\,\partial_{(i}\,\Lambda_{\,j)}\,, \tag{2.28}$$

thus showing that the *generalised transversality constraints*

$$\partial_{(i}\,\Lambda_{\,j)}\;=\;0\,, \tag{2.29}$$

summarizing in our compact notation the relations (1.11), suffice to guarantee gauge invariance of the corresponding Lagrangian

$$\mathcal{L}\;=\;\frac{1}{2}\,\varphi\,(\Box\,-\,\partial^{\,i}\,\partial_i)\,\varphi\;\equiv\;\frac{1}{2}\,\varphi\,M\,\varphi\,. \tag{2.30}$$

Actually the general solution granting the vanishing of (2.28), taken at face value, would subject the parameters to a weaker set of conditions, namely

$$\partial^{\,i}\,\partial^{\,j}\,\partial_{(i}\,\Lambda_{\,j)}\;=\;0\,, \tag{2.31}$$

and it is not manifestly obvious that the two conditions (2.29) and (2.31) should be regarded as equivalent. Indeed, the summation over family indices in (2.31), that we shall refer to as defining "weak constraints", leads to an equation that would admit in principle additional solutions as compared to (2.29), here referred to as "strong constraints", thus raising an issue about the effective amount of gauge symmetry possessed by the Lagrangian (2.30).

A first indication of the eventual equivalence of the two conditions obtains from the analysis of gauge-for-gauge transformations associated to (2.27), which take the general form

$$\delta\,\Lambda_{\,i}\;=\;\partial^{\,k}\,\Lambda_{\,[ik]}\,, \tag{2.32}$$

where square brackets denote antisymmetrisations, and in our notation indicate that the two missing indices in $\Lambda_{\,[ik]}$ cannot belong to the same family. Exploiting (2.32) it is not hard to show that the weak form of the transversality constraints (2.31) does not impose additional conditions on the parameters of gauge-for-gauge transformations:

$$\partial^{\,i}\,\partial^{\,j}\,\partial_i\,\partial^{\,k}\,\Lambda_{\,[jk]}\;=\;2\,\Box\,\partial^{\,i}\,\partial^{\,j}\,\Lambda_{\,[ij]}\;+\;\partial^{\,i}\,\partial^{\,j}\,\partial^{\,k}\,\partial_i\,\Lambda_{\,[jk]}\;\equiv\;0\,, \tag{2.33}$$

where in particular the two contributions on the r.h.s. vanish independently, due to their symmetry properties[9]. On the other hand, the strong constraints (2.29) do impose restrictions on the gauge-for-gauge parameters, summarized by the relations

$$\partial^{\,k}\,\partial_i\,\Lambda_{\,[jk]}\;+\;\partial^{\,k}\,\partial_j\,\Lambda_{\,[ik]}\;=\;0\,. \tag{2.34}$$

[9]Actually the result might be expected, since the weak form of the constraints reproduces directly the variation of φ within the equations of motion, and as such must vanish identically under the full set of second-generation variations (2.32).

Thus we see that the potentially wider set of solutions to weak constraints is subject to a bigger invariance under gauge-for-gauge transformations, while the latter provide less severe conditions for solutions to strong constraints, on account of (2.34).

A complete argument to conclude on the equivalence of weak and strong conditions can be formulated in two steps, considering the equations of motion in momentum space and discussing separately the cases $p^2 \neq 0$ and $p^2 = 0$. In the former case, saturating all indices of (2.26) by divergences, it is possible to iteratively prove that, whenever φ solves (2.26), all its double-divergences vanish[10]

$$p_i\, p_j\, \varphi \,=\, 0\,. \tag{2.35}$$

This implies that $p_i\, \varphi$ is transverse, and thus solving for φ in (2.26),

$$\varphi \,=\, \frac{p^i}{p^2}\, p_i\, \varphi\,, \tag{2.36}$$

one can conclude that only pure gauge solutions are available, where the effective parameter $\tilde{\Lambda}_i \,=\, \frac{p_i}{p^2}\, \varphi$ naturally satisfies *strong* constraints, thus making manifest that for $p^2 \neq 0$ there is no room for additional gauge transformations to be effective on the field. For $p^2 \,=\, 0$ it is convenient to go to the frame where $p_\mu \,=\, p_+$, where it is possible to immediately observe that all components transverse to the light-cone (i.e. the physical ones) are gauge invariant, regardless of the conditions to be imposed on the parameters. The issue is then to show that parameters subject to strong constraints actually suffice, in conjunction with the equations of motion, to remove all components longitudinal to the light-cone, thus implicitly showing that weak constraints must eventually be equivalent to the former, since they could not remove anyway additional polarisations. We discuss in detail this aspect of the proof in section 2.1.2, where we show that the spectrum of the transverse-invariant theory defined by (2.30) coincides with the particle content of the reduced Fierz system

$$\begin{aligned} \Box\, \varphi \,&=\, 0\,, \\ \partial_i\, \varphi \,&=\, 0\,, \end{aligned} \tag{2.37}$$

with appropriate residual gauge invariance, with Λ_i also satisfying a system analogous to (2.37).

Let us mention that the same kind of issue might be raised when discussing the nature of the constraints in the Labastida formulation [6]. In that framework indeed the gauge variation of the basic kinetic tensor

$$\mathcal{F} \,=\, M\, \varphi + \frac{1}{2}\, \partial^i \partial^j\, T_{ij}\, \varphi \tag{2.38}$$

[10] One can make sure that the corresponding condition is gauge invariant solving the system for the double divergences of (2.29), $p_k\, p_{(i}\, \Lambda_{j)} \,=\, 0$, $p_j\, p_{(k}\, \Lambda_{i)} \,=\, 0$, $p_i\, p_{(j}\, \Lambda_{k)} \,=\, 0$, with $i \neq j \neq k$. (The other cases are trivial.) Equivalently, it is possible to observe that of the two projections admissible in principle, $p_k\, p_i\, \Lambda_j \,\sim\, \ydiagram{2,1} \oplus \ydiagram{3}$, only the first one does not vanish directly after imposing (2.29). This implies the equality $p_k\, p_i\, \Lambda_j \,=\, \frac{2}{3}\,(p_k\, p_i\, \Lambda_j \,-\, \frac{1}{2}\, p_j\, p_{(i}\, \Lambda_{k)})$, allowing in its turn to conclude again that all double divergences of the parameters vanish, once more due to the transversality conditions (2.29).

is proportional to a triple gradient of the traces of the field,

$$\delta\,\mathcal{F}\,=\,\frac{1}{6}\,\partial^{\,i}\,\partial^{\,j}\,\partial^{\,k}\,T_{(\,ij}\,\Lambda_{\,k\,)}\,, \tag{2.39}$$

so that the natural condition granting gauge invariance of the equations of motion would look

$$\partial^{\,i}\,\partial^{\,j}\,\partial^{\,k}\,T_{(\,ij}\,\Lambda_{\,k\,)}\,=\,0\,, \tag{2.40}$$

rather than its strong form, where all symmetrised traces are directly required to vanish:

$$T_{(\,ij}\,\Lambda_{\,k\,)}\,=\,0\,. \tag{2.41}$$

Even for this case, however, a reasoning similar to the one we presented above should allow to conclude for the eventual equivalence of the two possibilities, on account of the different amount of gauge-for-gauge invariance available in the two cases.

Thus, from now on we will always assume that the transversality constraints are satisfied in their strong form (2.29), which is easier to manipulate. In section 2.1.2 we will show that starting from reducible $GL(D)-$tensors the propagating polarisations are of the form

$$\varphi_{i^1{}_1\cdots\, i^1{}_{s_1},\,\cdots\,,\, i^N{}_1\cdots\, i^N{}_{s_N}}\,, \tag{2.42}$$

with indices $i^j{}_l = 1, \ldots , D-2$, here displayed for additional clarity, taking values along the directions transverse to the light-cone. The resulting expression can be first decomposed in diagrams of $GL(D-2)$, each carrying in its turn a reducible particle content described by the corresponding branching in irreps of $O(D-2)$, so that working with multi-symmetric tensors leads to a spectrum that is two-fold reducible, in a sense, and which can be directly compared with the one emerging from the component expansion of string field theory.

However, it is also possible to stay closer in spirit to the more customary examples of low spin and choose φ in an irreducible representation of $GL(D)$ or even of $O(D)$, enforcing (2.24) and possibly (2.25), without otherwise spoiling the general scheme of the construction. Indeed, the Maxwell operator defined in (2.26) commutes with the operators $S^{\,i}{}_j$,

$$[\,M,\,S^{\,i}{}_j\,]\,=\,0\,, \tag{2.43}$$

and thus also with the operator projecting a multi-symmetric tensor to an irrep of $GL(D)$,[11] thus ensuring invariance of the form of the Lagrangian and of the equations of

[11]The projectors take the schematic form $\Pi = \mathbb{I} + S^n$, where S^n denotes a product of $S^i{}_j$ operators with all indices contracted; for instance, the multi-symmetric tensor $\varphi_{\mu_1\mu_2,\,\nu}$ can be projected to the hook diagram of $GL(D)$ by the following operator:

$$Y_{\{2,1\}}\,\varphi\,=\,\frac{4}{3}\left(\mathbb{I}\,-\,\frac{1}{4}\,S^{\,1}{}_2\,S^{\,2}{}_1\right)\varphi\,.$$

Let us observe that this also implies that the $S^i{}_j$ commute with the projectors themselves. From a different perspective, one can observe that acting with $S^i{}_j$ on a Young diagram one obtains a sum of vectors in the same irreducible space.

motion even in the irreducible case. Similar considerations are also valid for the Labastida Lagrangian [6], which retains indeed the same form in both cases, as discussed for instance in section 4 of [7]. Adding the condition of tracelessness (2.25), on the other hand, as required to describe the propagation of a single massless particle with a given symmetry pattern, again would not spoil the form of (2.30), similarly to what we already observed for the symmetric case in the previous section. What would be different in this latter case would be the actual form of the equations of motion, attaineable from (2.26) performing the appropriate traceless projection. Moreover, for fields belonging to irreps of $O(D)$, one should supplement the transversality constraints (2.29) with the appropriate conditions on the traces of the parameters and, correspondingly, on the gauge-for-gauge parameters. In agreement with our previous discussion, in the following we will assume them to take their simplest form, so that in particular for all generations gauge parameters can be supposed to be traceless.

To summarise, for the three cases under consideration, namely $GL(D)-$reducible, $GL(D)-$irreducible and $O(D)-$irreducible tensors of arbitrary type, eq. (2.30) provides a gauge-invariant Lagrangian on condition that the gauge parameters satisfy the symmetrised transversality constraint (2.29), together with appropriate restrictions on their traces for irreps of the orthogonal group.

In our opinion it is rather remarkable that a consistent Lagrangian for general tensor fields of any symmetry type can be formulated in such a simple form as (2.30). To better appreciate this point it might be useful to compare (2.30), with Λ_i constrained as in (2.29), with the form of the Labastida Lagrangian for the *two-family* case[12] [6], that in our notation reads[13]

$$\mathcal{L} \,=\, \frac{1}{2}\,\varphi\,\{\,\mathcal{F} \,-\, \frac{1}{2}\,\eta^{ij}\,T_{ij}\,\mathcal{F} \,+\, \frac{1}{36}\,\eta^{ij}\,\eta^{kl}\left(\,2\,T_{ij}\,T_{kl} \,-\, T_{i\,(\,k}\,T_{l\,)\,j}\,\right)\mathcal{F}\,\}\,, \tag{2.44}$$

where $\mathcal{F}$ is the Fronsdal-Labastida tensor (2.38), φ is subject to a generalised double-trace constraint,

$$T_{(\,ij}\,T_{kl\,)}\,\varphi \,=\, 0\,, \tag{2.45}$$

while, as already recalled, gauge invariance holds if the symmetrised traces of Λ_i are constrained to vanish, as in (2.41).

For what concerns tensionless strings in the sector of mixed-symmetry fields, generalised “triplets” were introduced in [35] and there shown to describe the full spectrum of the free open string collapsed to zero mass. To establish contact with our formulation we refer to the form of corresponding equations of motion for the multi-tensor φ transforming as in (2.27), rephrased in our present notation as

$$M\,\varphi \,+\, \frac{1}{2}\,\partial^{\,i}\,\partial^{\,j}\,\mathcal{D}_{\,(ij)} \,=\, 0\,, \tag{2.46}$$

[12]For an arbitrary number of families the result gets more and more complicated, and in particular the number of terms in the corresponding Lagrangians grows linearly with the number of families. We refer the reader to section 3 of [7] for more details.

[13]As already recalled, this form of the Lagrangian applies both to multi-tensors and to irreps of $GL(D)$.

where the set of fields $\mathcal{D}_{ij}$ transform according to

$$\delta\,\mathcal{D}_{ij}\,=\,\partial_i\,\Lambda_j\,, \tag{2.47}$$

while also being subject to the constraint $\partial_k\,\mathcal{D}_{ij}\,=\,\partial_i\,\mathcal{D}_{kj}$, so as to prevent their gauge-invariant combinations to propagate unwanted degrees of freedom. It should be noted that the tensors $\mathcal{D}_{ij}$ are *not* symmetric in their family indices, so that while their combination appearing in (2.46) displays the proper gauge transformation as required for consistency with (2.28), the comparison of the two formulations at the Lagrangian level is less straightforward and should be the object of a separate investigation. Here we limit ourselves to stress that working with divergence-free parameters, in the sense of (2.29), allows to bypass completely the need for the additional set of auxiliary fields and corresponding gauge parameters introduced in [35], leading also with respect to the latter formulation to a sizeable simplification.

Let us also mention that, similarly to the symmetric case, an alternative presentation of (2.26) obtains also in this case introducing the following generalisation of the first connexion of [32]

$$\Gamma_{\rho,\,\mu^1{}_{s_1},\,\cdots,\,\mu^N{}_{s_N}}\,=\,\partial_\rho\,\varphi_{\mu^1{}_{s_1},\,\cdots,\,\mu^N{}_{s_N}}\,-\,\sum_{i\,=\,1}^{N}\,\partial_{\mu^i}\,\varphi_{\mu^1{}_{s_1},\,\cdots,\,\mu^i{}_{s_i-1}\,\rho,\,\cdots,\,\mu^N{}_{s_N}}\,, \tag{2.48}$$

where the index $\mu^i{}_{s_i}$ is a shortcut for a group of s_i indices in the $i-$th family, consistently with the notation used for (2.17). (See footnote 8.) Making use of (2.48) it is then possible to recast the equations of motion (2.26) in the form

$$(M\,\varphi)_{\mu^1{}_{s_1},\,\cdots,\,\mu^N{}_{s_N}}\,=\,\partial^\alpha\,\Gamma_{\alpha,\,\mu^1{}_{s_1},\,\cdots,\,\mu^N{}_{s_N}}\,, \tag{2.49}$$

along the lines of their counterparts (2.18) for the case of symmetric tensors.

2.2 (A)dS backgrounds

In this section we show how to build transverse-invariant Lagrangians in (A)dS backgrounds. For symmetric tensors the construction mirrors the procedure exploited in Minkowski space-time, for both reducible and irreducible cases. The case of mixed-symmetry tensors is technically more involved and conceptually more subtle, due to the unconventional branching of the corresponding irreducible representations in terms of $O(D-2)$ ones [23, 24, 25]. For this class of fields we focus on single-particle Lagrangians, while still allowing the corresponding tensors to be of arbitrary symmetry type.

2.2.1 Symmetric tensors

We would like to construct the simplest deformation of Lagrangian (2.1) to the case of maximally symmetric backgrounds, the corresponding spectra are discussed in section

3.2.1. For this reason, starting with the covariantised version of the operator (2.2), we compute the gauge variation of the corresponding tensor

$$M\,\varphi \,\equiv\, (\,\Box\,-\,\nabla\,\nabla\cdot\,)\,\varphi\,, \tag{2.50}$$

under the divergence-free gauge transformations

$$\delta\varphi \,=\, \nabla\Lambda\,, \qquad\qquad \nabla\cdot\Lambda \,=\, 0\,, \tag{2.51}$$

obtaining[14]

$$M\delta\varphi \,=\, \frac{1}{L^2}\,\{\,[\,(s-2)(D+s-3)-s\,]\,\nabla\Lambda -\, 2\,g\,\nabla\Lambda'\,\}\,, \tag{2.52}$$

where g is the AdS metric. Its gauge-invariant completion is then easily found to be

$$M_L\,\varphi \,\equiv\, M\,\varphi \,-\, \frac{1}{L^2}\,\{\,[\,(s-2)(D+s-3)\,-\,s\,]\,\varphi\,-\,2\,g\,\varphi'\,\}\,, \tag{2.53}$$

displaying the same spin-dependent "mass term" as the covariantised Fronsdal theory [48], up to a sign-flip in the trace part due to the different roles played by the variation of φ' in the two cases[15]. The corresponding equations of motion,

$$M_L\,\varphi \,=\, 0\,, \tag{2.54}$$

are obtained from the Lagrangian

$$\mathcal{L} \,=\, \frac{1}{2}\,\varphi\,M_L\,\varphi\,, \tag{2.55}$$

which provides a smooth deformation of (2.1) to (A)dS space. In the absence of additional assumptions eq. (2.54) propagates a reducible spectrum of (A)dS massless particles of the same kind as its flat counterpart (2.1).

However, it can be also interesting to further restrict the relevant tensors in (2.55) to be traceless:

$$\varphi' \,=\, 0\,, \qquad\qquad\qquad \Lambda' \,=\, 0\,, \tag{2.56}$$

thus providing the (A)dS extension of the irreducible system in flat backgrounds. The proper Lagrangian under these assumptions is still given by (2.55), with the proviso that now the kinetic tensor M_L does not contain contributions involving the trace of φ, while the corresponding equations of motion

$$M\,\varphi \,-\, \frac{1}{L^2}\,[\,(s-2)(D+s-3)\,-\,s\,]\,\varphi \,+\, \frac{2}{D+2\,(s-2)}\,g\,\nabla\cdot\nabla\cdot\,\varphi \,=\, 0\,, \tag{2.57}$$

[14] The basic technical device needed is the commutator of two covariant derivatives acting on a vector

$$[\,\nabla_\mu\,,\nabla_\nu\,]\,V_\rho \,=\, \frac{1}{L^2}\,(\,g_{\nu\rho}\,V_\mu\,-\,g_{\mu\rho}\,V_\nu\,)\,,$$

where for definiteness we refer to the Anti-de Sitter case, with L denoting the radius and g the metric of a D-dimensional AdS space. With the substitution $L \to iL$ one recovers the commutator on dS; this modification would not affect our manipulations so that our results formally apply to the dS case as well.

[15] In the Fronsdal case with traceless parameter one has $\delta\,\varphi' \,=\, 2\,\nabla\cdot\Lambda$; divergence-free parameters we have in general $\delta\,\varphi' \,=\, \nabla\,\Lambda'$.

can be shown to propagate only the massless polarisations of spin s. In this sense, eqs. (2.55) (with $\varphi' = 0$) and (2.57) build an alternative to Fronsdal's theory in (A)dS [48], involving a minimal number of off-shell field components.

2.2.2 Mixed-symmetry tensors

As a starting point for our analysis we compute the gauge transformation of the covariantised form of (2.26),

$$M\,\varphi \,\equiv\, \left(\Box - \nabla^i\nabla_i\right)\varphi\,, \tag{2.58}$$

where φ is a multi-symmetric tensor with covariantised gauge variation

$$\delta\varphi \,=\, \nabla^i\Lambda_{\,i}\,, \tag{2.59}$$

trying to identify the compensating terms needed to make (2.58) gauge invariant. With the help of the commutators collected in appendix A.2 one can obtain

$$\begin{aligned} M\,\delta\varphi \,=\, & -\frac{1}{L^2}\left\{\,(D-1)\,\nabla^i\Lambda_i \,-\, (D-N-3)\,\nabla^i S^j{}_i\,\Lambda_j \,-\, \nabla^i S^j{}_k S^k{}_i\,\Lambda_j\,\right\} \\ & -\,\frac{1}{2}\,\nabla^i\nabla^j\nabla_{(\,i}\Lambda_{\,j)} + \frac{1}{L^2}\left\{\,2\,g^{ij}\,\nabla_{(\,i}\Lambda_{\,j)} \,+\, g^{ij}S^k{}_i\nabla_{[\,j}\Lambda_{\,k]} \,-\, 2\,\nabla^i g^{jk}\,T_{ij}\,\Lambda_k\,\right\}, \end{aligned} \tag{2.60}$$

where N denotes the number of families. From the resulting expression, still rather involved even after imposing the transversality conditions

$$\nabla_{(i}\Lambda_{\,j)} \,=\, 0\,, \tag{2.61}$$

it is possible to appreciate the difficulties met in extending the flat gauge invariance to the (A)dS case, already visible for the case of tensors with two families of indices. Indeed, rewriting (2.60) for these fields in a more explicit notation as

$$\begin{aligned} (M\,\delta\,\varphi)_{\,\mu_s,\,\nu_r} \,=\, & \frac{1}{L^2}\,\Big\{[(s-1)(D+s-3)-(D+2s-3)]\,\nabla_\mu\,\Lambda_{\,\mu_{s-1},\,\nu_r} \\ & +[(r-1)(D+r-3)-(D+2r-3)]\,\nabla_\nu\,\lambda_{\,\mu_s,\,\nu_{r-1}} \\ & +\nabla_\mu\,\Lambda_{\,\nu\,\mu_{s-2},\,\mu\,\nu_{r-1}} \,+\, \nabla_\nu\,\lambda_{\,\mu_{s-1}\,\nu,\,\mu\,\nu_{r-2}} \\ & +\,(D+s+r-5)\left[\,\nabla_\nu\,\Lambda_{\,\mu_{s-1},\,\mu\,\nu_{r-1}} \,+\, \nabla_\mu\,\lambda_{\,\nu\,\mu_{s-1},\,\nu_{\,r-1}}\,\right] + \,\cdots\Big\}\ , \end{aligned} \tag{2.62}$$

where the dots stand for terms involving traces or divergences of the parameters while like indices are understood to be symmetrised, one can recognise that, for $s \neq r$, there is no way of compensating the first two terms in (2.62) with contributions linear in $\varphi_{\,\mu_s,\,\nu_r}$ of any sort[16] so that mixed-symmetry tensors in (A)dS are bound to possess a smaller gauge

[16] Considering counterterms involving exchanges of indices would not help. Indeed the variation of the generic term

$$\begin{aligned} \delta\varphi_{\mu_{s-n}\,\nu_n\,,\,\mu_n\,\nu_{r-n}} = & \ \nabla_\mu\,\Lambda_{\mu_{s-n-1}\,\nu_n,\,\mu_n\,\nu_{r-n}} \,+\, \nabla_\nu\,\Lambda_{\mu_{s-n}\,\nu_{n-1},\,\mu_n\,\nu_{r-n}} \\ & +\,\nabla_\nu\,\lambda_{\mu_{s-n}\,\nu_n,\,\mu_n\,\nu_{r-n-1}} \,+\, \nabla_\mu\,\lambda_{\mu_{s-n}\,\nu_n,\,\mu_{n-1}\,\nu_{r-n}}\,, \end{aligned}$$

makes it manifest that no simultaneous compensation of the first two terms in (2.62) is possible in general.

symmetry than their flat-space counterparts. In fact, with hindsight, this phenomenon is maybe not so surprising, given that already for the one-family case, involving symmetric tensors only, the gauge invariant completion of the (A)dS operator (2.53) depends on the length of the corresponding row.

This observation does not imply that multi-symmetric tensors cannot be given any Lagrangian formulation in (A)dS spaces, however it renders those maximally reducible objects less palatable, in the absence of simple criteria allowing to identify the proper gauge symmetry to be implemented off-shell. Indeed, we found it simpler to exploit tensors transforming irreducibly under permutations of their space-time indices, also in order to deal more efficiently with the complications introduced by the operators ${S^i}_j$, and in the remainder of this section we shall focus on this latter option. This means that in the following

$$\varphi_{{\mu^1}_1 \cdots {\mu^1}_{s_1},\, \cdots \,,\, {\mu^N}_1 \cdots {\mu^N}_{s_N}} \,\equiv\, Y_{\{s_1,\ldots,\,s_N\}}\, \varphi_{{\mu^1}_1 \cdots {\mu^1}_{s_1},\, \cdots \,,\, {\mu^N}_1 \cdots {\mu^N}_{s_N}} \,, \tag{2.63}$$

where $Y_{\{s_1,\ldots,\,s_N\}}$ denotes the projector onto the $GL(D)$ representation labelled by the Young diagram[17] $\{s_1, \ldots, s_N\}$, with $s_1 \geq s_2 \geq \cdots \geq s_N$, a condition that can be expressed in terms of the ${S^i}_j$ operators as

$${S^i}_j\, \varphi \,=\, 0\,, \quad \text{for} \ \ i < j\,. \tag{2.64}$$

Eventually, we shall show that for *traceless* fields satisfying (2.64) part of the gauge symmetry of the Maxwell-like Lagrangian" (2.30) can be restored in (A)dS choosing in

$$\mathcal{L} \,=\, \frac{1}{2}\, \varphi \left\{ \Box - \nabla^i \nabla_i \,-\, m^2 \right\} \varphi \tag{2.65}$$

a suitable "mass-term", leading to the formulation of candidate single-particle Lagrangians.

Irreducible gauge fields in Minkowski backgrounds transform with irreducible parameters obtained stripping one box from the corresponding tableau, in all admissible ways [50]. However, even in this case one can conveniently study the gauge variation of the Maxwell-like Lagrangians in (A)dS starting from (2.60): one has only to take into account that the multi-symmetric Λ_i are no longer independent due to (2.64). In our formalism we can recover the structure of the irreducible parameters analyzing the solutions of the variation of (2.64) given by the set of relations

$${S^i}_j\, \Lambda_k \,+\, {\delta^i}_k\, \Lambda_j \,=\, 0\,, \quad \text{for} \ \ i < j\,. \tag{2.66}$$

As we discuss more in detail in appendix B, the conditions (2.66) select the irreducible components carried by each Λ_k, that can be decomposed as

$$\Lambda_k \,=\, \sum_{n\,=\,k}^{N} \left(1 - \delta_{s_n,\, s_{n+1}}\right) Y_{\{s_1,\ldots,\, s_n-1,\ldots,\, s_N\}}\, \Lambda_k \,\equiv\, \sum_{n\,=\,k}^{N} \left(1 - \delta_{s_n,\, s_{n+1}}\right) \Lambda_k^{(n)}\,, \tag{2.67}$$

[17] We identify Young diagrams by ordered lists of the lengths of their rows enclosed between braces. See [49] and references therein for some introductory material on the representations of linear and orthogonal groups.

where, in particular, no components labelled by $n < k$ are present in Λ_k, while the factor between parentheses makes it manifest that if $s_n = s_{n+1}$ then $\{\ldots, s_n - 1, s_{n+1}, \ldots\}$ is not an admissible Young diagram. Moreover, eqs. (2.66) also imply that all $\Lambda_k^{(n)}$ with the same label (n) are proportional, as one can realise setting $i = k$ and acting with the proper Young projector so as to obtain

$$\Lambda_j^{(n)} \, = \, - \, S^k{}_j \, \Lambda_k^{(n)} \, , \qquad \text{for fixed } k < j \, . \tag{2.68}$$

This result (where no summation over k is implicit) also rests on the fact that the operators $S^i{}_j$ commute with Young projectors, as discussed in section 2.1.2. Therefore, one could identify the irreducible parameters with proper linear combinations of the $\Lambda_k^{(n)}$ associated to the same Young diagram. However, in the following it will be more convenient to preserve the redundancy of (2.59), that in the irreducible case one can rewrite more explicitly as

$$\delta \varphi \, = \, \sum_{n\,=\,1}^{N} \left(1 - \delta_{s_n ,\, s_{n+1}}\right) \sum_{i\,=\,1}^{n} \, \nabla^i \Lambda_i^{(n)} \, , \tag{2.69}$$

with the proviso that one can treat separately the various irreducible components labelled by (n), but not the different parameters labelled by i.

The key to analize the gauge variation (2.60) of the Maxwell operator is then that M commutes with all $S^i{}_j$. Therefore, for any fixed irreducible component carried by the parameters the structure of the gradient terms in $M\delta\varphi$ should agree with (2.69) in order to be compatible with (2.64). On the other hand, the irreducibility condition cannot fix the relative coefficients in the sum over n because any addendum is annihilated independently by all $S^i{}_j$ with $i < j$. As a result, using the relations (2.66) it should be possible to recast (2.60) in the form

$$M \, \delta \varphi \, = \, \sum_{n\,=\,1}^{N} \, k_n \left(1 - \delta_{s_n ,\, s_{n+1}}\right) \sum_{i\,=\,1}^{n} \, \nabla^i \Lambda_i^{(n)} \, + \text{divergences and traces.} \tag{2.70}$$

This argument is supported by an explicit computation in appendix B, where we also fix the coefficients k_n obtaining

$$k_n \, = \, \frac{1}{L^2} \left[(s_n - n - 1)(D + s_n - n - 2) - \sum_{k\,=\,1}^{N} s_k \right] . \tag{2.71}$$

Let us now mention that – even if one works with a traceful φ – the terms displayed explicitly in (2.70) clearly cannot receive any correction from the gauge variation of traces of the field. Therefore, one can only cancel them with a counterterm involving φ, so that, for Young-projected fields, the only possibility is to define

$$M_L \, \varphi \, \equiv \, \left(\Box - \nabla^i \nabla_i \, \right) \varphi \, - \, m^2 \, \varphi \, , \tag{2.72}$$

since all alternative counterterms must be of the type

$$\Delta \, \varphi \, \equiv \left(a_1 \, S^i{}_j S^j{}_i \, + \sum_k \, a_k \, S^i{}_{j_1} S^{j_1}{}_{j_2} \cdots \, S^{j_k}{}_i \right) \varphi \tag{2.73}$$

in order to preserve the index structure of $M\varphi$. However, Δ commutes with all $S^i{}_j$ and, as a result, it acts as a multiple of the identity on any irreducible representation of the $gl(N)$ algebra generated by them (see (A.14)). On the other hand, eq. (2.64) implies that φ is a highest-weight state, that as such uniquely specifies an irreducible representation of $gl(N)$. Therefore, Δ acts diagonally on any φ satisfying (2.64), and in our present setup can only shift the coefficient m^2 in (2.72). One can make this property manifest by casting, for instance, the first addendum of (2.73) (corresponding to the quadratic Casimir of $gl(N)$) in the form[18]

$$\mathcal{C} \,=\, \sum_{i\,=\,1}^{N} S^i{}_i\left(\,S^i{}_i + N - 2i + 1\,\right) \,+\, 2\sum_{i\,=\,1}^{N-1}\sum_{j\,=\,i+1}^{N} S^j{}_i S^i{}_j \tag{2.74}$$

where in particular the second term vanishes on account of (2.64). As a consequence, one can only tune a single parameter in M_L, whereas in general all k_n in $M\delta\varphi$ are different. Therefore, one can cancel at most the gradient terms corresponding to a single irreducible component by suitably tuning m^2 in (2.72), while it remains to be verified whether the leftover terms in (2.60) induce extra constraints.

Let us start from the divergence terms in (2.60),

$$M\,\delta\varphi \,=\, \cdots - \frac{1}{2}\,\nabla^i\nabla^j\nabla_{(\,i}\Lambda_{\,j)} + \frac{1}{L^2}\left\{\,2\,g^{ij}\,\nabla_{(\,i}\Lambda_{\,j)} \,+\, g^{ij}S^k{}_i\nabla_{[\,j}\Lambda_{\,k]}\,\right\} \,+\, \cdots\,, \tag{2.75}$$

since also for this class of contributions the discussion applies to both traceless and traceful fields. The novelty with respect to the symmetric case is the term containing the antisymmetric combination $\nabla_{[\,j}\Lambda_{\,k]}$, that does not vanish manifestly even after forcing the constraint (2.61). Indeed, the vanishing of the divergence terms in (2.60) requires that the surviving irreducible parameter be fully divergenceless,

$$\nabla_i\,\Lambda_j^{(n)} \,=\, 0\,, \qquad \text{for } n \text{ fixed and } \forall\; i,j\,, \tag{2.76}$$

although, as we show in appendix B, when a single irreducible gauge parameter is present this condition is already implied by the constraints (2.61). Therefore, in the gauge variation of the deformed Maxwell-like equation (2.72) only the term

$$M_L\,\delta\varphi \,=\, -\,\frac{2}{L^2}\,\nabla^i g^{jk}\,T_{ij}\,\Lambda_k \tag{2.77}$$

remains to be discussed, and at this stage working with or without trace constraints makes a notable difference. The simplest possibility is to impose

$$T_{ij}\,\varphi \,=\, 0\,. \tag{2.78}$$

[18] We can also illustrate this fact displaying explicitly the space-time indices in a simple example. First of all, the action of $S^i{}_jS^j{}_i$ preserves the lengths of the groups of symmetrised indices, but displaces their position. For a field $\varphi_{\mu\nu,\,\rho}$ the only alternative is $\varphi_{\rho\,(\mu\,,\,\nu)}$, which results from the action of $(S^1{}_2S^2{}_1 - 2\cdot\mathbb{1})$ on $\varphi_{\mu\nu,\,\rho}$. However, with a simple direct calculation one can show that

$$Y_{\{2,1\}}\varphi_{\mu\nu,\,\rho} \,=\, \frac{1}{2}\left(\,2\,\varphi_{\mu\nu,\,\rho} - \varphi_{\rho\,(\mu\,,\,\nu)}\right) \quad\Rightarrow\quad Y_{\{2,1\}}\varphi_{\rho\,(\mu\,,\,\nu)} \,=\, -\,Y_{\{2,1\}}\varphi_{\mu\nu,\,\rho}\,.$$

At the level of field equations this extra condition would require to project (2.72) on its traceless component, but we can discuss gauge invariance directly at the level of the Lagrangian. There the contraction with another traceless field avoid the need for a projection and the self-adjointness of M_L implies

$$\delta\,\mathcal{L} \,=\, \varphi\, M_L\, \delta\varphi \,=\, \frac{s_i s_j s_k}{L^2}\,(\,\nabla_i\, T_{jk}\,\varphi\,)\, T_{ij}\,\Lambda_k \,=\, 0\,. \tag{2.79}$$

In conclusion, if φ satisfies (2.64) and (2.78) then the Maxwell-like Lagrangian

$$\mathcal{L} \,=\, \frac{1}{2}\,\varphi\left\{\Box \,-\, \nabla^i\,\nabla_i \,-\, \frac{1}{L^2}\left[\,(s_n-n-1)(D+s_n-n-2) \,-\, \sum_{k\,=\,1}^{N} s_k\,\right]\right\}\varphi \tag{2.80}$$

is invariant under the gauge transformation generated by a *single* fully divergenceless $\{\ldots, s_n-1, \ldots\}$-projected parameter. Let us observe that the "masses" that we found coincide with those appearing in the on-shell system presented in [24], while for the particular case $N\,=\,1$ (2.80) reproduces our result for symmetric tensors discussed in section 2.2.1.

As is manifest in eq. (2.67), in the presence of blocks of rows of equal length one cannot choose n arbitrarily in the interval from 1 to N. The allowed values correspond to the rows at the end of each block: it could then be convenient to denote a general Young diagram by $\{(s_1,t_1),\ldots,(s_p,t_p)\}$ where the pair (s_k,t_k) denotes the dimensions of the $i-$th block, so that

$$\sum_{i\,=\,1}^{p} t_i \,=\, N\,. \tag{2.81}$$

A field transforming in the $\{(s_1,t_1),\ldots,(s_p,t_p)\}$ representation of $GL(D)$ thus admits p independent gauge parameters on Minkowski backgrounds, while in (A)dS backgrounds one can at most keep the invariance under the gauge transformation

$$\delta\varphi \,=\, \sum_{i\,=\,1}^{t_1+\,\cdots\,+t_k} \nabla^i \Lambda_i^{(t_1+\,\cdots\,+t_k)}\,, \tag{2.82}$$

for a given value of k. Stressing the existence of blocks of rows with equal length leads to rewrite the Lagrangian (2.80) in the form

$$\mathcal{L} \,=\, \frac{1}{2}\,\varphi\left\{\Box \,-\, \nabla^i\,\nabla_i \,-\, \frac{1}{L^2}\left[\,(s_k - \sum_{j\,=\,1}^{k} t_j -1)(D+s_k-\sum_{j\,=\,1}^{k} t_j -2) \,-\, \sum_{j\,=\,1}^{p} t_j s_j\,\right]\right\}\varphi\,. \tag{2.83}$$

3 Spectra

3.1 Flat backgrounds

In this section we investigate the spectra described by the equations (2.5) and (2.26). For the symmetric case we already showed in section 2.1.1 that the transverse-invariant equations of motion reduce to the (traceful) Fierz system (2.6). However, here we provide an independent counting of the degrees of freedom based on a light-cone analysis that is interesting in itself and constitutes our main tool for the study of the flat-space mixed-symmetry construction. In addition, for both cases, we also discuss some aspects of the Hamiltonian analysis of our systems, presenting in particular a simple argument to evaluate the number of first-class constraints associated to our divergence-free gauge symmetry.

In the ensuing discussion we shall work in momentum space in light-cone coordinates, denoting indices transverse to the light-cone directions with small Latin letters $i, j, k, \ldots$; for $p^2 \neq 0$ it is easy to prove that only pure gauge solutions exist: indeed, solving for φ in (2.5) one obtains

$$\varphi \,=\, \frac{p}{p^2}\, p \cdot \varphi \,, \tag{3.1}$$

where the combination $\frac{1}{p^2}\, p \cdot \varphi$ can play the role of a proper gauge parameter in the present framework, due to the condition (2.7) ensuring transversality of $p \cdot \varphi$. While this observation would allow one to restrict the analysis to null momenta, we prefer anyway to keep it slightly more general and show how the elimination of all components longitudinal to the light-cone works for the case of arbitrary momenta. Thus in the following we will only assume

$$p_+ \,\neq\, 0 \,, \tag{3.2}$$

which is always admissible for physical particles.

As a general observation let us mention that, because of the constraints (2.4) and (2.29) on the gauge parameters, it is not possible to fully reach the light-cone gauge off-shell, and we shall need to use the equations of motion to complete the elimination of components along both light-cone directions. The simplest example of this analysis is given by the spin-2 case that we review here explicitly for pedagogical reasons. The condition of transversality (2.4) on the vector parameter Λ_μ,

$$p \cdot \Lambda \,=\, -\,p_+\, \Lambda_- \,-\, p_-\, \Lambda_+ \,+\, p_i\, \Lambda_i \,=\, 0 \,, \tag{3.3}$$

implies that Λ_- is effectively determined in terms of the remaining $D-1$ components, Λ_+ and Λ_i, $i = 1, \ldots, D-2$. This implies that fixing the gauge completely (modulo singular gauge transformations) one can eliminate at most h_{++} and h_{+i}; from the corresponding equations of motion evaluated in this gauge one finds however $(p \cdot h)_+ = 0$ and $(p \cdot h)_i = 0$, which imply in their turn $h_{-+} = 0$ and $h_{-i} = \frac{p_j}{p_+}\, h_{ij}$. Finally, from the equation for h_{-+} one finds $h_{--} = \frac{1}{p_+^2}\, p_i\, p_j\, h_{ij}$, so that the only independent components of $h_{\mu\nu}$ are indeed the transverse ones, subject to the equation $p^2\, h_{ij} \,=\, 0$ and thus arbitrary on the light-cone $p^2 \,=\, 0$. These components describe an irreducible tensor of $GL\,(D-2)$, whose branching in terms of irreps of $O(D-2)$ identifies its particle content, as expected, with that of a massless spin-2 particle together with a massless scalar.

3.1.1 Symmetric tensors

In this section we will use the following notation[19]:

$$\varphi_{\underbrace{-\,\cdots\,-}_{l}\,\underbrace{+\,\cdots\,+}_{s-k-l}\,i_1\cdots i_k} \;\equiv\; \varphi_{-^l\,+^{s-k-l}\,i_k}\,. \tag{3.4}$$

The condition of trasversality on the gauge parameter

$$p\cdot\Lambda_{\,\mu_{s-2}} \;=\; -p_+\,\Lambda_{\,-\,\mu_{s-2}} - \,p_-\,\Lambda_{\,+\,\mu_{s-2}} \,+\, p_i\,\Lambda_{\,i\,\mu_{s-2}} \;=\; 0 \tag{3.5}$$

fixes all components of $\Lambda_{\mu_{s-1}}$ with at least one "$-$" index in terms of components of the form $\Lambda_{\,+^{s-k-1}\,i_k}$. Thus a complete gauge-fixing is reached setting

$$\varphi_{\,+^{s-k}\,i_k} \;=\; 0\,, \tag{3.6}$$

with k ranging from 0 to $s-1$, while in order to obtain conditions on components involving "$-$" indices we have to resort to the equations of motion. From (3.6) we obtain, recursively,

$$(M\,\varphi)_{\,+^{s-k}\,i_k} \;=\; 0 \qquad\quad \Rightarrow \qquad\quad (p\cdot\varphi)_{+^{s-k-1}\,i_k} \;=\; 0\,, \tag{3.7}$$

whose expansion allows to iteratively set to zero all components of φ with one index along the "$-$" direction and at least one index along the "+" direction:

$$\begin{aligned} &\varphi_{\,-\,+^{s-k-1}\,i_k} \;=\; 0\,, \\ &k \;=\; 0,\ldots,s\,-\,2\,, \end{aligned} \tag{3.8}$$

while also providing the relations

$$\varphi_{\,-\,i_{s-1}} \;=\; \frac{p_j}{p_+}\,\varphi_{\,j\,i_{s-1}}\,. \tag{3.9}$$

One can now repeat the procedure, exploiting the consequences of the equations of motion for the components of φ set to zero in (3.8). In analogy with the previous steps one obtains

$$(M\,\varphi)_{\,-\,+^{s-k-1}\,i_k} \;=\; 0 \qquad\quad \Rightarrow \qquad\quad (p\cdot\varphi)_{-\,+^{s-k-2}\,i_k} \;=\; 0\,, \tag{3.10}$$

with $k=0,\ldots s-2$. As a consequence one finds that all components with two "$-$" indices and at least one "+" index vanish

$$\begin{aligned} &\varphi_{\,-^2\,+^{s-k-2}\,i_k} \;=\; 0\,, \\ &k \;=\; 0,\ldots,s\,-\,3\,, \end{aligned} \tag{3.11}$$

together with an additional relation for the component with no "+" indices, to be combined with (3.9)

$$\varphi_{\,-^2\,i_{s-2}} \;=\; \frac{p_j}{p_+}\,\varphi_{\,-\,j\,i_{s-2}} \;=\; \frac{p_j p_k}{p_+^2}\,\varphi_{\,j\,k\,i_{s-2}}\,. \tag{3.12}$$

[19] In practice, we insert an exponent to indicate the number of times a specific "+" or "$-$" component appears, while we denote with a numerical label the total number of indices for components transverse to the light-cone.

The corresponding iterative pattern can be proven by induction and leads to

$$(M\,\varphi)_{\,-^{l}\,+^{s-k-l}\,i_k} \,=\, 0 \qquad \Rightarrow \qquad (p\cdot\varphi)_{-^{l}\,+^{s-k-l-1}\,i_k} \,=\, 0\,, \tag{3.13}$$

from which it is possible to deduce the following relations:

$$\begin{aligned}
&\varphi_{\,-^{l}\,+^{s-k-l}\,i_k} \,=\, 0\,,\\
&\varphi_{\,-^{l+1}\,i_{s-l-1}} \,=\, \frac{1}{(p_+)^{l+1}}\,p_{j_1}\cdots p_{j_{l+1}}\,\varphi_{\,j_1\cdots j_{l+1}\,i_{s-l-1}},\\
&k \,=\, 0,\ldots,s\,-\,l\,-\,1\,,\\
&l \,=\, 0,\ldots,\,s\,-\,1\,,
\end{aligned} \tag{3.14}$$

essentially stating that the only independent components of φ are those containing just indices transverse to the light-cone, $\varphi_{\,i_s} \equiv \varphi_{\,i_1\cdots i_s}$, which satisfy the equations

$$p^2\,\varphi_{\,i_1\cdots i_s} \,=\, 0\,, \tag{3.15}$$

and thus describe a set of massless particles carrying spin $s, s-2, s-4,\ldots,$ down to 1 or 0.

From the perspective of the Hamiltonian analysis [51, 52] the peculiarity of transverse-invariant systems is found in the unusual counting of the corresponding first-class constraints, associated to the presence of higher generations of constraints besides the primary and secondary ones present in more conventional situations. (See [41] for a discussion of the spin-2 case and [5] for the case of symmetric and traceless tensors.) In general, on a Cauchy surface, one has to assign independently the values of a given component of the gauge parameter and of its time derivatives, up to the highest order appearing in the variation of the gauge field, thus implying that they have to be counted as independent constraints; thus, for instance, for conventional theories with parameters entering with one derivative in $\delta\varphi$, and in the absence of additional constraints, each gauge component has to be counted twice, since its first time derivative provide an additional independent condition to be imposed on the system.

Our observation is that for transverse-invariant theories there is a simple procedure allowing to compute the number of components of the parameters, including their time derivatives, that have to be counted as independent on a given Cauchy surface. Indeed, solving the transversality constraint (2.4) with respect to the time derivative one finds

$$\partial^{\alpha}\,\Lambda_{\,\alpha\,\mu_2\cdots\mu_{s-1}} \,=\, 0 \qquad \Rightarrow \qquad \dot{\Lambda}_{\,0\,\mu_2\cdots\mu_{s-1}} \,=\, \vec{\nabla}\,\cdot\,\Lambda_{\,\mu_2\cdots\mu_{s-1}}\,, \tag{3.16}$$

where in the r.h.s. the divergence is computed along the spatial directions. One can thus appreciate that for all components of $\Lambda_{\,\mu_1\,\mu_2\cdots\mu_{s-1}}$ carrying at least one temporal index the time derivatives are not to be regarded as independent, in view of the condition (3.16). This means that the total number of first-class constraints is twice the number of components of the parameters with only spatial indices, $\Lambda_{\,a_1\,a_2\cdots a_{s-1}}$, $\;a_k = 1,\ldots,D-1$, since for the latter their time derivatives are really independent, and only once the number of components possessing at least one temporal index, in view of the previous observation. Thus, for the case of rank$-(s-1)$ symmetric parameters discussed in this section the total number of first class constraints is given by the formula

$$\#\,\text{1st class} \,=\, 2\,\underbrace{\binom{D+s-3}{s-1}}_{\Lambda_{a_1\,a_2\,\cdots\,a_{s-1}}} + \underbrace{\binom{D+s-3}{s-2}}_{\Lambda_{0\,\mu_2\,\cdots\,\mu_{s-1}}}\,. \tag{3.17}$$

In the absence of second-class constraints one can use (3.17) to directly compute the propagating degrees of freedom of the transverse-invariant system using the formula [52]

$$\#\,\text{d.o.f.} \,=\, \#\,(\text{components in } \varphi) \,-\, \#\,\text{1st class}\,, \tag{3.18}$$

finding agreement with our result (3.15). The light-cone analysis in its turn implicitly provides a proof of the absence of second-class constraints, thus dispensing the need to study the full Hamiltonian system of constraints associated with (2.1).

3.1.2 Mixed-symmetry tensors

Having discussed in some detail the counting of degrees of freedom for the case of symmetric tensors we are now in the position to extend our proof to the more general case of $GL(D)-$tensors subject to (2.26), (2.27) and (2.29).

Similarly to what we saw for the symmetric case also in this context it would be possible to distinguish the two cases $p^2 \neq 0$ and $p^2 = 0$. In particular in the former case one can solve for φ in the equation of motion (2.26) obtaining

$$\varphi \,=\, \frac{p^i}{p^2}\, p\,\cdot\,\varphi \tag{3.19}$$

thus implying that φ only contains pure gauge components, provided one also shows that under the same conditions all double divergences vanish,

$$p_i\, p_j\, \varphi \,=\, 0\,, \tag{3.20}$$

which, in its turn, can be proven iteratively. On the other hand, for light-like momenta it is possible to work in the reference frame where only p_+ is non-vanishing, which in particular implies that the components transverse to the light-cone are to be gauge invariant, and indeed one can show that they are actually arbitrary on the light-cone. It would remain to prove that the transverse gauge invariance (2.29) suffices to remove all components longitudinal to the light-cone, when (2.26) holds. However, it is also possible to discuss the elimination of unphysical components simultaneously for arbitrary momenta, and in the following we shall abide by this latter option when discussing the general case of $N-$family tensors, so as to illustrate the procedure in its generality.

We conclude this section evaluating the number of first-class constraints for the case of two-family tensors, both reducible and irreducible, while also commenting on the role of gauge-for-gauge transformations in our light-cone computation.

As a first step let us elaborate on the meaning of the conditions (2.29) on the gauge parameters, that we write again here for clarity:

$$\partial_{(i}\, \Lambda_{j)} \,=\, 0\,. \tag{3.21}$$

From the "diagonal" sector of (3.21), given by $i = j$, we obtain

$$-\, p_+\, \Lambda_{i\,(-)_i} \,-\, p_-\, \Lambda_{i\,(+)_i} \,+\, p_K\, \Lambda_{i\,(K)_i} \,=\, 0\,, \tag{3.22}$$

where for instance with the notation

$$\Lambda_{i\,(-)_i} \tag{3.23}$$

we denoted a component of the gauge parameter Λ_i with one "$-$" index in the $i-$th family, while in order to distinguish family indices $i, j, k, \cdots$ from transverse component indices here we denote the latter with capital Latin letters from the same part of the alphabet: $I, J, K, \cdots$. It is not hard, then, to recognize that (3.22) imposes on each parameter Λ_i a condition analogous to (3.5) for the symmetric case, essentially stating that the components with "$-$" indices in the $i-$th family are not independent, and thus cannot be used to gauge fix some components of φ, once all components of Λ_i with "+" and transverse indices in the $i-$th family have been used.

Now let us consider the role of the "off-diagonal" constraints

$$\partial_i\, \Lambda_j \,+\, \partial_j\, \Lambda_i \,=\, 0\,, \qquad\qquad i \,<\, j\,, \tag{3.24}$$

which can be expanded as

$$-\, p_+\, (\Lambda_{i\,(-)_j} \,+\, \Lambda_{j\,(-)_i}) - \,p_-\, (\Lambda_{i\,(+)_j} \,+\, \Lambda_{j\,(+)_i}) \,+\, p_K\, (\Lambda_{i\,(K)_j} \,+\, \Lambda_{j\,(K)_i}) \,=\, 0\,, \tag{3.25}$$

where for instance the notation

$$\Lambda_{i\,(-)_j} \tag{3.26}$$

identifies a gauge parameter with one index less in the $i-$th family, possessing at least one "$-$" component in the $j-$th family. It is then possible to make use of all parameters exploiting systematically (3.22) and (3.25) as follows: first, we perform a gauge-fixing using the parameter Λ_1, avoiding to make use of its components with "$-$" indices in the first family that are not independent due to (3.22); when it comes to Λ_2, together with similar limitations on the use of components with "$-$" indices in the second family we must take into account (3.25) as well, which implies that also the components of Λ_2 with "$-$" indices in the *first* family are now fixed and cannot be used. Similarly, once both Λ_1 and Λ_2 have been completely fixed, we will be in the position to fully use Λ_3 up to components involving "$-$" indices in any of the first three families: the third because of the diagonal condition (3.22), and the first and second family because of (3.25), and so forth. Although other gauge-fixings are certainly possible, the procedure we are suggesting allows to take into account all conditions in a systematic fashion, and reduces indeed the counting of degrees of freedom to a relatively direct extension of the procedure implemented for the symmetric case.

More explicitly, computing the variation of the components of φ with s_1-k "+" indices and no "−" indices in the first family gives

$$\delta\,\varphi_{\,(+^{s_1-k}\,I_k)_1}\;=\;(s_1-k)\,p_+\,\Lambda_{1\,(+^{s_1-k-1}\,I_k)_1}\;+\;p_I\,\Lambda_{1\,(+^{s_1-k}\,I_{k-1})_1}\;+\;\sum_{j\,=\,2}^{N}\,p^j\,\Lambda_j\,, \tag{3.27}$$

where in the last summation all parameters are understood to carry the same components in the first family as φ. Thus we see that, at the price of solving completely for Λ_1, it is possible to reach the gauge

$$\begin{aligned}&\varphi_{\,(+^{s_1-k}\,I_k)_1}\;=\;0\,,\\&k\;=\;0,\ldots,s_1-1\,.\end{aligned} \tag{3.28}$$

As a consequence, in the equations of motion for the components of φ gauge-fixed as in (3.28), only some terms in the divergence involving the first family survive; explicitly:

$$\begin{aligned}(M\,\varphi)_{\,(+^{s-k}\,I_k)_1}\;=&\;-\;(s-k)\,p_+\,p_1\,\varphi_{(+^{s_1-k-1}\,I_k)_1}\;-\;p^I\,p_1\,\varphi_{(+^{s_1-k}\,I_{k-1})_1}\\&-\sum_{j\,=\,2}^{N}\,p^j\,p_j\,\varphi_{(+^{s_1-k}\,I_k)_1}\;=\;0\,,\end{aligned} \tag{3.29}$$

where indeed all terms in the second line are zero due to (3.28). A simple iterative argument allows to conclude that

$$(M\,\varphi)_{\,(+^{s-k}\,I_k)_1}\;=\;0\qquad\Rightarrow\qquad(p_1\,\varphi)_{(+^{s_1-k-1}\,I_k)_1}\;=\;0\,, \tag{3.30}$$

a set of conditions analogous to (3.8) and (3.9) for the symmetric case, essentially stating that components of φ carrying one "−" index in the first family are not independent:

$$\begin{aligned}&\varphi_{\,(-\,+^{s_1-k-1}\,I_k)_1}\;=\;0\,,\\&k\;=\;0,\ldots,s_1\,-\,2\,,\\&\varphi_{\,(-\,I_{s_1-1})_1}\;=\;\frac{1}{p_+}\,p_J\,\varphi_{\,(J\,I_{s_1-1})_1}\,.\end{aligned} \tag{3.31}$$

The key observation allowing to proceed further, and actually the clue for the whole analysis, is to notice that for the first family of indices one can reproduce the same analysis as for the symmetric case, while the remaining families are invisible for our purposes. Thus, following the steps from (3.7) to (3.14) performed for symmetric tensors one can prove by induction that the following conditions hold:

$$\begin{aligned}&\varphi_{\,(-^l\,+^{s_1-k-l}\,I_k)_1}\;=\;0\,,\\&\varphi_{\,(-^{l+1}\,I_{s_1-l-1})_1}\;=\;\frac{1}{(p_+)^{l+1}}\,p_{J_1}\cdots p_{J_{l+1}}\,\varphi_{\,(J_1\cdots J_{l+1}\,I_{s_1-l-1})_1}\,,\\&k\;=\;0,\ldots,s_1\,-\,l\,-\,1\,,\\&l\;=\;0,\ldots,\,s_1-1\,.\end{aligned} \tag{3.32}$$

As a result, only the components of φ with transverse indices in this set are really independent, so that in the following we shall assume that no "+" nor "−" indices are present in the first family.

We can now proceed to analyse the gauge fixing and its consequences for the second family. Consistently with our general scheme the idea is to exploit the parameter Λ_2 to eliminate all components of φ with no "−" indices in the second family, in agreement with (3.22), and only transverse indices in the first family, due to (3.32)[20]. More explicitly, let us consider the variation of $\varphi_{\,(+^{s_2-k}\,I_k)_2}$,

$$\begin{aligned}\delta\,\varphi_{\,(I_{s_1})_1,(+^{s_2-k}\,I_k)_2} =& (s_2-k)\,p_+\,\Lambda_{2\,(I_{s_1})_1,\,(+^{s_2-k-1}\,I_k)_2} + p_I\,\Lambda_{2\,(I_{s_1})_1,\,(+^{s_2-k}\,I_{k-1})_2}\\ &+\,p^1\,\Lambda_1 + \sum_{j\,=\,3}^{N} p^j\,\Lambda_j\,,\end{aligned} \tag{3.33}$$

written with emphasis on the relevant indices in the first and second families for φ and Λ_2, while for the variations involving the other parameters we are using the compact notation; we see that solving for Λ_2 it is possible to fix the gauge

$$\begin{aligned}&\varphi_{\,(I_{s_1})_1,\,(+^{s_2-k}\,I_k)_2} \,=\, 0\,,\\ &k \,=\, 0,\ldots,s_2-1\,.\end{aligned} \tag{3.34}$$

Inserting (3.33) in the equations of motion we obtain a condition on a set of divergences computed in the second family,

$$(M\,\varphi)_{\,(I_{s_1})_1,\,(+^{s_2-k}\,I_k)_2} \,=\, 0 \qquad \Rightarrow \qquad (p_2\,\varphi)_{\,(I_{s_1})_1,\,(+^{s_2-k-1}\,I_k)_2} \,=\, 0\,, \tag{3.35}$$

analogous to (3.30). As a matter of fact at this level the index content of the families other than the second can be ignored, and one can proceed focussing on the second family as if the field φ were effectively a symmetric tensor. It is then possible to prove recursively that the following conditions hold:

$$\begin{aligned}&\varphi_{\,(I_{s_1})_1,\,(-^l\,+^{s_2-k-l}\,I_k)_2} \,=\, 0\,,\\ &\varphi_{\,(I_{s_1})_1,\,(-^{l+1}\,I_{s_2-l-1})_2} \,=\, \frac{1}{(p_+)^{l+1}}\,p_{J_1}\cdots p_{J_{l+1}}\,\varphi_{\,(I_{s_1})_1,\,(J_1\cdots J_{l+1}\,I_{s_2-l-1})_2}\,,\\ &k \,=\, 0,\ldots,s_2\,-\,l\,-\,1\,,\\ &l \,=\, 0,\ldots,\,s_2-1\,,\end{aligned} \tag{3.36}$$

stating that for the first two families only the components of φ with transverse indices are really independent.

The proof for the general case can be obtained by induction: we assume it is possible to obtain a set of conditions like (3.36) for the first $N-1$ families, fixing the parameters

[20] Because of (3.25) we cannot use components of Λ_2 with "−" indices in the first family since Λ_1 is now completely fixed, while further transformations involving "+" components of Λ_2 in the first family can be interpreted as particular gauge-for-gauge transformations; see appendix C.

$\Lambda_1,\ \Lambda_2\,,\ldots,\ \Lambda_{N-1}$ and exploiting the consequences of the equations of motion for the gauge-fixed components of φ, as explicitly verified for $N=1,2$. It is then possible to proceed for the $N-$th family using the parameter Λ_N to gauge fix all components of φ with transverse indices in the first $N-1$ families and at least one "+" index, possibly together with transverse ones, (but no "−" indices) in the $N-$th family:

$$\begin{aligned}\delta\,\varphi_{\,(I_{s_1})_1,\,(I_{s_2})_2,\cdots,\,(+^{s_N-k}\,I_k)_N}\ &=\sum_{j=1}^{N-1}\,p^j\,\Lambda_j\\ &+\ (s_N-k)\,p_+\,\Lambda_{N\,(+^{s_N-k-1}\,I_k)_N}\ +\ p_I\,\Lambda_{N\,(+^{s_N-k}\,I_{k-1})_N}\,,\end{aligned}\tag{3.37}$$

where the first $N-1$ parameters at this stage are fixed while the indices in the first $N-1$ families of Λ_N are congruent with those of φ. Indeed, consistently with the constraints (3.22) and (3.25) on Λ_N, we can choose a gauge such that

$$\begin{aligned}&\varphi_{\,(I_{s_1})_1,(I_{s_2})_2,\cdots,\,(+^{s_N-k}\,I_k)_N}\ =\ 0\,,\\ &k\ =\ 0,\ldots,s_N-1\,.\end{aligned}\tag{3.38}$$

As for the previous steps, the clue to complete our proof is to observe that we can now manipulate the components in the $N-$th family as if the tensor were symmetric, obtaining the following consequences of the equations of motion

$$(M\,\varphi)_{\,(I_{s_1})_1,(I_{s_2})_2,\cdots,\,(+^{s_N-k}\,I_k)_N}\ =\ 0\qquad\Rightarrow\qquad(p_N\,\varphi)_{(I_{s_1})_1,(I_{s_2})_2,\cdots,(+^{s_N-k-1}\,I_k)_N}\ =\ 0\,,\tag{3.39}$$

which imply in their turn

$$\begin{aligned}&\varphi_{\,(I_{s_1})_1,\,(I_{s_2})_2,\cdots,\,(-^l\,+^{s_N-l-k}\,I_k)_N}\ =\ 0\,,\\ &\varphi_{\,(I_{s_1})_1,\,\cdots,\,(-^{l+1}\,I_{s_2-l-1})_N}\ =\ \frac{1}{(p_+)^{l+1}}\,p_{J_1}\cdots p_{J_{l+1}}\,\varphi_{\,(I_{s_1})_1,\,\cdots,\,(J_1\cdots J_{l+1}\,I_{s_2-l-1})_N}\,,\\ &k\ =\ 0,\ldots,s_N\ -\ l\ -\ 1\,,\\ &l\ =\ 0,\ldots,\ s_N-1\,.\end{aligned}\tag{3.40}$$

As a consequence of (3.40) only transverse indices in the $N-$th family define independent components, and are otherwise arbitrary on the mass-shell $p^2=0$. All in all, we are left with a $GL(D-2)-$reducible tensor with N families of symmetric indices,

$$\varphi_{\,(I_{s_1})_1,(I_{s_2})_2,\cdots,\,(I_{s_N})_N}\,,\tag{3.41}$$

whose branching in $O(D-2)$ irreps describes the full content of massless particles propagating in (2.26).

For what concerns the role of gauge-for-gauge transformations let us observe that, since we are performing explicit gauge fixings, component by component in φ, in our procedure we never have to deal with transformations that, by definition, cannot alter any components of the field. However, one can keep track of their presence observing that, once our gauge-fixing procedure is completed, one is left with a number of components of the various parameters that have not been used. In appendix C we discuss explicitly from this perspective the example of a reducible (1, 1) field.

We conclude this section extending the counting of first-class constraints to the case of two-family fields, for all classes of theories discussed in section 2.1.2, i.e. for reducible and irreducible $GL(D)-$tensors and for irreducible $O(D)-$tensors. As in the symmetric case analysed at the end of section 3.1.1, one has to count the number of independent components of the (gauge-for-)gauge parameters and of their time derivatives on a given Cauchy surface. Let us begin studying a reducible $GL(D)-$field $\varphi_{\mu_1\cdots\,\mu_s,\,\nu_1\cdots\,\nu_r}$ which admits two multi-symmetric gauge parameters, $\Lambda_{\mu_1\cdots\,\mu_{s-1},\,\nu_1\cdots\,\nu_r}$ and $\lambda_{\mu_1\cdots\,\mu_s,\,\nu_1\cdots\,\nu_{r-1}}$, and one multi-symmetric gauge-for-gauge parameter $\theta_{\mu_1\cdots\,\mu_{s-1},\,\nu_1\cdots\,\nu_{r-1}}$. The constraints (2.29) can be expanded as in (3.16) to give

$$\begin{aligned}
&\dot{\Lambda}_{0\,\mu_1\cdots\,\mu_{s-2},\,\nu_1\cdots\,\nu_r} = \partial^a\Lambda_{a\,\mu_1\cdots\,\mu_{s-2},\,\nu_1\cdots\,\nu_r}\,,\\
&\dot{\Lambda}_{\mu_1\cdots\,\mu_{s-1},\,0\,\nu_1\cdots\,\nu_{r-1}} + \dot{\lambda}_{0\,\mu_1\cdots\,\mu_{s-1},\,\nu_1\cdots\,\nu_{r-1}} = \partial^a\left(\Lambda_{\mu_1\cdots\,\mu_{s-1},\,a\,\nu_1\cdots\,\nu_{r-1}} + \lambda_{a\,\mu_1\cdots\,\mu_{s-1},\,\nu_1\cdots\,\nu_{r-1}}\right),\\
&\dot{\lambda}_{\mu_1\cdots\,\mu_s,\,0\,\nu_1\cdots\,\nu_{r-2}} = \partial^a\lambda_{\mu_1\cdots\,\mu_s,\,a\,\nu_1\cdots\,\nu_{r-2}}\,,
\end{aligned} \tag{3.42}$$

while for the corresponding constraints on the gauge-for-gauge parameter, eq. (2.34), we assume they also hold in their strong form

$$\begin{aligned}
\dot{\theta}_{0\,\mu_1\cdots\,\mu_{s-2},\,\nu_1\cdots\,\nu_{r-1}} &= \partial^a\,\theta_{a\,\mu_1\cdots\,\mu_{s-2},\,\nu_1\cdots\,\nu_{r-1}}\,,\\
\dot{\theta}_{\mu_1\cdots\,\mu_{s-1},\,0\,\nu_1\cdots\,\nu_{r-2}} &= \partial^a\,\theta_{\mu_1\cdots\,\mu_{s-1},\,a\,\nu_1\cdots\,\nu_{r-2}}\,,
\end{aligned} \tag{3.43}$$

where the index a runs over spatial directions, $a = 1, \ldots, D-1$. In this case not all components of $\dot{\Lambda}$ and $\dot{\lambda}$ with a single temporal index depend on the spatial ones, but assigning the value 0 to an additional index in the second of (3.42) one realises that all components with at least two temporal indices are not to be regarded as independent.

Concerning the gauge-for-gauge parameter θ note that one has to consider as independent on the given Cauchy surface its time derivatives up to $\ddot{\theta}$, since the latter appears in the gauge variation of the first-time derivatives of the parameters Λ and λ. In this respect, the only independent components of $\dot{\theta}$ and $\ddot{\theta}$ after imposing the constraints are the spatial ones, as one can see computing a time derivative of (3.43). As a result, the number of first-class constraints can be computed as

$$\begin{aligned}
\#\,\text{1st class} = &\ \#\,\Lambda_{\mu_1\cdots\,\mu_{s-1},\,\nu_1\cdots\,\nu_r} + \#\,\lambda_{\mu_1\cdots\,\mu_s,\,\nu_1\cdots\,\nu_{r-1}}\\
&+ \#\,\Lambda_{a_1\cdots\,a_{s-1},\,a_1\cdots\,a_r} + \#\,\lambda_{a_1\cdots\,a_s,\,a_1\cdots\,a_{r-1}} + \#\,\Lambda_{a_1\cdots\,a_{s-1},\,0\,a_1\cdots\,a_{r-1}}\\
&- \#\,\theta_{\mu_1\cdots\,\mu_{s-1},\,\nu_1\cdots\,\nu_{r-1}} - 2\,\#\,\theta_{a_1\cdots\,a_{s-1},\,a_1\cdots\,a_{r-1}}\,,
\end{aligned} \tag{3.44}$$

where we denoted by e.g. $\#\Lambda$ the number of independent components of the multi-symmetric tensor Λ and where Greek indices take values form 0 to $D-1$, while Latin indices take values only in the spatial directions. Subtracting the result to the number of components of φ as in (3.18) one obtains the number of components of a reducible (s,k) tensor of $GL(D-2)$. Also in this case the light-cone analysis thus provides an implicit proof of the absence of second-class constraints for these systems. Under the same hypothesis one can also compute the degrees of freedom propagated in the irreducible theories: it is indeed possible to verify that our differential constraints lead also in these cases to a counting of first-class constraints that is formally identical to (3.44), but where $\#\,\Lambda_{\mu_1\cdots\,\mu_{s-1},\,\nu_1\cdots\,\nu_r}$ now denotes the dimension of the $\{s-1,r\}$ irrep of $GL(D)$ or $O(D)$, while $\#\,\Lambda_{a_1\cdots\,a_{s-1},\,a_1\cdots\,a_r}$ denotes the dimension of the corresponding irrep of $GL(D-1)$ or $O(D-1)$. In both cases, subtracting the result to the number of independent components of the field gives the dimension of the $\{s,r\}$ irrep of $GL(D-2)$ or $O(D-2)$.

3.2 (A)dS backgrounds

Our analysis of the spectra in (Anti-)de Sitter backgrounds relies on one assumption: under "smooth" deformation of a Lagrangian gauge theory in Minkowski space to a Lagrangian gauge theory in (A)dS space the number of degrees of freedom is unchanged. The deformation is termed "smooth" if it keeps the number of gauge symmetries. In Hamiltonian terms this statement is essentially equivalent to saying that a smooth deformation cannot introduce second-class constrains into the (A)dS system that were not already present in the flat one. We are not aware of any general proof of this otherwise reasonable[21] conclusion, and in the following we shall abide by the conventional wisdom of assuming its validity. Therefore, we shall show that our (A)dS Lagrangians always define smooth deformations of flat-space Lagrangians whose degrees of freedom are under control, while also providing occasionally a few additional independent arguments in support of our conclusions.

3.2.1 Symmetric tensors

For symmetric tensors in (A)dS background the Lagrangian (2.55) retains the same number of unbroken independent gauge symmetries as its Minkowskian counterpart (2.1), which is in fact true for both reducible and irreducible cases. Thus, the number of propagating polarisations is expected to coincide with that of the flat case. To provide further support to this conclusion let us also discuss a couple of independent arguments to the same effect.

For the irreducible case we have to analyse the content of the equations (2.57),

$$M\,\varphi\,-\,\frac{1}{L^2}\,[\,(s-2)(D+s-3)\,-\,s\,]\,\varphi\,+\,\frac{2}{D+2\,(s-2)}\,g\,\nabla\cdot\nabla\cdot\,\varphi\,=\,0\,. \tag{3.45}$$

[21]Under such a deformation the number of primary constraints clearly does not change. However, it is a general result that whenever second-class constraints are present at least one of them should appear among primary constraints [51]. Thus, assuming the flat theory to be free of second-class constraints, the possibility that they appear in the deformed (A)dS system would imply that some of the primary constraints changed their nature under the deformation, without the overall number of gauge generators being modified.

Since the number of first-class constraints is the same as for the Minkowsian case we can at least conclude that the degrees of freedom associated to the (A)dS equation cannot exceed those of the flat theory. The latter, on the other hand, also coincide with the propagating polarisations described by the Fierz system in (A)dS (see [53] and [54] for dS and AdS backgrounds, respectively):

$$\begin{aligned} &\left\{\Box \,-\, \frac{1}{L^2}\,[\,(s-2)(D+s-3)\,-\,s\,]\right\}\varphi \,=\, 0\,, \\ &\nabla\cdot\varphi \,=\, 0\,, \\ &\varphi' \,=\, 0\,. \end{aligned} \tag{3.46}$$

Thus, in order to prove that (2.57) propagates the degrees of freedom of a single massless particle of spin s it will be sufficient to show explicitly that it possesses all the solutions to (3.46). Indeed in our framework $\varphi' = 0$ by assumption, while for fields in the kernel of the Klein-Gordon operator, i.e. for φ s.t.

$$\left\{\Box \,-\, \frac{1}{L^2}\,[\,(s-2)(D+s-3)\,-\,s\,]\right\}\varphi \,=\, 0, \tag{3.47}$$

computing n divergences of (3.45) we obtain

$$\left\{\frac{n\,(n-1)}{L^2}\,\rho_{n+3}\,\rho_{n+4}\,(\nabla\cdot)^{\,n} \,+\, \rho_{2n+4}\,\nabla\,(\nabla\cdot)^{\,n+1} \,-\, 2\,g\,(\nabla\cdot)^{\,n+2}\right\}\varphi \,=\, 0\,, \tag{3.48}$$

where we defined $\rho_n \,=\, D + 2s - n$. It is then possible to observe that (3.48) recursively sets to zero all multiple divergences of φ in decreasing order, finally leading to $\nabla\cdot\varphi \,=\, 0$.

For the case of traceful tensors, described by the equations of motion (2.53)

$$M_L\,\varphi \,\equiv\, M\,\varphi \,-\, \frac{1}{L^2}\,\{\,[\,(s-2)(D+s-3)\,-\,s\,]\,\varphi \,-\, 2\,g\,\varphi'\,\}\,, \tag{3.49}$$

while it is still true that the number of first class constraints is the same as the flat reducible theory, however it is not obvious what should be the proper “Fierz system” with which to compare our equations in order to prove that the degrees of freedom actually match those of the flat case (2.5). The naive guess suggested by the flat-space example (2.6) would be to reproduce the first two conditions in (3.46) while keeping the trace undetermined. However, it is simple to observe that, as a consequence of (3.49), the first two conditions in (3.46) would anyway imply $\varphi' = 0$ thus leading to the contradictory conclusion that (3.49) describes the same degrees of freedom as the irreducible case. The reason behind this difference with respect to the case of flat background is that massless fields in (A)dS have mass-like terms depending on the spin, so that the various propagating components in φ actually satisfy different equations of motion.

However, the effective particle content associated to Lagrangian (2.55) can be identified comparing with the unconstrained Lagrangian for (A)dS triplets of [35, 38, 55]. In that context the relevant equations after eliminating an auxiliary field are

$$\begin{aligned} M_L\,\varphi \,&=\, -\,2\,\nabla^{\,2}\,\mathcal{D} \,+\, \frac{8}{L^{\,2}}\,g\,\mathcal{D}\,, \\ \hat{M}_L\,\mathcal{D} \,&=\, \nabla\cdot\nabla\cdot\,\varphi \,-\, \frac{4}{L^{\,2}}\,\varphi^{\,\prime}\,, \end{aligned} \tag{3.50}$$

where $\hat{M}_L$ is a deformation of the flat-space kinetic operator for $\mathcal{D}$ (2.14),

$$\hat{M}_L \,=\, 2\,\Box \,+\, \nabla\,\nabla\cdot \,-\, \frac{2}{L^{\,2}}\,\left[(s\,-\,1)\,(D\,+\,s\,-\,3)\,+\,3\right] \,+\, \frac{4}{L^{\,2}}\,g\,T\,, \tag{3.51}$$

while gauge invariance obtains choosing $\delta\,\varphi \,=\, \nabla\,\Lambda$ and $\delta\,\mathcal{D} \,=\, \nabla\cdot\Lambda$. To make contact with our constrained theory, as already observed for the flat case, we remove the transversality constraint (2.51) à la Stueckelberg, performing the gauge-invariant redefinition

$$\varphi \,\longrightarrow\, \varphi \,-\, \nabla\,\theta \tag{3.52}$$

where $\delta\,\theta \,=\, \Lambda$. The resulting Lagrangian

$$\mathcal{L} \,=\, \frac{1}{2}\,\varphi\,M_L\,\varphi \,+\, \varphi\,\left(\nabla^{\,2} \,-\, \frac{4}{L^{\,2}}\,g\right)\,\nabla\cdot\theta \,-\, 2\binom{s}{2}\,\nabla\cdot\theta\,\hat{M}_L\,\nabla\cdot\theta\,, \tag{3.53}$$

actually coincides, upon renaming $\nabla\cdot\theta \equiv \mathcal{D}$, with the (A)dS triplet Lagrangian leading to (3.50), whose particle content was shown in [38, 55] to correspond to that of the flat space-time reducible system here computed in section 3.1, thus completing our check. In section 4.2 we show how to decompose the field φ in order to identify in (2.55) the propagating modes, each described by a single-particle Lagrangian leading to equations of the form (2.57).

3.2.2 Mixed-symmetry tensors

In this section we discuss the spectrum of the theory described by the Lagrangian (2.83), corresponding to the the AdS-unitary choice of keeping the gauge parameter lacking one box in the first rectangular block:

$$\mathcal{L} \,=\, \frac{1}{2}\,\varphi\left\{\Box \,-\, \nabla^{i}\,\nabla_{i} \,-\, \frac{1}{L^2}\left[\,(s_1 - t_1 - 1)(D + s_1 - t_1 - 2) \,-\, \sum_{j\,=\,1}^{p} t_j s_j\,\right]\right\}\varphi\,. \tag{3.54}$$

Here φ carries a representation of $O(D)$ described by a diagram with p rectangular blocks. The reduced amount of gauge invariance available for (A)dS tensors with mixed symmetry introduces additional complications if compared to more standard situations. For instance, for symmetric tensors on flat space-time the variation of the divergence of the field in transverse-invariant theories is proportional to the D'Alembertian of the parameter:

$$\delta\,\partial\cdot\varphi \,=\, \Box\,\Lambda\,, \tag{3.55}$$

thus implying that the transverse part of $\partial \cdot \varphi$ can be removed upon partial gauge-fixing and need not be eliminated manipulating the equations of motion. To appreciate the differences met in our present case it suffices to consider the simplest $O(D)$-hook field $\varphi_{\,\mu\nu,\,\rho}$, whose divergence varies according to

$$\delta\, \nabla^{\,\alpha} \varphi_{\,\alpha\nu,\,\rho} \, = \, \left(\Box \, - \, \frac{D-2}{L^2} \right) \Lambda_{\,\nu,\,\rho} \, , \tag{3.56}$$

where $\Lambda_{\,\nu,\,\rho}$ is the antisymmetric parameter dictated by the general analysis of [23, 24]. It is then manifest that only the antisymmetric projection of $\nabla^{\,\alpha} \varphi_{\,\alpha\nu,\,\rho}$ can be gauged away, while its symmetric projection, being gauge invariant, now has to be eliminated by the equations of motion themselves. Our strategy to avoid dealing directly with these complications consists in constructing the Stueckelberg Lagrangian [18, 19] for the degrees of freedom of interest, to then discuss the corresponding gauge fixing to our Lagrangian (3.54). An additional virtue of the Stueckelberg procedure is that it allows to form an intuitive picture of the essential peculiarities of (A)dS gauge fields with mixed symmetry if compared to other more conventional classes of free fields.

Indeed, for a gauge field φ in a given irrep of $O(D)$ a natural road to its (A)dS deformation would be to covariantise its flat gauge transformation,

$$\delta\, \varphi \, = \, \nabla^{\,i} \, \Lambda_{\,i} \, , \tag{3.57}$$

to then try and construct the corresponding gauge-invariant kinetic operator. To get a deeper insight into the reasons for the absence of general solutions to this program, here explicitly observed in section 2.2.2, one can appreciate a related difficulty whose clarification also bears the essence of its solution: *in (A)dS backgrounds gauge-for-gauge invariance is unavoidably broken*; indeed, the transformations of the parameters

$$\delta\, \Lambda_{\,i} \, = \, \nabla^{\,j} \, \Lambda_{\,[ij]} \, , \tag{3.58}$$

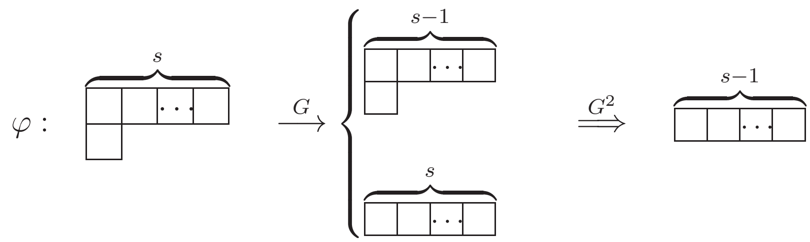
that in flat space would leave φ unaltered, now produce a variation of the field itself according to

$$\delta\, \varphi \, = \, \frac{1}{2} \, [\nabla^{\,i} , \, \nabla^{\,j}] \, \Lambda_{\,[ij]} \, . \tag{3.59}$$

Given that in a quantitative analysis of the consequences of (3.59) one should take into account all generations of broken gauge-for-gauge transformations, it is anyway clear that in (A)dS the gauge-for-gauge parameters $\Lambda_{\,[ij]}$, instead of providing a convenient bookkeeping for those combinations of the parameters $\Lambda_{\,i}$ that do not affect the gauge field, encode instead true additional gauge redundancies whose presence would eventually affect the counting of degrees of freedom of the resulting theory.

However, it might still be possible to propagate the polarisations of the $O(D-2)$ irrep associated with φ provided one “neutralizes” the effect of those broken gauge-for-gauge transformations encoded in (3.59) by *promoting them to play the role of standard gauge parameters for new fields (and corresponding new degrees of freedom) to be introduced in* *the theory*. This is the basic idea underlying the Stueckelberg construction and, in our opinion, it also provides an interesting alternative insight into the mechanism encoded in the BMV pattern [25], showing how the degrees of freedom carried by individual (A)dS massless particles distribute over *multiplets* of flat-space particles of zero mass.

Considering for instance the case of $\{s,1\}$ tensors of $O(D)$, their standard description as gauge fields in flat space would entail two gauge parameters with tableaux $\{s\}$ and $\{s-1,1\}$ respectively, and one gauge-for-gauge parameter given by a symmetric tensor of rank $s-1$:

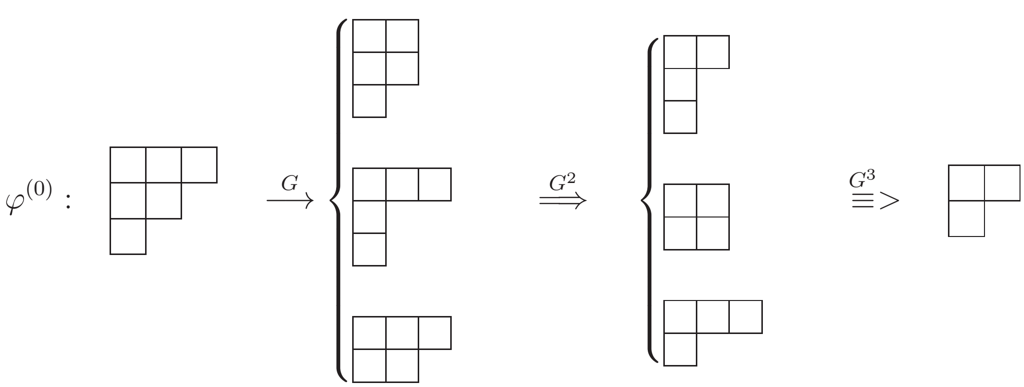


(3.60)

According to our previous discussion, in (A)dS the latter has to play the role of a standard parameter for an additional field, that might be either of the form $\{s-1,1\}$ or $\{s\}$, where the second option is the only one eventually resulting in a unitary theory in AdS. The corresponding Stueckelberg Lagrangian smoothly deforms the sum of the two flat Lagrangians for the $\{s,1\}$ and for the $\{s\}$ representations, thus providing a description of the same $O(D-2)$ degrees of freedom. However, from the AdS vantage point, those degrees of freedom are to be viewed as corresponding to a single massless particle with the symmetries of the $\{s,\,1\}$-tableau.

As an additional example of the general construction let us also discuss the case of an $O(D)$–tensor with three families of indices and tableau structure $\{3,2,1\}$, that we denote $\varphi^{(0)}$, which is instructive in particular due to the presence of one more generation of broken gauge-for-gauge invariances.

The first consequence of covariantising derivatives in the flat gauge transformation of $\varphi^{(0)}$ is the appearance of three broken gauge-for-gauge parameters, with diagram structure $\{2,1,1\}$, $\{2,2\}$ and $\{3,1\}$ respectively; in addition, there is a third-generation gauge parameter with tableau $\{2,1\}$ to be discussed later. The full pattern of gauge generations, including the first one, associated to $\varphi^{(0)}$ is summarized in the following scheme:



(3.61)

As a first step, in order to deal with the additional gauge freedom emerging from the breaking of gauge-for-gauge invariance, we would include in our description two Stueckelberg fields, $\varphi^{(1,\,1)}$ and $\varphi^{(1,\,2)}$, with tableau structure $\{3,1,1\}$ and $\{3,2\}$, while following [23, 24] we descard at this level the other possible choice of an additional Stueckelberg field $\varphi^{(1,\,3)}$ with structure $\{2,2,1\}$, assuming that it would lead to a non-unitary theory. Looking at (3.61) this choice is tantamount to promoting the two lost first-generation parameters to play the role of Stueckelberg fields. A pictorial synopsis of the pattern of gauge generations for each of these two fields is provided in the following schemes:

$$\varphi^{(1,1)}: \quad \yng(3,1,1) \quad \xrightarrow{G} \begin{cases} \yng(2,1,1) \\ \\ \yng(3,1) \end{cases} \quad \overset{G^2}{\Longrightarrow} \begin{cases} \yng(2,1) \\ \\ \yng(3) \end{cases} \quad \overset{G^3}{\equiv\!>} \quad \yng(2) \tag{3.62}$$

$$\varphi^{(1,2)}: \quad \yng(3,2) \quad \xrightarrow{G} \begin{cases} \yng(2,2) \\ \\ \yng(3,1) \end{cases} \quad \overset{G^2}{\Longrightarrow} \quad \yng(2,1) \tag{3.63}$$

Let us notice that the gauge variations of $\varphi^{(1,\,1)}$ and $\varphi^{(1,\,2)}$ comprise the three gauge-for-gauge parameters associated with $\varphi^{(0)}$, together with an additional parameter with the tableau structure $\{3,1\}$, needed to ensure that eventually both $\varphi^{(1,\,1)}$ and $\varphi^{(1,\,2)}$ propagate only their flat-space physical degrees of freedom, and not more. These Stueckelberg fields in their turn generate two broken gauge-for-gauge symmetries, both of the form $\{2,1\}$, while in relation with the $\{3,1,1\}$-tensor one should also take into account the existence of a third-order transformation with a rank-two symmetric tensor parameter.

At this point an important novelty with respect to the two-family case manifests itself: as already observed, in the complete pattern of flat gauge transformations associated with $\varphi^{(0)}$ there is also a third-generation gauge parameter whose tableau structure is $\{2,1\}$. In flat-space it would indicate the existence of combinations of the first-generation parameters apparently hampered by the existence of gauge-for-gauge transformations, but at a closer look effective on $\varphi^{(0)}$. Thus, in order for the matching between field-components and gauge parameters to be exact, in our Stueckelberg construction we have to accommodate an additional gauge freedom with tableau structure $\{2,1\}$. This means in practice that, of the two broken gauge-for-gauge parameters with structure $\{2,1\}$ associated to $\varphi^{(1,\,1)}$ and $\varphi^{(1,\,2)}$ one combination has to be left free, exactly to account for the part of the gauge symmetry of the initial field that has been removed after introducing the Stueckelberg fields $\varphi^{(1,\,1)}$ and $\varphi^{(1,\,2)}$. Thus, the presence of the pair of $\{2,1\}$ parameters in (3.62) and (3.63) calls for the introduction of only one second-generation Stueckelberg field $\varphi^{(2,\,1)}$, whose structure is again fixed by unitarity to be that of a $\{3,1\}$ tensor (while possible alternative options like tableaux $\{2,2\}$ or $\{2,1,1\}$ are discarded) and whose gauge pattern is given as follows:

$$\varphi^{(2,1)} : \quad \ydiagram{3,1} \quad \xrightarrow{G} \begin{cases} \ydiagram{2,1} \\ \\ \ydiagram{3} \end{cases} \quad \xRightarrow{G^2} \quad \ydiagram{2} \tag{3.64}$$

Finally, let us notice that the additional gauge symmetry provided by the second-generation parameter in (3.64) is just what is needed to account for the third generation of broken gauge-for-gauge symmetry of $\varphi^{(1,\,1)}$, and thus the pattern of Stueckelberg fields that one needs to introduce does not include an additional rank-three symmetric tensor. The resulting system of $O(D-2)$ tableaux of the form $\{3,2,1\}$, $\{3,1,1\}$, $\{3,2\}$ and $\{3,1\}$ matches the degrees of freedom of the massless AdS particle with the symmetries of the diagram $\{3,2,1\}$ as resulting from the BMV conjecture [25, 26].

These two examples should convey the general idea behind our interpretation of the BMV phenomenon while also suggesting the concrete procedure to build the Stueckelberg Lagrangian for the degrees of freedom of a given (A)dS massless particle with mixed symmetry.

In the general case $\varphi^{(0)}$ can be an $O(D)$-tableau with N rows (that for simplicity one might assume as being of different lengths) whose hierarchy of flat gauge-transformations comprises N gauge parameters, $\binom{N}{2}$ gauge-for-gauge parameters, $\binom{N}{3}$ third generation parameters and so on. To deal with the first instance of gauge-for-gauge breaking we would introduce $N-1$ first-generation Stueckelberg fields $\varphi^{(1,k)}$, $k=1,\ldots,N-1$, (i.e. all possible Stueckelberg fields whose first row has the same length as that of $\varphi^{(0)}$, effectively corresponding to all first-generation gauge parameters with first row of maximal length) to which one can associate an equivalent pattern of broken reducible gauge transformations. The generation of new fields will stop as soon as the overall gauge symmetry of the system will match that of its flat-space counterpart, accounting in particular for the full pattern of reducible gauge transformations for each mixed-symmetry field introduced in the spectrum.

Let us stress once more that insofar as gauge symmetry alone is concerned the pattern would not be uniquely determined: at each step different choices of Stueckelberg gauge fields would be indeed consistent with the additional gauge parameters emerging at the previous level. In AdS all ambiguities are fixed performing at each step the unitary choice dictated by the analysis of [23, 24], which amounts to choosing as allowed Stueckelberg fields only tableaux whose first row has the same length as that of $\varphi^{(0)}$. We expect that, pursuing the construction of the corresponding Lagrangian for different choices, at least some of the Stueckelberg fields would eventually appear with kinetic terms of wrong signs.

Having discussed in some detail the motivations and the general structure of our approach, in the remainder of this section we shall construct the Stueckelberg Lagrangian for the case of two-family $O(D)$-tensors in (A)dS, smoothly deforming the flat-space transverse-invariant Lagrangians for the corresponding fields presented in section 2.1.2, thus also providing an application of our general construction in Minkowski space. Showing that the final result admits an off-shell gauge-fixing to our Lagrangian (2.80) will constitute our proof that the degrees of freedom associated to the latter are those described by the corresponding Metsaev equations.

According to the general discussion of the previous paragraphs, starting with an $O(D)$ tensor in the $\{s, k\}$ representation we expect our Lagrangian to involve a total of k additional fields with tableau structure $\{s,\, k-i\}$:

$$\varphi^{(i)} \;\sim\; \{s,\, k-i\}; \qquad i = 1, \ldots, k\,. \tag{3.65}$$

Each of the fields $\varphi^{(i)}$ included in the system experiences gauge-for-gauge breaking phenomenon involving parameters having structure $\{s-1,\, k-i-1\}$, which is taken care of by the gauge transformation of the next field in the resulting hierarchy, $\varphi^{(i-1)}$.

The general form of the resulting Stueckelberg Lagrangian is

$$\mathcal{L} \,=\, \frac{1}{2}\sum_{i=0}^{k} \varphi^{(i)} \left(M - \frac{m_i}{L^2}\right) \varphi^{(i)} \,+\, \sum_{i=0}^{k-1} \frac{c_i}{L}\, \varphi^{(i+1)}\, \nabla_2\, \varphi^{(i)}\,, \tag{3.66}$$

where the quadratic part in $\varphi^{(i)}$ defines the deformation of the transverse-invariant flat Lagrangian (2.30) for the corresponding representation, while the off-diagonal terms provide the only possible couplings available with less than two derivatives (as required in order for their flat limit to vanish) for traceless tensors, and involve a divergence of the tensor $\varphi^{(i)}$ with respect to its second family, here denoted ∇_2 in accordance with the general conventions for our index-free notation.

However, working with two-family tensors allows for a more explicit notation already used in section 2.1.1, according to which tensors of the form $\{s, k\}$ will be denoted by

$$\varphi_{\,\mu_1 \cdots \mu_s,\, \nu_1 \cdots \nu_k} \;\equiv\; \varphi_{\,\mu_s,\, \nu_k}\,, \tag{3.67}$$

while when computing products of tensors we will make use of the same symbols for indices that are meant to be totally symmetrised; e.g.

$$\partial_{\,(\mu_1}\, \varphi_{\,\mu_2 \cdots \mu_{s+1}),\, \nu_1 \cdots \nu_k} \;\equiv\; \partial_{\,\mu}\, \varphi_{\,\mu_s,\, \nu_k}\,. \tag{3.68}$$

In this notation all rules for symmetric calculus collected in (A.1) apply independently for the two sets of indices, while additional prescriptions for contracting indices belonging to different families are not difficult to derive case by case.

Taking into account the whole set of gauge parameters available within the system we can write the general form of the gauge transformation of each field as

$$
\begin{aligned}
\delta\varphi^{(i)}{}_{\mu_s,\,\nu_{k-i}} \;=\; & \nabla_\mu\,\Lambda^{(i)}{}_{\mu_{s-1},\,\nu_{k-i}} \,+\, \nabla_\nu\,\lambda^{(i)}{}_{\mu_s,\,\nu_{k-i-1}} - \frac{1}{s-k+i+1}\,\nabla_\mu\,\lambda^{(i)}{}_{\mu_{s-1}\,\nu,\,\nu_{k-i-1}} \\
& + \frac{\alpha_i}{L}\lambda^{(i-1)}{}_{\mu_s,\nu_{k-i}} + \frac{\beta_i}{L}\left\{2g_{\mu\mu}\Lambda^{(i+1)}{}_{\mu_{s-2}\nu,\nu_{k-i-1}} - (s-k+i)g_{\mu\nu}\Lambda^{(i+1)}{}_{\mu_{s-1},\nu_{k-i-1}}\right\} \\
& + \frac{\gamma_i}{L}\left\{\,2\,g_{\mu\mu}\,\lambda^{(i+1)}{}_{\mu_{s-2}\,\nu\nu,\,\nu_{k-i-2}} - (s-k+i+1)\,g_{\mu\nu}\,\lambda^{(i+1)}{}_{\mu_{s-1}\,\nu,\,\nu_{k-i-2}}\right. \\
& \qquad \left. +\,(s-k+i+1)(s-k+i+2)\,g_{\nu\nu}\,\lambda^{(i+1)}{}_{\mu_s,\,\nu_{k-i-2}}\,\right\}, \qquad (3.69)
\end{aligned}
$$

where it is possible to appreciate that, besides the parameters present in the flat gauge transformation, $\Lambda^{(i)}{}_{\mu_{s-1},\,\nu_{k-i}}$ and $\lambda^{(i)}{}_{\mu_s,\,\nu_{k-i-1}}$, a number of additional contributions can also enter the variation of $\varphi^{(i)}{}_{\mu_s,\,\nu_{k-i}}$ (in the combinations needed to recover the corresponding $\{s,\,k-i\}$-projection, here collected in braces), exploiting parameters entering the system from the flat variation of other fields in the multiplet. It is due to this mixing of gauge transformations that gauge-for-gauge breaking at the level of a single field can be in principle reabsorbed in the whole system. Indeed, along with (3.69) one can define the following transformations of the parameters:

$$
\begin{aligned}
\delta\Lambda^{(i)}{}_{\mu_{s-1},\,\nu_{k-i}} =\; & \nabla_\nu\,\Theta^{(i)}{}_{\mu_{s-1},\,\nu_{k-i-1}} - \frac{1}{s-k+i}\,\nabla_\mu\,\Theta^{(i)}{}_{\mu_{s-2}\,\nu,\,\nu_{k-i-1}} + \frac{b_i}{L}\,\Theta^{(i-1)}{}_{\mu_{s-1},\,\nu_{k-i}} \\
& + \frac{d_i}{L}\left\{\,2\,g_{\mu\mu}\,\Theta^{(i+1)}{}_{\mu_{s-3}\,\nu\nu,\,\nu_{k-i-2}} - (s-k+i)\,g_{\mu\nu}\,\Theta^{(i+1)}{}_{\mu_{s-2}\,\nu,\,\nu_{k-i-2}}\right. \\
& \qquad \left. +\,(s-k+i)(s-k+i+1)\,g_{\nu\nu}\,\Theta^{(i+1)}{}_{\mu_{s-1},\,\nu_{k-i-2}}\,\right\}, \qquad (3.70\text{a})
\end{aligned}
$$

$$
\begin{aligned}
\delta\lambda^{(i)}{}_{\mu_s,\,\nu_{k-i-1}} =\; & -\,\frac{s-k+i+1}{s-k+i}\,\nabla_\mu\,\Theta^{(i)}{}_{\mu_{s-1},\,\nu_{k-i-1}} \\
& + \frac{e_i}{L}\left\{\,2\,g_{\mu\mu}\,\Theta^{(i+1)}{}_{\mu_{s-2}\,\nu,\,\nu_{k-i-2}} - (s-k+i+1)\,g_{\mu\nu}\,\Theta^{(i+1)}{}_{\mu_{s-1},\,\nu_{k-i-2}}\,\right\}, \qquad (3.70\text{b})
\end{aligned}
$$

smoothly deforming the gauge-for-gauge transformations of the flat theory. Finally, the transversality conditions get also deformed by the Stuecklberg construction according to

$$
\begin{aligned}
& \nabla\cdot\Lambda^{(i)}{}_{\mu_{s-2},\nu_{k-i}} - \frac{1}{s-k+i+1}\nabla\cdot\lambda^{(i)}{}_{\mu_{s-2}\nu,\nu_{k-i-1}} \qquad (3.71) \\
& + \frac{\beta_i}{L}(D+s+k-i-4)\Lambda^{(i+1)}{}_{\mu_{s-2}\nu,\nu_{s-k-i-1}} + \frac{\gamma_i}{L}\left[D+2(k-i-3)\right]\lambda^{(i+1)}_{\mu_{s-2}\,\nu\nu,\,\nu_{k-i-2}} = 0\,,
\end{aligned}
$$

as expected on general grounds, besides being directly related to the need of keeping gauge-invariant the condition of tracelessness of each field $\varphi^{(i)}{}_{\mu_s,\,\nu_{k-i}}$.

Our goal is to look for values of the coefficients in (3.66), (3.69) and (3.70) so that the following conditions are simultaneously satisfied:

- the Lagrangian (3.66) is gauge invariant,
- the fields $\varphi^{(i)}{}_{\mu_s,\,\nu_{k-i}}$ are invariant under the transformations of the parameters in (3.70),

thus ensuring that the Stueckelberg Lagrangian possesses the same net amount of gauge symmetry as the sum of flat, transverse-invariant Lagrangians for the same set of fields.

After showing that the corresponding equations admit solutions, which is non-trivial a priori since it is easy to verify that the system to be solved is over-determined, to complete our argument we will need to discuss the possibility of gauge fixing the Stueckelberg Lagrangian to our form (3.54), while also recovering the transversality conditions (2.76) on the remaining parameter.

Asking for gauge invariance of Lagrangian (3.66) allows at first to express the coefficients in the gauge transformations (3.69) in terms of the couplings c_i according to

$$\alpha_i \;=\; -\,c_{i-1}\,, \tag{3.72a}$$

$$\beta_i \;=\; -\,\frac{c_i}{(k-i)(s-k+i+2)(D+s+k-i-6)}\,, \tag{3.72b}$$

$$\gamma_i \;=\; \frac{2\,c_i}{(k-i)(s-k+i+2)^2[D+2(k-i-4)]}\,, \tag{3.72c}$$

while for the coefficients m_i one finds:

$$m_i \,=\, \widehat{m}_0 \,-\, i \,+\, \frac{(s-k+i-1)(D+s+k-i-4)}{(k-i+1)(s-k+i+1)(D+s+k-i-5)}\,c_{i-1}^2\,. \tag{3.73}$$

Here $\widehat{m}_0$ denotes the physical mass of the $\varphi^{(0)}$ field, that for a tableau with structure $\{s,k\}$ reads

$$\widehat{m}_0 \,=\, s+k-(s-2)(D+s-3)\,, \tag{3.74}$$

in agreement with our general result in (2.80). Let us also notice that the "physical" mass associated to a field with symmetry $\{s,k-i\}$ corresponds to the combination $\widehat{m}_0-i$, thus indicating that in our general solution only $\varphi^{(0)}$ will appear in the Lagrangian with its own physical kinetic operator while the other values of m_i do not define the "masses" of the corresponding AdS fields. The remaining equations involve the squares of the c_i and are solved by

$$c_i^2 \,=\, \frac{(i+1)(k-i)(s-k+i+2)(D+2k-i-6)(D+s+k-i-6)}{D+2(k-i-3)}\,, \tag{3.75}$$

which, upon substitution in (3.73) allows to express the coefficients m_i in the relatively simple form:

$$m_i \,=\, \widehat{m}_0 \,-\, i \,+\, i\,\frac{(s-k+i-1)(D+s+k-i-4)(D+2k-i-5)}{D+2(k-i-2)}\,. \tag{3.76}$$

However, we still have to impose invariance under the gauge-for-gauge transformations

(3.70). Solving the corresponding equations fixes the remaining coefficients to the forms

$$b_i \;=\; \frac{s-k+i}{s-k+i-1}\,\alpha_i\,, \tag{3.77a}$$

$$d_i \;=\; \frac{2\,\beta_i \,+\, (s-k+i+2)(s-k+i+3)\,\gamma_i}{(s-k+i+1)^2}\,, \tag{3.77b}$$

$$e_i \;=\; \frac{(s-k+i+2)\,\gamma_i \,-\, (s-k+i)\,\beta_i}{s-k+i+1}\,, \tag{3.77c}$$

while also imposing the following consistency conditions

$$\alpha_i\, e_{i-1} \,+\, \beta_i\, b_{i+1} \;=\; \frac{s-k+i+1}{s-k+i}\,, \tag{3.78a}$$

$$d_{i+1}\,\beta_i \,+\, e_{i+1}\,\gamma_i \;=\; 0\,, \tag{3.78b}$$

that are also satisfied by the expressions found in (3.72), thus showing existence and uniqueness of the Stueckelberg Lagrangian we were after.

The gauge-fixing procedure to eliminate the Stueckelberg fields ${\varphi^{(i)}}_{\mu_s,\,\nu_{k-i}}$ for all values of $i \neq 0$ can be discussed iteratively, starting from the highest values of i:

$$\delta {\varphi^{(k)}}_{\mu_s} \;=\; \nabla_\mu\, {\Lambda^{(k)}}_{\mu_{s-1}} + \frac{\alpha_k}{L}\, {\lambda^{(k-1)}}_{\mu_s}, \tag{3.79}$$

$$\begin{aligned}\delta {\varphi^{(k-1)}}_{\mu_s,\,\nu} \;=\;& \nabla_\mu\, {\Lambda^{(k-1)}}_{\mu_{s-1},\,\nu} \,+\, \nabla_\nu\, {\lambda^{(k-1)}}_{\mu_s} - \frac{1}{s}\,\nabla_\mu\, {\lambda^{(k-1)}}_{\mu_{s-1}\,\nu} \\ &+ \frac{\alpha_{k-1}}{L}\, {\lambda^{(k-2)}}_{\mu_s,\,\nu} \,+\, \frac{\beta_{k-1}}{L}\left\{ 2\, g_{\mu\mu}\, {\Lambda^{(k)}}_{\mu_{s-2}\,\nu} - (s-1)\, g_{\mu\nu}\, {\Lambda^{(k)}}_{\mu_{s-1}} \right\},\end{aligned} \tag{3.80}$$

$$\begin{aligned}\delta {\varphi^{(k-2)}}_{\mu_s,\,\nu\nu} \;=\;& \nabla_\mu\, {\Lambda^{(k-2)}}_{\mu_{s-1},\,\nu\nu} \,+\, \nabla_\nu\, {\lambda^{(k-2)}}_{\mu_s,\,\nu} - \frac{1}{s-1}\,\nabla_\mu\, {\lambda^{(k-2)}}_{\mu_{s-1}\,\nu,\,\nu} \\ &+ \frac{\alpha_{k-2}}{L}\, {\lambda^{(k-3)}}_{\mu_s,\,\nu\nu} \,+\, \frac{\beta_{k-2}}{L}\left\{ 2\, g_{\mu\mu}\, {\Lambda^{(k-1)}}_{\mu_{s-2}\,\nu,\,\nu} - (s-2)\, g_{\mu\nu}\, {\Lambda^{(k-1)}}_{\mu_{s-1},\,\nu} \right\} \\ &+ \frac{\gamma_{k-2}}{L}\left\{2g_{\mu\mu}{\lambda^{(k-1)}}_{\mu_{s-2}\nu\nu} - (s-1)g_{\mu\nu}{\lambda^{(k-1)}}_{\mu_{s-1}\nu} + s(s-1)g_{\nu\nu}{\lambda^{(k-1)}}_{\mu_s}\right\},\end{aligned} \tag{3.81}$$

$$\cdots\,.$$

From (3.79) one sees that the elimination of $\varphi^{(k)}$ is indeed possible due to $\lambda^{(k-1)}$, but actually makes use of both parameters $\Lambda^{(k)}$ and $\lambda^{(k-1)}$ up to gauge-for-gauge transformations that, as such, do not affect the other fields as well. Similarly, when it comes to eliminating $\varphi^{(k-1)}$, we see from (3.80) that a complete gauge-fixing of this field is reached making use of $\Lambda^{(k-1)}$ and $\lambda^{(k-2)}$, given that at this level $\Lambda^{(k)}$ and $\lambda^{(k-1)}$ can no more affect $\varphi^{(k-1)}$, and so on and so forth. In this fashion all fields $\varphi^{(k-i)}$, with $i \neq k$ can be set to zero performing at each step gauge fixings that make use of two parameters, $\Lambda^{(k-i)}$ and $\lambda^{(k-i-1)}$, whose leftover freedom only amounts to irrelevant gauge-for-gauge transformations, while the remaining parameters entering in principle the variation of $\varphi^{(k-i)}$ have been already exploited to gauge fix the fields $\varphi^{(k-i+l)}$ for all values of $l = 0 \cdots i-1$. After eliminating in this way all Stueckelberg fields the resulting Lagrangian clearly coincides with our result (2.80) (or (3.54) in this section), with $\varphi^{(0)}$ subject to an effective transformation where only one parameter appears,

$$\delta\varphi^{(0)}{}_{\mu_s,\,\nu_k} \;=\; \nabla_\mu\, \Lambda^{(0)}{}_{\mu_{s-1},\,\nu_k}\,, \tag{3.82}$$

while we see from (3.71) that the latter is now subject to the proper transversality condition required in this setting,

$$\nabla\cdot \Lambda^{(0)}{}_{\mu_{s-1},\,\nu_k} \;=\; 0\,, \tag{3.83}$$

thus completing our argument.

4 Diagonalisation of reducible theories

Besides the analysis of the spectrum, we would also like to discern the proper combinations of the components of φ associated to each of the irreducible representations identified by our preceding analysis. Focussing on the case of symmetric tensors, in this section we present a systematic way to construct the field redefinitions needed to decompose the Lagrangians (2.1) and (2.55) in their block-diagonal form, where each block provides an action suitable for the description of the irreducible polarisations of a given spin. In our opinion this latter approach retains some specific advantages: first, the resulting Lagrangians display at a glance both number and nature of the irreducible propagating degrees of freedom, including the relative signs among the various kinetic terms making manifest the absence of ghosts; moreover, it allows in principle to interpret possible non-linear deformations of (2.1) in terms of couplings among single-particle fields. For the unconstrained versions of our Lagrangians provided by triplet systems the corresponding diagonalisation was discussed in [56, 55].

4.1 Symmetric tensors in flat backgrounds

Our starting point is a formal decomposition of φ involving fields of decreasing spins:

$$\varphi \;=\; \phi_s \,+\, O_{s-2}\,\phi_{s-2} \,+\, O_{s-4}\,\phi_{s-4} \,+\, \cdots \,+\, O_{s-2k}\,\phi_{s-2k} \,+\, \cdots\,, \tag{4.1}$$

where ϕ_{s-2k} is a symmetric tensors of rank $(s-2k)$, while the associated operators O_{s-2k} are to be chosen so that when (4.1) is inserted in (2.1) the latter decomposes into a sum of decoupled Lagrangians. Each of these Lagrangians will enjoy transverse gauge invariance and must ultimately describe irreducible, massless spin-$(s-2k)$ degrees of freedom, with $k = 0, 1, \ldots, [\frac{s}{2}]$; as discussed in section 2.1.1, this requires that the fields ϕ_{s-2k}, and the corresponding gauge parameters, be *traceless*.

More explicitly, inserting (4.1) into (2.1) one gets

$$\mathcal{L} \,=\, \frac{1}{2} \sum_{k,l\,=\,0}^{[\frac{s}{2}]} O_{s-2k}\, \phi_{s-2k}\, M\, O_{s-2l}\, \phi_{s-2l}\,, \tag{4.2}$$

where $O_s \,=\, \mathbb{I}$ and where contraction of indices is understood between $O_{s-2k}\, \phi_{s-2k}$ and $M\, O_{s-2l}\, \phi_{s-2l}$. From the previous expression one can see that the diagonalisation obtains if

$$O_{s-2k}\, \phi_{s-2k}\, M\, O_{s-2l}\, \phi_{s-2l} \,\sim\, \delta_{\,k,l}\, \phi_{s-2k}\, M\, \phi_{s-2l}\,, \tag{4.3}$$

and we will show that the latter condition holds indeed if the operators O_{s-2k} satisfy the equation

$$M\, O_{s-2k} \,=\, \eta^{\,k}\, M\,. \tag{4.4}$$

In general eq. (4.4) possesses several solutions, due to the invariance of the Maxwell-like operator M under the gauge transformation

$$\delta\, O_{s-2k} \,=\, \partial\, \Lambda_k\,, \tag{4.5}$$

where Λ_k is itself an operator satisfying the transversality condition $\partial \cdot \Lambda_k = 0$[22]. Nonetheless, we shall see that whenever (4.4) is satisfied the diagonalisation conditions (4.3) holds as well, so that the explicit form of the operators O_{s-2k} is not really needed for our present purposes. At any rate, it is possible to conclude on general grounds that all solutions to (4.4) are to involve non-local operators, as we discuss in appendix D where we also exhibit an explicit solution.

Let us make use of (4.4) in (4.2) assuming in addition, without loss of generality, $k \geq l$:

$$\begin{aligned} O_{s-2k}\, \phi_{s-2k}\, M\, O_{s-2l}\, \phi_{s-2l} \,&=\, O_{s-2k}\, \phi_{s-2k}\, \eta^{\,l} M\, \phi_{s-2l} \\ &= c_l\, \left\{ M\, T^{\,l}\, O_{s-2k}\, \phi_{s-2k} \right\}\, \phi_{s-2l} \\ &= c_l\, \left\{ [M, T^{\,l}]\, O_{s-2k}\, \phi_{s-2k} \,+\, T^{\,l}\, \eta^{\,k}\, M\, \phi_{s-2k} \right\} \phi_{s-2l}\,, \end{aligned} \tag{4.6}$$

where we exploited both the self-adjointness of M (up to total derivatives) and (4.4), and where

$$c_l \,=\, (2\,l \,-\, 1)!!\, \binom{s}{2l} \tag{4.7}$$

is a combinatorial factor coming from the contraction of the l powers of η, leading to the l traces in the second line of (4.6), here denoted in operatorial notation as $T^{\,l}$. Let us evaluate separately the two terms in the third line of (4.6).

In the first term, the commutator of M and $T^{\,l}$ is proportional to a double divergence; more precisely:

$$[M, T^{\,l}] \,=\, 2\,l\, T^{\,l-1}\, \partial \cdot \partial \cdot\,, \tag{4.8}$$

[22]The solution would be unique if for some reasons there were no candidates for a divergenceless Λ_k; while this is not the case in general, it happens indeed for a special subset of the operators O_{s-2k}, as we shall see in appendix D.

as can be verified recursively starting from $[M, T] = 2\,\partial\cdot\partial\cdot$ and taking into account that traces and divergences commute. In addition, the divergence of (4.4) gives

$$\partial\cdot\partial\cdot O_{s-2k} \;=\; \eta^{\,k}\,\partial\cdot\partial\cdot \,-\, \eta^{\,k-1}\,M\,, \tag{4.9}$$

where we factored out an overall gradient. Let us stress that (4.9) allows us to dispense with the detailed structure of the operators O_{s-2k}, which otherwise would make the general proof significantly more involved. All in all, we have to evaluate

$$2\,l\,T^{\,l-1}\left\{\,\eta^{\,k}\,\partial\cdot\partial\cdot\,-\,\eta^{\,k-1}\,M\,\right\}\phi_{s-2k}\,\phi_{s-2l}\,, \tag{4.10}$$

where, due to the tracelessness of ϕ_{s-2l}, for $k \geq l$ the first term never contributes[23] while the second term can be conveniently rewritten as

$$-\,2\,l\,T^{\,l-1}\,\eta^{\,k-1}\,M\,\phi_{s-2k}\,\phi_{s-2l} \;=\; -\,2\,l\,\tilde{c}_{\,l,k}\,M\,\phi_{s-2k}\,T^{\,k-1}\,\eta^{\,l-1}\,\phi_{s-2l}\,, \tag{4.11}$$

up to an overall combinatorial coefficient $\tilde{c}_{\,l,k}$, that we do not need to evaluate in general since (4.11) contributes only for $k = l$ when the coefficient itself is trivial ($\tilde{c}_{\,k,k} = 1$). For the same reason in the second term to be evaluated,

$$T^{\,l}\,\eta^{\,k}\,M\,\phi_{s-2k}\,\phi_{s-2l} \;=\; \hat{c}_{\,l,k}\,M\,\phi_{s-2k}\,T^{\,k}\,\eta^{\,l}\phi_{s-2l}\,, \tag{4.12}$$

the only contribution obtains for $k = l$; in both cases the relevant quantity to compute is

$$T^{\,k}\,\eta^{\,l}\phi_{s-2l} \;=\; \delta_{\,k,l}\,\prod_{i\,=\,0}^{k-1}\left[D + 2\,(s-2k+i)\right]\phi_{s-2k}\,. \tag{4.13}$$

Substituting (4.13) in (4.11) and (4.12), and then inserting the corresponding expressions in (4.6), we finally obtain

$$\mathcal{L} \;=\; \frac{1}{2}\,\sum_{k,l\,=\,0}^{[\frac{s}{2}]}\,O_{s-2k}\,\phi_{s-2k}\,M\,O_{s-2l}\,\phi_{s-2l} \;=\; \frac{1}{2}\,\sum_{k\,=\,0}^{[\frac{s}{2}]}\,c_{\,k}\,b_{\,k,s,D}\,\phi_{s-2k}\,M\,\phi_{s-2k}\,, \tag{4.14}$$

where $c_{\,k}$ was given in (4.7) and where we defined

$$b_{\,k,s,D} \;=\; \prod_{i\,=\,0}^{k-1}\left[D + 2\,(s-2k+i-1)\right]. \tag{4.15}$$

This proves that the redefinition (4.1) in conjunction with the defining property (4.4) of the operators O_{s-2k} actually diagonalise (4.2). Each of the decoupled Lagrangians involves traceless fields and displays transverse gauge invariance with traceless parameters, as required for them to propagate each a single particle of a given spin. The fact that all relative signs are equal confirms the absence of ghosts, while an additional rescaling would be needed in order to assign to the various fields their canonical normalization.

[23] More explicitly: $(T^{\,l-1}\,\eta^{\,k}\,\partial\cdot\partial\cdot\phi_{s-2k})\phi_{s-2l}\;\sim\;\partial\cdot\partial\cdot\phi_{s-2k}(T^{\,k}\,\eta^{\,l-1}\,\phi_{s-2l}) = 0.$

4.2 Symmetric tensors in (A)dS backgrounds

The diagonalisation of the Lagrangian (2.55) follows closely the corresponding procedure just presented for the flat case, and for this reason we shall limit ourselves to recalling its main steps while stressing a few additional peculiar features of the (A)dS case. We first introduce a set of traceless tensors of decreasing spins ϕ_{s-2k} via

$$\varphi \,=\, \phi_s \,+\, O^L_{s-2}\,\phi_{s-2} \,+\, O^L_{s-4}\,\phi_{s-4} \,+\, \cdots \,+\, O^L_{s-2k}\,\phi_{s-2k} \,+\cdots\,, \tag{4.16}$$

and then look for operators O^L_{s-2k} implementing the diagonalisation condition for (2.55)[24]

$$O^L_{s-2k}\,\phi_{s-2k}\,M^{(s)}_L\,O^L_{s-2l}\,\phi_{s-2l} \,\sim\, \delta_{k,l}\,\phi_{s-2k}\,M^{(s-2l)}_L\,\phi_{s-2l}\,. \tag{4.17}$$

The key to the whole procedure is to assume that the operators O^L_{s-2k} satisfy the basic identity

$$M^{(s)}_L\,O^L_{s-2k} \,=\, g^k\,M^{(s-2k)}_L\,, \tag{4.18}$$

which allows to write

$$\begin{aligned} O^L_{s-2k}\,\phi_{s-2k}\,M_L\,O^L_{s-2l}\,\phi_{s-2l} \,&=\, O^L_{s-2k}\,\phi_{s-2k}\,g^{\,l}\,M^{(s-2l)}_L\,\phi_{s-2l} \\ &=\, c_l\,M^{(s-2l)}_L\,T^{\,l}\,O^L_{s-2k}\,\phi_{s-2k}\,\phi_{s-2l} \\ &= c_l\,\Big\{\,[M^{(s-2l)}_L, T^{\,l}]\,O^L_{s-2k}\,\phi_{s-2k} \\ &+\, T^{\,l}\,(M^{(s-2l)}_L \,-\, M^{(s)}_L)\,O^L_{s-2k}\,\phi_{s-2k} \\ &+T^{\,l}g^{\,k}\,M^{(s-2k)}_L\,\phi_{s-2k}\Big\}\,\phi_{s-2l}\,, \end{aligned} \tag{4.19}$$

where the combinatorial coefficient c_l is given in (4.7). Computing the commutator in (4.19) gives

$$[\,M^{(s-2k)}_L, T^{\,l}\,] \,=\, 2\,l\,T^{\,l-1}\,\nabla\cdot\nabla\cdot\,-\,\frac{2}{L^2}\,l\,[D+2\,(s-l-1)]\,T^{\,l}\,, \tag{4.20}$$

so that, after some manipulations, one finds that the term involving the commutator and the following one in (4.19) sum up to

$$2\,l\,T^{\,l-1}\left(\nabla\cdot\nabla\cdot\,-\,\frac{4}{L^2}\,T\right)O^L_{s-2k}\,\phi_{s-2k}\,\phi_{s-2l}\,. \tag{4.21}$$

To evaluate (4.21) we make use of the identity

$$\left(\nabla\cdot\nabla\cdot\,-\,\frac{4}{L^2}\,T\right)O^L_{s-2k} \,=\, -\,g^{\,k-1}\,M^{(s-2k)}_L \,+\, g^{\,k}\left(\nabla\cdot\nabla\cdot\,-\,\frac{4}{L^2}\,T\right) \tag{4.22}$$

[24] At the risk of being pedantic, here we add a label to specify the value of s in the spin-dependent part of the kinetic operators; thus $M^{(s)}_L$ corresponds to M_L as defined in (2.53), while $M^{(s-2k)}_L$ can be obtained from (2.53) by the substitution $s \to s-2k$. It might be also useful to stress that the operators O^L_{s-2k} in (4.16) depend on the rank of φ, so that if $\text{rank}(\varphi) = s$ they are assumed to satisfy (4.18) only under the action of $M^{(s)}_L$.

which in itself is a consequence of the divergence of (4.18). Assuming for simplicity $k \geq l$ and completing the computation as in section 2.1.1 it is then possible to conclude that the redefinition (4.16) decomposes the Maxwell-like Lagrangian (2.55) on AdS as

$$\mathcal{L} = \frac{1}{2}\,\varphi\,M_L^{(s)}\,\varphi = \frac{1}{2}\sum_{k=0}^{[\frac{s}{2}]} c_k\,b_{k,s,D}\,\phi_{s-2k}\,M_L^{(s-2k)}\,\phi_{s-2k}\,, \tag{4.23}$$

with the same combinatorial coefficients as for the flat case of (4.7) and (4.15), respectively.

5 Discussion

In this work we performed a systematic exploration of theories describing massless bosons of arbitrary spin and symmetry under conditions of transversality on the corresponding gauge parameters, obtaining Lagrangians that are typically simpler than their more conventional counterparts.

Higher-spin free Lagrangians have been intensively studied from several perspectives; in the metric-like approach, with second-order kinetic operators, the various options can be viewed as different solutions to the problem of dressing the D'Alembertian wave operator so that the resulting theory possesses a given amount of gauge invariance. As a necessary condition, the latter has to grant at least the elimination of field components whose presence would spoil the consistency of the theory. Aside from this requirement, however, stressing additional features can lead to different realizations of the same program, according to whether one aims to simplicity of the resulting action, to minimality –in terms of number of field components to be kept off-shell– or to the possibility of formulating the theory in terms of quantities amenable of a geometric interpretation, just to mention a few possible ancillary criteria. Clearly, the general goal lying on the background would be to prepare the stage for an investigation of interactions displaying in itself some advantages, either technical or conceptual, with respect to other known approaches.

Without entering into a detailed illustration of the various directions explored so far, let us observe that at least some of them can be pictorially organized according to whether they refer more directly to the spin-two model of linearised gravity or to the spin-one example of Maxwell's theory. In both cases the corresponding higher-spin extension can be implemented with or without additional constraints, and both equations of motion and Lagrangians admit a formulation either in terms of suitably defined "connexions" or, when constraints are removed, more geometrically in terms of higher-spin curvatures [1].

The Fronsdal-Labastida theory [4, 6], together with its minimal unconstrained extensions [27, 7], can be safely placed in the first category due to the formal similarity of the corresponding kinetic operators with the linearised Ricci tensor. The resulting equations, together with their non-local extensions formulated in terms of higher-spin curvatures [29, 28, 46], naturally provide irreducible descriptions of free higher-spin propagation.

Differently, as we discussed at length in the previous sections, the Maxwell-like theories that we explored in this work allow more naturally for the description of reducible higher-spin spectra, insofar as trace constraints are not imposed. On the other hand, considering the same Lagrangians on the restricted space of traceless gauge potentials leads to alternative formulations of irreducible theories that are somehow "minimal" with respect to their off-shell field content. The unconstrained extensions of Maxwell-like theories are attractive in their own right, since in their local form they bear a direct relation with free open strings in their tensionless regime [34, 30, 35], while their geometric incarnations display actions as simple as squares of curvatures, thus adding a piece of pictorial evidence to their formal relation with spin-one systems [36].

The simplification obtained focussing on the Maxwell operator allowed us to extend the scope of our construction to the case of mixed-symmetry fields in (A)dS backgrounds, providing a complete one-particle Lagrangian description of the corresponding representations in the general case. For this latter setting, far less explored in the literature if compared to the cases of flat backgrounds or symmetric tensors in (A)dS, its unconstrained extensions and their possible relation to tensionless strings are at present not known.

The construction of a corresponding scheme for fermions appears to be less direct to implement. Indeed, if we were to follow closely the analogy with the bosonic case, starting from the Fang-Fronsdal equations for single massless fermions of spin $s + \frac{1}{2}$ [57],

$$\mathcal{S} \,=\, i\,(\not{\partial}\,\psi \,-\, \partial\,\not{\psi}) \,=\, 0\,, \tag{5.1}$$

the simplest candidate to play the role of kinetic operator for a reducible theory in this case would seem to be the Dirac operator $\mathrm{D} \,= \not{\partial}$. However, in order to allow for gauge invariance of the corresponding equation under $\delta\psi = \partial\,\epsilon$ one should also impose $\not{\partial}\,\epsilon = 0$, thus implying that only on-shell gauge invariance would be admissible. The counterpart of this observation from the point of view of fermionic triplets [30, 35] is that for those systems, differently from the bosonic ones, there are no fields satisfying purely algebraic equations of motion, so that the reduction to a simpler local system seemingly implies either to keep some auxiliary fields off-shell, or to impose constraints on the gauge parameters somehow stronger than the condition of transversality at the basis of our present construction. We leave to future work a more detailed analysis of the possible constrained theories for systems of reducible fermions.

In perspective, the main issue to investigate concerns the possibility that transverse-invariance might allow for a systematic study of higher-spin interactions while also retaining at least part of the advantages met for the free theory. To begin with, one might ask whether the simplicity of Maxwell-like Lagrangians survives in some forms when interactions are turned on. At the level of cubic vertices, and with the proviso that only explicit calculations can really clarify the issue, one can expect the answer to be in the affirmative, given the minimal form of the completion needed in this case to promote the known, leading on-shell term in cubic interactions to a full off-shell form[25].

After all, the existence of non-linear theories for unimodular gravity indicates that the transversality constraint should not represent an obstacle to this programme. On the other hand, one should anyway expect that the constraint (2.4), and generalizations thereof, be properly deformed at the non-linear level, and indeed uncovering the systematics of this deformation might represent one of the clues to the whole construction.

In addition, it would be interesting to investigate what would be at level of vertices the implications of the non-local redefinitions needed to diagonalise the reducible systems, described in appendix D. Indeed, given the existence of local interactions for single-particle couplings (at least to cubic order, insofar as flat space is considered), one would naturally expect the (cubic) couplings for reducible theories to reproduce the former, after diagonalising the quadratic part. However, in order for the resulting vertices among single particles to stay local after the redefinitions, some non-trivial cancellations ought to occur whose systematics is yet to be explored. Let us mention that the issue does not appear to be related to the choice of flat background, given that in the field redefinition we found to diagonalise the (A)dS system the issue of non-locality appears even more severe than for its flat-space counterpart, and in this sense it can not be interpreted as a manifestation of yet another pathology of higher-spin interactions in Minkowski space-time.

One could also investigate directly the structure of couplings deforming single-particle Lagrangians, exploiting traceless fields. Once again, given the simplified kinematical setting at the level of fields involved, the possible complications are likely to come from the preservation or deformation of the constraints, and it could well be that, at the end, the final balance would not especially favor transverse-invariance as a starting point for investigating interactions. However, an additional reason to explore this path is that, starting from Lagrangians (2.80), one has in principle the possibility to address in a systematic and more direct fashion the interactions among bosonic gauge fields of mixed-symmetry on (Anti-)de Sitter backgrounds.

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[25]See e.g. [58] for various approaches to the systematics of cubic vertices for higher-spin bosonic fields.

A Notation and useful formulae

A.1 Symmetric tensors

We work with mostly-positive metric in D space-time dimensions. If not otherwise specified, symmetrised indices are left implicit, while symmetrisation is understood with no weight factors. Thus, for instance, the symmetrised product $A\,B$ of two vectors A_μ and B_ν here stands for $A_\mu\,B_\nu + A_\nu\,B_\mu$, without additional factors of $1/2$. Traces can be denoted by "primes", by numbers in square brackets or even by means of the operator T: $\varphi' \equiv T\,\varphi$ is thus the trace of φ, φ'' is its double trace and $\varphi^{[n]} \equiv T^n\,\varphi$ represents its $n-$th trace. Multiple gradients are denoted by symbols like $\partial^{\,k}$, while for divergences we use the symbol "$\partial\cdot$". The relevant combinatorics is summarised in the following rules [29]:

$$\begin{aligned}
(\partial^{\,p}\,\varphi)^{\,\prime} \;&=\; \Box\,\partial^{\,p-2}\,\varphi \;+\; 2\,\partial^{\,p-1}\,\partial\cdot\varphi \;+\; \partial^{\,p}\,\varphi^{\,\prime}\,, \\
\partial^{\,p}\,\partial^{\,q} \;&=\; \binom{p+q}{p}\partial^{\,p+q}\,, \\
\partial\cdot(\partial^{\,p}\;\varphi) \;&=\; \Box\;\partial^{\,p-1}\;\varphi \;+\; \partial^{\,p}\;\partial\cdot\varphi\,, \\
\partial\cdot(\eta^{\,k}\,\varphi) \;&=\; \partial\,\eta^{\,k-1}\,\varphi + \eta^{\,k}\,\partial\cdot\varphi, \\
\left(\eta^{k}\,\varphi\right)^{\,\prime} \;&=\; [\,D \,+\, 2\,(s+k-1)\,]\;\eta^{\,k-1}\,\varphi \;+\; \eta^{\,k}\,\varphi^{\,\prime}\,, \\
(\varphi\,\psi)^{\,\prime} \;&=\; \varphi^{\,\prime}\,\psi \,+\, \varphi\,\psi^{\,\prime} \,+\, 2\,\varphi\cdot\psi\,, \\
\eta\,\eta^{\,n-1} \;&=\; n\,\eta^{\,n}\,.
\end{aligned} \tag{A.1}$$

Switching to (A)dS backgrounds requires the substitutions

$$\partial \,\to\, \nabla\,, \qquad\qquad \eta \,\to\, g\,, \tag{A.2}$$

where g denotes the (A)dS metric, while also taking into account the following commutators,

$$[\,\nabla\cdot\,,\,\nabla\,]\,\varphi \,=\, \Box\,\varphi \,-\, \frac{1}{L^2}\,\{\,s(D+s-2)\,\varphi \,-\, 2\,g\,\varphi'\,\}\;, \tag{A.3}$$

$$[\,\Box\,,\,\nabla\,]\,\varphi \,=\, -\,\frac{1}{L^2}\,\{\,(D+2s-1)\,\nabla\varphi \,-\, 4\,g\,\nabla\cdot\varphi\,\}\;, \tag{A.4}$$

$$[\,\nabla\cdot\,,\,\Box\,]\,\varphi \,=\, -\,\frac{1}{L^2}\,\{\,(D+2s-3)\,\nabla\cdot\varphi \,-\, 2\,\nabla\varphi'\,\}\;. \tag{A.5}$$

In several manipulations it is convenient to make use of the Lichnerowicz operator [59]

$$\Box_L\,\varphi \,\equiv\, \Box\,\varphi \,+\, \frac{1}{L^2}\,\{\,s\,(D+s-2)\,\varphi \,-\, 2\,g\,\varphi^{\,\prime}\,\}\,, \tag{A.6}$$

defined so as to satisfy

$$[\,\Box_L,\,\nabla\,]\,\varphi \,=\, 0\,. \tag{A.7}$$

A.2 Mixed-symmetry tensors

Unless otherwise specified we work with tensors $\varphi_{\mu_1\cdots\mu_{s_1},\nu_1\cdots\nu_{s_2},\cdots}$ often simply denoted by φ possessing several "families" of symmetric indices, with no additional symmetry properties relating different sets. In this sense they define reducible $GL(D)$ tensors, here often also referred to as "multi-symmetric" tensors. In order to keep our formulas readable usually we do not display space-time indices, while we introduce family indices denoted by small-case Latin letters. We are thus able to identify tensors carrying a different number of indices in some sets as compared to the basic field φ, while also keeping track of index-reshuffling among different families. Thus, for instance, a gradient carrying a space-time index to be symmetrised with indices belonging to the $i-$th group is denoted by

$$\partial^{\,i}\,\varphi\;\equiv\;\partial_{(\,\mu^i{}_1|}\,\varphi_{\cdots,\,|\,\mu^i{}_2\cdots\mu^i{}_{s_i+1}),\,\cdots}\,, \tag{A.8}$$

with parentheses to signify symmetrization with no additional overall factors, while for a divergence contracting an index in the $i-$th group we use the notation

$$\partial_{\,i}\,\varphi\;\equiv\;\partial^{\,\lambda}\,\varphi_{\cdots,\,\lambda\,\mu^i{}_1\cdots\mu^i{}_{s_i-1}\,,\,\cdots}\,. \tag{A.9}$$

Thus, as a basic rule, the position of the family indices carries information on their role, so that *lower* family indices are associated to operators removing Lorentz indices, while *upper* family indices are associated to operators adding Lorentz indices, to be symmetrized with their peers belonging to the group identified by the family label, as shown in (A.8). In a similar spirit, the gauge parameters are denoted by $\Lambda_{\,i}$ to indicate that they carry one index less than the gauge field φ in the $i-$th family. The Einstein convention for summing over pairs of them is used throughout, although one should be careful not to confuse saturation in family indices with contraction between space-time indices. A notable example is the gauge transformation of φ (2.27),

$$\delta\,\varphi\;=\;\partial^{\,i}\,\Lambda_{\,i}\,, \tag{A.10}$$

given by a sum of symmetrised gradients, each for any of the families of φ. Another important class of operators is defined by the following equations:

$$S^{i}{}_{i}\,\varphi\;\equiv\;s_i\;\varphi_{\cdots,\,\mu^i_1\cdots\mu^i_{s_i},\,\cdots}\,, \tag{A.11a}$$

$$S^{i}{}_{j}\,\varphi\;\equiv\;\varphi_{\cdots,\,(\mu^i_1\cdots\mu^i_{s_i}|,\,\cdots,\,|\mu^i_{s_i+1})\,\mu^j_1\cdots\mu^j_{s_j-1},\,\cdots}\quad\text{for}\quad i\neq j\,, \tag{A.11b}$$

whose effect for $i\neq j$ is thus to displace indices from one family to another, while also implementing the corresponding symmetrization. For more general maximally symmetric backgrounds the flat metric η^{ij} gets replaced by the (A)dS metric g^{ij}, while $\delta_k{}^i$ simply denotes a Kronecker δ-function in family space. In the following list we collect some useful

(A)dS commutators, whose flat limit clearly obtains for $L^2 \to \infty$.

$$[\, S^i{}_j\,,\, \nabla^k\,] \;=\; \nabla^i\, \delta_j{}^k\,, \tag{A.12}$$

$$[\, \nabla_k\,,\, S^i{}_j\,] \;=\; \delta_k{}^i\, \nabla_j\,, \tag{A.13}$$

$$[\, S^i{}_j\,,\, S^k{}_l\,] \;=\; \delta_j{}^k S^i{}_l \,-\, \delta_l{}^i S^k{}_j\,, \tag{A.14}$$

$$[\, T_{ij}\,,\, \nabla^k\,] \;=\; \nabla_{(i}\, \delta_{j)}{}^k\,, \tag{A.15}$$

$$[\, T_{ij}\,,\, g^{kl}\,] \;=\; \frac{D}{2}\, \delta_i{}^{(k} \delta_j{}^{l)} \,+\, \frac{1}{2}\left(\delta_i{}^{(k} S^{l)}{}_j \,+\, \delta_j{}^{(k} S^{l)}{}_i\right)\,, \tag{A.16}$$

$$[\, \nabla_k\,,\, g^{ij}\,] \;=\; \frac{1}{2}\, \delta_k{}^{(i}\, \nabla^{j)}\,, \tag{A.17}$$

$$[\, \nabla^i\,,\, \nabla^j\,] \;=\; -\,\frac{2}{L^2}\, g^{k[i}\, S^{j]}{}_k\,, \tag{A.18}$$

$$[\, \nabla_i\,,\, \nabla^j\,] \;=\; \Box\, \delta_i{}^j - \frac{1}{L^2}\, \left\{(D-N-1)\, S^j{}_i \,+\, S^j{}_k\, S^k{}_i\right\} + \frac{2}{L^2}\, g^{jk}\, T_{ik}\,, \tag{A.19}$$

$$[\, \Box\,,\, \nabla^i\,] \;=\; -\,\frac{1}{L^2}\left\{\,(D-1)\, \nabla^i \,+\, 2\, \nabla^j S^i{}_j\,\right\} \,+\, \frac{4}{L^2}\, g^{ij}\, \nabla_j\,, \tag{A.20}$$

$$[\, \nabla_i\,,\, \Box\,] \;=\; -\,\frac{1}{L^2}\left\{\,(D-1)\, \nabla_i \,+\, 2\, S^j{}_i \nabla_j\,\right\} \,+\, \frac{2}{L^2}\, \nabla^j\, T_{ij}\,. \tag{A.21}$$

B Variation of the Maxwell-like tensor in (A)dS

In section 2.2.2 we argued that for *irreducible* fields on (A)dS the gauge variation (2.60) of the Maxwell-like tensor (2.58) should take the form

$$M\, \delta\varphi \;=\; \sum_{n\,=\,1}^{N} \sum_{i\,=\,1}^{n} k_n\, \nabla^i \Lambda_i^{(n)} \,+\, \text{divergences and traces.} \tag{B.1}$$

We are now going to prove that this is the correct structure of $M\delta\varphi$ and to compute the coefficients k_n in order to prove eq. (2.71). At the end of this appendix we also prove that the constraints (2.61) imply the vanishing of all divergences of the single surviving irreducible gauge parameter.

The key of the proof is the possibility to treat independently contributions proportional to different irreducible components of the gauge parameters (labelled by (n) in (B.1)). A crucial ingredient is thus the decomposition of the reducible gauge parameters presented in eq. (2.67),

$$\Lambda_k \;=\; \sum_{n\,=\,k}^{N} \left(1 - \delta_{s_n,\, s_{n+1}}\right) Y_{\{s_1,\ldots,\, s_n - 1,\ldots,\, s_N\}}\, \Lambda_k \;\equiv\; \sum_{n\,=\,k}^{N} \left(1 - \delta_{s_n,\, s_{n+1}}\right) \Lambda_k^{(n)}\,, \tag{B.2}$$

that determines the extrema of the sum over i in (B.1). We therefore begin by showing how to derive the decomposition (B.2) from the conditions (2.66), that we recall here for the reader convenience:

$$S^i{}_j\,\Lambda_k\,+\,\delta^i{}_k\,\Lambda_j\,=\,0\,,\qquad \text{for } i<j\,. \tag{B.3}$$

In order to illustrate the meaning of eqs. (B.3) it might be useful to first focus on the case of two families, where they take the explicit form

$$\begin{aligned} &S^1{}_2\,\Lambda_1\,+\,\Lambda_2\,=\,0\,,\\ &S^1{}_2\,\Lambda_2\,=\,0\,. \end{aligned} \tag{B.4}$$

The second of (B.4) is the condition of irreducibility for Λ_2, allowing to identify the latter with its homologous diagram:

$$\Lambda_2\,=\,Y_{\{s_1,\,s_2-1\}}\,\Lambda_2\,\equiv\,\Lambda_2^{(2)}\,, \tag{B.5}$$

while from the first one we can now induce that, among all possible projections contained in Λ_1, only two of them survive, namely $\Lambda_1^{(1)}\sim\{s_1-1,\,s_2\}$, in the kernel of $S^1{}_2$, and $\Lambda_1^{(2)}\sim\{s_1,\,s_2-1\}$, related to $\Lambda_2^{(2)}$ by

$$S^1{}_2\,\Lambda_1^{(2)}\,+\,\Lambda_2^{(2)}\,=\,0\,. \tag{B.6}$$

In the special case $s_1\,=\,s_2$ there is no $\Lambda_1^{(1)}$ projection, since the corresponding diagram does not exist, and the only independent parameter lives in the $\{s_1,\,s_2-1\}$ representation, with the corresponding components of Λ_1 and Λ_2 related as in (B.6).

In the general case it is also convenient to analyse eqs. (B.3) starting from the highest value of the family label carried by the parameters: for $k=N$ these relations imply that Λ_N is irreducible since it is annihilated by all $S^i{}_j$ with $i<j$. As a result, it coincides with $\Lambda_N^{(N)}$ in agreement with (B.2). On the other hand, if one decomposes the multi-symmetric parameter Λ_{N-1} in all its irreducible components the (B.3) imply[26]

$$S^i{}_j\,\Lambda_{N-1}^{(n)}\,=\,0\,,\qquad \text{for } n<N \text{ and } i<j\,, \tag{B.7}$$

while for $n=N$ the parameter is annihilated only by the $S^i{}_j$ with $i<N-1$ and

$$S^{N-1}{}_N\,\Lambda_{N-1}^{(N)}\,=\,-\,\Lambda_N^{(N)}\,. \tag{B.8}$$

To obtain these relations we used once more the fact that the operators $S^i{}_j$ commute with Young projectors, as discussed in section 2.1.2. The system of equations (B.7) is solved only by a tensor whose associated diagram has the same manifest symmetries, and we can thus conclude that Λ_{N-1} admits two irreducible components: the $\Lambda_{N-1}^{(N-1)}$ and the $\Lambda_{N-1}^{(N)}$

[26]The multi-symmetric tensor Λ_{N-1} carries additional components with respect to those that we labelled by the index (n) in (B.2). However, the argument showing that those with $n<N-1$ are not compatible with (B.3) applies also to those that we did not recall explicitly in eq. (B.7) to simplify the presentation.

related to $\Lambda_N^{(N)}$ via (B.8). It should now be clear that one can show by induction that a generic Λ_k satisfies

$$S^i{}_j\,\Lambda_k^{(n)}\;=\;0\,,\qquad \text{for } n\leq k \text{ and } i<j\,,\tag{B.9}$$

while the components with $n>k$ are related to the Λ_i with $i>k$ via eqs. (2.68), that generalise (B.8).

We can now exploit (B.1) in (2.60), focussing on the variation induced by a single irreducible component so as to obtain

$$\begin{aligned} M\delta_{(n)}\varphi \;=&\;-\,\frac{1}{L^2}\sum_{i\,=\,1}^{n}\nabla^i\Big\{(D+\sum_{l\,=\,1}^{N}s_l-2)\,\Lambda_i^{(n)}-\sum_{j\,=\,1}^{n}\Big((D-3)\,S^j{}_i+\sum_{k\,=\,1}^{n}S^k{}_iS^j{}_k\Big)\Lambda_j^{(n)}\Big\}\\ &+\text{divergences and traces,}\end{aligned}\tag{B.10}$$

where we also used (A.14) to change the order of $S^i{}_j$ operators and we fixed the estrema of the sums according to (B.2). In order to proceed it is convenient to distinguish when the contracted indices are smaller, equal or greater than i. We shall thus treat separately

$$\alpha_i^{(n)}\;\equiv\;(D+\sum_{l\,=\,1}^{N}s_l-2)\,\Lambda_i^{(n)}-\sum_{j\,=\,1}^{i}\Big((D-3)\,S^j{}_i+\sum_{k\,=\,1}^{i}S^k{}_i\,S^j{}_k\Big)\Lambda_j^{(n)}\,,\tag{B.11}$$

that can be reduced to the form (B.1) simply by exploiting (2.68), and

$$\beta_i^{(n)}\;\equiv\;(D-3)\sum_{j\,=\,i+1}^{n}S^j{}_i\,\Lambda_j^{(n)}\;+\sum_{j\,=\,1}^{i}\sum_{k\,=\,i+1}^{n}S^k{}_i\,S^j{}_k\,\Lambda_j^{(n)}\,,\tag{B.12a}$$

$$\gamma_i^{(n)}\;\equiv\;\sum_{j\,=\,i+1}^{n}\sum_{k\,=\,1}^{n}S^k{}_i\,S^j{}_k\,\Lambda_j^{(n)}\,,\tag{B.12b}$$

that require a more sophisticated discussion.

With the help of (A.11a) and (2.68), eq. (B.11) can be cast in the form

$$\begin{aligned}\alpha_i^{(n)}&=\{\,D+s_{tot}-(s_i-i)(D+s_i-4)-2\,\}\,\Lambda_i^{(n)}-\sum_{k\,=\,1}^{i-1}[\,S^k{}_i\,,S^i{}_k\,]\,\Lambda_i^{(n)}-\sum_{j,k\,=\,1}^{i-1}S^k{}_iS^j{}_k\,\Lambda_j^{(n)}\\ &=\{\,D+s_{tot}-(s_i-i)(D+s_i-4)+(i-1)(s_i-2)-2\,\}\,\Lambda_i^{(n)}\\ &-\sum_{j\,=\,1}^{i-1}\left(\sum_{k\,=\,1}^{j-1}[\,S^k{}_i\,,S^j{}_k\,]\,+\sum_{k\,=\,j+1}^{i-1}S^k{}_iS^j{}_k\right)\Lambda_j^{(n)}\,,\end{aligned}\tag{B.13}$$

where we introduced the shorthand $s_{tot}=\sum_{l=1}^{N}s_l$. Using again (2.68) one can show

$$\sum_{j\,=\,1}^{i-1}\left(\sum_{k\,=\,1}^{j-1}[\,S^k{}_i\,,S^j{}_k\,]\,+\sum_{k\,=\,j+1}^{i-1}S^k{}_iS^j{}_k\right)\Lambda_j^{(n)}\;=\;(i-1)(i-2)\,\Lambda_i^{(n)}\,,\tag{B.14}$$

and eventually conclude

$$\alpha_i^{(n)} \,=\, -\,\{\,(s_i-i-1)(D+s_i-i-2)-s_{tot}\,\}\,\Lambda_i^{(n)}\,. \tag{B.15}$$

If one supposes that (B.1) holds this computation suffices to fix the coefficients k_n since the term $\nabla^n\Lambda_n^{(n)}$ cannot receive further corrections. At any rate, we shall proceed by evaluating also the remaining contributions collected in (B.12).

Using (2.68), eq. (B.12a) can be cast in the form

$$\beta_i^{(n)} \,=\, (D-i-3)\sum_{j\,=\,i+1}^{n} S^j{}_i\,\Lambda_j^{(n)}\,. \tag{B.16}$$

One cannot eliminate the remaining $S^j{}_i$ with (2.68), but one can use it to build a portion of the quadratic $gl(N)$ Casimir that was introduced in (2.74):

$$\mathcal{C} \,=\, \chi \,+\, 2\sum_{i\,=\,1}^{N-1}\sum_{j\,=\,i+1}^{N} S^j{}_iS^i{}_j\,, \quad \text{with} \;\; \chi \,=\, \sum_{i\,=\,1}^{N} S^i{}_i\left(S^i{}_i+N-2i+1\right). \tag{B.17}$$

Using (2.68) one can indeed add a $S^i{}_j$ operator in (B.16) which becomes

$$\beta_i^{(n)} \,=\, -\,(D-i-3)\sum_{j\,=\,i+1}^{n} S^j{}_i\,S^i{}_j\,\Lambda_i^{(n)} \,=\, -\,\frac{1}{2}\,(D-i-3)\,(\,\mathcal{C}-\chi\,)\,\Lambda_i^{(n)}\,. \tag{B.18}$$

One can now observe that $\Lambda_n^{(n)}$, like φ, is a highest-weight state in a representation of the $gl(N)$ algebra generated by all $S^i{}_j$ (see (A.14)). This follows from its irreducibility that translates in

$$S^i{}_j\,\Lambda_n^{(n)} \,=\, 0\,, \quad \text{for} \;\; i<j\,, \tag{B.19}$$

while (2.68) implies that all $\Lambda_i^{(n)}$ belong to the same representation. As a result $\mathcal{C}$ takes the same value on all $\Lambda_i^{(n)}$, and one can conveniently compute it on $\Lambda_n^{(n)}$. On the other hand, χ acts diagonally on any tensor and this leads to

$$(\,\mathcal{C}-\chi\,)\,\Lambda_i^{(n)} \,=\, 2\,(\,s_i-s_n+n-i\,)\,\Lambda_i^{(n)}\,, \tag{B.20}$$

and eventually to

$$\beta_i^{(n)} \,=\, -\,(D-i-3)(s_i-s_n+n-i)\,\Lambda_i^{(n)}\,. \tag{B.21}$$

The leftover term (B.12b) can be simplified with a similar strategy: we shall build again $(\mathcal{C}-\chi)$ with the help of (2.68). To this end, one can begin by distinguishing various contributions in the sum over k:

$$\begin{aligned}\gamma_i^{(n)} = \sum_{j\,=\,i+1}^{n}\Bigg\{ &\sum_{k\,=\,1}^{i-1}[\,S^k{}_i\,,\,S^j{}_k\,]\,\Lambda_j^{(n)} \,+\, (s_i+s_j-2)\,S^j{}_i\,\Lambda_j^{(n)} \\ &-\sum_{k\,=\,j+1}^{n} S^k{}_i\,\Lambda_k^{(n)} \,-\, \sum_{k\,=\,i+1}^{j-1} S^k{}_iS^j{}_kS^k_j\,\Lambda_k^{(n)}\Bigg\}.\end{aligned} \tag{B.22}$$

In (B.22) we already used (2.68) to manipulate the two sums in the second line. There the extrema of the sums over k depend on j, but they can be reorganized such that

$$\gamma_i^{(n)} = \sum_{j=i+1}^{n} (s_i + s_j - j)\, S^j{}_i\, \Lambda_j^{(n)} \;-\; \sum_{k=i+1}^{n} S^k{}_i \left(\sum_{j=k+1}^{n} S^j{}_k S^k{}_j\, \Lambda_k^{(n)} \right) . \tag{B.23}$$

While one cannot built $(\mathcal{C} - \chi)$ in the first sum due to the $j-$dependent coefficients, the terms between parentheses in the second one can be substituted by $(\mathcal{C} - \chi)\Lambda_i^{(n)}$. The result is

$$\begin{aligned} \gamma_i^{(n)} &= \sum_{j=i+1}^{n} \{(s_i + s_j - j) - (s_j - s_n + n - j)\}\, S^j{}_i\, \Lambda_j^{(n)} \\ &= -\,(s_i + s_n - n)(s_i - s_n + n - i)\, \Lambda_i^{(n)}. \end{aligned} \tag{B.24}$$

In conclusion, summing (B.15), (B.21) and (B.24) one obtains

$$\begin{aligned} M\, \delta_{(n)}\varphi \;&=\; -\frac{1}{L^2} \sum_{i=1}^{n} \nabla^i \left(\alpha_i^{(n)} - \beta_i^{(n)} - \gamma_i^{(n)} \right) \\ &=\; \frac{1}{L^2}\, \{\, (s_n - n - 1)(D + s_n - n - 2) - s_{tot}\, \} \sum_{i=1}^{n} \nabla^i \Lambda_i^{(n)} . \end{aligned} \tag{B.25}$$

As expected we obtained an overall coefficient that depends on the chosen irreducible component and coincides with the one appearing in (B.15) for $i = n$.

The previous discussion suffices to conclude that the Lagrangian (2.80) is invariant under transformations generated by a single irreducible and fully divergenless parameter. In section 3.2.2 we also checked that this amount of gauge symmetry suffices to remove the unphysical components, at least in the two-family case. However, here we also would like to show that the vanishing of all divergences of the residual parameter is already forced by the apparently weaker condition (2.61), that in this case reads

$$\nabla_i\, \Lambda_i^{(n)} \;=\; 0\,, \tag{B.26a}$$

$$\nabla_i\, \Lambda_j^{(n)} \;+\; \nabla_j\, \Lambda_i^{(n)} \;=\; 0\,, \quad \text{for } i < j\,, \tag{B.26b}$$

while if different irreducible components were present they would mix in (B.26). Eqs. (2.68) and (A.13) then imply

$$\nabla_j\, \Lambda_i^{(n)} \;=\; -\, S^k{}_j\, \nabla_i\, \Lambda_k^{(n)}\,, \qquad \text{for fixed } \; k < j\,, \tag{B.27}$$

where we recalled that no summation over k is implied. Thus, one can choose the value $k = i$ and then exploit (B.26a) to conclude that

$$\nabla_j\, \Lambda_i^{(n)} \;=\; 0\,, \qquad \text{for fixed } \; i \leq n < j\,. \tag{B.28}$$

The remaining divergences can be shown to vanish with a recursive argument that relies again on (B.20). In fact, combining this result with (2.68) enables one to obtain[27]

$$\Lambda_i^{(n)} \,=\, -\,\frac{1}{s_i - s_n + n - i}\,\sum_{j\,=\,i+1}^{n} S^j{}_i\,\Lambda_j^{(n)} \tag{B.29}$$

which in its turn, upon substitution in eq. (B.26b) and with the help of (A.13), gives

$$(s_i - s_n + n - i - 1)\,\nabla_i\,\Lambda_n^{(n)} \,-\, \sum_{j\,=\,i+1}^{n} S^j{}_i\,\nabla_n\,\Lambda_j^{(n)}\,. \tag{B.30}$$

For $i \,=\, n-1$ eq. (B.30) implies $\nabla_{n-1}\,\Lambda_n^{(n)} \,=\, 0$ and, a posteriori, also $\nabla_n\,\Lambda_{n-1}^{(n)} \,=\, 0$. Increasing the value of i taking into account the previous outcomes eventually implies the vanishing of all divergences of all $\Lambda_i^{(n)}$.

C Light-cone gauge-fixing and gauge-for-gauge

As is well known, for gauge theories involving tensors of mixed symmetry the gauge algebra is generically reducible. However, in discussing the spectrum of transverse-invariant theories for multi-symmetric tensors in section 3.1.2 we do not really need to come to terms with this phenomenon. Indeed, we only perform explicit gauge-fixings that, as such, can only make use of components of the parameters that do have an effect on the gauge field. Consistently, we should be also able to identify (combinations of) components of the various parameters that do not affect the gauge potential and therefore do not enter the gauge-fixing procedure.

The simplest illustration of this aspect is provided by the case of reducible $(1,1)-$tensors, that we analyze in the following restoring indices for better clarity. Under the gauge transformation

$$\delta\,\varphi_{\,\mu,\nu} \,=\, \partial_{\,\mu}\,\Lambda_{\,\nu} \,+\, \partial_{\,\nu}\,\lambda_{\,\mu} \tag{C.1}$$

the field φ is left invariant if

$$\begin{aligned} \delta\,\Lambda_{\,\nu} \,&=\, \partial_{\,\nu}\,\rho\,, \\ \delta\,\lambda_{\,\mu} \,&=\, -\,\partial_{\,\mu}\,\rho\,, \end{aligned} \tag{C.2}$$

providing the simplest example of gauge-for-gauge transformation. Thus, of the $D+D$ components of the parameters available in principle we must expect to be able to make use of only $D+D-1$ of them, to account for the invariance under (C.2).

Let us consider first the Labastida case: $\Lambda_{\,\mu}$ and $\lambda_{\,\nu}$ are not constrained and we can try to reach the light-cone gauge directly, without invoking the equations of motion. Indeed,

[27] This relation also allows to rewrite the gauge variation (B.1) only in terms of the irreducible $\Lambda_n^{(n)}$, that have the right structure to be identified with the parameters of [50].

the number of components in $\varphi_{+,\nu}$ and $\varphi_{\mu,+}$ matches the number of independent gauge transformations, and even from the detailed form of their transformations,

$$\begin{aligned}
\delta\,\varphi_{+,+} &= p_+\,(\Lambda_+ + \lambda_+)\,, \\
\delta\,\varphi_{+,i} &= p_+\,\Lambda_i + p_i\,\lambda_+\,, \\
\delta\,\varphi_{+,-} &= p_+\,\Lambda_- + p_-\,\lambda_+\,, \\
\delta\,\varphi_{i,+} &= p_i\,\Lambda_+ + p_+\,\lambda_i\,, \\
\delta\,\varphi_{-,+} &= p_-\,\Lambda_+ + p_+\,\lambda_-\,,
\end{aligned} \tag{C.3}$$

we see that $\varphi_{+,\nu} = 0 = \varphi_{\mu,+}$ is a possible gauge. It would seem that one parameter is still left out, in particular since only the combination $\Lambda_+ + \lambda_+$ enters to remove $\varphi_{+,+}$. However, due to (C.2) we see that, for instance, we can freely set λ_+ to zero via $\lambda_+ \to \lambda_+ + p_+\,\rho$ (while all other components transform accordingly, to keep φ unchanged), thereby implying that once the light-cone gauge is reached there are no residual gauge transformations affecting φ, and the gauge is effectively completely fixed.

Alternatively, in the transverse-invariant situation where Λ_μ and λ_ν are related by the constraint

$$p\cdot(\Lambda + \lambda) = -\,p_+\,(\Lambda_- + \lambda_-) - p_-\,(\Lambda_+ + \lambda_+) + \sum_i p_i\,(\Lambda_i + \lambda_i) = 0 \tag{C.4}$$

the number of independent components of the parameters is further reduced to $D+D-2$. As a consequence it is not possible to reach the light-cone gauge off-shell and one needs to make use of the equations of motion, as explained in section 3.1.2, in order to eliminate one residual "+" component surviving after gauge fixing. In the general case some care is needed when counting the number of components of the parameters available for gauge-fixing, since the gauge-for-gauge transformations might be themselves affected by the constraints (2.29). For instance, for multi-symmetric (2, 1) tensors there is in principle a vector parameter of gauge-for-gauge transformations, due to the invariance of $\delta\,\varphi_{\mu\nu,\,\rho} = \partial_{(\mu}\,\Lambda_{\nu),\,\rho} + \partial_\rho\,\lambda_{\mu\nu}$ under

$$\begin{aligned}
\delta\,\Lambda_{\nu,\,\rho} &= \partial_\rho\,\alpha_\nu \\
\delta\,\lambda_{\mu\nu} &= -\,\partial_{(\mu}\,\alpha_{\nu)}\,.
\end{aligned} \tag{C.5}$$

However, because of the transversality conditions (2.29) the divergence of α_μ is itself forced to vanish, thus reducing to $D-1$ the number of independent components in the gauge-for-gauge transformation.

D Explicit forms of diagonal Lagrangians

In this section we discuss explicit solutions to (4.4), that we report here for the sake of clarity:

$$M\,O_{s-2k} = \eta^k\,M\,, \tag{D.1}$$

where the dimensionless operators O_{s-2k} appearing in the redefinition of φ (4.1)

$$\varphi \,=\, \phi_s \,+\, O_{s-2}\,\phi_{s-2} \,+\, O_{s-4}\,\phi_{s-4} \,+\, \cdots \,+\, O_{s-2k}\,\phi_{s-2k} \,+\cdots\,, \tag{D.2}$$

consist of linear combinations of monomials involving the metric tensor η, suitable powers of gradients and divergences, together with the appropriate inverse powers of the D'Alembertian operator.

We see from (D.1) that the operators O_{s-2k} satisfy a Maxwell equation sourced by $\eta^k M$; the general solution is thus expected to be of the form

$$O_{s-2k} \,=\, {O^o}_{s-2k} \,+\, {O^*}_{s-2k}\,, \tag{D.3}$$

with ${O^*}_{s-2k}$ a particular solution to (D.1), while among the solutions of the homogeneous equation there should be pure-gauge operators of the form

$$ {O^o}_{s-2k} \,=\, \partial\,\Lambda^o\,, \tag{D.4}$$

with Λ^o satisfying the transversality condition

$$\partial\cdot\Lambda^o \,=\, 0\,. \tag{D.5}$$

While these observations imply that the general solution to (D.1) is not unique, in special circumstances it might happen that it is not possible to construct an operator with the properties of Λ^o. Let us consider for instance the spin-2 case,

$$\varphi \,=\, \phi_2 \,+\, O_0\,\phi_0\,, \tag{D.6}$$

and let us construct a solution ${O_0}^*$ to (D.1) by iteration:

$$\begin{aligned}
{O_0}^{(0)} &= \eta && \longrightarrow \quad M\,{O_0}^{(0)} \,=\, [M,\,\eta] \,+\, \eta\,M \,=\, -\,2\,\partial^2 \,+\, \eta\,M\,, && \text{(D.7)}\\
{O_0}^{(1)} &= \eta \,+\, a\,\frac{\partial^2}{\Box} && \longrightarrow \quad M\,{O_0}^{(1)} \,=\, -\,(a\,+\,2)\,\partial^2 \,+\, \eta\,M\,, && \text{(D.8)}\\
{O_0}^{*} &= \eta \,-\, 2\,\frac{\partial^2}{\Box}\,. && && \text{(D.9)}
\end{aligned}$$

In this particular case the solution is thus completely fixed, since in the various steps of the construction there was never the possibility to choose among alternative options. In view of the previous observations we can interpret this result as due to the impossibility of building the gauge parameter Λ^o for this special case. Indeed Λ^o should be a rank-1 operator acting on scalars, thus implying that neither η nor $\partial\cdot$ can appear in its definition, while a pure gradient is also excluded in force of the transversality condition (D.5). Actually this is a manifestation of a general phenomenon valid for all even spins, when it comes to solving the equation (D.1) for the rank$-s$ operator O_0, since in all those cases it is impossible to build the corresponding gauge parameter. However, for the general case parameters Λ^o can be constructed, leading to solutions depending on a number of arbitrary coefficients. In the following we will exhibit a particular solution to (D.1), which is tantamount to choosing a specific gauge.

To begin with, we would like to expand the operators O_{s-2k} in their monomial components, so as to translate (D.1) into an explicit system for the coefficients of those terms. Each coefficient can be identified by means of three labels: $a_{\,i}^{(m,k)}$, where

$$\begin{aligned} & m \;\rightarrow \text{ denotes the power of } \eta\,; \\ & k \;\rightarrow \text{ is related to the rank of the operator } O_{\,s-2k} \;: rk\{O_{\,s-2k}\} = 2k\,; \\ & i \;\rightarrow \text{ denotes the number of divergences}\,, \end{aligned} \tag{D.10}$$

so that in general $O_{\,s-2k}$ can be cast in the form

$$\begin{aligned} O_{\,s-2k} \;&=\; \sum_{i=0}^{s-2k} a_{\,i}^{(0,k)}\,\frac{\partial^{\,2k+i}}{\Box^{\,k+i}}\,\partial\cdot^{\,i} \,+\, \eta\,\sum_{i=0}^{s-2k} a_{\,i}^{(1,k)}\,\frac{\partial^{\,2(k-1)+i}}{\Box^{\,k-1+i}}\,\partial\cdot^{\,i} \,+\cdots \\ &+\eta^{\,m}\,\sum_{i=0}^{s-2k} a_{\,i}^{(m,k)}\,\frac{\partial^{\,2(k-m)+i}}{\Box^{\,k-m+i}}\,\partial\cdot^{\,i} \,+\,\cdots \\ &=\sum_{i=0}^{s-2k}\,\sum_{m=0}^{k+\left[\frac{i}{2}\right]} a_{\,i}^{(m,k)}\,\eta^{m}\,\frac{\partial^{\,2(k-m)+i}}{\Box^{\,k-m+i}}\,\partial\cdot^{\,i}\,. \end{aligned} \tag{D.11}$$

Moreover, it is understood that

$$2\,(k-m)\,+\,i\,\geq 0\,, \tag{D.12}$$

otherwise the corresponding coefficients are simply not present. We fix the set of initial data

$$a_{\,0}^{(k,k)} \;=\; 1\,, \tag{D.13}$$

corresponding to a choice for the normalization of the ϕ_{s-2k}'s convenient for our manipulations. In terms of these definitions (D.1) translates into the system

$$[2(k-m)+i-1]\,\Big\{\,a_i^{(m,k)} + [2(k-m)+i]\,a_i^{(m+1,k)}\,\Big\} + [2(k-m)+i]\,a_{i-1}^{(m,k)} \;=\; 0\,, \tag{D.14}$$

that for $m \neq k$ provides a set of conditions $\forall\, i$, while it applies only for $i > 1$ for $m = k$. In particular for $i = 0$ we have the initial datum $a_{\,0}^{(k,k)} \;=\; 1$, and for $i = 1$ we just get

$$a_{\,1}^{(k,k)} \,-\, a_{\,1}^{(k,k)} \,-\, a_{\,0}^{(k,k)} \,-\, 0\cdot a_{\,1}^{(k+1,k)} \;=\; -\,1\,, \tag{D.15}$$

thus ensuring that the two terms with zero divergences and one divergence respectively correctly recombine to give M.

Since the coefficients depend on m and k only through the combination $k-m$, it is convenient to define

$$n \;=\; k-m\,, \tag{D.16}$$

and to introduce the shorthand

$$a_{i,n} \;=\; a_i^{(k-n,k)}\,, \tag{D.17}$$

so that (D.14) simplifies to

$$(2n+i-1)\,\{\,a_{i,n}+(2n+i)\,a_{i,n-1}\,\}+(2n+i)\,a_{i-1,n} \;=\; 0 \tag{D.18}$$

with $n \leq k$ (corresponding to $m \geq 0$), with the proviso that for $n<0$ one has $i \geq -2n$, while for $n>0$ one has $i \geq 0$.

Eq. (D.18) simplifies for the minimum values of i admitted for a given n. For $n \geq 0$ and $i=0$ it becomes

$$a_{0,n} + \,2n\,a_{0,n} \;=\; 0\,. \tag{D.19}$$

With the initial condition $a_{0,0}=1$ this recursion relation is solved by

$$a_{0,n} \;=\; (-1)^n\,(2n)!!\,. \tag{D.20}$$

For $n<0$ and $i=-\,2n$ eq. (D.18) implies

$$a_{-2n,n} \;=\; 0\,. \tag{D.21}$$

For generic values of i the structure of (D.18) and of the conditions (D.20) and (D.21) suggest to consider the ansatz

$$a_{i,n} \;=\; (-1)^{n+i}\,k_i\,(2n+i)\,(2(n+i-1))!!\,. \tag{D.22}$$

It manifestly satisfies the condition (D.21) due to the factor $(2n+i)$ and it reduces to (D.20) for $i=0$. Moreover, it enables one to factor out various terms so that (D.18) becomes

$$(-1)^{n+i}(2n+i-1)(2n+i)(2(n+i-2))!!\,\{\,2(n+i-1)k_i-(2n+i-2)k_i-k_{i-1}\,\} \;=\; 0 \tag{D.23}$$

and reduces to

$$i\,k_i \;-\; k_{i-1} \;=\; 0 \quad\Rightarrow\quad k_i \;=\; \frac{1}{i!}\,. \tag{D.24}$$

Notice that the structure of the double factorial was chosen in order to let k_{i-1} contribute only through a constant term. In conclusion, a particular solution of eq. (D.1) is provided by (D.11) with the coefficients

$$a_i^{(k-n,k)} \;=\; (-1)^{n+i}\,2^{n+i-1}\,(2n+i)\,\frac{(n+i-1)!}{i!}\,. \tag{D.25}$$

For the (A)dS case we expect to be able to find solutions for the operators O^L_{s-2k} as deformations of any flat solution by terms of $\mathcal{O}\,(\frac{1}{L^2})$. It is interesting that, at least for spin 2, the operator O^L_0 satisfying

$$M_L^{(2)}\,O_0^L \;=\; g\,M_L^{(0)}\,, \tag{D.26}$$

actually coincides with its flat counterpart, up to covariantization of the derivatives:

$$O_0^L \,=\, g \,-\, 2\,\frac{\nabla^2}{\Box_L}\,, \tag{D.27}$$

where in particular in the construction of the corresponding projector we make use of the Lichnerowicz operator.

However, for tensors of higher ranks the naive covariantization of the flat-space O_{s-2k} does not solve (4.18), that is the (A)dS counterpart of (D.1). The correct deformation involves infinite series of terms with growing powers of the inverse Lichnerowicz operator. This phenomenon can be conveniently illustrated in the simplest example given by the O_1 operator, that suffices to complete the decomposition of a rank-3 field. In this case the general solution of (D.1) contains a free parameter and reads

$$O_1 \,=\, \eta \,-\, 2\,\frac{\partial^2}{\Box} \,+\, a\,\eta\,\frac{\partial}{\Box}\,\partial\cdot \,+\, 3(1-a)\,\frac{\partial^3}{\Box^2}\,\partial\cdot\;. \tag{D.28}$$

It coincides with (D.25) for $a = -1$, but for any value of the parameter (4.18) can be solved by deforming (D.28) with an infinite number of terms that are proportional to negative powers of $L^2\,\Box_L$:

$$O_1^L \,=\, g - \sum_{k=0}^{\infty} \frac{1}{L^{2k}\Box_L^k}\left\{ 2\,[\,2(s-2)(D+s-4)]^k\,\frac{\nabla^2}{\Box_L} \,+\, a_k\,g\,\frac{\nabla}{\Box_L}\,\nabla\cdot \,+\, b_k\,\frac{\nabla^3}{\Box_L^2}\,\nabla\cdot \right\} \tag{D.29}$$

where the coefficients a_k and b_k satisfy

$$\begin{aligned} &3a_k \,+\, b_k \,-\, [(s-2)(D+s-4)+(s-3)(D+s-5)]\,b_{k-1} \\ &=\, -\,3\,[\,2(s-2)(D+s-4)]^k\,. \end{aligned} \tag{D.30}$$

The free parameters thus reside only in the divergence terms, as in flat space, while the infinite tower of contributions in (D.29) appears unavoidable. However, before drawing a definite conclusion, it would be adviceable to explore alternative deformations of the inverse D'Alembertian other then the inverse of the Lichnerowicz operator, here used to avoid order ambiguities.

References

[1] D. Francia, Prog. Theor. Phys. Suppl. **188** (2011) 94 [arXiv:1103.0683 [hep-th]].

[2] M. A. Vasiliev, Int. J. Mod. Phys. D **5** (1996) 763 [hep-th/9611024], In *Shifman, M.A. (ed.): The many faces of the superworld* 533-610 [hep-th/9910096]; D. Sorokin, AIP Conf. Proc. **767** (2005) 172 [hep-th/0405069]; N. Bouatta, G. Compere and A. Sagnotti, hep-th/0409068; A. Sagnotti, E. Sezgin and P. Sundell, hep-th/0501156; X. Bekaert, S. Cnockaert, C. Iazeolla and M. A. Vasiliev, hep-th/0503128; D. Francia and A. Sagnotti, J. Phys. Conf. Ser. **33** (2006) 57 [hep-th/0601199]; A. Campoleoni, arXiv:1110.5841 [hep-th]; A. Sagnotti, arXiv:1112.4285 [hep-th].

[3] M. Fierz, Helv. Phys. Acta **12** (1939) 3.

[4] C. Fronsdal, Phys. Rev. D **18** (1978) 3624.

[5] E. D. Skvortsov and M. A. Vasiliev, Phys. Lett. B **664** (2008) 301 [arXiv:hep-th/0701278].

[6] J. M. F. Labastida, Nucl. Phys. B **322** (1989) 185.

[7] A. Campoleoni, D. Francia, J. Mourad and A. Sagnotti, Nucl. Phys. B **815** (2009) 289 [arXiv:0810.4350 [hep-th]].

[8] A. Campoleoni, D. Francia, J. Mourad and A. Sagnotti, Nucl. Phys. B **828** (2010) 405 [arXiv:0904.4447 [hep-th]].

[9] A. Campoleoni, arXiv:0905.1472 [hep-th].

[10] A. Campoleoni, Riv. Nuovo Cim. **033** (2010) 123 [arXiv:0910.3155 [hep-th]].

[11] T. Curtright, Phys. Lett. B **165** (1985) 304; C. S. Aulakh, I. G. Koh and S. Ouvry, Phys. Lett. B **173** (1986) 284; W. Siegel and B. Zwiebach, Nucl. Phys. B **282** (1987) 125.

[12] E. D. Skvortsov, Nucl. Phys. B **808** (2009) 569 [arXiv:0807.0903 [hep-th]]; E. D. Skvortsov and Y. .M. Zinoviev, Nucl. Phys. B **843** (2011) 559 [arXiv:1007.4944 [hep-th]].

[13] C. Burdik, A. Pashnev and M. Tsulaia, Mod. Phys. Lett. A **16** (2001) 731 [hep-th/0101201]; K. B. Alkalaev, M. Grigoriev and I. Y. .Tipunin, Nucl. Phys. B **823** (2009) 509 [arXiv:0811.3999 [hep-th]]; D. Cherney, E. Latini and A. Waldron, Phys. Lett. B **682** (2010) 472 [arXiv:0909.4578 [hep-th]]; I. L. Buchbinder and A. Reshetnyak, Nucl. Phys. B **862** (2012) 270 [arXiv:1110.5044 [hep-th]].

[14] M. A. Vasiliev, Phys. Lett. B **243** (1990) 378.

[15] N. Boulanger, E. D. Skvortsov and Y. .M. Zinoviev, J. Phys. A A **44** (2011) 415403 [arXiv:1107.1872 [hep-th]]; N. Boulanger and E. D. Skvortsov, JHEP **1109** (2011) 063 [arXiv:1107.5028 [hep-th]].

[16] Y. .M. Zinoviev, hep-th/0211233; P. de Medeiros, Class. Quant. Grav. **21** (2004) 2571 [hep-th/0311254]. C. Burdik and A. Reshetnyak, J. Phys. Conf. Ser. **343** (2012) 012102 [arXiv:1111.5516 [hep-th]].

[17] K. B. Alkalaev, O. V. Shaynkman and M. A. Vasiliev, JHEP **0508** (2005) 069 [hep-th/0501108], hep-th/0601225.

[18] Y. .M. Zinoviev, Nucl. Phys. B **821** (2009) 21 [arXiv:0904.0549 [hep-th]].

[19] Y. .M. Zinoviev, Nucl. Phys. B **826** (2010) 490 [arXiv:0907.2140 [hep-th]].

[20] K. Alkalaev, JHEP **1103** (2011) 031 [arXiv:1011.6109 [hep-th]].

[21] E. D. Skvortsov, J. Phys. A A **42** (2009) 385401 [arXiv:0904.2919 [hep-th]], JHEP **1001** (2010) 106 [arXiv:0910.3334 [hep-th]].

[22] K. B. Alkalaev and M. Grigoriev, Nucl. Phys. B **835** (2010) 197 [arXiv:0910.2690 [hep-th]], Nucl. Phys. B **853** (2011) 663 [arXiv:1105.6111 [hep-th]]; M. Grigoriev, arXiv:1204.1793 [hep-th].

[23] R. R. Metsaev, Phys. Lett. B **354** (1995) 78.

[24] R. R. Metsaev, arXiv:hep-th/9810231.

[25] L. Brink, R. R. Metsaev and M. A. Vasiliev, Nucl. Phys. B **586** (2000) 183 [arXiv:hep-th/0005136].

[26] N. Boulanger, C. Iazeolla and P. Sundell, JHEP **0907** (2009) 013 [arXiv:0812.3615 [hep-th]], JHEP **0907** (2009) 014 [arXiv:0812.4438 [hep-th]].

[27] D. Francia and A. Sagnotti, Phys. Lett. B **624** (2005) 93 [arXiv:hep-th/0507144].

[28] D. Francia, J. Mourad and A. Sagnotti, Nucl. Phys. B **773** (2007) 203 [arXiv:hep-th/0701163], Nucl. Phys. B **804** (2008) 383 [arXiv:0803.3832 [hep-th]].

[29] D. Francia and A. Sagnotti, Phys. Lett. B **543** (2002) 303 [arXiv:hep-th/0207002].

[30] D. Francia and A. Sagnotti, Class. Quant. Grav. **20** (2003) S473 [arXiv:hep-th/0212185].

[31] D. Francia, Nucl. Phys. B **796** (2008) 77 [arXiv:0710.5378 [hep-th]], Fortsch. Phys. **56** (2008) 800 [arXiv:0804.2857 [hep-th]].

[32] B. de Wit and D. Z. Freedman, Phys. Rev. D **21** (1980) 358.

[33] A. Pashnev and M. Tsulaia, Mod. Phys. Lett. A **13** (1998) 1853 [hep-th/9803207]; I. L. Buchbinder, A. Pashnev and M. Tsulaia, Phys. Lett. B **523** (2001) 338 [hep-th/0109067]; K. Hallowell and A. Waldron, Nucl. Phys. B **724** (2005) 453 [hep-th/0505255]; G. Barnich and M. Grigoriev, JHEP **0608** (2006) 013 [hep-th/0602166]; F. Bastianelli, O. Corradini and E. Latini, JHEP **0702** (2007) 072 [hep-th/0701055]; I. L. Buchbinder, A. V. Galajinsky and V. A. Krykhtin, Nucl. Phys. B **779** (2007) 155 [hep-th/0702161]; R. Marnelius, arXiv:0906.2084 [hep-th].

[34] S. Ouvry and J. Stern, Phys. Lett. B **177** (1986) 335; A. K. H. Bengtsson, Phys. Lett. B **182** (1986) 321; M. Henneaux and C. Teitelboim, in "Quantum Mechanics of Fundamental Systems, 2", eds. C. Teitelboim and J. Zanelli (Plenum Press, New York, 1988), p. 113.

[35] A. Sagnotti and M. Tsulaia, Nucl. Phys. B **682** (2004) 83 [arXiv:hep-th/0311257].

[36] D. Francia, Phys. Lett. B **690** (2010) 90 [arXiv:1001.5003 [hep-th]].

[37] G. Bonelli, Nucl. Phys. B **669** (2003) 159 [hep-th/0305155].

[38] A. Fotopoulos, M. Tsulaia, Int. J. Mod. Phys. **A24** (2009) 1 [arXiv:0805.1346 [hep-th]].

[39] D. P. Sorokin and M. A. Vasiliev, Nucl. Phys. B **809** (2009) 110 [arXiv:0807.0206 [hep-th]].

[40] A. Einstein, Siz. Preuss. Acad. Scis. 1919, english translation in "The principle of relativity", A. Einstein et al. eds., Dover; J. J. van der Bij, H. van Dam and Y. J. Ng, Physica **116A** (1982) 307; W. Buchmuller and N. Dragon, Phys. Lett. B **207** (1988) 292;

[41] W. G. Unruh, Phys. Rev. D **40** (1989) 1048; M. Henneaux and C. Teitelboim, Phys. Lett. B **222** (1989) 195.

[42] E. Alvarez, D. Blas, J. Garriga and E. Verdaguer, Nucl. Phys. B **756** (2006) 148 [arXiv:hep-th/0606019]; E. Alvarez, A. F. Faedo and J. J. Lopez-Villarejo, JHEP **0810** (2008) 023 [arXiv:0807.1293 [hep-th]]; E. Alvarez and R. Vidal, Phys. Rev. D **81** (2010) 084057 [arXiv:1001.4458 [hep-th]].

[43] F. Bastianelli and R. Bonezzi, JHEP **0903** (2009) 063 [arXiv:0901.2311 [hep-th]]; F. Bastianelli, R. Bonezzi and C. Iazeolla, arXiv:1204.5954 [hep-th].

[44] D. Francia, J. Phys. Conf. Ser. **222** (2010) 012002 [arXiv:1001.3854 [hep-th]].

[45] P. de Medeiros and C. Hull, JHEP **0305** (2003) 019 [hep-th/0303036].

[46] X. Bekaert and N. Boulanger, Commun. Math. Phys. **245** (2004) 27 [hep-th/0208058], Commun. Math. Phys. **271** (2007) 723 [hep-th/0606198].

[47] R. Manvelyan, K. Mkrtchyan, W. Ruhl and M. Tovmasyan, Phys. Lett. B **699** (2011) 187 [arXiv:1102.0306 [hep-th]].

[48] C. Fronsdal, Phys. Rev. D **20** (1979) 848.

[49] X. Bekaert and N. Boulanger, arXiv:hep-th/0611263.

[50] J. M. F. Labastida and T. R. Morris, Phys. Lett. B **180** (1986) 101.

[51] D. M. Gitman and I. V. Tyutin, "Quantization of fields with constraints," Berlin, Germany: Springer (1990).

[52] M. Henneaux, C. Teitelboim and J. Zanelli, Nucl. Phys. B **332** (1990) 169, M. Henneaux and C. Teitelboim, "Quantization of gauge systems," Princeton, USA: Princeton Univ. Pr. (1992).

[53] A. Higuchi, J. Math. Phys. **28** (1987) 1553 [Erratum-ibid. **43** (2002) 6385].

[54] R. R. Metsaev, Class. Quant. Grav. **11** (1994) L141.

[55] A. Fotopoulos and M. Tsulaia, arXiv:1007.0747 [hep-th].

[56] A. Fotopoulos and M. Tsulaia, JHEP **0910** (2009) 050 [arXiv:0907.4061 [hep-th]].

[57] J. Fang and C. Fronsdal, Phys. Rev. D **18** (1978) 3630.

[58] R. R. Metsaev, Nucl. Phys. B **759** (2006) 147 [hep-th/0512342]; I. L. Buchbinder, A. Fotopoulos, A. C. Petkou and M. Tsulaia, Phys. Rev. D **74** (2006) 105018 [hep-th/0609082]; A. Fotopoulos and M. Tsulaia, Phys. Rev. D **76** (2007) 025014 [arXiv:0705.2939 [hep-th]]; N. Boulanger, S. Leclercq and P. Sundell, JHEP **0808** (2008) 056 [arXiv:0805.2764 [hep-th]]; X. Bekaert, E. Joung and J. Mourad, JHEP **0905** (2009) 126 [arXiv:0903.3338 [hep-th]]; R. Manvelyan, K. Mkrtchyan and W. Ruhl, Nucl. Phys. B **836** (2010) 204 [arXiv:1003.2877 [hep-th]]; A. Sagnotti and M. Taronna, Nucl. Phys. B **842** (2011) 299 [arXiv:1006.5242 [hep-th]]; R. Manvelyan, K. Mkrtchyan and W. Ruehl, Phys. Lett. B **696** (2011) 410 [arXiv:1009.1054 [hep-th]]; D. Polyakov, Phys. Rev. D **84** (2011) 126004 [arXiv:1106.1558 [hep-th]]; M. Taronna, JHEP **1204** (2012) 029 [arXiv:1107.5843 [hep-th]]; M. A. Vasiliev, Nucl. Phys. B **862** (2012) 341 [arXiv:1108.5921 [hep-th]]; E. Joung and M. Taronna, Nucl. Phys. B **861** (2012) 145 [arXiv:1110.5918 [hep-th]]; E. Joung, L. Lopez and M. Taronna, arXiv:1203.6578 [hep-th]; P. Dempster and M. Tsulaia, arXiv:1203.5597 [hep-th]; R. R. Metsaev, arXiv:1205.3131 [hep-th].

[59] A. Lichnerowicz, Publ. Math. IHES **10** (1961) 5.