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Mathieu Moonshine and Orbifold K3s*

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Abstract: The current status of 'Mathieu Moonshine', the idea that the Mathieu group $\mathbb{M}_{24}$ organises the elliptic genus of K3, is reviewed. While there is a consistent decomposition of all Fourier coefficients of the elliptic genus in terms of Mathieu $\mathbb{M}_{24}$ representations, a conceptual understanding of this phenomenon in terms of K3 sigma-models is still missing. In particular, it follows from the recent classification of the automorphism groups of arbitrary K3 sigma-models that (i) there is no single K3 sigma-model that has $\mathbb{M}_{24}$ as an automorphism group; and (ii) there exist 'exceptional' K3 sigma-models whose automorphism group is not even a subgroup of $\mathbb{M}_{24}$. Here we show that all cyclic torus orbifolds are exceptional in this sense, and that almost all of the exceptional cases are realised as cyclic torus orbifolds. We also provide an explicit construction of a $\mathbb{Z}_5$ torus orbifold that realises one exceptional class of K3 sigma-models.

1 Introduction

In 2010, Eguchi, Ooguri and Tachikawa observed that the elliptic genus of K3 shows signs of an underlying Mathieu $\mathbb{M}_{24}$ group action [1]. In particular, they noted (see section 2 below for more details) that the Fourier coefficients of the elliptic genus can be written as sums of dimensions of irreducible $\mathbb{M}_{24}$ representations.† This intriguing observation is very reminiscent of the famous realisation of McKay and Thompson who noted that the Fourier expansion coefficients of the J-function can be written in terms of dimensions of representations of the Monster group [2, 3]. This led to a development that is now usually referred to as 'Monstrous Moonshine', see [4] for a nice review. One important

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†Actually, they did not just look at the Fourier coefficients themselves, but at the decomposition of the elliptic genus with respect to the elliptic genera of irreducible $\mathcal{N} = 4$ superconformal representations. They then noted that these expansion coefficients (and hence in particular the usual Fourier coefficients) are sums of dimensions of irreducible $\mathbb{M}_{24}$ representations.

upshot of that analysis was that the J-function can be thought of as the partition function of a self-dual conformal field theory, the 'Monster conformal field theory' [5, 6], whose automorphism group is precisely the Monster group. The existence of this conformal field theory explains many aspects of Monstrous Moonshine although not all — in particular, the genus zero property is rather mysterious from this point of view.

In the Mathieu case, the situation is somewhat different compared to the early days of Monstrous Moonshine. It is by construction clear that the underlying conformal field theory *is* a K3 sigma-model (describing string propagation on the target space K3). However, this does not characterise the corresponding conformal field theory uniquely as there are many inequivalent such sigma-models — in fact, there is an 80-dimensional moduli space of such theories, all of which lead to the same elliptic genus. The natural analogue of the 'Monster conformal field theory' would therefore be a special K3 sigma-model whose automorphism group coincides with $\mathbb{M}_{24}$. Unfortunately, as we shall review here (see section 3), such a sigma-model does not exist: we have classified the automorphism groups of all K3 sigma-models, and none of them contains $\mathbb{M}_{24}$ [7]. In fact, not even all automorphism groups are contained in $\mathbb{M}_{24}$: the exceptional cases are the possibilities (ii), (iii) and (iv) of the theorem in section 3 (see [7]), as well as case (i) for nontrivial G'. Case (iii) was already shown in [7] to be realised by a specific Gepner model that is believed to be equivalent to a torus orbifold by $\mathbb{Z}_3$. Here we show that also cases (ii) and (iv) are realised by actual K3s — the argument in [7] for this relied on some assumption about the regularity of K3 sigma-models — and in both cases the relevant K3s are again torus orbifolds. More specifically, case (ii) is realised by an asymmetric $\mathbb{Z}_5$ orbifold of $\mathbb{T}^4$ (see section 5),[‡] while for case (iii) the relevant orbifold is by $\mathbb{Z}_3$ (see section 6).

Cyclic torus orbifolds are rather special K3s since they always possess a quantum symmetry whose orbifold leads back to $\mathbb{T}^4$. Using this property of cyclic torus orbifolds, we show (see section 4) that the group of automorphisms of K3s that are cyclic torus orbifolds is always exceptional; in particular, the quantum symmetry itself is never an element of $\mathbb{M}_{24}$. Although some 'exceptional' automorphism groups (contained in case (i) of the classification theorem) can also arise in K3 models that are not cyclic torus orbifolds, our observation may go a certain way towards explaining why only $\mathbb{M}_{24}$ seems to appear in the elliptic genus of K3.

We should mention that Mathieu Moonshine can also be formulated in terms of a mock modular form that can be naturally associated to the elliptic genus of K3 [1, 9, 10, 11]; this point of view has recently led to an interesting class of generalisations [12]. There are also indications that, just as for Monstrous Moonshine, Mathieu Moonshine can possibly be understood in terms of an underlying Borcherds-Kac-Moody algebra [13, 14, 15, 16].

2 Mathieu Moonshine

Let us first review the basic idea of 'Mathieu Moonshine'. We consider a conformal field theory sigma-model with target space K3. This theory has $\mathcal{N} = (4, 4)$ superconformal symmetry on the world-sheet. As a consequence, the space of states can be decomposed into representations of the $\mathcal{N} = 4$ superconformal algebra, both for the left- and the

[‡]Since the orbifold action is asymmetric, this evades various no-go-theorems (see e.g. [8]) that state that the possible orbifold groups are either $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_4$, or $\mathbb{Z}_6$.

right-movers. (The left- and right-moving actions commute, and thus we can find a simultaneous decomposition.) The full space of states takes then the form

$$\mathcal{H} = \bigoplus_{i,j} N_{ij}\, \mathcal{H}_i \otimes \bar{\mathcal{H}}_j \ , \tag{2.1}$$

where i and j label the different $\mathcal{N} = 4$ superconformal representations, and $N_{ij} \in \mathbb{N}_0$ denote the multiplicities with which these representations appear. The $\mathcal{N} = 4$ algebra contains, apart from the Virasoro algebra L_n at $c = 6$, four supercharge generators, as well as an affine $\hat{\mathfrak{su}}(2)_1$ subalgebra at level one; we denote the Cartan generator of the zero mode subalgebra $\mathfrak{su}(2)$ by J_0.

The full partition function of the conformal field theory is quite complicated, and is only explicitly known at special points in the moduli space. However, there exists some sort of partial index that is much better behaved. This is the so-called *elliptic genus* that is defined by

$$\phi_{\rm K3}(\tau, z) = \mathrm{Tr}_{\rm RR}\Big(q^{L_0 - \frac{c}{24}}\, y^{J_0}\, (-1)^F\, \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}}\, (-1)^{\bar{F}}\Big) \equiv \phi_{0,1}(\tau, z) \ . \tag{2.2}$$

Here the trace is only taken over the Ramond-Ramond part of the spectrum (2.1), and the right-moving $\mathcal{N} = 4$ modes are denoted by a bar. Furthermore, $q = \exp(2\pi i\tau)$ and $y = \exp(2\pi i z)$, F and $\bar{F}$ are the left- and right-moving fermion number operators, and the two central charges equal $c = \bar{c} = 6$. Note that the elliptic genus does not actually depend on $\bar{\tau}$, although $\bar{q} = \exp(-2\pi i\bar{\tau})$ does; the reason for this is that, with respect to the right-moving algebra, the elliptic genus is like a Witten index, and only the right-moving ground states contribute. To see this one notices that states that are not annihilated by a supercharge zero mode appear always as a boson-fermion pair; the contribution of such a pair to the elliptic genus however vanishes because the two states contribute with the opposite sign (as a consequence of the $(-1)^{\bar{F}}$ factor). Thus only the right-moving ground states, i.e. the states that are annihilated by all right-moving supercharge zero modes, contribute to the elliptic genus, and the commutation relations of the $\mathcal{N} = 4$ algebra then imply that they satisfy $(\bar{L}_0 - \frac{\bar{c}}{24})\phi_{\rm ground} = 0$; thus it follows that the elliptic genus is independent of $\bar{\tau}$. Note that this argument does not apply to the left-moving contributions because of the y^{J_0} factor. (The supercharges are 'charged' with respect to the J_0 Cartan generator, and hence the two terms of a boson-fermion pair come with different powers of y. However, if we also set $y = 1$, the elliptic genus does indeed become a constant, independent of τ and $\bar{\tau}$.)

It follows from general string considerations that the elliptic genus defines a *weak Jacobi form of weight zero and index one* [17]. Recall that a weak Jacobi form of weight w and index m is a function [18]

$$\phi_{w,m} : \mathbb{H}_+ \times \mathbb{C} \to \mathbb{C} \ , \qquad (\tau, z) \mapsto \phi_{w,m}(\tau, z) \tag{2.3}$$

that satisfies

$$\phi_{w,m}\Big(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\Big) = (c\tau + d)^w\, e^{2\pi i m \frac{cz^2}{c\tau+d}}\, \phi_{w,m}(\tau, z) \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}) \ , \tag{2.4}$$

$$\phi(\tau, z + \ell\tau + \ell') = e^{-2\pi i m(\ell^2\tau + 2\ell z)}\phi(\tau, z) \qquad\qquad \ell, \ell' \in \mathbb{Z} \ , \tag{2.5}$$

and has a Fourier expansion

$$\phi(\tau,z) = \sum_{n\geq 0,\, \ell\in\mathbb{Z}} c(n,\ell) q^n y^\ell \tag{2.6}$$

with $c(n,\ell) = (-1)^w c(n,-\ell)$. Weak Jacobi forms have been classified, and there is only one weak Jacobi form with $w=0$ and $m=1$. Up to normalisation $\phi_{\rm K3}$ must therefore agree with this unique weak Jacobi form $\phi_{0,1}$, which can explicitly be written in terms of Jacobi theta functions as

$$\phi_{0,1}(\tau,z) = 8 \sum_{i=2,3,4} \frac{\vartheta_i(\tau,z)^2}{\vartheta_i(\tau,0)^2} \ . \tag{2.7}$$

Note that the Fourier coefficients of $\phi_{\rm K3}$ are integers; as a consequence they cannot change continuously as one moves around in the moduli space of K3 sigma-models, and thus $\phi_{\rm K3}$ must be actually independent of the specific K3 sigma-model that is being considered, i.e. independent of the point in the moduli space. Here we have used that the moduli space is connected. More concretely, it can be described as the double quotient

$$\mathcal{M}_{\rm K3} = {\rm O}(\Gamma^{4,20}) \setminus {\rm O}(4,20) \ / \ {\rm O}(4)\times {\rm O}(20) \ . \tag{2.8}$$

We can think of the Grassmannian on the right

$${\rm O}(4,20) \ / \ {\rm O}(4)\times {\rm O}(20) \tag{2.9}$$

as describing the choice of a positive-definite 4-dimensional subspace $\Pi\subset\mathbb{R}^{4,20}$, while the group on the left, ${\rm O}(\Gamma^{4,20})$, leads to discrete identifications among them. Here ${\rm O}(\Gamma^{4,20})$ is the group of isometries of a given fixed unimodular lattice $\Gamma^{4,20}\subset\mathbb{R}^{4,20}$. (In physics terms, the lattice $\Gamma^{4,20}$ can be thought of as the D-brane charge lattice of string theory on K3.)

Let us denote by $\mathcal{H}^{(0)}\subset\mathcal{H}_{\rm RR}$ the subspace of (2.1) that consists of those RR states for which the right-moving states are ground states. (Thus $\mathcal{H}^{(0)}$ consists of the states that contribute to the elliptic genus.) $\mathcal{H}^{(0)}$ carries an action of the left-moving $\mathcal{N}=4$ superconformal algebra, and at any point in moduli space, its decomposition is of the form

$$\mathcal{H}^{(0)} = 20\cdot\mathcal{H}_{h=\frac{1}{4},j=0} \ \oplus\ 2\cdot\mathcal{H}_{h=\frac{1}{4},j=\frac{1}{2}} \ \oplus\ \bigoplus_{n=1}^{\infty} D_n\,\mathcal{H}_{h=\frac{1}{4}+n,j=\frac{1}{2}} \ , \tag{2.10}$$

where $\mathcal{H}_{h,j}$ denotes the irreducible $\mathcal{N}=4$ representation whose Virasoro primary states have conformal dimension h and transform in the spin j representation of $\mathfrak{su}(2)$. The multiplicities D_n are *not* constant over the moduli space, but the above argument shows that

$$A_n = {\rm Tr}_{D_n}(-1)^{\bar F} \tag{2.11}$$

are (where D_n is now understood not just as a multiplicity, but as a representation of the right-moving $(-1)^{\bar F}$ operator that determines the sign with which these states contribute to the elliptic genus). In this language, the elliptic genus then takes the form

$$\phi_{\rm K3}(\tau,z) = 20\cdot\chi_{h=\frac{1}{4},j=0}(\tau,z) - 2\cdot\chi_{h=\frac{1}{4},j=\frac{1}{2}}(\tau,z) + \sum_{n=1}^{\infty} A_n\cdot\chi_{h=\frac{1}{4}+n,j=\frac{1}{2}}(\tau,z) \ , \tag{2.12}$$

where $\chi_{h,j}(\tau,z)$ is the 'elliptic' genus of the corresponding $\mathcal{N}=4$ representation,

$$\chi_{h,j}(\tau,z) = \mathrm{Tr}_{\mathcal{H}_{h,j}}\left(q^{L_0-\frac{c}{24}}\, y^{J_0}(-1)^F\right)\ , \tag{2.13}$$

and we have used that $(-1)^{\bar{F}}$ takes the eigenvalues $+1$ and -1 on the 20- and 2-dimensional multiplicity spaces of the first two terms in (2.10), respectively.

The key observation of Eguchi, Ooguri & Tachikawa (EOT) [1] was that the A_n are sums of dimensions of $\mathbb{M}_{24}$ representation, in striking analogy to the original Monstrous Moonshine conjecture of [3]; the first few terms are

$$\begin{aligned} A_1 &= 90 = \mathbf{45} + \overline{\mathbf{45}} && (2.14)\\ A_2 &= 462 = \mathbf{231} + \overline{\mathbf{231}} && (2.15)\\ A_3 &= 1540 = \mathbf{770} + \overline{\mathbf{770}}\ , && (2.16)\end{aligned}$$

where $\mathbf{N}$ denotes a representation of $\mathbb{M}_{24}$ of dimension N. Actually, they guessed correctly the first six coefficients; from A_7 onwards the guesses become much more ambiguous (since the dimensions of the $\mathbb{M}_{24}$ representations are not that large) and they actually misidentified the seventh coefficient in their original analysis. (We will come back to the question of why and how one can be certain about the 'correct' decomposition shortly, see section 2.2.) The alert reader will also notice that the first two coefficients in (2.10), namely 20 and -2, are not directly $\mathbb{M}_{24}$ representations; the correct prescription is to introduce virtual representations and to write

$$20 = \mathbf{23} - 3\cdot\mathbf{1}\ , \qquad -2 = -2\cdot\mathbf{1}\ . \tag{2.17}$$

Recall that $\mathbb{M}_{24}$ is a sporadic finite simple group of order

$$|\mathbb{M}_{24}| = 2^{10}\cdot 3^3\cdot 5\cdot 7\cdot 11\cdot 23 = 244\,823\,040\ . \tag{2.18}$$

It has 26 conjugacy classes (which are denoted by 1A, 2A, 3A, ..., 23A, 23B, where the number refers to the order of the corresponding group element) — see eqs. (2.19) and (2.20) below for the full list — and therefore also 26 irreducible representations whose dimensions range from $N=1$ to $N=10395$. The Mathieu group $\mathbb{M}_{24}$ can be defined as the subgroup of the permutation group S_{24} that leaves the extended Golay code invariant; equivalently, it is the quotient of the automorphism group of the $\mathfrak{su}(2)^{24}$ Niemeier lattice, divided by the Weyl group. Thought of as a subgroup of $\mathbb{M}_{24}\subset S_{24}$, it contains the subgroup $\mathbb{M}_{23}$ that is characterised by the condition that it leaves a given (fixed) element of $\{1,\ldots,24\}$ invariant.

2.1 Classical symmetries

The appearance of a Mathieu group in the elliptic genus of K3 is not totally surprising in view of the Mukai theorem [19, 20]. It states that any finite group of symplectic automorphisms of a K3 surface can be embedded into the Mathieu group $\mathbb{M}_{23}$. The symplectic automorphisms of a K3 surface define symmetries that act on the multiplicity spaces of the $\mathcal{N}=4$ representations, and therefore explain part of the above findings. However, it is also clear from Mukai's argument that they do not even account for the full $\mathbb{M}_{23}$ group. Indeed, every symplectomorphism of K3 has at least five orbits on the set $\{1,\dots,24\}$, and thus not all elements of $\mathbb{M}_{23}$ can be realised as a symplectomorphism. More specifically, of the 26 conjugacy classes of $\mathbb{M}_{24}$, 16 have a representative in $\mathbb{M}_{23}$, namely

$$\text{repr. in } \mathbb{M}_{23}: \qquad \begin{array}{ll} \text{1A, 2A, 3A, 4B, 5A, 6A, 7A, 7B, 8A} & \text{(geometric)} \\ \text{11A, 14A, 14B, 15A, 15B, 23A, 23B} & \text{(non-geometric)}\ , \end{array} \tag{2.19}$$

where 'geometric' means that a representative can be (and in fact is) realised by a geometric symplectomorphism (i.e. that the representative has at least five orbits when acting on the set $\{1,\dots,24\}$), while 'non-geometric' means that this is not the case. The remaining conjugacy classes do *not* have a representative in $\mathbb{M}_{23}$, and are therefore not accounted for geometrically via the Mukai theorem

$$\text{no repr. in } \mathbb{M}_{23}: \qquad \text{2B, 3B, 4A, 4C, 6B, 10A, 12A, 12B, 21A, 21B}\ . \tag{2.20}$$

The classical symmetries can therefore only explain the symmetries in the first line of (2.19). Thus an additional argument is needed in order to understand the origin of the other symmetries; we shall come back to this in section 3.

2.2 Evidence for Moonshine

As was already alluded to above, in order to determine the 'correct' decomposition of the A_n multiplicity spaces in terms of $\mathbb{M}_{24}$ representations, we need to study more than just the usual elliptic genus. By analogy with Monstrous Moonshine, the natural objects to consider are the analogues of the McKay Thompson series [21]. These are obtained from the elliptic genus upon replacing

$$A_n = \dim R_n \ \rightarrow\ \mathrm{Tr}_{R_n}(g)\ , \tag{2.21}$$

where $g \in \mathbb{M}_{24}$, and R_n is the $\mathbb{M}_{24}$ representation whose dimension equals the coefficient A_n; the resulting functions are then (compare (2.12))

$$\phi_g(\tau,z) = \mathrm{Tr}_{\mathbf{23}-3\cdot\mathbf{1}}(g)\,\chi_{h=\frac{1}{4},j=0}(\tau,z) - 2\,\mathrm{Tr}_{\mathbf{1}}(g)\,\chi_{h=\frac{1}{4},j=\frac{1}{2}}(\tau,z) + \sum_{n=1}^{\infty} \mathrm{Tr}_{R_n}(g)\,\chi_{h=\frac{1}{4}+n,j=\frac{1}{2}}(\tau,z)\ . \tag{2.22}$$

The motivation for this definition comes from the observation that if the underlying vector space $\mathcal{H}^{(0)}$, see eq. (2.10), of states contributing to the elliptic genus were to carry an action of $\mathbb{M}_{24}$, $\phi_g(\tau,z)$ would equal the 'twining elliptic genus', i.e. the elliptic genus twined by the action of g

$$\phi_g(\tau,z) = \mathrm{Tr}_{\mathcal{H}^{(0)}}\Big(g\, q^{L_0-\frac{c}{24}}\, y^{J_0}\, (-1)^F\, \bar{q}^{\bar{L}_0-\frac{\bar{c}}{24}}\, (-1)^{\bar{F}}\Big)\ . \tag{2.23}$$

Obviously, a priori, it is not clear what the relevant R_n in (2.21) are. However, we have some partial information about them:

(i) For any explicit realisation of a symmetry of a K3 sigma-model, we can calculate (2.23) directly. (In particular, for some symmetries, the relevant twining genera had already been calculated in [22].)

(ii) The observation of EOT determines the first six coefficients explicitly.

(iii) The twining genera must have special modular properties.

Let us elaborate on (iii). Assuming that the functions $\phi_g(\tau, z)$ have indeed an interpretation as in (2.23), they correspond in the usual orbifold notation of string theory to the contribution

$$\phi_g(\tau, z) \longleftrightarrow e \ \underset{g}{\square} \tag{2.24}$$

where e is the identity element of the group. Under a modular transformation it is believed that these twining and twisted genera transform (up to a possible phase) as

$$h \ \underset{g}{\square} \quad \xrightarrow{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} \quad h^d g^c \ \underset{g^a h^b}{\square} \tag{2.25}$$

The twining genera (2.24) are therefore invariant (possibly up to a phase) under the modular transformations with

$$\gcd(a, o(g)) = 1 \qquad \text{and} \qquad c = 0 \mod o(g), \tag{2.26}$$

where $o(g)$ is the order of the group element g and we used that for $\gcd(a, o(g)) = 1$, the group element g^a is in the same conjugacy class as g or g^{-1}. (Because of reality, the twining genus of g and g^{-1} should be the same.) Since $ad - bc = 1$, the second condition implies the first, and we thus conclude that $\phi_g(\tau, z)$ should be (up to a possible multiplier system) a weak Jacobi form of weight zero and index one under the subgroup of $\mathrm{SL}(2, \mathbb{Z})$

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}) \ : \ c = 0 \bmod (N) \right\} , \tag{2.27}$$

where $N = o(g)$. This is a relatively strong condition, and knowing the first few terms (for a fixed multiplier system) determines the function uniquely. In order to use this constraint, however, it is important to know the multiplier system. An ansatz (that seems to work, see below) was made in [23]

$$\phi_g\Big(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\Big) = e^{\frac{2\pi i cd}{Nh}} \, e^{\frac{2\pi i \, cz^2}{c\tau + d}} \, \phi_g(\tau, z) \ , \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \ , \tag{2.28}$$

where N is again the order of g and $h | \gcd(N, 12)$. The multiplier system is trivial ($h = 1$) if and only if g contains a representative in $\mathbb{M}_{23} \subset \mathbb{M}_{24}$. For the other conjugacy classes, the values are tabulated in table 1. It was noted in [10] that h equals the length of the shortest cycle (when interpreted as a permutation in S_{24}, see table 1 of [10]).

Using this ansatz, explicit expressions for all twining genera were determined in [23]; independently, the same twining genera were also found (using guesses based on the cycle shapes of the corresponding S_{24} representations) in [24]. (Earlier partial results had been obtained in [9] and [25].)

Class	2B	3B	4A	4C	6B	10A	12A	12B	21AB
h	2	3	2	4	6	2	2	12	3

Table 1: Value of h for the conjugacy classes in (2.28).

These explicit expressions for the twining genera then allow for a very non-trivial check of the EOT proposal. As is clear from their definition in (2.22), they determine the coefficients

$$\mathrm{Tr}_{R_n}(g) \qquad \text{for all } g \in \mathbb{M}_{24} \text{ and all } n \geq 1. \tag{2.29}$$

This information is therefore sufficient to *determine* the representations R_n, i.e. to calculate their decomposition into irreducible $\mathbb{M}_{24}$ representations, for all n. We have worked out the decompositions explicitly for the first 500 coefficients, and we have found that each R_n can be written as a direct sum of $\mathbb{M}_{24}$ representations with non-negative integer multiplicities [23]. (Subsequently [24] tested this property for the first 600 coefficients, and apparently Tachikawa has also checked it for the first 1000 coefficients.) Terry Gannon has informed us that this information is sufficient to prove that the same will then happen for all n [26]. In some sense this then proves the EOT conjecture.

3 Symmetries of K3 models

While the above considerations establish in some sense the EOT conjecture, they do not offer any insight into why the elliptic genus of K3 should exhibit an $\mathbb{M}_{24}$ symmetry. This is somewhat similar to the original situation in Monstrous Moonshine, after Conway and Norton had found the various Hauptmodules by somewhat similar techniques. Obviously, in the case of Monstrous Moonshine, many of these observations were subsequently explained by the construction of the Monster CFT (that possesses the Monster group as its automorphism group) [5, 6]. So we should similarly ask for a microscopic explanation of these findings.

In some sense it is clear what the analogue of the Monster CFT in the current context should be: we know that the function in question *is* the elliptic genus of K3. However, there is one problem with this. As we mentioned before, there is not just one K3 sigma-model, but rather a whole moduli space (see eq. (2.8)) of such CFTs. So the simplest explanation of the EOT observation would be if there is (at least) one special K3 sigma-model that has $\mathbb{M}_{24}$ as its automorphism group. Actually, the relevant symmetry group should commute with the action of the $\mathcal{N} = (4,4)$ superconformal symmetry (since it should act on the multiplicity spaces in $\mathcal{H}^{(0)}$, see eq. (2.10)). Furthermore, since the two $\mathcal{N} = 4$ representations with $h = \frac{1}{4}$ and $j = \frac{1}{2}$ are singlets — recall that the coefficient -2 transforms as $-2 = -2 \cdot \mathbf{1}$, see (2.17) — the automorphism must act trivially on the 4 RR ground states that transform in the $(\mathbf{2}, \mathbf{2})$ representation of the $\mathfrak{su}(2)_L \times \mathfrak{su}(2)_R$ subalgebra of $\mathcal{N} = (4,4)$. Note that these four states generate the simultaneous half-unit spectral flows in the left- and the right-moving sector; the requirement that the symmetry leaves them invariant therefore means that spacetime supersymmetry is preserved.

Recall from (2.8) that the different K3 sigma-models are parametrised by the choice of a positive-definite 4-dimensional subspace $\Pi \subset \mathbb{R}^{4,20}$, modulo some discrete identifications. Let us denote by G_Π the group of symmetries of the sigma-model described by Π that commute with the action of $\mathcal{N} = (4,4)$ and preserve the RR ground states in the $(\mathbf{2},\mathbf{2})$ (see above). It was argued in [7] that G_Π is precisely the subgroup of $\mathrm{O}(\Gamma^{4,20})$ consisting of those elements that leave Π pointwised fixed. The possible symmetry groups G_Π can then be classified following essentially the paradigm of the Mukai-Kondo argument for the symplectomorphisms of K3 surfaces [19, 20]. The outcome of the analysis can be summarised by the following theorem [7]:

Theorem: *Let G be the group of symmetries of a non-linear sigma-model on $K3$ preserving the $\mathcal{N} = (4,4)$ superconformal algebra as well as the spectral flow operators. One of the following possibilities holds:*

(i) $G = G'.G''$, *where G' is a subgroup of $\mathbb{Z}_2^{11}$, and G'' is a subgroup of $\mathbb{M}_{24}$ with at least four orbits when acting as a permutation on* $\{1,\dots,24\}$

(ii) $G = 5^{1+2} : \mathbb{Z}_4$

(iii) $G = \mathbb{Z}_3^4 : A_6$

(iv) $G = 3^{1+4} : \mathbb{Z}_2.G''$, *where G'' is either trivial, $\mathbb{Z}_2$ or $\mathbb{Z}_2^2$.*

Here $G = N.Q$ means that N is a normal subgroup of G, and $G/N \cong Q$; when G is the semidirect product of N and Q, we denote it by $N : Q$. Furthermore, p^{1+2n} is an extra-special group of order p^{1+2n}, which is an extension of $\mathbb{Z}_p^{2n}$ by a central element of order p.

We will give a sketch of the proof below (see section 3.1), but for the moment let us comment on the implications of this result. First of all, our initial expectation from above is not realised: none of these groups $G \equiv G_\Pi$ contains $\mathbb{M}_{24}$. In particular, the twining genera for the conjugacy classes 12B, 21A, 21B, 23A, 23B of $\mathbb{M}_{24}$ cannot be realised by any symmetry of a K3 sigma-model. Thus we cannot give a direct explanation of the EOT observation along these lines.

Given that the elliptic genus is constant over the moduli space, one may then hope that we can explain the origin of $\mathbb{M}_{24}$ by 'combining' symmetries from different points in the moduli space. As we have mentioned before, this is also similar to what happens for the geometric symplectomorphisms of K3: it follows from the Mukai theorem that the Mathieu group $\mathbb{M}_{23}$ is the smallest group that contains all symplectomorphisms, but there is no K3 surface where all of $\mathbb{M}_{23}$ is realised, and indeed, certain generators of $\mathbb{M}_{23}$ can never be symmetries, see (2.19). However, also this explanation of the EOT observation is somewhat problematic: as is clear from the above theorem, not all symmetry groups of K3 sigma-models are in fact subgroups of $\mathbb{M}_{24}$. In particular, none of the cases (ii), (iii) and (iv) (as well as case (i) with G' non-trivial) have this property, as can be easily seen by comparing the prime factor decompositions of their orders to (2.18). The smallest group that contains all groups of the theorem is the Conway group Co_1, but as far as we are aware, there is no evidence of any 'Conway Moonshine' in the elliptic genus of K3.

One might speculate that, generically, the group G must be a subgroup of $\mathbb{M}_{24}$, and that the models whose symmetry group is not contained in $\mathbb{M}_{24}$ are, in some sense, special or 'exceptional' points in the moduli space. In order to make this idea precise, it is useful to analyse the exceptional models in detail. In [7], some examples have been provided of case (i) with non-trivial G' (a torus orbifold $\mathbb{T}^4/\mathbb{Z}_2$ or the Gepner model 2^4, believed to be equivalent to a $\mathbb{T}^4/\mathbb{Z}_4$ orbifold), and of case (iii) (the Gepner model 1^6, which is believed to be equivalent to a $\mathbb{T}^4/\mathbb{Z}_3$ orbifold, see also [27]). For the cases (ii) and (iv), only an existence proof was given. In section 5, we will improve the situation by constructing in detail an example of case (ii), realised as an asymmetric $\mathbb{Z}_5$-orbifold of a torus $\mathbb{T}^4$. Furthermore, in section 6 we will briefly discuss the $\mathbb{Z}_3$-orbifold of a torus and the explicit realisation of its symmetry group, corresponding to cases (ii) and (iv) for any G''.

Notice that all the examples of exceptional models known so far are provided by torus orbifolds. In fact, we will show below (see section 4) that all cyclic torus orbifolds have exceptional symmetry groups. Conversely, we will prove that the cases (ii)–(iv) of the theorem are always realised by (cyclic) torus orbifolds. On the other hand, as we shall also explain, some of the exceptional models in case (i) are not cyclic torus orbifolds.

3.1 Sketch of the proof of the Theorem

In this subsection, we will describe the main steps in the proof of the above theorem; the details can be found in [7].

It was argued in [7] that the supersymmetry preserving automorphisms of the non-linear sigma-model characterised by Π generate the group $G \equiv G_\Pi$ that consists of those elements of $O(\Gamma^{4,20})$ that leave Π pointwise fixed. Let us denote by L^G the sublattice of G-invariant vectors of $L \equiv \Gamma^{4,20}$, and define L_G to be its orthogonal complement that carries a genuine action of G. Since $L^G \otimes \mathbb{R}$ contains the subspace Π, it follows that L^G has signature $(4, d)$ for some $d \geq 0$, so that L_G is a negative-definite lattice of rank $20 - d$. In [7], it is proved that, for any consistent model, L_G can be embedded (up to a change of sign in its quadratic form) into the Leech lattice Λ, the unique 24-dimensional positive-definite even unimodular lattice containing no vectors of squared norm 2. Furthermore, the action of G on L_G can be extended to an action on the whole of Λ, such that the sublattice $\Lambda^G \subset \Lambda$ of vectors fixed by G is the orthogonal complement of L_G in Λ. This construction implies that G must be a subgroup of $\mathrm{Co}_0 \equiv \mathrm{Aut}(\Lambda)$ that fixes a sublattice Λ^G of rank $4 + d$. Conversely, it can be shown that any such subgroup of $\mathrm{Aut}(\Lambda)$ is the symmetry group of some K3 sigma-model.

This leaves us with characterising the possible subgroups of the finite group $\mathrm{Co}_0 \equiv \mathrm{Aut}(\Lambda)$ that stabilise a suitable sublattice; problems of this kind have been studied in the mathematical literature before. In particular, the stabilisers of sublattices of rank at least 4 are, generically, the subgroups of $\mathbb{Z}_2^{11} : \mathbb{M}_{24}$ described in case (i) of the theorem above. The three cases (ii), (iii), (iv) arise when the invariant sublattice Λ^G is contained in some $\mathcal{S}$-lattice $S \subset \Lambda$. An $\mathcal{S}$-lattice S is a primitive sublattice of Λ such that each vector of S is congruent modulo $2S$ to a vector of norm 0, 4 or 6. Up to isomorphisms, there are only three kind of $\mathcal{S}$-lattices of rank at least four; their properties are summarised in the following table:

Name	type	rk S	Stab(S)	Aut(S)
$(A_2 \oplus A_2)'(3)$	$2^9 3^6$	4	$\mathbb{Z}_3^4 : A_6$	$\mathbb{Z}_2 \times (S_3 \times S_3).\mathbb{Z}_2$
$A_4^*(5)$	$2^5 3^{10}$	4	$5^{1+2} : \mathbb{Z}_4$	$\mathbb{Z}_2 \times S_5$
$E_6^*(3)$	$2^{27} 3^{36}$	6	$3^{1+4} : \mathbb{Z}_2$	$\mathbb{Z}_2 \times W(E_6)$.

Here, S is characterised by the type $2^p 3^q$, which indicates that S contains p pairs of opposite vectors of norm 4 (type 2) and q pairs of opposite vectors of norm 6 (type 3).

The group $\mathrm{Stab}(S)$ is the pointwise stabiliser of S in Co_0 and $\mathrm{Aut}(S)$ is the quotient of the setwise stabiliser of S modulo its pointwise stabiliser $\mathrm{Stab}(S)$. The group $\mathrm{Aut}(S)$ always contains a central $\mathbb{Z}_2$ subgroup, generated by the transformation that inverts the sign of all vectors of the Leech lattice. The lattice of type $2^{27}3^{36}$ is isomorphic to the weight lattice (the dual of the root lattice) of E_6 with quadratic form rescaled by 3 (i.e. the roots in $E_6^*(3)$ have squared norm 6), and $\mathrm{Aut}(S)/\mathbb{Z}_2$ is isomorphic to the Weyl group $W(E_6)$ of E_6. Similarly, the lattice of type $2^5 3^{10}$ is the weight lattice of A_4 rescaled by 5, and $\mathrm{Aut}(S)/\mathbb{Z}_2$ is isomorphic to the Weyl group $W(A_4) \cong S_5$ of A_4. Finally, the type $2^9 3^6$ is the 3-rescaling of a lattice $(A_2 \oplus A_2)'$ obtained by adjoining to the root lattice $A_2 \oplus A_2$ an element $(e_1^*, e_2^*) \in A_2^* \oplus A_2^*$, with e_1^* and e_2^* of norm 2/3. The latter $\mathcal{S}$-lattice can also be described as the sublattice of vectors of $E_6^*(3)$ that are orthogonal to an $A_2(3)$ sublattice of $E_6^*(3)$. The group $\mathrm{Aut}(S)/\mathbb{Z}_2$ is the product $(S_3 \times S_3).\mathbb{Z}_2$ of the Weyl groups $W(A_2) = S_3$, and the $\mathbb{Z}_2$ symmetry that exchanges the two A_2 and maps e_1^* to e_2^*.

The cases (ii)–(iv) of the above theorem correspond to Λ^G being isomorphic to $A_4^*(5)$ (case ii), to $(A_2 \oplus A_2)'(3)$ (case iii) or to a sublattice of $E_6^*(3)$ different from $(A_2 \oplus A_2)'(3)$ (case iv). In the cases (ii) and (iii), G is isomorphic to $\mathrm{Stab}(S)$. In case (iv), $\mathrm{Stab}(S)$ is, generically, a normal subgroup of G, and $G'' \cong G/\mathrm{Stab}(S)$ is a subgroup of $\mathrm{Aut}(S) \cong \mathbb{Z}_2 \times W(E_6)$ that fixes a sublattice $\Lambda^G \subseteq E_6^*(3)$, with $\Lambda^G \neq (A_2 \oplus A_2)'(3)$, of rank at least 4. The only non-trivial subgroups of $\mathbb{Z}_2 \times W(E_6)$ with these properties are $G'' = \mathbb{Z}_2$, which corresponds to Λ^G being orthogonal to a single vector of norm 6 in $E_6^*(3)$ (a rescaled root), and $G'' = \mathbb{Z}_2^2$, which corresponds to Λ^G being orthogonal to two orthogonal vectors of norm 6.[§] If Λ^G is orthogonal to two vectors $v_1, v_2 \in E_6^*(3)$ of norm 6, with $v_1 \cdot v_2 = -3$, then $\Lambda^G \cong (A_2 \oplus A_2)'(3)$ and case (iii) applies.

4 Symmetry groups of torus orbifolds

In this section we will prove that all K3 sigma-models that are realised as (possibly left-right asymmetric) orbifolds of $\mathbb{T}^4$ by a cyclic group have an 'exceptional' group of symmetries, i.e. their symmetries are not a subgroup of $\mathbb{M}_{24}$. Furthermore, these torus orbifolds account for most of the exceptional models (in particular, for all models in the cases (ii)–(iv) of the theorem). On the other hand, as we shall also explain, there are exceptional models in case (i) that are not cyclic torus orbifolds.

Our reasoning is somewhat reminiscent of the construction of [28, 29] in the context of Monstrous Moonshine. Any $\mathbb{Z}_n$-orbifold of a conformal field theory has an automorphism g of order n, called the *quantum symmetry*, which acts trivially on the untwisted sector and by multiplication by the phase $\exp(\frac{2\pi i k}{n})$ on the k-th twisted sector. Furthermore, the orbifold of the orbifold theory by the group generated by the quantum symmetry g, is equivalent to the original conformal field theory [30]. This observation is the key for characterising K3 models that can be realised as torus orbifolds:

A K3 model $\mathcal{C}$ is a $\mathbb{Z}_n$-orbifold of a torus model if and only if it has a symmetry g of order n such that $\mathcal{C}/\langle g \rangle$ is a consistent orbifold equivalent to a torus model.

[§] The possibility $G'' = \mathbb{Z}_4$ that has been considered in [7] has to be excluded, since there are no elements of order 4 in $W(E_6)$ that preserve a four-dimensional sublattice of $E_6^*(3)$.

In order to see this, suppose that $\mathcal{C}_{K3}$ is a K3 sigma-model that can be realised as a torus orbifold $\mathcal{C}_{K3} = \tilde{\mathcal{C}}_{\mathbb{T}^4}/\langle \tilde{g} \rangle$, where $\tilde{g}$ is a symmetry of order n of the torus model $\tilde{\mathcal{C}}_{\mathbb{T}^4}$. Then $\mathcal{C}_{K3}$ possesses a 'quantum symmetry' g of order n, such that the orbifold of $\mathcal{C}_{K3}$ by g describes again the original torus model, $\tilde{\mathcal{C}}_{\mathbb{T}^4} = \mathcal{C}_{K3}/\langle g \rangle$.

Conversely, suppose $\mathcal{C}_{K3}$ has a symmetry g of order n, such that the orbifold of $\mathcal{C}_{K3}$ by g is consistent, i.e. satisfies the level matching condition — this is the case if and only if the twining genus ϕ_g has a trivial multiplier system — and leads to a torus model $\mathcal{C}_{K3}/\langle g \rangle = \tilde{\mathcal{C}}_{\mathbb{T}^4}$. Then $\mathcal{C}_{K3}$ itself is a torus orbifold since we can take the orbifold of $\tilde{\mathcal{C}}_{\mathbb{T}^4}$ by the quantum symmetry associated to g, and this will, by construction, lead back to $\mathcal{C}_{K3}$.

Thus we conclude that $\mathcal{C} \equiv \mathcal{C}_{K3}$ can be realised as a torus orbifold if and only if $\mathcal{C}$ contains a symmetry g such that (i) ϕ_g has a trivial multiplier system; and (ii) the orbifold of $\mathcal{C}$ by g leads to a torus model $\tilde{\mathcal{C}}_{\mathbb{T}^4}$. It is believed that the orbifold of $\mathcal{C}$ by any $\mathcal{N} = (4,4)$-preserving symmetry group, if consistent, will describe a sigma-model with target space either a torus $\mathbb{T}^4$ or a K3 manifold. The two cases can be distinguished by calculating the elliptic genus; in particular, if the target space is a torus, the elliptic genus vanishes. Actually, since the space of weak Jacobi forms of weight zero and index one is 1-dimensional, this condition is equivalent to the requirement that the elliptic genus $\tilde{\phi}(\tau, z)$ of $\tilde{\mathcal{C}} = \mathcal{C}/\langle g \rangle$ vanishes at $z = 0$.

Next we recall that the elliptic genus of the orbifold by a group element g of order $n = o(g)$ is given by the usual orbifold formula

$$\tilde{\phi}(\tau, z) = \frac{1}{n} \sum_{i,j=1}^{n} \phi_{g^i, g^j}(\tau, z) \ , \tag{4.1}$$

where $\phi_{g^i, g^j}(\tau, z)$ is the twining genus for g^j in the g^i-twisted sector; this can be obtained by a modular transformation from the untwisted twining genus $\phi_{g^d}(\tau, z)$ with $d = \gcd(i, j, n)$. As we have explained above, it is enough to evaluate the elliptic genus for $z = 0$. Then

$$\phi_{g^d}(\tau, z = 0) = \mathrm{Tr}_{\mathbf{24}}(g^d) \ , \tag{4.2}$$

where $\mathrm{Tr}_{\mathbf{24}}(g^d)$ is the trace of g^d over the 24-dimensional space of RR ground states, and since (4.2) is constant (and hence modular invariant) we conclude that

$$\tilde{\phi}(\tau, 0) = \frac{1}{n} \sum_{i,j=1}^{n} \mathrm{Tr}_{\mathbf{24}}(g^{\gcd(i,j,n)}) \ . \tag{4.3}$$

According to the theorem in section 3, all symmetry groups of K3 sigma-models are subgroups of Co_0 and, in fact, $\mathrm{Tr}_{\mathbf{24}}(g^d)$ coincides with the trace of $g^d \in \mathrm{Co}_0$ in the standard 24-dimensional representation of Co_0. Thus, the elliptic genus of the orbifold model $\tilde{\mathcal{C}} = \mathcal{C}/\langle g \rangle$ only depends on the conjugacy class of g in Co_0. The group Co_0 contains 167 conjugacy classes, but only 42 of them contain symmetries that are realised by some K3 sigma-model, i.e. elements that fix at least a four-dimensional subspace in the standard 24-dimensional representation of Co_0. If $\mathrm{Tr}_{\mathbf{24}}(g) \neq 0$ (this happens for 31 of the above 42 conjugacy classes), the twining genus $\phi_g(\tau, z)$ has necessarily a trivial multiplier system, and the orbifold $\mathcal{C}/\langle g \rangle$ is consistent. These classes are listed in the following table, together with the dimension of the space that is fixed by g, the trace over the 24-dimensional representation, and the elliptic genus $\tilde{\phi}(\tau, z = 0)$ of the orbifold model $\tilde{\mathcal{C}}$ (we underline the classes that restrict to $\mathbb{M}_{24}$ conjugacy classes):

Co_0-class	$\underline{\text{1A}}$	$\underline{\text{2B}}$	2C	$\underline{\text{3B}}$	3C	4B	$\underline{\text{4E}}$	4F	$\underline{\text{5B}}$	5C	6G	6H	6I	$\underline{\text{6K}}$	6L	6M	$\underline{\text{7B}}$
dim fix	24	16	8	12	6	8	10	6	8	4	6	6	6	8	4	4	6
$\mathrm{Tr}_{\mathbf{24}}(g)$	24	8	-8	6	-3	8	4	-4	4	-1	-4	4	5	2	-2	-1	3
$\tilde{\phi}(\tau, 0)$	24	24	0	24	0	24	24	0	24	0	0	24	24	24	0	0	24

Co_0-class	8D	$\underline{\text{8G}}$	8H	9C	10F	10G	10H	$\underline{\text{11A}}$	12I	12L	12N	12O	$\underline{\text{14C}}$	$\underline{\text{15D}}$
dim fix	4	6	4	4	4	4	4	4	4	4	4	4	4	4
$\mathrm{Tr}_{\mathbf{24}}(g)$	4	2	-2	3	-2	2	3	2	2	1	-2	2	1	1
$\tilde{\phi}(\tau, 0)$	24	24	0	24	0	24	24	24	24	24	0	24	24	24

Note that the elliptic genus of the orbifold theory $\tilde{\mathcal{C}}$ is always 0 or 24, corresponding to a torus or a K3 sigma-model, respectively. Out of curiosity, we have also computed the putative elliptic genus $\tilde{\phi}(\tau, 0)$ for the 11 classes of symmetries g with $\mathrm{Tr}_{24}(g) = 0$ for which we do not expect the orbifold to make sense — the corresponding twining genus ϕ_g will typically have a non-trivial multiplier system, and hence the orbifold will not satisfy level-matching. Indeed, for almost none of these cases is $\tilde{\phi}(\tau, 0)$ equal to 0 or 24, thus signaling an inconsistency of the orbifold model:

Co_0-class	$\underline{\text{2D}}$	$\underline{\text{3D}}$	4D	$\underline{\text{4G}}$	$\underline{\text{4H}}$	6O	$\underline{\text{6P}}$	8C	8I	$\underline{\text{10J}}$	$\underline{\text{12P}}$
dim fix	12	8	4	8	6	6	4	4	4	4	4
$\mathrm{Tr}_{\mathbf{24}}(g)$	0	0	0	0	0	0	0	0	0	0	0
$\tilde{\phi}(\tau, 0)$	12	8	0	12	6	12	4	12	6	12	12

The only exception is the class 4D, which might define a consistent orbifold (a torus model). It follows that a K3 model $\mathcal{C}$ is the $\mathbb{Z}_n$-orbifold of a torus model if and only if it contains a symmetry g in one of the classes

$$\begin{aligned} &\text{2C, 3C, 4F, 5C, 6G, 6L, 6M, 8H, 10F, 12N,} \\ &\text{4B, 4D, 6H, 6I, 8C, 8D, 9C, 10G, 10H, 12I, 12L, 12O} \end{aligned} \tag{4.4}$$

of Co_0.¶ Here we have also included (in the second line) those classes of elements $g \in \mathrm{Co}_0$ for which $\mathcal{C}/\langle g^i \rangle$ is a torus model, for some power $i > 1$. Our main observation is now that none of the Co_0 classes in (4.4) restricts to a class in $\mathbb{M}_{24}$, i.e.

All K3 models that are realised as $\mathbb{Z}_n$-orbifolds of torus models are exceptional. In particular, the quantum symmetry is not an element of $\mathbb{M}_{24}$.

One might ask whether the converse is also true, i.e. whether all exceptional models are cyclic torus orbifolds. This is not quite the case: for example, the classification theorem

¶We should emphasise that for us the term 'orbifold' always refers to a conformal field theory (rather than a geometrical) construction. Although a non-linear sigma-model on a geometric orbifold $M/\mathbb{Z}_N$ always admits an interpretation as a CFT orbifold, the converse is not always true. In particular, there exist asymmetric orbifold constructions that do not have a direct geometric interpretation, see for example section 5.

of section 3 predicts the existence of models with a symmetry group $G \cong GL_2(3)$ (the group of 2×2 invertible matrices on the field $\mathbb{F}_3$ with 3 elements). The group G contains no elements in the classes (4.4), so the model is not a cyclic torus orbifold; on the other hand, G contains elements in the class 8I of Co_0, which does not restrict to $\mathbb{M}_{24}$. A second counterexample is a family of models with a symmetry g in the class 6O of Co_0. A generic point of this family is not a cyclic torus model (although some special points are), since the full symmetry group is generated by g and contains no elements in (4.4). Both these counterexamples belong to case (i) of the general classification theorem. In fact, we can prove that

The symmetry group G of a K3 sigma-model $\mathcal{C}$ contains a subgroup $3^{1+4} : \mathbb{Z}_2$ (cases (iii) and (iv) of the theorem) if and only if $\mathcal{C}$ is a $\mathbb{Z}_3$-orbifold of a torus model. Furthermore, $G = 5^{1+2} : \mathbb{Z}_4$ (case (ii)) if and only if $\mathcal{C}$ is a $\mathbb{Z}_5$-orbifold of a torus model.

The proof goes as follows. All subgroups of Co_0 of the form $3^{1+4} : \mathbb{Z}_2$ (respectively, $5^{1+2} : \mathbb{Z}_4$) contain an element in the class 3C (resp., 5C), and therefore the corresponding models are $\mathbb{Z}_3$ (resp., $\mathbb{Z}_5$) torus orbifolds. Conversely, consider a $\mathbb{Z}_3$-orbifold of a torus model. Its symmetry group G contains the quantum symmetry g in class 3C of Co_0. (It must contain a symmetry generator of order three whose orbifold leads to a torus, and 3C is then the only possibility.) The sublattice $\Lambda^{\langle g\rangle} \subset \Lambda$ fixed by g is the $\mathcal{S}$-lattice $2^{27}3^{36}$ [31]. From the classification theorem, we know that G is the stabiliser of a sublattice $\Lambda^G \subset \Lambda$ of rank at least 4. Since $\Lambda^G \subseteq \Lambda^{\langle g\rangle}$, G contains as a subgroup the stabiliser of $\Lambda^{\langle g\rangle}$, namely $3^{1+4} : \mathbb{Z}_2$.

Analogously, a $\mathbb{Z}_5$ torus orbifold always has a symmetry in class 5C, whose fixed sublattice $\Lambda^{\langle g\rangle}$ is the $\mathcal{S}$-lattice $2^5 3^{10}$ [31]. Since $\Lambda^{\langle g\rangle}$ has rank 4 and is primitive, $\Lambda^G = \Lambda^{\langle g\rangle}$ and the symmetry group G must be the stabiliser $5^{1+2} : \mathbb{Z}_4$ of $\Lambda^{\langle g\rangle}$.

It was shown in [7] that the Gepner model $(1)^6$ corresponds to the case (ii) of the classification theorem. It thus follows from the above reasoning that it must indeed be equivalent to a $\mathbb{Z}_3$-orbifold of $\mathbb{T}^4$, see also [27]. (We shall also study these orbifolds more systematically in section 6.) In the next section, we will provide an explicit construction of a $\mathbb{Z}_5$-orbifold of a torus model and show that its symmetry group is $5^{1+2} : \mathbb{Z}_4$, as predicted by the above analysis.

5 A K3 model with symmetry group $5^{1+2} : \mathbb{Z}_4$

In this section we will construct a supersymmetric sigma-model on $\mathbb{T}^4$ with a symmetry g of order 5 commuting with an $\mathcal{N} = (4,4)$ superconformal algebra and acting asymmetrically on the left- and on the right-moving sector. The orbifold of this model by g will turn out to be a well-defined SCFT with $\mathcal{N} = (4,4)$ (in particular, the level matching condition is satisfied) that can be interpreted as a non-linear sigma-model on K3. We will argue that the group of symmetries of this model is $G = 5^{1+2}.\mathbb{Z}_4$, one of the exceptional groups considered in the general classification theorem.

5.1 The torus model

Let us consider a supersymmetric sigma-model on the torus $\mathbb{T}^4$. Geometrically, we can characterise the theory in terms of a metric and a Kalb-Ramond field, but it is actually more convenient to describe it simply as a conformal field theory that is generated by the following fields: four left-moving $u(1)$ currents $\partial X^a(z)$, $a = 1, \ldots, 4$, four free fermions $\psi^a(z)$, $a = 1, \ldots, 4$, their right-moving analogues $\bar\partial X^a(\bar z), \tilde\psi^a(\bar z)$, as well as some winding-momentum fields $V_\lambda(z, \bar z)$ that are associated to vectors λ in an even unimodular lattice $\Gamma^{4,4}$ of signature $(4,4)$. The mode expansions of the left-moving fields are

$$\partial X^a(z) = \sum_{n\in\mathbb{Z}} \alpha_n z^{-n-1} \ , \qquad \psi^a = \sum_{n\in\mathbb{Z}+\nu} \psi_n z^{-n-\frac{1}{2}} \ , \tag{5.1}$$

where $\nu = 0, 1/2$ in the R- and NS-sector, respectively. Furthermore, we have the usual commutation relations

$$[\alpha^a_m, \alpha^b_n] = m\, \delta^{ab}\, \delta_{m,-n} \qquad\qquad \{\psi^a_m, \psi^b_n\} = \delta^{ab}\, \delta_{m,-n} \ . \tag{5.2}$$

Analogous statements also hold for the right-moving modes $\tilde\alpha_n$ and $\tilde\psi_n$. The vectors $\lambda \equiv (\lambda_L, \lambda_R) \in \Gamma^{4,4}$ describe the eigenvalues of the corresponding state with respect to the left- and right-moving zero modes α^a_0 and $\tilde\alpha^a_0$, respectively. In these conventions the inner product on $\Gamma^{4,4}$ is given as

$$(\lambda, \lambda') = \lambda_L \cdot \lambda'_L - \lambda_R \cdot \lambda'_R \ . \tag{5.3}$$

5.1.1 Continuous and discrete symmetries

Any torus model contains an $\hat{\mathfrak{su}}(2)_1 \oplus \hat{\mathfrak{su}}(2)_1 \oplus \hat{\mathfrak{u}}(1)^4$ current algebra, both on the left and on the right. Here, the $\hat{\mathfrak{u}}(1)^4$ currents are the ∂X^a, $a = 1, \ldots, 4$, while $\hat{\mathfrak{su}}(2)_1 \oplus \hat{\mathfrak{su}}(2)_1 = \hat{\mathfrak{so}}(4)_1$ is generated by the fermionic bilinears

$$a^3 := \bar\psi^{(1)}\psi^{(1)} + \bar\psi^{(2)}\psi^{(2)} \qquad a^+ := \bar\psi^{(1)}\bar\psi^{(2)} \qquad a^- := -\psi^{(1)}\psi^{(2)} \ , \tag{5.4}$$

$$\hat a^3 := \bar\psi^{(1)}\psi^{(1)} - \bar\psi^{(2)}\psi^{(2)} \qquad \hat a^+ := \bar\psi^{(1)}\psi^{(2)} \qquad \hat a^- := -\psi^{(1)}\bar\psi^{(2)} \ , \tag{5.5}$$

where

$$\psi^{(1)} = \tfrac{1}{\sqrt{2}}\,(\psi^1 + i\psi^2) \qquad\qquad \psi^{(2)} = \tfrac{1}{\sqrt{2}}\,(\psi^3 + i\psi^4) \tag{5.6}$$

$$\bar\psi^{(1)} = \tfrac{1}{\sqrt{2}}\,(\psi^1 - i\psi^2) \qquad\qquad \bar\psi^{(2)} = \tfrac{1}{\sqrt{2}}\,(\psi^3 - i\psi^4) \ . \tag{5.7}$$

At special points in the moduli space, where the $\Gamma^{4,4}$ lattice contains vectors of the form $(\lambda_L, 0)$ with $\lambda_L^2 = 2$, the bosonic $\mathfrak{u}(1)^4$ algebra is enhanced to some non-abelian algebra $\mathfrak{g}$ of rank 4. There are generically 16 (left-moving) supercharges; they form four $(\mathbf{2},\mathbf{2})$ representations of the $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ zero mode algebra from (5.4) and (5.5). Altogether, the chiral algebra at generic points is a *large* $\mathcal{N} = 4$ superconformal algebra.

We want to construct a model with a symmetry g of order 5, acting non-trivially on the fermionic fields, and commuting with the *small* $\mathcal{N} = 4$ subalgebras both on the left and on the right. A small $\mathcal{N} = 4$ algebra contains an $\hat{\mathfrak{su}}(2)_1$ current algebra and four supercharges in two doublets of $\mathfrak{su}(2)$. The symmetry g acts by an $\mathrm{O}(4, \mathbb{R})$ rotation on the left-moving fermions ψ^a, preserving the anti-commutation relations (5.2). Without loss of generality, we may assume that $\psi^{(1)}$ and $\bar{\psi}^{(1)}$ are eigenvectors of g with eigenvalues ζ and ζ^{-1}, where ζ is a primitive fifth root of unity

$$\zeta^5 = 1 \; , \tag{5.8}$$

and that the $\hat{\mathfrak{su}}(2)_1$ algebra preserved by g is (5.4). This implies that g acts on all the fermionic fields by

$$\psi^{(1)} \mapsto \zeta\, \psi^{(1)} \; , \quad \bar{\psi}^{(1)} \mapsto \zeta^{-1}\bar{\psi}^{(1)} \; , \quad \psi^{(2)} \mapsto \zeta^{-1}\psi^{(2)} \; , \quad \bar{\psi}^{(2)} \mapsto \zeta\, \bar{\psi}^{(2)} \; . \tag{5.9}$$

Note that the action of g on the fermionic fields can be described by $e^{\frac{2\pi i k}{5}\hat{a}^3_0}$ for some $k = 1, \ldots, 4$, where $\hat{a}^3$ is the current in the algebra (5.5). The four g-invariant supercharges can then be taken to be

$$\sqrt{2}\sum_{i=1}^{2} J^{(i)}\bar{\psi}^{(i)} \; , \quad \sqrt{2}\sum_{i=1}^{2} \bar{J}^{(i)}\psi^{(i)} \; , \quad \sqrt{2}(\bar{J}^{(1)}\bar{\psi}^{(2)} - \bar{J}^{(2)}\bar{\psi}^{(1)}) \; , \quad \sqrt{2}(J^{(1)}\psi^{(2)} - J^{(2)}\psi^{(1)}) \; , \tag{5.10}$$

where $J^{(1)}, \bar{J}^{(1)}, J^{(2)}, \bar{J}^{(2)}$ are suitable (complex) linear combinations of the left-moving currents ∂X^a, $a = 1, \ldots, 4$. In order to preserve the four supercharges, g must act with the same eigenvalues on the bosonic currents

$$J^{(1)} \mapsto \zeta\, J^{(1)} \; , \quad \bar{J}^{(1)} \mapsto \zeta^{-1}\bar{J}^{(1)} \; , \quad J^{(2)} \mapsto \zeta^{-1} J^{(2)} \; , \quad \bar{J}^{(2)} \mapsto \zeta\, \bar{J}^{(2)} \; . \tag{5.11}$$

A similar reasoning applies to the right-moving algebra with respect to an eigenvalue $\tilde{\zeta}$, with $\tilde{\zeta}^5 = 1$. For the symmetries with a geometric interpretation, the action on the left- and right-moving bosonic currents is induced by an $\mathrm{O}(4,\mathbb{R})$-transformation on the scalar fields X^a, $a, = 1, \ldots, 4$, representing the coordinates on the torus; then ζ and $\tilde{\zeta}$ are necessarily equal. In our treatment, we want to allow for the more general case where $\zeta \neq \tilde{\zeta}$.

The action of g on J^a and $\tilde{J}^a$ induces an $\mathrm{O}(4,4,\mathbb{R})$-transformation on the lattice $\Gamma^{4,4}$. The transformation g is a symmetry of the model if and only if it induces an automorphism on $\Gamma^{4,4}$. In particular, it must act by an invertible integral matrix on any lattice basis. The requirement that the trace of this matrix (and of any power of it) must be integral, leads to the condition that

$$2(\zeta^i + \zeta^{-i} + \tilde{\zeta}^i + \tilde{\zeta}^{-i}) \in \mathbb{Z} \; , \tag{5.12}$$

for all $i \in \mathbb{Z}$. For g of order 5, this condition is satisfied by

$$\zeta = e^{\frac{2\pi i}{5}} \qquad \text{and} \qquad \tilde{\zeta} = e^{\frac{4\pi i}{5}} \; , \tag{5.13}$$

and this solution is essentially unique (up to taking powers of it or exchanging the definition of ζ and ζ^{-1}). Eq. (5.13) shows that a supersymmetry preserving symmetry of order 5 is necessarily left-right asymmetric, and hence does not have a geometric interpretation.

It is now clear how to construct a torus model with the symmetries (5.9) and (5.11). First of all, we need an automorphism g of $\Gamma^{4,4}$ of order five. Such an automorphism must have eigenvalues $\zeta, \zeta^2, \zeta^3, \zeta^4$, each corresponding to two independent eigenvectors $v^{(1)}_{\zeta^i}, v^{(2)}_{\zeta^i}$, $i = 1, \ldots, 4$, in $\Gamma^{4,4} \otimes \mathbb{C}$. Given the discussion above, see in particular (5.13), we now require that the vectors

$$v^{(1)}_{\zeta^1}\ , \quad v^{(2)}_{\zeta^1}\ , \quad v^{(1)}_{\zeta^4}\ , \quad v^{(2)}_{\zeta^4} \tag{5.14}$$

span a positive-definite subspace of $\Gamma^{4,4} \otimes \mathbb{C}$ (i.e. correspond to the left-movers), while the vectors

$$v^{(1)}_{\zeta^2}\ , \quad v^{(2)}_{\zeta^2}\ , \quad v^{(1)}_{\zeta^3}\ , \quad v^{(2)}_{\zeta^3} \tag{5.15}$$

span a negative-definite subspace of $\Gamma^{4,4} \otimes \mathbb{C}$ (i.e. correspond to the right-movers).

An automorphism g with the properties above can be explicitly constructed as follows. Let us consider the real vector space with basis vectors $x_1, \ldots, x_4$, and $y_1, \ldots, y_4$, and define a linear map g of order 5 by

$$g(x_i) = x_{i+1}\ , \qquad g(y_i) = y_{i+1}\ , \qquad\qquad i = 1, \ldots, 3\ , \tag{5.16}$$

and

$$g(x_4) = -(x_1 + x_2 + x_3 + x_4)\ , \qquad g(y_4) = -(y_1 + y_2 + y_3 + y_4)\ . \tag{5.17}$$

A g-invariant bilinear form on the space is uniquely determined by the conditions

$$(x_i, x_i) = 0 = (y_i, y_i)\ , \qquad i = 1, \ldots, 4 \tag{5.18}$$

and

$$(x_1, x_2) = 1\ , \quad (x_1, x_3) = (x_1, x_4) = -1\ , \qquad (y_1, y_2) = 1\ , \quad (y_1, y_3) = (y_1, y_4) = -1\ , \tag{5.19}$$

as well as

$$(x_1, y_1) = 1\ , \quad \text{and} \quad (x_i, y_1) = 0\ , \quad (i = 2, 3, 4)\ . \tag{5.20}$$

The lattice spanned by these basis vectors is an indefinite even unimodular lattice of rank 8 and thus necessarily isomorphic to $\Gamma^{4,4}$. The g-eigenvectors can be easily constructed in terms of the basis vectors and one can verify that the eigenspaces have the correct signature.

This torus model has an additional $\mathbb{Z}_4$ symmetry group that preserves the superconformal algebra and normalises the group generated by g. The generator h of this group acts by

$$h(x_i) := g^{1-i}(x_1 + x_4 + 2y_1 + y_2 + y_3 + y_4)\ , \tag{5.21}$$

$$h(y_i) := g^{1-i}(-2x_1 - x_2 - x_3 - x_4 - y_1 - y_3 - y_4)\ , \qquad i = 1, \ldots, 4\ , \tag{5.22}$$

on the lattice vectors. The g-eigenvectors $v^{(a)}_{\zeta^i}$, $a = 1, 2$, $i = 1, \ldots, 4$ can be defined as

$$v^{(1)}_{\zeta^i} := \sum_{j=0}^{4} \zeta^{-ij} g^j(x_1 + h(x_1))\ , \qquad v^{(2)}_{\zeta^i} := \sum_{j=0}^{4} \zeta^{-ij} g^j(x_1 - h(x_1))\ , \tag{5.23}$$

so that

$$h(v^{(1)}_{\zeta^i}) = -v^{(2)}_{\zeta^{-i}}\ , \qquad h(v^{(2)}_{\zeta^i}) = v^{(1)}_{\zeta^{-i}}\ . \tag{5.24}$$

Correspondingly, the action of h on the left-moving fields is

$$\psi^{(1)} \mapsto -\psi^{(2)} \ , \quad \psi^{(2)} \mapsto \psi^{(1)} \ , \quad \bar{\psi}^{(1)} \mapsto -\bar{\psi}^{(2)} \ , \quad \bar{\psi}^{(2)} \mapsto \bar{\psi}^{(1)} \ , \tag{5.25}$$

$$J^{(1)} \mapsto -J^{(2)} \ , \quad J^{(2)} \mapsto J^{(1)} \ , \quad \bar{J}^{(1)} \mapsto -\bar{J}^{(2)} \ , \quad \bar{J}^{(2)} \mapsto \bar{J}^{(1)} \ , \tag{5.26}$$

and the action on the right-moving fields is analogous. It is immediate to verify that the generators of the superconformal algebra are invariant under this transformation.

5.2 The orbifold theory

Next we want to consider the orbifold of this torus theory by the group $\mathbb{Z}_5$ that is generated by g.

5.2.1 The elliptic genus

The elliptic genus of the orbifold theory can be computed by summing over the $\mathrm{SL}(2,\mathbb{Z})$ images of the untwisted sector contribution, which in turn is given by

$$\phi^U(\tau, z) = \frac{1}{5} \sum_{k=0}^{4} \phi_{1,g^k}(\tau, z) \ , \tag{5.27}$$

where

$$\phi_{1,g^k}(\tau, z) = \mathrm{Tr}_{\mathrm{RR}}(g^k \, q^{L_0 - \frac{c}{24}} \bar{q}^{\tilde{L}_0 - \frac{\tilde{c}}{24}} y^{2J_0} (-1)^{F+\tilde{F}}) \ . \tag{5.28}$$

The $k = 0$ contribution, i.e. the elliptic genus of the original torus theory, is zero. Each g^k-contribution, for $k = 1, \ldots, 4$, is the product of a factor coming from the ground states, one from the oscillators and one from the momenta

$$\phi_{1,g^k}(\tau, z) = \phi^{\mathrm{gd}}_{1,g^k}(\tau, z) \, \phi^{\mathrm{osc}}_{1,g^k}(\tau, z) \, \phi^{\mathrm{mom}}_{1,g^k}(\tau, z) \ . \tag{5.29}$$

These contributions are, respectively,

$$\phi^{\mathrm{gd}}_{1,g^k}(\tau, z) = y^{-1}(1 - \zeta^k y)(1 - \zeta^{-k} y)(1 - \zeta^{2k})(1 - \zeta^{-2k}) = 2y^{-1} + 2y + 1 \ , \tag{5.30}$$

$$\phi^{\mathrm{osc}}_{1,g^k}(\tau, z) = \prod_{n=1}^{\infty} \frac{(1 - \zeta^k y q^n)(1 - \zeta^{-k} y q^n)(1 - \zeta^k y^{-1} q^n)(1 - \zeta^{-k} y^{-1} q^n)}{(1 - \zeta^k q^n)^2 (1 - \zeta^{-k} q^n)^2} \ , \tag{5.31}$$

and

$$\phi^{\mathrm{mom}}_{1,g^k}(\tau, z) = 1 \ , \tag{5.32}$$

where the last result follows because the only g-invariant state of the form (k_L, k_R) is the vacuum $(0, 0)$. Thus we have

$$\phi_{1,g^k}(\tau, z) = 5 \, \frac{\vartheta_1(\tau, z + \frac{k}{5}) \, \vartheta_1(\tau, z - \frac{k}{5})}{\vartheta_1(\tau, \frac{k}{5}) \, \vartheta_1(\tau, -\frac{k}{5})} \ , \tag{5.33}$$

where

$$\vartheta_1(\tau, z) = -i q^{1/8} y^{-\frac{1}{2}} (y - 1) \prod_{n=1}^{\infty} (1 - q^n)(1 - y q^n)(1 - y^{-1} q^n) \ , \tag{5.34}$$

is the first Jacobi theta function. Modular transformations of $\phi_{1,g^k}(\tau,z)$ reproduce the twining genera in the twisted sector

$$\phi_{g^l,g^k}(\tau,z) = \mathrm{Tr}_{\mathcal{H}^{(l)}}\left(g^k\, q^{L_0-\frac{c}{24}}\, \bar{q}^{\tilde{L}_0-\frac{\tilde{c}}{24}} y^{2J_0}(-1)^{F+\tilde{F}}\right) , \tag{5.35}$$

and using the modular properties of the theta function we obtain

$$\phi_{g^l,g^k}(\tau,z) = 5\, \frac{\vartheta_1(\tau, z+\frac{k}{5}+\frac{l\tau}{5})\,\vartheta_1(\tau, z-\frac{k}{5}-\frac{l\tau}{5})}{\vartheta_1(\tau, \frac{k}{5}+\frac{l\tau}{5})\,\vartheta_1(\tau, -\frac{k}{5}-\frac{l\tau}{5})} , \tag{5.36}$$

for $k,l \in \mathbb{Z}/5\mathbb{Z}$, $(k,l)\neq(0,0)$. The elliptic genus of the full orbifold theory is then

$$\phi_{\rm orb}(\tau,z) = \frac{1}{5}\sum_{k,l\in\mathbb{Z}/5\mathbb{Z}} \phi_{g^l,g^k}(\tau,z) = \sum_{\substack{k,l\in\mathbb{Z}/5\mathbb{Z}\\(k,l)\neq(0,0)}} \frac{\vartheta_1(\tau, z+\frac{k}{5}+\frac{l\tau}{5})\,\vartheta_1(\tau, z-\frac{k}{5}-\frac{l\tau}{5})}{\vartheta_1(\tau, \frac{k}{5}+\frac{l\tau}{5})\,\vartheta_1(\tau, -\frac{k}{5}-\frac{l\tau}{5})} . \tag{5.37}$$

Since $\phi_{g^k,g^l}(\tau,0)=5$ for all $(k,l)\neq(0,0)$, we have

$$\phi_{\rm orb}(\tau,0) = \frac{1}{5}\sum_{\substack{k,l\in\mathbb{Z}/5\mathbb{Z}\\(k,l)\neq(0,0)}} 5 = 24 , \tag{5.38}$$

which shows that the orbifold theory is a non-linear sigma-model on K3. In particular, the untwisted sector has 4 RR ground states, while each of the four twisted sectors contains 5 RR ground states. For the following it will be important to understand the structure of the various twisted sectors in detail.

5.2.2 The twisted sectors

In the g^k-twisted sector, let us consider a basis of g-eigenvectors for the currents and fermionic fields. For a given eigenvalue ζ^i, $i\in\mathbb{Z}/5\mathbb{Z}$, of g^k, the corresponding currents $J^{i,a}$ and fermionic fields $\psi^{i,b}$ (where a,b labels distinct eigenvectors with the same eigenvalue) have a mode expansion

$$J^{i,a}(z) = \sum_{n\in\frac{i}{5}+\mathbb{Z}} \alpha^{i,a}_n z^{-n-1} , \qquad \psi^{i,a}(z) = \sum_{r\in\frac{i}{5}+\nu+\mathbb{Z}} \psi^{i,a}_r z^{-r-1/2} , \tag{5.39}$$

where $\nu=1/2$ in the NS- and $\nu=0$ in the R-sector. The ground states of the g^k-twisted sector are characterised by the conditions

$$\alpha^{i,a}_n|m,k\rangle = \tilde{\alpha}^{i,a}_n|m,k\rangle = 0 , \qquad \forall\ n>0,\ i,\ a , \tag{5.40}$$

$$\psi^{i,b}_r|m,k\rangle = \tilde{\psi}^{i,b}_r|m,k\rangle = 0 , \qquad \forall\ r>0,\ i,\ b , \tag{5.41}$$

where $|m,k\rangle$ denotes the $m^{\rm th}$ ground state in the g^k-twisted sector. Note that since none of the currents are g-invariant, there are no current zero modes in the g^k-twisted sector, and similarly for the fermions. For a given k, the states $|m,k\rangle$ have then all the same conformal dimension, which can be calculated using the commutation relation $[L_{-1},L_1]=2L_0$ or read off from the leading term of the modular transform of the twisted character (5.35). In the g^k-twisted NS-NS-sector the ground states have conformal dimension

$$\text{NS-NS } g^k\text{-twisted:} \qquad h = \frac{k}{5} \qquad \text{and} \qquad \tilde{h} = \frac{2k}{5} \ , \tag{5.42}$$

while in the RR-sector we have instead

$$\text{R-R } g^k\text{-twisted:} \qquad h = \tilde{h} = \frac{1}{4} \ . \tag{5.43}$$

In particular, level matching is satisfied, and thus the asymmetric orbifold is consistent [32]. The full g^k-twisted sector is then obtained by acting with the negative modes of the currents and the fermionic fields on the ground states $|m, k\rangle$.

Let us have a closer look at the ground states of the g^k twisted sector; for concreteness we shall restrict ourselves to the case $k = 1$, but the modifications for general k are minor (see below). The vertex operators $V_\lambda(z, \bar{z})$ associated to $\lambda \in \Gamma^{4,4}$, act on the ground states $|m, 1\rangle$ by

$$\lim_{z\to 0} V_\lambda(z, \bar{z})|m, 1\rangle = e_\lambda |m, 1\rangle \ , \tag{5.44}$$

where e_λ are operators commuting with all current and fermionic oscillators and satisfying

$$e_\lambda \, e_\mu = \epsilon(\lambda, \mu) \, e_{\lambda+\mu} \ , \tag{5.45}$$

for some fifth root of unity $\epsilon(\lambda, \mu)$. The vertex operators V_λ and V_μ must be local relative to one another, and this is the case provided that (see the appendix)

$$\frac{\epsilon(\lambda, \mu)}{\epsilon(\mu, \lambda)} = C(\lambda, \mu) \ , \tag{5.46}$$

where

$$C(\lambda, \mu) = \prod_{i=1}^{4} (\zeta^i)^{(g^i(\lambda), \mu)} = \zeta^{(P_g(\lambda), \mu)} \qquad \text{with} \qquad P_g(\lambda) = \sum_{i=1}^{4} i g^i(\lambda) \ . \tag{5.47}$$

The factor $C(\lambda, \mu)$ has the properties

$$C(\lambda, \mu_1 + \mu_2) = C(\lambda, \mu_1) \, C(\lambda, \mu_2) \ , \qquad C(\lambda_1 + \lambda_2, \mu) = C(\lambda_1, \mu) \, C(\lambda_2, \mu) \ , \tag{5.48}$$

$$C(\lambda, \mu) = C(\mu, \lambda)^{-1} \ , \tag{5.49}$$

$$C(\lambda, \mu) = C(g(\lambda), g(\mu)) \ . \tag{5.50}$$

Because of (5.48), $C(\lambda, 0) = C(0, \lambda) = 1$ for all $\lambda \in \Gamma^{4,4}$, and we can set

$$e_0 = 1 \ , \tag{5.51}$$

so that $\epsilon(0, \lambda) = 1 = \epsilon(\lambda, 0)$. More generally, for the vectors λ in the sublattice

$$R := \{\lambda \in \Gamma^{4,4} \mid C(\lambda, \mu) = 1 \ , \text{ for all } \mu \in \Gamma^{4,4}\} \subset \Gamma^{4,4} \ , \tag{5.52}$$

we have $C(\lambda + \mu_1, \mu_2) = C(\mu_1, \mu_2)$, for all $\mu_1, \mu_2 \in \Gamma^{4,4}$, so that we can set

$$e_{\mu+\lambda} = e_\mu \ , \qquad \forall \lambda \in R \ , \quad \mu \in \Gamma^{4,4} \ . \tag{5.53}$$

Thus, we only need to describe the operators corresponding to representatives of the group $\Gamma^{4,4}/R$. The vectors $\lambda \in R$ are characterised by

$$(P_g(\lambda), \mu) \equiv 0 \mod 5 \;, \qquad \text{for all } \mu \in \Gamma^{4,4} \;, \tag{5.54}$$

and since $\Gamma^{4,4}$ is self-dual this condition is equivalent to

$$P_g(\lambda) \in 5\,\Gamma^{4,4} \;. \tag{5.55}$$

Since g has no invariant subspace, we have the identity

$$1 + g + g^2 + g^3 + g^4 = 0 \tag{5.56}$$

that implies (see (5.47))

$$P_g \circ (1-g) = (1-g) \circ P_g = -5 \cdot \mathbf{1} \;. \tag{5.57}$$

Thus, $\lambda \in R$ if and only if

$$P_g(\lambda) = P_g \circ (1-g)(\tilde{\lambda}) \;, \tag{5.58}$$

for some $\tilde{\lambda} \in \Gamma^{4,4}$, and since P_g has trivial kernel (see (5.57)), we finally obtain

$$R = (1-g)\,\Gamma^{4,4} \;. \tag{5.59}$$

Since also $(1-g)$ has trivial kernel, R has rank 8 and $\Gamma^{4,4}/R$ is a finite group. Furthermore,

$$|\Gamma^{4,4}/R| = \det(1-g) = 25 \;, \tag{5.60}$$

and, since $5\,\Gamma^{4,4} \subset R$, the group $\Gamma^{4,4}/R$ has exponent 5. The only possibility is

$$\Gamma^{4,4}/R \cong \mathbb{Z}_5 \times \mathbb{Z}_5 \;. \tag{5.61}$$

Let $x, y \in \Gamma^{4,4}$ be representatives for the generators of $\Gamma^{4,4}/R$. By (5.49), we know that $C(x,x) = C(y,y) = 1$, so that $C(x,y) \neq 1$ (otherwise C would be trivial over the whole $\Gamma^{4,4}$), and we can choose x, y such that

$$C(x,y) = \zeta \;. \tag{5.62}$$

Thus, the ground states form a representation of the algebra of operators generated by e_x, e_y, satisfying

$$e_x^5 = 1 = e_y^5 \;, \qquad e_x e_y = \zeta e_y e_x \;. \tag{5.63}$$

The group generated by e_x and e_y is the extra-special group 5^{1+2}, and all its non-abelian irreducible representations[‖] are five dimensional.

In particular, for the representation on the g-twisted ground states, we can choose a basis of e_x-eigenvectors

$$|m;1\rangle \quad \text{with} \quad e_x|m;1\rangle = \zeta^m |m;1\rangle \;, \qquad m \in \mathbb{Z}/5\mathbb{Z} \;, \tag{5.64}$$

[‖]We call a representation non-abelian if the central element does not act trivially.

and define the action of the operators e_y by

$$e_y|m;1\rangle = |m+1;1\rangle \ . \tag{5.65}$$

For any vector $\lambda \in \Gamma^{4,4}$, there are unique $a, b \in \mathbb{Z}/5\mathbb{Z}$ such that $\lambda = ax + by + (1-g)(\mu)$ for some $\mu \in \Gamma^{4,4}$ and we define**

$$e_\lambda := e_x^a e_y^b \ . \tag{5.66}$$

Since $g(\lambda) = ax + by + (1-g)(g(\mu) - ax - by)$, by (5.53) we have

$$e_{g(\lambda)} = e_\lambda \ , \tag{5.67}$$

so that, with respect to the natural action $g(e_\lambda) := e_{g(\lambda)}$, the algebra is g-invariant. This is compatible with the fact that all ground states have the same left and right conformal weights h and $\tilde{h}$, so that the action of $g = e^{2\pi i(h-\tilde{h})}$ is proportional to the identity.

The construction of the g^k-twisted sector, for $k = 2, 3, 4$, is completely analogous to the g^1-twisted case, the only difference being that the root ζ in the definition of $C(\lambda, \mu)$ should be replaced by ζ^k. Thus, one can define operators $e_x^{(k)}, e_y^{(k)}$ on the g^k-twisted sector, for each $k = 1, \ldots, 4$, satisfying

$$(e_x^{(k)})^5 = 1 = (e_y^{(k)})^5 \ , \qquad e_x^{(k)} e_y^{(k)} = \zeta^k e_y^{(k)} e_x^{(k)} \ . \tag{5.68}$$

The action of these operators on the analogous basis $|m;k\rangle$ with $m \in \mathbb{Z}/5\mathbb{Z}$ is then

$$e_x^{(k)}|m;k\rangle = \zeta^m|m;k\rangle \ , \quad e_y^{(k)}|m;k\rangle = |m+k;k\rangle \ . \tag{5.69}$$

5.2.3 Spectrum and symmetries

The spectrum of the actual orbifold theory is finally obtained from the above twisted sectors by projecting onto the g-invariant states; technically, this is equivalent to restricting to the states for which the difference of the left- and right- conformal dimensions is integer, $h - \tilde{h} \in \mathbb{Z}$. In particular, the RR ground states (5.69) in each (twisted) sector have $h = \tilde{h} = 1/4$, so that they all survive the projection. Thus, the orbifold theory has 4 RR ground states in the untwisted sector (the spectral flow generators), forming a $(\mathbf{2}, \mathbf{2})$ representation of $\mathfrak{su}(2)_L \oplus \mathfrak{su}(2)_R$, and 5 RR ground states in each twisted sector, which are singlets of $\mathfrak{su}(2)_L \oplus \mathfrak{su}(2)_R$. In total there are therefore 24 RR ground states, as expected for a non-linear sigma-model on K3. (Obviously, we are here just reproducing what we already saw in (5.38).)

Next we want to define symmetry operators acting on the orbifold theory. First we can construct operators e_λ associated to $\lambda \in \Gamma^{4,4}$, that will form the extra special group 5^{1+2}. They are defined to act by $e_\lambda^{(k)}$ on the g^k-twisted sector. The action of the untwisted sector preserves the subspaces $\mathcal{H}_m^U$, $m \in \mathbb{Z}/5\mathbb{Z}$, of states with momentum of the form $\lambda = nx + my + (1-g)(\mu)$, for some $n \in \mathbb{Z}$ and $\mu \in \Gamma^{4,4}$. Let us denote by $T_{m;k}$ a generic vertex operator associated with a g^k-twisted state, $k = 1, \ldots, 4$, with e_x-eigenvalue ζ^m,

**The ordering of e_x and e_y in this definition is arbitrary; however, any other choice corresponds to a redefinition $\tilde{e}_\lambda = c(\lambda)e_\lambda$, for some fifth root of unity $c(\lambda)$, that does not affect the commutation relations $\tilde{e}_\lambda \tilde{e}_\mu = C(\lambda, \mu)\tilde{e}_\mu \tilde{e}_\lambda$.

$m \in \mathbb{Z}/5\mathbb{Z}$, and by $T_{m;0}$ a vertex operator associated with a state in $\mathcal{H}^U_m$. Consistency of the OPE implies the fusion rules

$$T_{m;k} \times T_{m';k'} \to T_{m+m';k+k'} \ . \tag{5.70}$$

These rules are preserved by the maps

$$T_{m;k} \mapsto e_\lambda \, T_{m;k} \, e_\lambda^{-1} \ , \qquad \lambda \in \Gamma^{4,4} \ , \tag{5.71}$$

which therefore define symmetries of the orbifold theory. As we have explained above, these symmetries form the extra-special group 5^{1+2}.

Finally, the symmetries (5.21), (5.22), (5.25) and (5.26) of the original torus theory induce a $\mathbb{Z}_4$-group of symmetries of the orbifold. Since $h^{-1}gh = g^{-1}$, the space of g-invariant states of the original torus theory is stabilised by h, so that h restricts to a well-defined transformation on the untwisted sector of the orbifold. Furthermore, h maps the g^k- to the g^{5-k}-twisted sector. Eqs. (5.21) and (5.22) can be written as

$$h(x_1) = 2x_1 + (1-g)(-x_1 - x_2 - x_3 + y_1 + y_2 + y_3 + y_4) \ , \tag{5.72}$$
$$h(y_1) = 2y_1 + (1-g)(-x_1 - x_2 - x_3 - x_4 - 2y_1 - y_2 - y_3 - y_4) \ . \tag{5.73}$$

It follows that the action of h on the operators $e_\lambda^{(k)}$, $k = 1, \ldots, 4$ must be

$$h e_x^{(k)} h^{-1} = e_{2x}^{(5-k)} \ , \qquad h e_y^{(k)} h^{-1} = e_{2y}^{(5-k)} \ , \tag{5.74}$$

and it is easy to verify that this transformation is compatible with (5.68). Correspondingly, the action on the twisted sector ground states is

$$h|m;k\rangle = |3m; 5-k\rangle \ , \tag{5.75}$$

and it is consistent with (5.70).

Thus the full symmetry group is the semi-direct product

$$G = 5^{1+2} : \mathbb{Z}_4 \ , \tag{5.76}$$

where the generator $h \in \mathbb{Z}_4$ maps the central element $\zeta \in 5^{1+2}$ to ζ^{-1}.

All of these symmetries act trivially on the superconformal algebra and on the spectral flow generators, and therefore define symmetries in the sense of the general classification theorem. Indeed, G agrees precisely with the group in case (ii) of the theorem. Thus our orbifold theory realises this possibility.

6 Models with symmetry group containing $3^{1+4} : \mathbb{Z}_2$

Most of the torus orbifold construction described in the previous section generalises to symmetries g of order different than 5. In particular, one can show explicitly that orbifolds of $\mathbb{T}^4$ models by a symmetry g of order 3 contain a group of symmetries $3^{1+4} : \mathbb{Z}_2$, so that they belong to one of the cases (iii) and (iv) of the theorem, as expected from the discussion in section 4.

We take the action of the symmetry g on the left-moving currents and fermionic fields to be of the form (5.9) and (5.11), where ζ is a now a third root of unity; analogous transformations hold for the right-moving fields with respect to a third root of unity $\tilde{\zeta}$. In this case, eq.(5.12) can be satisifed by

$$\zeta = \tilde{\zeta} = e^{\frac{2\pi i}{3}} , \tag{6.1}$$

so that the action is left-right symmetric and g admits an interpretation as a geometric $\mathrm{O}(4,\mathbb{R})$-rotation of order 3 of the torus $\mathbb{T}^4$. For example, the torus $\mathbb{R}^4/(A_2 \oplus A_2)$, where A_2 is the root lattice of the $su(3)$ Lie algebra, with vanishing Kalb-Ramond field, admits such an automorphism.

The orbifold by g contains 6 RR ground states in the untwisted sector. In the k^{th} twisted sector, $k = 1, 2$, the ground states form a representation of an algebra of operators $e^{(k)}_\lambda$, $\lambda \in \Gamma^{4,4}$, satisfying the commutation relations

$$e^{(k)}_\lambda e^{(k)}_\mu = C(\lambda,\mu)^k e^{(k)}_\mu e^{(k)}_\lambda , \tag{6.2}$$

where

$$C(\lambda,\mu) = \zeta^{(P_g(\lambda),\mu)} , \qquad P_g = g + 2g^2 . \tag{6.3}$$

As discussed in section 5.2.2, we can set

$$e^{(k)}_{\lambda+\mu} = e^{(k)}_\mu , \qquad \forall \lambda \in R, \mu \in \Gamma^{4,4} , \tag{6.4}$$

where

$$R = (1-g)\Gamma^{4,4} . \tag{6.5}$$

(Note that $\Gamma^{4,4}$ contains no g-invariant vectors). The main difference with the analysis of section 5.2.2 is that, in this case,

$$\Gamma^{4,4}/R \cong \mathbb{Z}_3^4 . \tag{6.6}$$

In particular, we can find vectors $x_1, x_2, y_1, y_2 \in \Gamma^{4,4}$ such that

$$C(x_i, y_j) = \zeta^{\delta_{ij}} , \qquad C(x_i, x_j) = C(y_i, y_j) = 1 . \tag{6.7}$$

The corresponding operators obey the relations

$$e^{(k)}_{x_i} e^{(k)}_{y_j} = \zeta^{k\delta_{ij}} e^{(k)}_{y_j} e^{(k)}_{x_i} , \quad e^{(k)}_{x_i} e^{(k)}_{x_j} = e^{(k)}_{x_j} e^{(k)}_{x_i} , \quad e^{(k)}_{y_i} e^{(k)}_{y_j} = e^{(k)}_{y_j} e^{(k)}_{y_i} , \tag{6.8}$$

as well as

$$(e^{(k)}_{x_i})^3 = 1 = (e^{(k)}_{y_i})^3 . \tag{6.9}$$

These operators generate the extra-special group 3^{1+4} of exponent 3, and the k^{th}-twisted ground states form a representation of this group. We can choose a basis $|m_1, m_2; k\rangle$, with $m_1, m_2 \in \mathbb{Z}/3\mathbb{Z}$, of simultaneous eigenvectors of $e^{(k)}_{x_1}$ and $e^{(k)}_{x_2}$, so that

$$e^{(k)}_{x_i} |m_1, m_2; k\rangle = \zeta^{m_i} |m_1, m_2; k\rangle , \qquad e^{(k)}_{y_i} |m_1, m_2; k\rangle = |m_1 + k\delta_{1i}, m_2 + k\delta_{2i}; k\rangle . \tag{6.10}$$

The resulting orbifold model has 9 RR ground states in each twisted sector, for a total of $6 + 9 + 9 = 24$ RR ground states, as expected for a K3 model. As in section 5.2.3, the group 3^{1+4} generated by $e_\lambda^{(k)}$ extends to a group of symmetries of the whole orbifold model. Furthermore, the $\mathbb{Z}_2$-symmetry that flips the signs of the coordinates in the original torus theory induces a symmetry h of the orbifold theory, which acts on the twisted sectors by

$$h\,|m_1, m_2; k\rangle = |-m_1, -m_2; k\rangle \ . \tag{6.11}$$

We conclude that the group G of symmetries of any torus orbifold $\mathbb{T}^4/\mathbb{Z}_3$ contains a subgroup $3^{1+4} : \mathbb{Z}_2$, and is therefore included in the cases (iii) or (iv) of the classification theorem. This obviously ties in nicely with the general discussion of section 4.

7 Conclusions

In this paper we have reviewed the current status of the EOT conjecture concerning a possible $\mathbb{M}_{24}$ symmetry appearing in the elliptic genus of K3. We have explained that, in some sense, the EOT conjecture has already been proven since twining genera, satisfying the appropriate modular and integrality properties, have been constructed for all conjugacy classes of $\mathbb{M}_{24}$. However, the analogue of the Monster conformal field theory that would 'explain' the underlying symmetry has not yet been found. In fact, no single K3 sigma-model will be able to achieve this since none of them possesses an automorphism group that contains $\mathbb{M}_{24}$.

Actually, the situation is yet further complicated by the fact that there are K3 sigma-models whose automorphism group is *not even a subgroup of* $\mathbb{M}_{24}$; on the other hand, the elliptic genus of K3 does not show any signs of exhibiting 'Moonshine' with respect to any larger group. As we have explained in this paper, most of the exceptional automorphism groups (i.e. automorphism groups that are not subgroups of $\mathbb{M}_{24}$) appear for K3s that are torus orbifolds. In fact, all cyclic torus orbifolds are necessarily exceptional in this sense, and (cyclic) torus orbifolds account for all incarnations of the cases (ii) – (iv) of the classification theorem of [7] (see section 4). We have checked these predictions by explicitly constructing an asymmetric $\mathbb{Z}_5$ orbifold that realises case (ii) of the theorem (see section 5), and a family of $\mathbb{Z}_3$ orbifolds realising cases (iii) and (iv) of the theorem (see section 6). Incidentally, these constructions also demonstrate that the exceptional cases (ii)-(iv) actually appear in the K3 moduli space — in the analysis of [7] this conclusion relied on some assumption about the regularity of K3 sigma-models.

The main open problem that remains to be understood is why precisely $\mathbb{M}_{24}$ is 'visible' in the elliptic genus of K3, rather than any smaller (or indeed bigger) group. Recently, we have constructed (some of) the twisted twining elliptic genera of K3 [33], i.e. the analogues of Simon Norton's generalised Moonshine functions [34]. We hope that they will help to shed further light on this question.

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A Commutation relations in the twisted sector

The vertex operators $V_\lambda(z,\bar z)$ in the g-twisted sector can be defined in terms of formal exponentials of current oscillators

$$E^{\pm}_\lambda(z,\bar z) := \exp\Big(\sum_{\substack{r\in\frac{1}{5}\mathbb{Z}\\ \pm r>0}} (\lambda_L\cdot\alpha)^{(r)}_r \frac{z^{-r}}{r}\Big) \exp\Big(\sum_{\substack{r\in\frac{1}{5}\mathbb{Z}\\ \pm r>0}} (\lambda_R\cdot\tilde\alpha)^{(r)}_r \frac{\bar z^{-r}}{r}\Big) \ , \tag{A.1}$$

where $(\lambda_L\cdot\alpha)^{(r)}_r$ and $(\lambda_R\cdot\tilde\alpha)^{(r)}_r$ are the r-modes of the currents

$$(\lambda_L\cdot\partial X)^{(r)} := \frac{1}{5}\sum_{i=0}^{4}\zeta^{5ir}\lambda_L\cdot g^i(\partial X) = \frac{1}{5}\sum_{i=0}^{4}\zeta^{5ir}g^{-i}(\lambda_L)\cdot\partial X \ , \tag{A.2}$$

$$(\lambda_R\cdot\bar\partial X)^{(r)} := \frac{1}{5}\sum_{i=0}^{4}\bar\zeta^{5ir}\lambda_R\cdot g^i(\bar\partial X) = \frac{1}{5}\sum_{i=0}^{4}\bar\zeta^{5ir}g^{-i}(\lambda_R)\cdot\bar\partial X \ . \tag{A.3}$$

Then we can define

$$V_\lambda(z,\bar z) := E^-_\lambda(z,\bar z)\,E^+_\lambda(z,\bar z)\,e_\lambda \ , \tag{A.4}$$

where the operators e_λ commute with all current oscillators and satisfy

$$e_\lambda\,e_\mu = \epsilon(\lambda,\mu)\,e_{\lambda+\mu} \ , \tag{A.5}$$

for some fifth root of unity $\epsilon(\lambda,\mu)$. The commutator factor

$$C(\lambda,\mu) := \frac{\epsilon(\lambda,\mu)}{\epsilon(\mu,\lambda)} \ , \tag{A.6}$$

can be determined by imposing the locality condition

$$V_\lambda(z_1,\bar z_1)\,V_\mu(z_2,\bar z_2) = V_\mu(z_2,\bar z_2)\,V_\lambda(z_1,\bar z_1) \ . \tag{A.7}$$

To do so, we note that the commutation relations between the operators $E^\pm_\lambda$ can be computed, as in [35], using the Campbell-Baker-Hausdorff formula

$$\begin{aligned} E^+_\lambda(z_1,\bar z_1)E^-_\mu(z_2,\bar z_2) &= E^-_\mu(z_2,\bar z_2)E^+_\lambda(z_1,\bar z_1) \\ &\prod_{i=0}^{4}[(1-\zeta^{-i}(\tfrac{z_1}{z_2})^{\frac{1}{5}})^{g^i(\lambda)_L\cdot\mu_L}(1-\bar\zeta^{-i}(\tfrac{\bar z_1}{\bar z_2})^{\frac{1}{5}})^{g^i(\lambda)_R\cdot\mu_R}] \ . \end{aligned} \tag{A.8}$$

Using (A.8) and $e_\lambda e_\mu = C(\lambda,\mu)e_\mu e_\lambda$, the locality condition is then equivalent to

$$C(\lambda,\mu)\prod_{i=0}^{4}\frac{(1-\zeta^{-i}(\frac{z_1}{z_2})^{\frac{1}{5}})^{g^i(\lambda)_L\cdot\mu_L}(1-\bar\zeta^{-i}\frac{\bar z_1^{1/5}}{\bar z_2^{1/5}})^{g^i(\lambda)_R\cdot\mu_R}}{(1-\zeta^{i}(\frac{z_2}{z_1})^{\frac{1}{5}})^{g^i(\lambda)_L\cdot\mu_L}(1-\bar\zeta^{i}(\frac{\bar z_2}{\bar z_1})^{\frac{1}{5}})^{g^i(\lambda)_R\cdot\mu_R}} = 1 \ , \tag{A.9}$$

that is

$$C(\lambda,\mu)\Big(-\frac{z_1^{1/5}}{z_2^{1/5}}\Big)^{\sum_i g^i(\lambda)_L\cdot\mu_L}\Big(-\frac{\bar z_1^{1/5}}{\bar z_2^{1/5}}\Big)^{\sum_i g^i(\lambda)_R\cdot\mu_R}\prod_{i=0}^{4}[(\zeta^{-i})^{g^i(\lambda)_L\cdot\mu_L}(\bar\zeta^{-i})^{g^i(\lambda)_R\cdot\mu_R}] = 1 \ . \tag{A.10}$$

Since $\Gamma^{4,4}$ has no g-invariant vector, we have the identities

$$\sum_{i=0}^{4} g^i(\lambda)_L = 0 = \sum_{i=0}^{4} g^i(\lambda)_R \ , \tag{A.11}$$

and hence finally obtain

$$C(\lambda,\mu) = \prod_{i=0}^{4} (\zeta^i)^{g^i(\lambda_L)\cdot\mu_L - g^i(\lambda_R)\cdot\mu_R} \ . \tag{A.12}$$

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