

Creation of a quantum oscillator by classical control

Stefan Danilishin,^{1,2} Helge Müller-Ebhardt,¹ Henning Rehbein,¹ Kentaro Somiya,^{3,4} Roman Schnabel,¹ Karsten Danzmann,¹ Thomas Corbitt,⁵ Christopher Wipf,⁵ Nergis Mavalvala,⁵ and Yanbei Chen^{3,4}

¹Max-Planck Institut für Gravitationsphysik (Albert-Einstein-Institut) and Leibniz Universität Hannover, Callinstr. 38, 30167 Hannover, Germany

²Physics Faculty, Moscow State University, Moscow 119991, Russia

³Theoretical Astrophysics 130-33, California Institute of Technology, Pasadena, CA 91125, USA

⁴Max-Planck Institut für Gravitationsphysik (Albert-Einstein-Institut), Am Mühlenberg 1, 14476 Golm, Germany

⁵LIGO Laboratory, NW22-295, Massachusetts Institute of Technology, Cambridge, MA 02139, USA

As a pure quantum state is being approached via linear feedback, and the occupation number approaches and eventually goes below unity, optimal control becomes crucial. We obtain theoretically the optimal feedback controller that minimizes the uncertainty for a general linear measurement process, and show that even in the absence of classical noise, a pure quantum state is not always achievable via feedback. For Markovian measurements, the deviation from minimum Heisenberg Uncertainty is found to be closely related to the extent to which the device beats the free-mass Standard Quantum Limit for force measurement. We then specialize to optical Markovian measurements, and demonstrate that a slight modification to the usual input-output scheme — either injecting frequency independent squeezed vacuum or making a homodyne detection at a non-phase quadrature — allows controlled states of kilogram-scale mirrors in future LIGO interferometers to reach occupation numbers significantly below unity.

Motivation. The detectors of the Laser Interferometer Gravitational-wave Observatory (LIGO) [1] are currently operating at a factor of 10 above the Standard Quantum Limit (SQL) [2] at closest approach, limited by classical noise below about 100 Hz, and by quantum shot noise above 100 Hz [3]. This low noise performance allows for probing and manipulating LIGO mirrors (10 kg each and suspended as pendulums with resonant frequency $\omega_p = 2\pi \times 0.7$ Hz and typical damping rate $\gamma_p \sim 10^{-6} \text{ sec}^{-1}$) close to scales set by the Heisenberg Uncertainty Principle. Recently, electronic feedback control was used to shift the resonant frequency of the pendulum mode up to ~ 140 Hz and damp it, leading to an effective occupation number of 234 for the kg-scale oscillator [4]. Semiclassical calculations estimate an effective occupation number of $N_{\text{eff}} \approx Q_{\text{eff}} S_x(\Omega_{\text{eff}}) / S_{\text{SQL}}(\Omega_{\text{eff}})$, where S_x is the detector’s position-referred noise spectral density, $S_{\text{SQL}} = 2\hbar/(m\Omega^2)$ is the free-mass SQL for position measurement, Ω_{eff} is the eigenfrequency of the controlled oscillator, and Q_{eff} is its quality factor. Extrapolation to $N_{\text{eff}} < 1$ favors $Q_{\text{eff}} \ll 1$; yet this is exactly where the approximation fails, and a more careful treatment of optimal control is required. Moreover, we must answer the following question: *does the strong continuous position measurement required for making the stiff electro-optical potential always produce significant decoherence of the oscillator’s quantum state?*

The cold damping technique used in Ref. [4] was first proposed by Mancini et al. [5] and demonstrated experimentally by Cohadon et al. [6]. Subsequent experiments have used cold damping to cool mechanical oscillators with the goal of reaching the quantum ground state [4, 6, 7, 8, 9, 10, 11, 12], typically using simple (proportional plus derivative) filters. Little attention has

been paid to the question of whether these filters are optimal. Furthermore, feedback control was used only for damping in these experiments, and not for shifting the resonant frequency of the oscillator, as in the LIGO experiment [4]. In this Letter, we obtain the optimal state-preparation control scheme for a general linear measurement process (possibly non-Markovian), and study prospects of quantum-state preparation for kg-scale test masses in future LIGO detectors via feedback control. Our general theory also applies to other mechanical structures [6, 7, 8, 9, 10, 11, 12].

It turns out that occupation number is not necessarily a good measure of “quantum-ness”: squeezed states, for example, can have high occupation number, yet they are more “quantum” than a vacuum state. Moreover, the definition of occupation number requires a well-defined real-valued eigenfrequency, which can be ambiguous for two reasons: (i) the controller can modify the oscillator’s original eigenfrequency ω_p ; and (ii) for oscillators with a finite quality factor Q_{eff} , the choice for an effective real eigenfrequency Ω_{eff} would be ambiguous by $\sim \Omega_{\text{eff}}/Q_{\text{eff}}$. Instead we use the *purity*, defined as

$$U \equiv \sqrt{V_{xx}V_{pp} - V_{xp}^2} \geq \hbar/2, \quad (1)$$

where V_{xx} , V_{pp} and V_{xp} are uncertainties of oscillator’s position and momentum along with their cross-correlation. For steady states, which have $V_{xp} = 0$, U can be converted to an effective occupation number,

$$N_{\text{eff}} \equiv U/\hbar - 1/2, \quad (2)$$

which is the *minimum* occupation number one could obtain when the same mirror state is put into a quadratic potential well with an arbitrary eigenfrequency Ω :

$$N_{\text{eff}} + 1/2 = \min_{\Omega} [(V_{pp}/(2m) + m\Omega^2 V_{xx}/2) / (\hbar\Omega)], \quad (3)$$

achieved at $\Omega = \sqrt{V_{pp}/(m^2 V_{xx})}$. Since Ω may be very different from the oscillator eigenfrequency, the resulting quantum state tends to be *position squeezed* if $\Omega > \omega_p$, and *momentum squeezed* if $\Omega < \omega_p$ [17]. N_{eff} also determines the von Neumann entropy of the state: $S = (N_{\text{eff}} + 1) \log(N_{\text{eff}} + 1) - N_{\text{eff}} \log N_{\text{eff}}$.

General Optimal Controller. A block diagram of the entire measurement-control system is shown in Fig. 1, where x is the position of the oscillator, y is the output field we measure, H is the measurement transfer function, $\tilde{R}_{xx} = -1/[(\Omega - \omega_p + i\gamma_p)(\Omega + \omega_p + i\gamma_p)]$ is the response function of the oscillator (with $\gamma_p = \omega_p/Q_p$ its relaxation rate; the tilde denotes a frequency-domain quantity), F and Z are force and sensing noises, G is a possible classical force acting on the oscillator, and C is the feedback kernel. We write the closed-loop position and momentum of the oscillator as

$$\tilde{x}_{\text{ctrl}} = \tilde{x}_0 - \tilde{K}_{\text{ctrl}} \tilde{y}_0, \quad \tilde{p}_{\text{ctrl}} = \tilde{p}_0 + i\Omega \tilde{K}_{\text{ctrl}} \tilde{y}_0 \quad (4)$$

where x_0, p_0 are the open-loop evolution of the oscillator position and momentum (we set its mass equal to 1, and use $\tilde{p} = -i\Omega \tilde{x}$), y_0 is the open-loop out-going field, and

$$\tilde{K}_{\text{ctrl}} = \tilde{R}_{xx} \tilde{C} / (1 + \tilde{R}_{xx} \tilde{C} \tilde{H}). \quad (5)$$

The closed-loop dynamics is *stable* and the feedback \tilde{C} *proper*, if and only if (i) \tilde{K}_{ctrl} is causal (i.e., no poles in the upper-half complex plane); and (ii) $\lim_{\Omega \rightarrow \infty} \Omega \tilde{K}_{\text{ctrl}}(\Omega) = 0$. The closed-loop response of the oscillator's position to external force is given by

$$\tilde{R}_{xx}^{\text{eff}} = \tilde{R}_{xx} (1 - \tilde{H} \tilde{K}_{\text{ctrl}}). \quad (6)$$

In Eq. (4), closed-loop quantities are viewed as subtracting the open-loop readout field from open-loop quantities. On the other hand, causal Wiener filters \tilde{K}_a ($a = x, p$) can be constructed based on the cross spectral density between x_0, p_0 and y_0 , to yield the best (least-mean-square) estimates of x_0 and p_0 based on past measurement of y_0 [16]. In terms of these filters, we have

$$\begin{bmatrix} V_{xx}^{\text{ctrl}} \\ V_{pp}^{\text{ctrl}} \end{bmatrix} = \begin{bmatrix} V_{xx}^c \\ V_{pp}^c \end{bmatrix} + \int_0^\infty \frac{d\Omega}{2\pi} \begin{bmatrix} |\tilde{K}_{\text{ctrl}} - \tilde{K}_x|^2 \\ |i\Omega \tilde{K}_{\text{ctrl}} - \tilde{K}_p|^2 \end{bmatrix} S_{yy}, \quad (7)$$

where S_{yy} is the single-sided spectral density of y_0 [18]. Minimizing over all stable closed-loop systems, we obtain

$$U_{\text{opt}} \equiv \min_{K_{\text{ctrl}}} \sqrt{V_{xx}^{\text{ctrl}} V_{pp}^{\text{ctrl}}} = \sqrt{V_{xx}^c V_{pp}^c} + V_{xp}^c, \quad (8)$$

which is achieved by a unique controller with

$$\tilde{K}_{\text{ctrl}} = \frac{1}{\tilde{\phi}_+(\Omega)} \left[\tilde{G}_x(\Omega) - \frac{G_x(0)}{\rho - i\Omega} \right], \quad \rho = \sqrt{\frac{V_{pp}^c}{V_{xx}^c}}. \quad (9)$$

Here $\tilde{\phi}_+^{-1}(\Omega)$ is a *causal whitening filter* for the output field y_0 , with $\tilde{\phi}_+^* \tilde{\phi}_+ = S_{yy}$ and both $\tilde{\phi}_+$ and $\tilde{\phi}_+^{-1}$

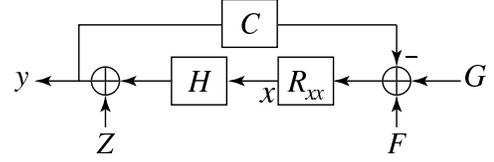


FIG. 1: Block diagram of the feedback control system.

analytic in the upper-half complex plane, while $\tilde{G}_x \equiv [S_{xy}/\phi_+^*]_+$, where S_{xy} is the cross spectral density between x_0 and y_0 , and $[\tilde{F}]_+$ denotes *extracting the causal part of \tilde{F}* while F^* stands for the complex conjugate of F . More specifically, under the scaling transform $\{x', p'\} = \{x/\sqrt{V_{xx}^c}, p/\sqrt{V_{pp}^c}\}$, the error ellipse of the optimally controlled state becomes a circle with $V_{x'x'}^{\text{ctrl}} = V_{p'p'}^{\text{ctrl}} = 1 + V_{x'p'}^c$, which is in turn equal to the larger eigenvalue of the re-scaled conditional covariance matrix. Thus the error ellipse of the optimally controlled state is the one with the minimum area among those that (i) totally encompass the conditional-state error ellipse; and (ii) are consistent with $V_{xp}^{\text{ctrl}} = 0$. This means the controlled state is always a mixed state, unless $V_{xp}^c = 0$ [19].

General Markovian measurements. We consider the open-loop system (Fig. 1)

$$y = Z + x, \quad x = \tilde{R}_{xx}(F + G) \quad (10)$$

where we have set $H = 1$ (without loss of generality), and Z and F are now characterized by *real and constant* (single-sided) cross spectral densities S_{ZZ} , S_{ZF} and S_{FF} satisfying the Heisenberg Uncertainty Relation [2]

$$\sqrt{S_{ZZ} S_{FF} - S_{ZF}^2} \equiv \mu \hbar, \quad (11)$$

where $\mu \geq 1$ measures the *purity* of the measurement process (with $\mu = 1$ corresponding to quantum-limited measurement). This describes measurement systems with white sensing and force noises, e.g., measurement of mirror location using a broadband Fabry-Perot cavity, with frequency independent input squeezing and homodyne detection. We recast S_{yy} into a *causally factorized form*:

$$\begin{aligned} S_{yy} &= S_{ZZ} + 2\text{Re}(\tilde{R}_{xx}) S_{ZF} + S_{FF} |\tilde{R}_{xx}|^2 \\ &\equiv S_{ZZ} L L^* / (P P^*). \end{aligned} \quad (12)$$

Here $P \equiv -1/\tilde{R}_{xx}$, $L L^* \equiv \Omega^4 - 2A\omega_p^2 \Omega^2 + B^2 \omega_p^4$,

$$A \equiv 1 + \frac{1}{\omega_p^2} \frac{S_{ZF}}{S_{ZZ}}, \quad B^2 \equiv 1 + \frac{2}{\omega_p^2} \frac{S_{ZF}}{S_{ZZ}} + \frac{1}{\omega_p^4} \frac{S_{FF}}{S_{ZZ}}. \quad (13)$$

L is defined here in such a way that it only has roots in the upper-half complex plane [20]. Using the definition for \tilde{K}_a and Eqs. (12), (13), the conditional covariance matrix can be obtained:

$$\mathbf{V} = \frac{\hbar \mu}{2} \begin{bmatrix} \frac{1}{\omega_p} \sqrt{\frac{2}{A+B}} & \sqrt{\frac{B-A}{B+A}} \\ \sqrt{\frac{B-A}{B+A}} & \omega_p \sqrt{\frac{2B^2}{A+B}} \end{bmatrix} \quad (14)$$

The conditional purity is given by

$$U_c = \sqrt{|\mathbf{V}|} = \mu\hbar/2, \quad (15)$$

which is *identical* to the ‘‘purity’’ of the measurement process; see Eq. (11). In the absence of classical noise, the conditional quantum state of the oscillator is always pure. With Eq. (8), we obtain

$$U_{\text{ctrl}}/(\hbar/2) = \mu(\sqrt{1 - A/B} + \sqrt{2})/\sqrt{1 + A/B} \quad (16)$$

which is achieved by the unique optimal filter [cf. (5) and (9)] with associated closed-loop response function

$$\tilde{C} = C_0(\Omega - C_1)/(\Omega - C_2), \quad (17)$$

$$\tilde{R}_{xx}^{\text{eff}} = -(\Omega - \Omega_4)/[(\Omega - \Omega_1)(\Omega - \Omega_2)(\Omega - \Omega_3)], \quad (18)$$

where $C_0 = -(\omega_p^2 + \Omega_4\Omega_3)$, $C_1 = (\Omega_3^3 + \omega_p^2\Omega_4)/(\omega_p^2 + \Omega_4\Omega_3)$, $C_2 = \Omega_4$ and $\Omega_{1,2}$ are roots of Q , namely

$$\Omega_{1,2} = \pm\omega_p\sqrt{(B+A)/2} - i\omega_p\sqrt{(B-A)/2} \quad (19)$$

while $\Omega_{3,4}$ are purely imaginary:

$$\Omega_3 = -i\sqrt{B}\omega_p, \quad \Omega_4 = -i[\sqrt{B} + \sqrt{2(B-A)}]\omega_p. \quad (20)$$

We note that: (i) the poles of the closed-loop dynamics, $\Omega_{1,2}$, are identical to the zeros of S_{yy} , i.e., the optimal controller ‘‘finds’’ the frequency of maximal sensitivity, and shifts the oscillator’s eigenfrequency there; (ii) the Ω that achieves N_{eff} in Eq. (3) and motivates N_{eff} as an occupation number in a harmonic potential, is equal to $|\Omega_{1,2}|$, the modulus of the closed-loop poles; (iii) the optimal controller (17) is in fact proportional feedback plus constant damping and simple band limiting (which is required for V_{pp}^{ctrl} to be finite), justifying previous choices [4, 6, 7, 8, 9, 10, 11, 12]; and (iv) a pure state is only strictly achievable when $\mu \approx 1$ (i.e., in the absence of classical noise) and $A \approx B$ (in general $|A| \leq B$).

Having $A \approx B$ corresponds to a high quality factor for the closed-loop dynamics [Cf. Eq. (19)], with

$$Q_{\text{eff}} = \sqrt{B+A}/(2\sqrt{B-A}). \quad (21)$$

This is consistent with our understanding that a low- Q oscillator cannot have a pure quantum state due to the Fluctuation Dissipation Theorem [2]. Moreover, $A \approx B$ also corresponds to S_{yy} having a very small minimum [in the limit of $A = B$, S_{yy} reaches 0; cf. Eq. (12)]. Let us consider the force (G)-referred noise spectrum,

$$S_G = S_{yy}/|\tilde{R}_{xx}|^2 = S_{ZZ}LL^*, \quad (22)$$

and compare it with the free-mass SQL for force detection $S_G^{\text{SQL}} = 2\Omega^2\hbar$. Taking the minimum over all frequencies, we obtain the factor by which the SQL is beaten:

$$\eta^2 \equiv [S_G/S_G^{\text{SQL}}]_{\text{min}} = \mu/(2Q_{\text{eff}}) \quad (23)$$

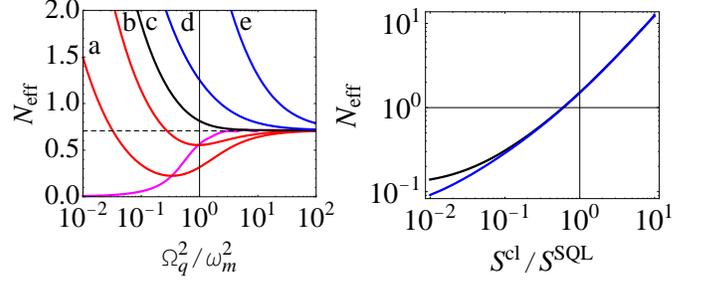


FIG. 2: *Left panel:* N_{eff} as a function of measurement strength Ω_q^2/ω_p^2 . Curves (a-e) represent N_{eff} at different temperatures θ : a) 0.1, b) 0.5, c) 1 (critical), d) 2, e) 10. *Right panel:* Minimal N_{eff} as a function of factor η_{cl}^2 for 10 dB squeezing (blue) and vacuum input (black).

which leads to [cf. Eq. (16)]

$$\frac{U_{\text{ctrl}}}{\hbar/2} = \eta^2 + \frac{\sqrt{2}\mu}{\sqrt{1+A/B}} \geq \eta^2 + 1, \quad N_{\text{eff}} \geq \eta^2/2. \quad (24)$$

This means an oscillator under measurement can only be converted into a quantum oscillator via control if it can beat the free-mass SQL in a force measurement, in which case the optimally-controlled closed-loop quality factor Q_{eff} would also far exceed unity.

Position measurement with light. In the realistic case with suspension and internal thermal noises and optical loss, we have (as in Ref. [14])

$$x = \tilde{R}_{xx}[\alpha a_1 + \xi_F + G] \quad (25a)$$

$$y = a_1 \sin \phi + \cos \phi [a_2 + \alpha/\hbar(x + \xi_x)] + \sqrt{\epsilon}n \quad (25b)$$

or in the notation of Eq. (10):

$$F = \alpha a_1 + \xi_F, \quad Z = \xi_x + \frac{a_1 \sin \phi + a_2 \cos \phi + \sqrt{\epsilon}n}{\alpha/\hbar \cos \phi}, \quad (26)$$

where $a_{1,2}$ are the input quadrature fields, ϕ is the read-out phase (0 for phase quadrature and $\pi/2$ for amplitude quadrature), α is the measurement strength ($\alpha = 4\sqrt{\hbar\omega_0 I_c}/(\tau c^2)$ for a Michelson interferometer with arm cavities, with ω_0 the carrier frequency, I_c the circulating power in the arms, τ the input-mirror power transmissivity, c the speed of light), ϵ is the optical loss, and n is the vacuum noise. ξ_x and ξ_F are the classical sensing and force noises, respectively, whose spectra cross the position- and force-SQL at frequencies Ω_q/ζ_x and $\Omega_q\zeta_F$, respectively, where $\Omega_q \equiv \alpha/\sqrt{\hbar}$ is the characteristic measurement frequency (as defined in Ref. [14]).

Viscosity noise alone and phase-quadrature readout. Here we consider the cold damped systems of Refs. [4, 6, 7, 8, 9, 10, 11, 12], i.e. $\epsilon = \zeta_x = \phi = 0$, $\omega_p \neq 0$, $\zeta_F^2 = 4\gamma_p k_B T/(\Omega_q^2 \hbar)$. Since a free mass under such measurements never beats the free-mass SQL, according to Eq. (24), it will always have $N_{\text{eff}} \geq 1/2$. In

order to have a lower N_{eff} , we have to rely on beating the free-mass SQL around the oscillator's original resonant frequency, which is only possible when $k_B T / (\hbar \omega_p Q_p) < 1/2$. Analytic results reveal a phase-transition-like situation: below a *critical temperature*, with

$$\theta \equiv T/T_c < 1, \quad T_c \equiv \hbar \omega_p Q_p / (2\sqrt{2}k_B), \quad (27)$$

the minimum occupation number

$$N_{\text{opt}}(\theta) = 2^{-3/2} [\sqrt{2 - \theta^2} + \sqrt{2\theta\sqrt{2 - \theta^2} + \theta - \sqrt{2}}] \quad (28)$$

$N_{\text{opt}} \sim 2^{-3/4} \theta^{1/2}$ as $\theta \rightarrow 0$, can be achieved with

$$\frac{\Omega_q}{\omega_p} = \sqrt{\frac{\theta^{1/2}(2 - \theta^2)^{3/4} - \theta}{\sqrt{2 - \theta^2} - \theta} - \frac{\theta}{\sqrt{2}}} \sim (\sqrt{2}\theta)^{1/4}, \quad (\theta \rightarrow 0), \quad (29)$$

while for $T > T_c$, a temperature-independent minimum of $N_{\text{eff}} = 1/\sqrt{2}$ is achieved with infinite measurement strength as indicated in Fig. 2 (left panel).

For a LIGO detector, $T \gg T_c$, so $N_{\text{eff}} \geq 1/\sqrt{2}$ even if viscous damping alone is considered. For the systems of Refs. [6, 7, 8, 9], prospects for surpassing $T_c \simeq 17 \text{ K} \times (Q_p/10^6) \times [\omega_p/(2\pi \times 1 \text{ MHz})]$ are much more promising.

Prospects for LIGO ($\omega_p = 0$). In order to evade the limitation of $N_{\text{eff}} \geq 1/\sqrt{2}$, we consider an arbitrary readout quadrature with squeezed light input. Given a classical noise budget, with vacuum input or fixed squeezing factor (10 dB), we optimize Ω_q , ϕ and the squeezing angle for a minimum N_{eff} . In the right panel of Fig. 2, we plot N_{eff} as a function of the factor $\eta_{\text{cl}}^2 \equiv [S^{\text{cl}}/S^{\text{SQL}}]_{\text{min}} = 2\zeta_F \zeta_x$ by which the total classical noise beats the SQL [14], while fixing the optical loss $\epsilon = 0.01$. The input squeezing appears to be not so crucial and almost the same results can be achieved without it. A detailed optimization will be presented elsewhere.

Summary In this Letter, we developed the general theory for optimal state-preparation via linear feedback control. The optimally controlled x - p error ellipse, achievable using the *unique* optimal controller, is the one with minimum area that still maintains $V_{xp} = 0$ and encompasses the conditional-state error ellipse. For a general Markovian measurement process, the conditional-state purity equals that of the measurement [cf. Eq. (15)], and the absence of classical noise guarantees a pure conditional state; yet a nearly pure controlled state requires additionally that correlations exist between sensing and force noises, in such a way that the device beats the free-mass SQL significantly in a force measurement. In this case, the optimal controller creates an oscillator with an eigenfrequency Ω_{eff} around the most sensitive frequency of the device, with quality factor $Q_{\text{eff}} \gg 1$ and an effective occupation number $N_{\text{eff}} \ll 1$. This N_{eff} is also meaningfully measured against a harmonic potential with real-valued eigenfrequency $\Omega \approx \Omega_{\text{eff}}$. Furthermore, restricting to conventional measurements applied to a viscously

damped oscillator, we found a critical temperature T_c , above which the oscillator is limited by $N_{\text{eff}} \geq 1/\sqrt{2}$, while below which, N_{eff} approaches 0 as $T/T_c \rightarrow 0$. Finally, for LIGO, the answer to the question raised at the beginning of this Letter is “yes” for conventional phase-quadrature readout, where the need to up-shift the oscillator's eigenfrequency does limit N_{eff} above $1/\sqrt{2}$, and “no” for an optimal choice of measurement strength and (non-phase) readout quadrature (possibly in combination with input squeezing), where the sub-SQL classical noise budget of future detectors will allow an optimally controlled kg-scale oscillator to achieve $N_{\text{eff}} \ll 1$.

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 - [17] In Refs. [4, 6, 7, 8, 9, 10, 11, 12], either kinetic or total energy is compared with quanta of the original oscillator; this will become ambiguous when Q_{eff} becomes low, and irrelevant when Ω_{eff} is shifted significantly.
 - [18] V_{ab}^c and V_{ab}^{ctrl} ($a, b = x, p$) stand for uncertainties of conditional and controlled states accordingly.
 - [19] $V_{xp} = 0$ corresponds to $G_x(0) = 0$, which in turn requires that no information about $x(t)$ be collected at time $t - 0$. This point will be elaborated in a future publication.
 - [20] We have set $\gamma_p \rightarrow 0$ in the definition of A and B .