Search for Continuous Gravitational Waves: simple criterion for optimal detector networks

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We derive a simple algebraic criterion to select the optimal detector network for a coherent wide parameter-space (all-sky) search for continuous gravitational waves. Optimality in this context is defined as providing the highest (average) sensitivity per computing cost. This criterion is a direct consequence of the properties of the multi-detector $\mathcal{F}$-statistic metric, which has been derived recently. Interestingly, the choice of the optimal network only depends on the noise-levels and duty-cycles of the respective detectors, and not on the available computing power.

1. Multi-detector matched filtering

The $\mathcal{F}$-statistic[2] is a coherent matched-filtering detection statistic for continuous gravitational waves (GWs). We follow the expressions and notation of our previous work[1] (Paper I) on the multi-detector $\mathcal{F}$-statistic metric. We consider a set of N detectors with (uncorrelated) noise power-spectra S_{X}, where X is the detector index, $\mathrm{X} = 1, ..., N$. Let T be the total observation time spanned by the data to be analyzed. The corresponding multi-detector scalar product for narrow-band continuous waves can be written as

$$(\boldsymbol{x}|\boldsymbol{y}) = T\mathcal{S}^{-1} \left\langle x\, y \right\rangle_S , \tag{1}$$

where boldface notation denotes multi-detector vectors, i.e. $\{\boldsymbol{x}(t)\}^{\mathrm{X}} = x^{\mathrm{X}}(t)$. We can allow for the fact that each detector will be in lock only for a duration $T_{\mathrm{X}} \le T$, so each detector can be characterized by a "duty cycle", $d_{\mathrm{X}} \equiv T_{\mathrm{X}}/T \le 1$. This is a slight, but straightforward generalization with respect to Paper I, and the corresponding noise-weighted time average $\langle . \rangle_S$ is defined as

$$\langle Q \rangle_S \equiv \frac{1}{T} \sum_{\mathrm{X}} w_{\mathrm{X}} \int_0^T Q^{\mathrm{X}}(t)\, dt , \tag{2}$$

where the weights w_{X} and the total inverse noise-power $\mathcal{S}^{-1}$ are defined as

$$w_{\mathrm{X}} \equiv d_{\mathrm{X}} \frac{S_{\mathrm{X}}^{-1}}{\mathcal{S}^{-1}} , \quad \text{where} \quad \mathcal{S}^{-1} \equiv \sum_{\mathrm{X}=1}^{N} d_{\mathrm{X}}\, S_{\mathrm{X}}^{-1} . \tag{3}$$

The importance of Eq. (1) is that it separates out the *scaling* with the total observation time T and the set of detectors (via $\mathcal{S}^{-1}$), from the averaged contribution $\langle x\, y \rangle_S$, which does not scale with T or the number of detectors. In terms of this scalar product (1), the optimal signal-to-noise ratio (SNR) for a perfectly-matched signal $\boldsymbol{s}(t)$ can be obtained as

$$\rho(0) = \sqrt{(\boldsymbol{s}|\boldsymbol{s})} = \sqrt{T\,\mathcal{S}^{-1}}\, \sqrt{\langle s^2 \rangle_S} . \tag{4}$$

It is obvious from this expression that the SNR increases when increasing the observation time T or the number of detectors N. However, here we are interested in the case of *wide parameter-space* searches, in which the highest achievable SNR is computationally limited. We therefore need to find the optimal sensitivity *per computing cost*.

2. Optimizing sensitivity per computing cost

For simplicity we only consider the sensitivity to an "average" sky-position, so we disregard the dependence of $\langle s^2 \rangle_S$ to both the sky-position as well as the relative orientation of the different detectors. Both should be small effects on average. In addition to Eq. (4) for the SNR, the second ingredient for the optimal network is the computing cost of a wide-parameter search. For the sake of example we consider a search for GWs from unknown isolated neutron stars, with unknown intrinsic GW frequency f, sky-position α, δ and one spindown-parameter $\dot{f}$. One can show[1] that in this case the number of required templates $\mathcal{N}_p$ scales (at least) as $\mathcal{N}_p \propto T^6$, which severely limits the computationally affordable observation time $T_{\max}$. Most importantly, however, the number of templates does *not* scale with the number N of detectors.[1] The corresponding computing cost $\mathcal{C}_p$ required to search these $\mathcal{N}_p$ templates can be estimated as $\mathcal{C}_p \propto N\,T^7$ for a "straightforward" computation, while it could be reduced down to about $\mathcal{C}_p \propto N\,T^6$ if the FFT-algorithm is used.[2] Generally, we can write

$$\mathcal{C}_p \propto N\,T^{\kappa}\,, \tag{5}$$

where typically $\kappa \sim 6-7$ for isolated neutron-star searches. The linear scaling with N comes from the fact that we need to compute the correlation of each template with each of the N detector time-series $x^{\mathrm{X}}(t)$.

The question we are trying to answer is the following: for given computing power $\mathcal{C}_p$ and a set of N detectors, which (sub)-set of $\hat{N} \le N$ detectors $\{\hat{\mathrm{X}}\} \subseteq \{\mathrm{X}\}$ yields the highest SNR? Using (5), we can express $T_{\max} \propto (\mathcal{C}_p/N)^{1/\kappa}$, and inserting this into (4), we find $\rho(0) \propto \mathcal{C}_p^{1/(2\kappa)}\,\sqrt{\gamma(\{\mathrm{X}\})}$, where the "gain function" γ is defined as

$$\gamma(\{\mathrm{X}\}) \equiv N^{-1/\kappa} \sum_{\mathrm{X}=1}^{N} d_{\mathrm{X}}\, S_{\mathrm{X}}^{-1}\,. \tag{6}$$

This simple algebraic function provides the sought-for criterion for the optimal detector-network $\{\hat{\mathrm{X}}\}$, depending only on the respective noise-floors S_{X} and duty-cycles d_{X}. The optimal detector network is simply the subset $\{\hat{\mathrm{X}}\}$ of detectors that maximizes the gain-function $\gamma(\{\hat{\mathrm{X}}\})$.

This optimal subset can be found in the following simple way: we label the detectors X in order of *decreasing* $d_{\mathrm{X}} S_{\mathrm{X}}^{-1}$, and include exactly the first $\hat{\mathrm{X}} = 1, ..., \hat{N}$ detectors in (6) where γ reaches a maximum. It is easy to see that this arrangement is optimal, as either adding further detectors, or replacing any term $d_{\hat{\mathrm{X}}} S_{\hat{\mathrm{X}}}^{-1}$ in the sum by another detector $\mathrm{X}' > \hat{N}$ reduces γ.

In the special case of identical detectors, the gain function γ is strictly monotonic with N, and so the optimal network simply consists of using as many detectors as possible, reducing the observation time T.

3. Example application

As an example, consider a set of "typical" detectors as given in Table 1. The assumed parameters are: LIGO (H1, H2, L1) at design sensitivity, with S5 duty-cycles, GEO (G1) at S5 sensitivity, and S4 duty-cycle, Virgo (V2) at design sensitivity, assuming a "typical" LIGO duty-cycle. We see that our simple criterion tells us that for a

Table 1. Example set of detectors with "typical" sensitivities and duty–cycles.

	Frequency	H1	L1	H2	G1	V2
$d_{\rm X}$	—	0.71	0.59	0.78	0.97	0.7
$\sqrt{S_{\rm X}}$ $[10^{-23}/\sqrt{\rm Hz}]$	$f = 200$ Hz	2.9	2.9	5.8	73	4.4
$\sqrt{S_{\rm X}}$ $[10^{-23}/\sqrt{\rm Hz}]$	$f = 600$ Hz	7.5	7.5	15	39	5.5

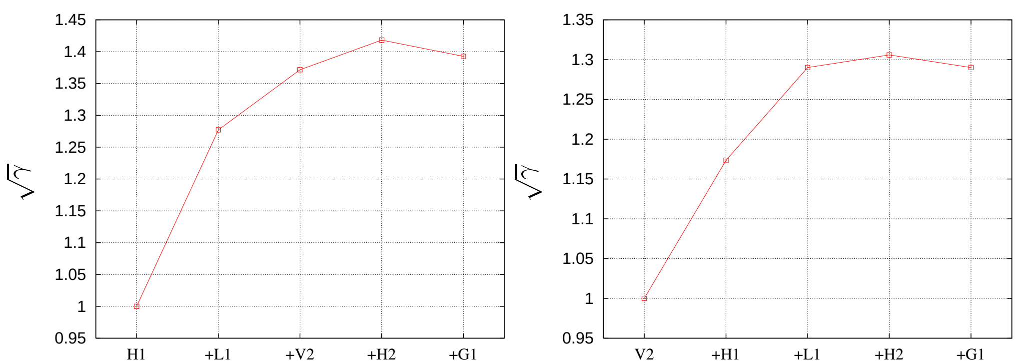

Fig. 1. SNR gain $\sqrt{\gamma(\{X\})}$ (assuming $\kappa = 6$) as a function of the detector network, normalized to the single-detector case *Left figure:* at f = 200 Hz. *Right figure:* at f = 600 Hz.

search at $f = 200$ Hz we should include H1, L1, V2, and H2 for the best all-sky sensitivity per computing cost, gaining on average a total of about 40% in SNR over H1 alone. Similarly, at $f = 600$ Hz, we find the same set of detectors to be optimal, with H2 providing a smaller marginal improvement.

References

1. R. Prix, *Phys. Rev. D.* **75**, p. 023004 (2007), (preprint gr-qc/0606088).
2. P. Jaranowski, A. Królak and B. F. Schutz, *Phys. Rev. D.* **58**, p. 063001 (1998).