Template-based searches for gravitational waves: efficient lattice covering of flat parameter spaces

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Abstract. The construction of optimal template banks for matched-filtering searches is an example of the *sphere covering problem.* For parameter spaces with constant-coefficient metrics a (near-) optimal template bank is achieved by the A_n^* lattice, which is the best lattice-covering in dimensions $n \leq 5$, and is close to the best covering known for dimensions $n \leq 16$. Generally this provides a *substantially* more efficient covering than the simpler hyper-cubic lattice. We present an algorithm for generating lattice template banks for constant-coefficient metrics and we illustrate its implementation by generating A_n^* template banks in $n = 2, 3, 4$ dimensions.

1. Introduction

The detection of gravitational waves (GWs) in the noisy data of detectors ideally requires the knowledge of the signal waveform, in order to coherently correlate the data with the expected signal by *matched filtering*. Depending on the type of astrophysical sources considered, however, one typically only knows a parametrized family of possible waveforms (or approximations thereof). The unknown parameters of these waveforms could be, for example, the frequency and sky-position of spinning neutron stars, or the masses and spins of inspiralling compact binary systems. Parameter spaces of such wide-parameter searches typically have between one and four dimensions, depending on computational constraints and the amount of astrophysical information available to constrain the search space a-priori. In the case of GWs from general binary systems, however, the number of dimensions of the parameter space could be as large as 17.

Obviously one can only search a finite subset of points in this parameter space, and this subset constitutes the "template bank". The templates must *cover* the parameter space, i.e., they must be placed densely enough that no signal in this space can lose more than a certain fraction of its power (called *mismatch*) at the closest template. However, coherently correlating the data with every template is computationally expensive and increases the expected number of statistical false-alarm candidates. An *optimal* template bank therefore consists of the smallest possible number of templates that still guarantees that the worst-case mismatch does not exceed a given limit.

It was realized early on that a geometric approach is very useful to construct template banks, in particular the introduction of a parameter-space *metric* [3, 11] based on the mismatch. This provides a natural measure of distance in parameter space and allows one to "correctly" place templates, in the sense that the maximal mismatch is not exceeded. Less attention, however, was devoted to the problem of *optimally* placing templates once the metric is known. Early works have sometimes used a hyper-cubic template grid for illustrative purposes [11], or the problem was incorrectly referred to as a "sphere packing problem" [12, 5]. We see in the following that constructing an optimal template bank is an instance of the *sphere covering* problem, which is somewhat "dual" to the sphere packing problem. The full solution to the sphere covering problem in Euclidean space is only known in $n = 2$ dimensions, partial solutions (restricted to lattices) are known in $n \leq 5$ dimensions, while an optimal solution for higher dimensions is unknown (cf. [8, 14]). The main motivation of the present work is to develop a general method for constructing efficient template banks in dimensions $n \lesssim 17$ by using the known results about Euclidean sphere covering.

Previous related work on template banks includes studies to optimally cover *non-flat* two-dimensional parameter spaces arising in searches for GWs from inspiralling compact binary systems [4, 2]. An interesting algorithm to construct a hexagonal (A_2^*) template bank for 2D inspiral searches was described recently in [7]. Various codes exist within the LIGO Scientific Collaboration to generate hyper-cubic lattices (`LALCreateFlatMesh()` [10]), two-dimensional grids for non-constant metrics (`LALCreateTwoDMesh()` [10]), and a tree-dimensional template bank based on the bcc-lattice (`LALInspiralSpinBank()` [10]), which is being used in a search for spinning binary inspirals on LIGO Data [1].

2. Template-based searches and parameter-space metric

A wide class of searches for GWs can be characterized as *template based*, in the sense that one searches for signals belonging to a family of waveforms $s(t;\boldsymbol{\lambda})$, which depend on a vector of parameters $\{\boldsymbol{\lambda}\}^i = \lambda^i$. The strain $x(t)$ measured by a detector contains (usually dominating) noise $n(t)$ in addition to possible weak GW signals $s(t;\boldsymbol{\lambda}_\mathrm{s})$, i.e., $x(t) = n(t) + s(t;\boldsymbol{\lambda}_\mathrm{s})$. One typically constructs a *detection statistic*, $\mathcal{F}(\boldsymbol{\lambda};x)$ say, namely a scalar characterizing the probability of a signal with parameters $\boldsymbol{\lambda}$ being present in the data $x(t)$. Due to the random noise fluctuations $n(t)$ in the data, the detection statistic is a random variable, and generally (assuming $\mathcal{F}$ is unbiased) its expectation value $\overline{\mathcal{F}}(\boldsymbol{\lambda};\boldsymbol{\lambda}_\mathrm{s}) \equiv E[\mathcal{F}(\boldsymbol{\lambda};x)]$ has a (local) maximum at the location of the signal $\boldsymbol{\lambda} = \boldsymbol{\lambda}_\mathrm{s}$, i.e.,

$$\left.\frac{\partial\overline{\mathcal{F}}(\boldsymbol{\lambda};\boldsymbol{\lambda}_\mathrm{s})}{\partial\boldsymbol{\lambda}}\right|_{\boldsymbol{\lambda}=\boldsymbol{\lambda}_\mathrm{s}} = 0\,. \tag{1}$$

Taylor-expanding the expected detection-statistic $\overline{\mathcal{F}}$ in small offsets $\Delta\boldsymbol{\lambda} = \boldsymbol{\lambda} - \boldsymbol{\lambda}_\mathrm{s}$ around the signal location $\boldsymbol{\lambda}_\mathrm{s}$ therefore reads as

$$\overline{\mathcal{F}}(\boldsymbol{\lambda};\boldsymbol{\lambda}_\mathrm{s}) = \overline{\mathcal{F}}(\boldsymbol{\lambda}_\mathrm{s};\boldsymbol{\lambda}_\mathrm{s}) + \frac{1}{2}\left.\frac{\partial^2\overline{\mathcal{F}}(\boldsymbol{\lambda};\boldsymbol{\lambda}_\mathrm{s})}{\partial\lambda^i\,\partial\lambda^j}\right|_{\boldsymbol{\lambda}_\mathrm{s}} \Delta\lambda^i\,\Delta\lambda^j + \O(\Delta\lambda^3)\,, \tag{2}$$

where the matrix of second derivatives of $\overline{\mathcal{F}}$ is negative definite. Here and in the following we use automatic summation over repeated parameter indices $i, j, \ldots$. We can introduce a *mismatch* m, which characterizes the fractional loss in the expected value of the detection statistic, $\overline{\mathcal{F}}$, at a parameter-space point $\boldsymbol{\lambda}$, with respect to the signal location $\boldsymbol{\lambda}_\mathrm{s}$, namely

$$m(\boldsymbol{\lambda};\boldsymbol{\lambda}_\mathrm{s}) \equiv \frac{\overline{\mathcal{F}}(\boldsymbol{\lambda}_\mathrm{s};\boldsymbol{\lambda}_\mathrm{s}) - \overline{\mathcal{F}}(\boldsymbol{\lambda};\boldsymbol{\lambda}_\mathrm{s})}{\overline{\mathcal{F}}(\boldsymbol{\lambda}_\mathrm{s};\boldsymbol{\lambda}_\mathrm{s})} \,. \tag{3}$$

Using the local expansion (2), we find

$$m(\boldsymbol{\lambda};\boldsymbol{\lambda}_\mathrm{s}) = g_{ij}(\boldsymbol{\lambda}_\mathrm{s})\,\Delta\lambda^i\Delta\lambda^j + \mathcal{O}(\Delta\lambda^3)\,, \tag{4}$$

where we defined the positive-definite *metric tensor* $g_{ij} \equiv -\frac{1}{2}\partial_i\partial_j\overline{\mathcal{F}}$, and $\partial_i \equiv \partial/\partial\lambda^i$. When searching a parameter space $\mathbb{P}(\lambda^i,\, g_{ij})$, we need to compute the detection statistic $\mathcal{F}(x;\boldsymbol{\lambda}_\xi)$ for a discrete set of templates $\boldsymbol{\lambda}_\xi \in \mathbb{P}$. Generally one can distinguish two different approaches to this problem: one is a random sampling of $\mathbb{P}$ using Markov-chain Monte-Carlo (MCMC) algorithms (e.g. see [6, 9]), and the other consists of constructing a template bank $\mathbb{T} \equiv \{\boldsymbol{\lambda}_\xi\} \subset \mathbb{P}$ that *covers* the whole of $\mathbb{P}$, in the sense that no point $\boldsymbol{\lambda} \in \mathbb{P}$ exceeds a given maximal mismatch m_max to its closest template $\boldsymbol{\lambda}_\xi \in \mathbb{T}$, i.e.,

$$\max_{\boldsymbol{\lambda}\in\mathbb{P}}\ \min_{\boldsymbol{\lambda}_\xi\in\mathbb{T}} m(\boldsymbol{\lambda};\boldsymbol{\lambda}_\xi) \leq m_\mathrm{max}\,. \tag{5}$$

Here we focus on the construction of *optimal* template banks, namely those satisfying (5) with the smallest possible number of templates $\boldsymbol{\lambda}_\xi$. In the local metric approximation (4), each template $\boldsymbol{\lambda}_\xi$ *covers* a region B_ξ of parameter space, namely

$$B_\xi = \left\{\boldsymbol{\lambda}\in\mathbb{P} : g_{ij}(\boldsymbol{\lambda}_\xi)\,\Delta\lambda^i\,\Delta\lambda^j \leq m_\mathrm{max}\,,\quad \Delta\boldsymbol{\lambda} \equiv \boldsymbol{\lambda} - \boldsymbol{\lambda}_\xi\right\}\,, \tag{6}$$

which is a *sphere* of radius $R = \sqrt{m_\mathrm{max}}$ in the metric space $\mathbb{P}(\lambda^i,\, g_{ij})$. We can therefore reformulate the definition of an optimal template bank as the set of (overlapping) spheres of covering radius R which *cover* the whole of $\mathbb{P}$ in the sense of (5) with the smallest number of spheres. This is known as the *sphere covering problem* [8], not to be confused with the somewhat dual *sphere packing problem*, which seeks to pack the largest number of non-overlapping "hard" spheres into a given volume.

3. The Euclidean sphere covering problem

In this section we summarize the current status of the sphere covering problem as far as relevant for the construction of optimal template banks. There has been impressive progress in the study of the covering problem in recent years, e.g. see [8] for a general overview and [14] for a more recent update. Unfortunately, all of these studies are restricted to Euclidean spaces $\mathbb{E}^n$, while the metric parameter spaces of GW searches are often curved. In the following we will therefore make the assumption that $\mathbb{P}(\lambda^i,\, g_{ij})$ can be treated as at least *approximately* flat, or can be broken into smaller pieces that can be treated as nearly flat. If the curvature of the metric is too strong, i.e., if the curvature radius is comparable to the covering radius, it will be difficult to make use of the Euclidean covering problem, and a different approach such as a stochastic template bank or an MCMC sampling might be more effective. We further assume that we have found a coordinate system of $\mathbb{P}$ such that the metric components are (approximately) constant, i.e., $g_{ij}(\boldsymbol{\lambda}) \approx \mathrm{const}_{ij}$, and for simplicity of notation we assume in this section (without loss of generality) that we have chosen coordinates x^i in which the constant-coefficient metric is Cartesian, i.e., $\mathbb{P} = \mathbb{E}^n(x^i,\, \delta_{ij})$.

A covering can consist of any arrangement of covering spheres, but currently all best coverings known are *lattices*, and we therefore restrict the following discussion to *lattice coverings*.

3.1. Basics on lattices

An n-dimensional lattice Λ can be defined as a discrete set of points $\boldsymbol{\nu}_\xi$ (forming an additive group) generated by

$$\boldsymbol{\nu}_\xi = \xi^i \, \boldsymbol{l}_{(i)} \,, \quad \text{with} \quad \xi^i \in \mathbb{Z} \,, \tag{7}$$

with summation over $i = 1, \ldots, n$, and where $\{\boldsymbol{l}_{(i)}\}_{i=1}^n$ is a *basis* of the lattice. Note that it is sometimes convenient to express the n basis vectors in a higher-dimensional Euclidean space, i.e., generally we can have $\boldsymbol{l}_{(i)} \in \mathbb{E}^m$ with $m \geq n$. When writing $\mathbb{E}^n$ in the following we refer to the subspace of $\mathbb{E}^m$ containing the n-dimensional lattice Λ. The $m \times n$ matrix $M^a{}_i \equiv l^a_{(i)}$ is called a *generator matrix* of the lattice, with the columns of M holding the m components of the n lattice basis vectors, so we can also write the lattice Λ as

$$\Lambda = \{\boldsymbol{\nu}_\xi : \boldsymbol{\nu}_\xi = M \, \boldsymbol{\xi} \,, \;\; \boldsymbol{\xi} \in \mathbb{Z}^n\} \,. \tag{8}$$

The $n \times n$ matrix $A \equiv M^{\mathrm{T}} M$ is called the *Gram matrix* (where T denotes the transpose), which is symmetric and positive definite, and $A_{ij} = \boldsymbol{l}_{(i)} \cdot \boldsymbol{l}_{(j)} = \delta_{ab} \, l^a_{(i)} \, l^b_{(j)}$, i.e., its coefficients are the mutual scalar products of lattice basis vectors. Each choice

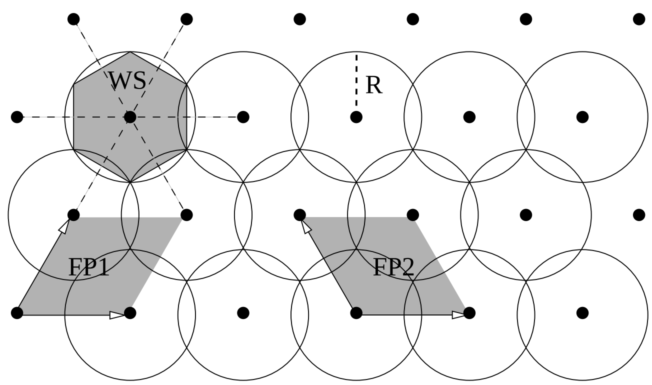

Figure 1. Hexagonal lattice (A_2^*) illustrating a 2-dimensional lattice covering. The shaded areas are different choices of fundamental regions for the lattice. FP1 and FP2 are fundamental polytopes (9) associated with different choices of lattice basis, WS is the Wigner-Seitz cell (11), and R is the covering radius.

of lattice basis $\{\boldsymbol{l}_{(i)}\}$ defines a corresponding fundamental parallelotope (FP), namely

$$\mathrm{FP}\left(\{\boldsymbol{l}_{(i)}\}\right) \equiv \left\{\boldsymbol{x} \in \mathbb{E}^n : \boldsymbol{x} = \theta^i \, \boldsymbol{l}_{(i)} \,, \;\; 0 \leq \theta^i < 1\right\} \,, \tag{9}$$

which is illustrated in figure 1. The FP is an example of a *fundamental region* for the lattice, i.e., a building block containing exactly one lattice point, which fills the whole space $\mathbb{E}^n$ when repeated. There are many different choices of basis and fundamental regions for the same lattice Λ, but they all have the same volume $\mathrm{vol}(\Lambda)$, given by

$$\mathrm{vol}(\Lambda) = \sqrt{\det A} \,, \tag{10}$$

and in the case where M is a square matrix we also have $\mathrm{vol}(\Lambda) = \det M$. One special choice of fundamental region is the *nearest-neighbor region*, often referred to as *Dirichlet-Voronoi* cell by mathematicians, and more commonly known as Wigner-Seitz cell or Brillouin zone by physicists, which is defined as

$$\mathrm{WS}(\Lambda) \equiv \{\boldsymbol{x} \in \mathbb{E}^n : \|\boldsymbol{x} - \boldsymbol{\nu}_0\| \leq \|\boldsymbol{x} - \boldsymbol{\nu}_\xi\|, \;\; \text{for all} \;\; \boldsymbol{\nu}_\xi \in \Lambda\} \,, \tag{11}$$

where $\|\boldsymbol{x}\| = \sqrt{\boldsymbol{x}\cdot\boldsymbol{x}}$ is the standard Euclidean norm in $\mathbb{E}^n$. The vertices of the Wigner-Seitz cell are by construction local maxima of the distance function of points in $\mathbb{E}^n$ from the nearest grid point. The maximum distance of any point in $\mathbb{E}^n$ to the nearest point of the lattice is called the *covering radius* R, which corresponds to the *circumradius* of WS, as seen in figure 1.

Two lattices Λ_1 and Λ_2 with generator matrices M_1 and M_2 are *equivalent* if they can be transformed into one another by a rotation, reflection and change of scale, namely if the generator matrices satisfy

$$M_2 = c\,B\,M_1\,U\,, \tag{12}$$

where $c \in \mathbb{R}$ is a scale-factor, U is integer-valued $\det U = \pm 1$, which accounts for different choices of basis vectors, and B is a real orthogonal matrix, i.e., $B^{\mathrm{T}}\,B = \mathbb{I}$. The associated Gram matrices are therefore related by

$$A_2 = c^2\,U^{\mathrm{T}}\,A_1\,U\,, \tag{13}$$

and the fundamental volumes (10) of the two lattices are

$$\mathrm{vol}(\Lambda_2) = c^n\,\mathrm{vol}(\Lambda_1)\,. \tag{14}$$

Let us consider as an example the 2-dimensional hexagonal lattice, illustrated in figure 1. An obvious generator matrix is

$$M_1 = \begin{pmatrix} 1 & 1/2 \\ 0 & \sqrt{3}/2 \end{pmatrix}, \tag{15}$$

corresponding to FP1 in figure 1. However, sometimes it is more convenient to work with a generator matrix of the form

$$M_2 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{pmatrix}, \tag{16}$$

which has simpler coefficients, but uses a 3-dimensional representation of the 2-dimensional lattice with all lattice points lying in the plane $x + y + z = 0$. One can verify that these two representations are equivalent in the sense of (12), namely with

$$c = \sqrt{2}\,,\quad U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\quad B = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{6} \\ -1/\sqrt{2} & -1/\sqrt{6} \\ 0 & \sqrt{2/3} \end{pmatrix}. \tag{17}$$

Such a higher-dimensional representation of the generator matrix will be useful later for the description of the n-dimensional A_n^* lattice.

3.2. Known results on optimal sphere covering

The efficiency of a sphere covering can be characterized by its *thickness* Θ (sometimes also referred to as the *covering density*), which measures the fractional amount of overlap between the covering spheres, or equivalently the average number of spheres covering any point in $\mathbb{E}^n$. This can be expressed as the ratio of the volume of one covering sphere to the volume of the fundamental region of the lattice, i.e.,

$$\Theta \equiv \frac{V_n\,R^n}{\mathrm{vol}(\Lambda)} \geq 1\,, \tag{18}$$

where R is the covering radius and V_n is the volume of the unit-sphere in n dimensions, namely $V_n = \pi^{n/2}/\Gamma(n/2+1)$. We also use the *normalized thickness* or *center density* θ, defined as

$$\theta \equiv \frac{\Theta}{V_n}\,, \tag{19}$$

which corresponds to the number of centers (i.e., templates) per unit volume in the case of $R = 1$. Note that under a lattice transformation (12), the covering radius R obviously scales as $R_2 = c\,R_1$, and we therefore see from (14) that the thickness (18) and (19) is an invariant property of a lattice, i.e., $\theta_2 = \theta_1$. The covering problem consists of finding the covering with the lowest center density θ.

Kershner showed in 1939 (see [8]) that in $n = 2$ dimensions the most economical arrangement of circles covering the plane is the hexagonal lattice, which is equivalent to an A_2^* lattice. In dimensions $n = 3, 4, 5$ only the best *lattice covering* is known, and is given by A_n^* in all three cases. In three dimensions, A_3^* is also known as the *body-centered-cubic* (bcc) lattice. Note that the best *packing* in $n = 2$ is also achieved by the hexagonal lattice, but for $n = 3$ the face-centered cubic (fcc) lattice provides a denser packing than bcc. In higher dimensions the best lattice coverings are currently still unknown, but the best coverings known can be found in see table 2 of [14], and [15] provides for an up-to-date online version. As will become clearer in the following, the A_n^* lattice, while no longer the "record holder" for most dimensions $5 < n \leq 17$, is still close to the best currently known covering in all cases. In the following we will therefore mostly focus on the A_n^* covering. The A_n^* lattice has a center density of

$$\theta(A_n^*) = \sqrt{n+1}\left\{\frac{n(n+2)}{12(n+1)}\right\}^{n/2}\,, \tag{20}$$

while for the hyper-cubic grid $\mathbb{Z}^n$ the Wigner-Seitz cell is a unit hypercube, so $\mathrm{vol}(\mathbb{Z}^n) = 1$, and the covering radius $R = \sqrt{n}/2$ is half the length of the diagonal. Therefore the center density (19) is found as $\theta(\mathbb{Z}^n) = 2^{-n}\,n^{n/2}$, which is dramatically worse than A_n^* in higher dimensions, as can be seen from the thickness ratio

$$\kappa(n) \equiv \frac{\theta(\mathbb{Z}^n)}{\theta(A_n^*)} = \frac{3^{n/2}}{\sqrt{n+1}}\left(\frac{n+1}{n+2}\right)^{n/2} \overset{n\to\infty}{\sim} \frac{3^{n/2}}{\sqrt{ne}}\,. \tag{21}$$

Table 1. Thickness ratio $\kappa(n) = \theta(\mathbb{Z}^n)/A_n^*$, and $\gamma(n) = \theta(\mathrm{best})/\theta(A_n^*)$ in dimensions $n \leq 17$.

n	2	3	4	5	6	7	8	9
$\kappa(n)$	1.3	1.9	2.8	4.3	6.8	10.9	17.7	28.9
$\gamma(n)$	1.0	1.0	1.0	1.0	0.97	0.95	0.86	0.97

n	10	11	12	13	14	15	16	17
$\kappa(n)$	47.4	78.2	130	216	359	601	1007	1692
$\gamma(n)$	0.98	0.88	0.99	0.86	0.82	0.86	1.0	0.68

There is a theoretical *lower* limit on the thickness of any covering, the Coxeter-Few-Rogers (CFR) bound τ_n (see [8]), i.e., $\theta_n \geq \tau_n/V_n$, where asymptotically $\tau_n \sim n/(e\sqrt{e})$ for $n \to \infty$. Figure 2 shows the normalized thickness θ as a function of dimension n for the A_n^* and hyper-cubic $\mathbb{Z}^n$ lattices, as well as the CFR bound and the best covering

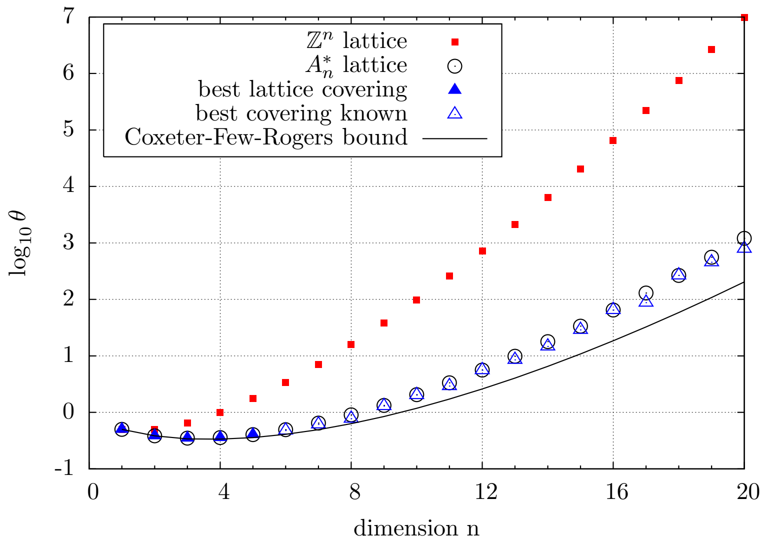

Figure 2. Normalized covering thickness θ as function of dimension n, for the hyper-cubic lattice ($\mathbb{Z}^n$), the A_n^* lattice, the theoretical lower bound (CFR), and the best lattice coverings *known*.

known. In table 1 we see that in dimensions $n > 5$ where A_n^* has been superseded as the best covering [15], the relative improvement $\gamma(n) \equiv \theta(\text{best})/\theta(A_n^*)$ in thickness is typically quite small. In particular, for $n \leq 16$ the improvement $\gamma(n)$ is typically less than 18%, while the advantage $\kappa(n)$ of A_n^* compared to the hyper-cubic grid $\mathbb{Z}^n$ grows large very rapidly, as seen in table 1 and figure 2. For practical simplicity we therefore propose to use A_n^* as the covering lattice of choice.

4. Lattice covering of template spaces

4.1. Template counting

The template spaces $\mathbb{P}(\lambda^i, g_{ij})$ with constant-coefficient metrics g_{ij} only differ from the Cartesian case of the previous section by a simple coordinate-transformation. An infinitesimal parameter-space region $d^n\lambda$ has a volume dV measured by the metric, namely $dV = \sqrt{g}\, d^n\lambda$, where $g \equiv \det g_{ij}$. The volume V of a finite region of parameter space is therefore

$$V = \int_{\mathbb{P}} dV = \sqrt{g} \int_{\mathbb{P}} d^n\lambda\,, \tag{22}$$

where we used the fact that g_{ij} is a constant-coefficient metric. The number of templates dN_p in dV is given by the inverse lattice volume, i.e.,

$$dN_p = \frac{dV}{\text{vol}(\Lambda)}\,. \tag{23}$$

Using the relation $R = \sqrt{m_{\text{max}}}$ together with (18), (19), we find

$$dN_p = \theta\, m_{\text{max}}^{-n/2}\, dV \quad \Longrightarrow \quad N_p = \theta\, m_{\text{max}}^{-n/2} \sqrt{g} \int_{\mathbb{P}} d^n\lambda\,, \tag{24}$$

which generalizes template counting [11, 12, 5] to arbitrary lattices.

4.2. Practical implementation of lattice covering

In this section we present a practical algorithm for generating lattices covering of given maximal mismatch $m_{\max}$. The approach described here works for any lattice generator M, but in practice (cf. section 3.2) we will be most interested in the A_n^* lattice. The generator for A_n^* can be expressed (cf. [8]) as an $(n+1) \times n$ matrix,

$$M^a{}_j(A_n^*) = \begin{pmatrix} 1 & 1 & \dots & 1 & \frac{-n}{n+1} \\ -1 & 0 & \dots & 0 & \frac{1}{n+1} \\ 0 & -1 & \dots & 0 & \frac{1}{n+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & -1 & \frac{1}{n+1} \\ 0 & 0 & \dots & 0 & \frac{1}{n+1} \end{pmatrix}, \tag{25}$$

where the columns of M hold the n lattice basis vectors $\boldsymbol{l}_{(j)}$ expressed in $\mathbb{E}^{n+1}$, i.e., $M^a{}_j = l^a_{(j)}$, with index conventions $i, j = 1, \dots, n$ and $a, b = 1, \dots, n+1$. The volume of the fundamental region and the covering radius for this generator are

$$\mathrm{vol}(A_n^*) = \frac{1}{\sqrt{n+1}}, \quad \text{and} \quad R(A_n^*) = \sqrt{\frac{n(n+2)}{12(n+1)}}, \tag{26}$$

which yields the (normalized) thickness $\theta(A_n^*)$ given in (20). In order to generate such a lattice in a parameter space $\mathbb{P}(\lambda^i, g_{ij})$, we need to express the generator $M^a{}_j$ in the λ^i coordinates, resulting in $\widetilde{M}^i{}_j$, say, such that the lattice of templates $\boldsymbol{\lambda}_\xi$ is generated by

$$\lambda^i_\xi = \widetilde{M}^i{}_j\, \xi^j, \quad \text{with} \quad \boldsymbol{\xi} \in \mathbb{Z}^n. \tag{27}$$

This coordinate transformation can be achieved in several steps:

(i) Reduce the $(n+1)\times n$ matrix $M^a{}_j$ to a full rank generator, $\widehat{M}^i{}_j$ say, by expressing the lattice basis vectors in a Euclidean basis spanning the n-dimensional subspace $\mathbb{E}^n$ of the lattice: a simple Gram-Schmidt procedure with respect to the Cartesian metric δ_{ab} is used on the $\{l^a_{(j)}\}$ to generate an orthonormal basis $\{e^a_{(j)}\}$ satisfying

$$\delta_{ab}\, e^a_{(i)}\, e^b_{(j)} = \delta_{ij}. \tag{28}$$

The full-rank generator $\widehat{M}^i{}_j$ is obtained from the components of the lattice vectors $\{l^a_{(i)}\}$ in this orthonormal basis, namely

$$\widehat{M}^i{}_j = \widehat{l}^i_{(j)} = l^a_{(j)}\, e^b_{(i)}\, \delta_{ab} = e_{(i)a}\, M^a{}_j. \tag{29}$$

(ii) Translate the full-rank generator $\widehat{M}^i{}_j$ from Cartesian coordinates into the coordinate system λ^i with metric g_{ij}. For this we use another Gram-Schmidt orthonormalization with respect to the metric g_{ij}, with the lattice vectors $\{\widehat{l}^j_{(i)}\}$ as input to find an orthonormal basis $\{d^j_{(i)}\}$ satisfying

$$g_{ij}\, d^i_{(l)}\, d^j_{(k)} = \delta_{lk}. \tag{30}$$

This representation of an orthonormal basis in coordinates λ^i allows us to express the lattice vectors in these coordinates as

$$\widetilde{l}^i_{(j)} = \widehat{l}^k_{(j)}\, d^i_{(k)} = d^i_{(k)}\, \widehat{M}^k{}_j. \tag{31}$$

(iii) Scale the generator to the desired covering radius $R = \sqrt{m_{\max}}$, and with (26) we find

$$\widetilde{M}^i{}_j = \sqrt{m_{\max}} \sqrt{\frac{12(n+1)}{n(n+2)}}\, \widetilde{l}^i_{(j)}\,, \tag{32}$$

which is a generator (27) for an A_n^* template lattice with maximal mismatch $m_{\max}$.

This algorithm has been implemented in `XLALFindCoveringGenerator()` in LAL [10], and some tests of this code are presented in the next section.

4.3. Tests of the implementation

In order to illustrate and test the implementation of this algorithm, we generate an A_n^* lattice in dimensions $n = 2, 3, 4$, respectively, with a maximal mismatch of $m_{\max} = 0.04$, i.e., a covering radius of $R = 0.2$. For generality we use a non-Cartesian metric $g_{ij} \neq \delta_{ij}$, as illustrated in the left panel of figure 3. We picked 100,000 points

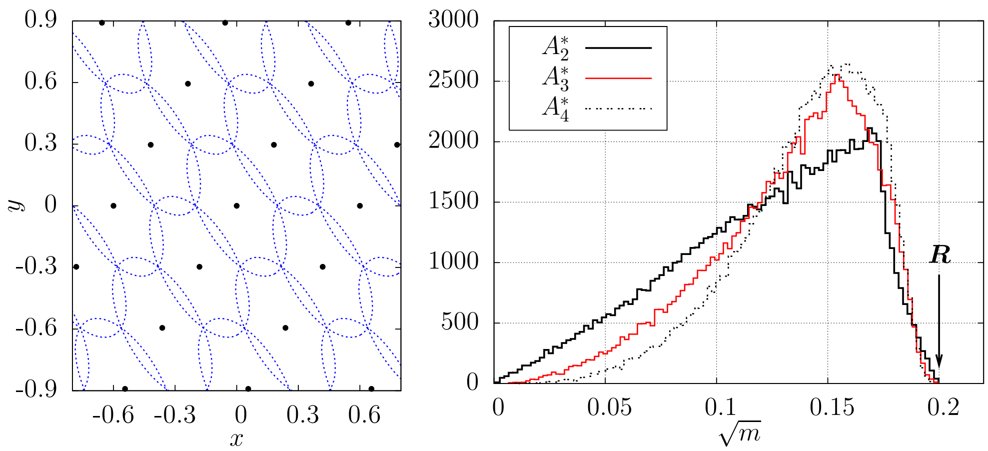

Figure 3. *Left panel:* Hexagonal (A_2^*) lattice covering in coordinates $\{x, y\}$ with metric $g_{ij} = [1,\ 0.4;\ 0.4,\ 0.5]$. *Right panel:* Histogram of measured distances $\sqrt{m}$ in a Monte-Carlo sampling of 100,000 points from an A_n^* covering in $n = 2, 3, 4$ dimensions, using non-Cartesian metrics g_{ij}. The nominal covering radius in all three cases was $R = \sqrt{m_{\max}} = 0.2$.

$\boldsymbol{\lambda} \in \mathbb{P}(\lambda^i,\, g_{ij})$ at random and computed their mismatch m (using the metric) to the nearest template $\boldsymbol{\lambda}_\xi$, which is a way of *measuring* the maximal mismatch of a template bank. The distribution of measured mismatch-distances $\sqrt{m}$ is plotted in the right-hand panel of figure 3, and we see that the mismatches are bounded by $\sqrt{m_{\max}} = 0.2$, satisfying (5). We can also measure the (normalized) thickness θ of the template bank, namely from the number of templates N_p in the covered parameter space $\Delta\boldsymbol{\lambda}^n$, we find using (24):

$$\theta = \frac{R^n}{\sqrt{g}}\, \frac{N_p}{\Delta\boldsymbol{\lambda}^n}\,. \tag{33}$$

These measured values of the thickness are found to agree to within 0.2% with the theoretical values (20) in all three cases $n = 2, 3, 4$. The generated template banks in this example have $N_p \sim \mathcal{O}(10^4)$ templates, and the error can most likely be attributed to boundary effects.

5. Discussion

Possible applications of this algorithm for GW searches can be found in template-based searches, such as for inspiralling compact binary systems and for "continuous waves", which in ground-based detectors refers mostly to signals from spinning neutron stars, and in the case of LISA includes white dwarf binaries, supermassive black hole binaries and extreme-mass ratio inspirals. The benefit of using this approach depends sensitively on the number of parameter-space dimensions, but can be estimated from table 1 at least in comparison to hypercubic grids.

However, the applicability of the lattice covering algorithm presented here is restricted to *explicitly flat* parameter spaces, which limits its usefulness to cases where we can find a coordinate system in which the parameter-space metric is (at least) *approximately* constant. The orbital metric approximation [13] for continuous GWs can be shown to be flat (work in progress), and would therefore be a natural case where this lattice covering could be used to greatest effect. One difficulty in this case, however, stems from that fact that the corresponding metric is found to be highly ill-conditioned, which results in the lattice-construction algorithm to fail due to numerical problems. One therefore needs to *analytically* "factor out" this near-degeneracy of the metric before this lattice-covering procedure can be safely applied. More work is also required to deal with non-trivial parameter-space boundaries, which complicates the n-dimensional filling algorithm.

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