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On Symmetry Enhancement in the $\mathfrak{psu}(1,1|2)$ Sector of $\mathcal{N}=4$ SYM

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Abstract

Strong evidence indicates that the spectrum of planar anomalous dimensions of $\mathcal{N}=4$ super Yang-Mills theory is given asymptotically by Bethe equations. A curious observation is that the Bethe equations for the $\mathfrak{psu}(1,1|2)$ subsector lead to very large degeneracies of 2^M multiplets, which apparently do not follow from conventional integrable structures. In this article, we explain such degeneracies by constructing suitable conserved nonlocal generators acting on the spin chain. We propose that they generate a subalgebra of the loop algebra for the $\mathfrak{su}(2)$ automorphism of $\mathfrak{psu}(1,1|2)$. Then the degenerate multiplets of size 2^M transform in irreducible tensor products of M two-dimensional evaluation representations of the loop algebra.

1 Introduction

Methods of integrability have become a central tool for investigating the dynamics of planar $\mathcal{N} = 4$ extended supersymmetric gauge theory and noninteracting strings on the $AdS_5 \times S^5$ background [1–3], cf. [4, 5] for reviews. Investigations of the S-matrix [6] have recently led to a highly nontrivial test of the AdS/CFT correspondence showing that it may correctly interpolate between weak and strong coupling [7]. The proposal has since been tested thoroughly, see [8].

Perturbative gauge theory in the planar limit can be cast into the form of a spin chain. This spin chain model has a $\mathfrak{psu}(2, 2|4)$ symmetry, and the spins transform in a noncompact module of the symmetry algebra. At leading order this spin chain model agrees with the standard nearest-neighbor integrable spin chain model based on this algebra and module [9, 2].

Dealing with perturbative corrections to the spin chain Hamiltonian and symmetry generators is however a formidable problem: With increasing order in perturbation theory the local interactions along the spin chain will act on more and more neighboring sites. Moreover, higher-order interactions change the length of the chain; they are dynamic [10]. Together with the infinite degrees of freedom at each site, the interactions become combinatorially almost intractable, even at relatively low perturbative orders. This holds true for obtaining them (through explicit evaluation of gauge theory Feynman diagrams or through clever construction) as well as for applying them to states. Furthermore, one can hardly rely on standard $\mathfrak{psu}(2, 2|4)$ representation theory because the algebra is not realized in a manifest way. Nevertheless, the commutation relations are essential in constraining the form of the corrections.

As a step toward the complete corrections at the first few loop orders one can restrict to certain subsectors. An apt choice is the $\mathfrak{psu}(1, 1|2)$ sector, which has complexity well balanced between realistic features and simplifications. It incorporates a noncompact spin representation whose components are quite simple to enumerate. Furthermore, the dynamic interactions are mostly frozen out: The generators change the length by a definite amount, either by one unit or not at all. Finally, the Hamiltonian is a nonseparable part of the symmetry algebra.

The construction of the higher-loop algebra for this sector was started in [11] (also see [12] for the two-loop dilatation generator of a $\mathfrak{sl}(2)$ subsector). A key simplification in this construction was based on some less obvious symmetries: In $\mathcal{N} = 4$ SYM the symmetry algebra of the sector contains two factors of $\mathfrak{psu}(1|1)$ in addition to the $\mathfrak{psu}(1, 1|2)$ algebra. They made it possible to find the Hamiltonian at the two-loop level and to represent it using simple building blocks. Beyond that order, the construction appears to be rather complex. However, it might be that some crucial insight is still lacking in order to extend the construction conveniently to higher orders.

For example, a curious observation made in [3] has not yet been explained or taken into account: The Bethe equations for the sector lead to a huge degeneracy of 2^M multiplets that is not explained by any known symmetries of the integral model. In this paper we would like to understand this degeneracy at the level of spin chain operators commuting with the Hamiltonian. These might be of help in the construction of higher-loop corrections to the algebra generators.

The degeneracy is partially explained by an $\mathfrak{su}(2)$ automorphism of the $\mathfrak{psu}(1,1|2)$ algebra, see e.g. [13]. The automorphism is not a part of the underlying $\mathfrak{psu}(2,2|4)$ algebra of $\mathcal{N}=4$ SYM. It is nevertheless an exact symmetry of the $\mathfrak{psu}(1,1|2)$ sector, i.e. it should apply also at finite N_c. The degenerate $\mathfrak{psu}(1,1|2)$ multiplets transform in a tensor product of $\mathfrak{su}(2)$ doublets, $\mathbf{2}^{\otimes M}$. However, such tensor products are reducible, and therefore the $\mathfrak{su}(2)$ automorphism alone cannot explain the full degeneracy.

With respect to $\mathfrak{su}(2)$, the multiplets transform in a reducible $\mathbf{2}^{\otimes M} = \mathbf{2}\otimes\ldots\otimes\mathbf{2}$ representation. This is reminiscent of the $\mathfrak{su}(n)$ Haldane-Shastry model [14], which also has degenerate states transforming in reducible tensor products of $\mathfrak{su}(n)$ representations [15]. There, the degeneracy is caused by a $\mathfrak{su}(n)$ Yangian algebra that commutes exactly with the Hamiltonian, even on a finite periodic chain. It is therefore conceivable that a $\mathfrak{su}(2)$ Yangian or a similar algebraic structure will explain the further degeneracy in our case as well. In the present paper we shall present evidence in favor of this conjecture.

In Section 2, we review the Bethe equations and transfer matrix and use them to observe this degeneracy. In Section 3, we review the leading-order spin representations for the $\mathfrak{psu}(1,1|2)$ and $\mathfrak{psu}(1|1)^2$ symmetry generators and present the $\mathfrak{su}(2)$ automorphism. To gain further intuition about the degeneracy, we study some degenerate spin chain states in Section 4. Finally, in Section 5 we explain the degeneracy by constructing an infinite set of nonlocal spin chain symmetry generators, at leading order. These generators are built from the $\mathfrak{psu}(1|1)^2$ generators and form a triplet of $\mathfrak{su}(2)$. We discuss how these new generators map between degenerate states and argue that they form a parabolic subalgebra of the loop algebra of $\mathfrak{su}(2)$. We also discuss the relation of this symmetry to the integrable model's Yangian symmetry. Directions for further research are given in Section 6. Appendix A contains the commutation relations for the extended $\mathfrak{psu}(1,1|2)$ and $\mathfrak{psu}(1|1)^2$ algebras, and in Appendix B we present relevant multilinear operators for the $\mathfrak{psu}(1,1|2)$ sector, including a cubic operator that is a $\mathfrak{su}(2)$-triplet and $\mathfrak{psu}(1,1|2)$ invariant. The proof that the nonlocal symmetry generators commute with the classical $\mathfrak{psu}(1,1|2)$ generators and the one-loop dilatation generator is given in Appendix C.

2 Symmetry Enhancement in the Bethe Ansatz

In this section, we describe the symmetries of the one-loop Bethe equations for the $\mathfrak{psu}(1,1|2)$ sector, as well as the resulting 2^M-fold degeneracies in the spectrum. Furthermore, we show that these degeneracies are also present for the transfer matrix, which provides the full set of local conserved charges of the integrable system.

2.1 Bethe Equations

The Bethe equations for the $\mathfrak{psu}(1,1|2)$ sector of planar $\mathcal{N}=4$ SYM at leading order take the form

$$1 = \prod_{j=1}^{K} \frac{v_k - u_j - \frac{i}{2}}{v_k - u_j + \frac{i}{2}}\,,$$

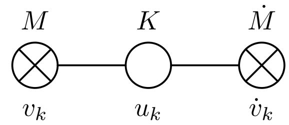

Figure 1: Dynkin diagram for $\mathfrak{psu}(1,1|2)$. The different flavors of Bethe roots and their overall numbers are indicated below/above the nodes, respectively.

$$1 = \left(\frac{u_k - \frac{i}{2}}{u_k + \frac{i}{2}}\right)^L \prod_{\substack{j=1\\j\neq k}}^{K} \frac{u_k - u_j + i}{u_k - u_j - i} \prod_{j=1}^{M} \frac{u_k - v_j - \frac{i}{2}}{u_k - v_j + \frac{i}{2}} \prod_{j=1}^{\dot{M}} \frac{u_k - \dot{v}_j - \frac{i}{2}}{u_k - \dot{v}_j + \frac{i}{2}},$$
$$1 = \prod_{j=1}^{K} \frac{\dot{v}_k - u_j - \frac{i}{2}}{\dot{v}_k - u_j + \frac{i}{2}}. \tag{2.1}$$

These are just the standard Bethe equations for a closed nearest-neighbor spin chain with $\mathfrak{psu}(1,1|2)$ symmetry (in the form determined by the Dynkin diagram in Fig. 1) and spins transforming in the $[0;1;0]$ representation. The three types of Bethe roots $v_{1,\ldots,M}$, $u_{1,\ldots,K}$ and $\dot{v}_{1,\ldots,\dot{M}}$ are associated to the three nodes of the Dynkin diagram in Fig. 1. The length of the spin chain is given by L.

The momentum and energy eigenvalues for eigenstates of this system are determined through the main Bethe roots $u_{1,\ldots,K}$ alone

$$\exp(iP) = \prod_{j=1}^{K} \frac{u_j + \frac{i}{2}}{u_j - \frac{i}{2}}, \qquad E = \sum_{j=1}^{K} \left(\frac{2i}{u_j + \frac{i}{2}} - \frac{2i}{u_j - \frac{i}{2}}\right). \tag{2.2}$$

2.2 Symmetries

The $\mathfrak{psu}(1,1|2)$ symmetry is realized in the standard way: One can add Bethe roots $v, u, \dot{v} = \infty$ to the set of Bethe roots for any eigenstate. It is easy to convince oneself that the Bethe equations (2.1) for the original roots as well as for the new root are satisfied. Moreover, the momentum and energy (2.2) are not changed by the introduction of the additional root. This means that the eigenstates come in highest-weight multiplets with degenerate momentum and energy eigenvalues. These multiplets are modules of the symmetry algebra $\mathfrak{psu}(1,1|2)$. Note that the Bethe roots $v, u, \dot{v} = \infty$ are allowed to appear in eigenstates more than one time, and thus even very large or infinite multiplets can be swept out with this symmetry.

Another type of symmetry that is very important to $\mathcal{N} = 4$ SYM exists only in the zero-momentum sector. Here one adds a single root $v = 0$ or $\dot{v} = 0$ to an eigenstate configuration while decreasing the length L by one unit [3]. The original Bethe equations are preserved, and the Bethe equation for $v = 0$ and $\dot{v} = 0$ is equal to the zero-momentum condition, cf. (2.2). As the momentum and energy eigenvalues depend explicitly on the main Bethe roots u_k only, they are not affected by this transformation. This symmetry leads to an additional fourfold degeneracy of states because each of the Bethe roots $v = 0$ and $\dot{v} = 0$ can only appear once at maximum. The associated algebra consists of

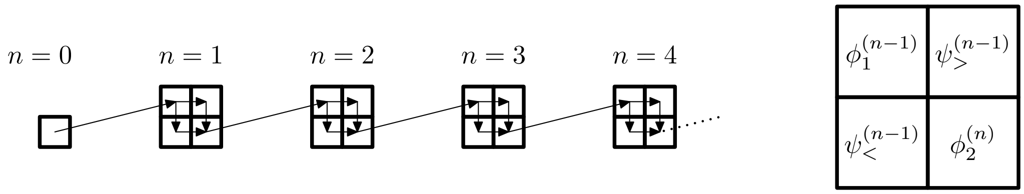

Figure 2: Structure of the spin representation (left). Each box represents one component of the module with the assignments shown on the right. Arrows represent simple roots of the algebra. The long diagonal arrows correspond to the middle node of the Dynkin diagram Fig. 1 while the short horizontal and vertical arrows correspond to the outer nodes.

two copies of $\mathfrak{su}(1|1)$ whose typical modules are two-dimensional. These two additional algebras are required for a consistent embedding of the spin chain into a larger model with $\mathfrak{psu}(2,2|4)$ symmetry [4]. Their generators were constructed in [4,11] at the leading order, and they transform one site of the spin chain into two or vice versa. We will present these generators in Section 3.

The third and most obscure type of symmetry was observed in [3]. The auxiliary Bethe roots v_k and $\dot{v}_k$ appear in the Bethe equations (2.1) completely symmetrically: The Bethe equation for v_k is exactly the same as the one for $\dot{v}_k$. Furthermore, the product in the Bethe equation for u_k involves a product over all v_j and $\dot{v}_j$ with the same form of factor. Therefore, we can freely interchange them

$$v_j \longleftrightarrow \dot{v}_{j'} \tag{2.3}$$

without violating the Bethe equations. As for the previous type of symmetry, modifying only the auxiliary Bethe roots does not change the momentum nor the energy. It is straightforward to convince oneself that this leads to a degeneracy of 2^{M_0} states where M_0 is the number of v_j roots which are distinct from $\dot{v}_j$ (in order to avoid coincident Bethe roots of the same type).

The closer investigation of this latter symmetry will be the main subject of the present paper.

2.3 Commuting Charges

A first question is whether the symmetry merely constitutes an accidental degeneracy of the momentum and energy spectrum or whether it is a symmetry of the full integrable structure. Therefore it is useful to look at the eigenvalues of the commuting charges of the integrable model. The eigenvalues of the higher local charges

$$Q_r = \frac{1}{r-1} \sum_{j=1}^{K} \left(\frac{i}{(u_j + \frac{i}{2})^{r-1}} - \frac{i}{(u_j - \frac{i}{2})^{r-1}} \right) \tag{2.4}$$

depend on the main Bethe roots u_j only, just like the momentum and energy (2.2). Consequently their spectrum displays this additional degeneracy.

However, this is not all there is to show; there are also nonlocal commuting charges whose invariance properties might lead to some additional clues. For instance, the local charge eigenvalues Q_r are accurate only for $r \leq L$. For $r > L$ these charges wrap the spin chain state fully, and they receive contributions from the auxiliary Bethe roots v_j and $\dot{v}_j$. This is best seen by considering the transfer matrix in the spin representation, which serves as a generating function for the local charges as

$$T_{\mathrm{spin}}(x) = \exp i \sum_{r=1}^{\infty} x^{r-1} Q_r. \tag{2.5}$$

A transfer matrix is a trace over a particular representation of the symmetry algebra. Therefore, its eigenvalues in a particular representation are typically written as a sum with as many terms as there are components in the representation. The eigenvalues of a transfer matrix can often be reverse engineered by a sort of analytic Bethe ansatz [16]. This requires some knowledge of the structure of the representations for which the transfer matrix is to be constructed. In particular, it is important to know what the components are and how they are connected by the simple roots of the algebra. The structure of the spin representation is depicted in Fig. 2. Now it is generally true that the transfer matrix has no dynamic poles, i.e. poles whose position depends on the Bethe roots. Conversely, the terms in the expression for the transfer matrix eigenvalue typically have many dynamic poles. These will have to cancel between the various terms once the Bethe equations are imposed. In particular, the Bethe equation for a particular type of Bethe root will have to ensure the cancellation of singularities between all terms that are related by the simple root associated to that Bethe root, cf. Fig. 1. We are then led to the following expression for the transfer matrix eigenvalue in the spin representation,

$$\begin{aligned} T_{\mathrm{spin}}(x) = \sum_{n=0}^{\infty} \left(\frac{x}{x-in}\right)^L \prod_{j=1}^{M} \frac{x-v_j}{x-v_j-in} \prod_{j=1}^{\dot{M}} \frac{x-\dot{v}_j}{x-\dot{v}_j-in} \\ \times \left(\delta_{n\neq 0} \prod_{j=1}^{K} \frac{x-u_j-i(n+\frac{1}{2})}{x-u_j-i(n-\frac{1}{2})} - 2\delta_{n\neq 0} + \prod_{j=1}^{K} \frac{x-u_j-i(n-\frac{1}{2})}{x-u_j-i(n+\frac{1}{2})} \right). \end{aligned} \tag{2.6}$$

We leave it as an exercise for the reader to confirm the cancellation of poles. This is true even if there are two coincident auxiliary Bethe roots $v_j = \dot{v}_{j'}$ in which case a potential double pole is fully eliminated. Furthermore, it is straightforward to show that the local charge eigenvalues (2.4) (for $r \leq L$) follow from (2.5,2.6) and that only the one term with $n = 0$ contributes for $r \leq L$.

This expression is clearly invariant under the degeneracy transformation (2.3). Therefore, the full transfer matrix obeys the enhanced symmetry, which is a clear hint that the integrable structure is compatible with the symmetry. It is however not fully invariant under it as the eigenvalues of transfer matrices in different representations show. For instance, for the fundamental and conjugate-fundamental representations it is easy to construct the transfer matrices

$$\begin{aligned} T_{\mathrm{fund}}(x) = &+ \left(\frac{x+\frac{i}{2}}{x}\right)^L \prod_{j=1}^{M} \frac{x-v_j-\frac{i}{2}}{x-v_j+\frac{i}{2}} \left(\prod_{j=1}^{K} \frac{x-u_j+i}{x-u_j} - 1 \right) \\ &+ \left(\frac{x-\frac{i}{2}}{x}\right)^L \prod_{j=1}^{\dot{M}} \frac{x-\dot{v}_j+\frac{i}{2}}{x-\dot{v}_j-\frac{i}{2}} \left(\prod_{j=1}^{K} \frac{x-u_j-i}{x-u_j} - 1 \right) \end{aligned} \tag{2.7}$$

and

$$T_{\overline{\rm fund}}(x) = +\left(\frac{x-\frac{i}{2}}{x}\right)^L \prod_{j=1}^{M} \frac{x-v_j+\frac{i}{2}}{x-v_j-\frac{i}{2}} \left(\prod_{j=1}^{K} \frac{x-u_j-i}{x-u_j} - 1\right)$$
$$+\left(\frac{x+\frac{i}{2}}{x}\right)^L \prod_{j=1}^{\dot M} \frac{x-\dot v_j-\frac{i}{2}}{x-\dot v_j+\frac{i}{2}} \left(\prod_{j=1}^{K} \frac{x-u_j+i}{x-u_j} - 1\right). \tag{2.8}$$

These expressions are clearly not invariant under the shuffling (2.3) of auxiliary Bethe roots. The violation of the symmetry may be related to the fact that the fundamental representations are centrally charged under $\mathfrak{su}(1,1|2)$ while the spin representation has zero central charge and thus belongs to $\mathfrak{psu}(1,1|2)$.[1]

Finally, we note that the transfer matrix in the spin representation (2.6) also has the degeneracy due to the $\mathfrak{psu}(1|1)$ symmetries (as do all of the $\mathcal{Q}_r$). Adding a v or $\dot v$ root at zero gives a factor of $x/(x-in)$ in each term of the sum. This is cancelled by decreasing L by one. However, again the degeneracy is not present for the transfer matrix in the fundamental or conjugate-fundamental representations.[2]

3 Symmetry Enhancement in the Lie Algebra

3.1 The Spin Representation

We begin by describing the spin representation on which the present spin chain model is based. By direct inspection of the explicit expressions we will uncover an additional $\mathfrak{su}(2)$ symmetry of the model.

The spin module with Dynkin labels $[0;1;0]$ is spanned by the states, cf. Fig. 2

$$|\phi_a^{(n)}\rangle, \qquad |\psi_{\mathfrak{a}}^{(n)}\rangle. \tag{3.1}$$

The Latin index a can take values $1,2$, the Gothic index $\mathfrak{a}$ can take the values '$<$', '$>$' and n is a nonnegative integer. The ϕ's are bosonic and the ψ's are fermionic. In $\mathcal{N}=4$ gauge theory, these states correspond to the fields with derivatives (in the notation of [4])

$$|\phi_a^{(n)}\rangle \simeq \frac{1}{n!}\,\mathcal{D}_{11}^n \Phi_{a3}, \qquad |\psi_>^{(n)}\rangle \simeq \frac{1}{n!\sqrt{n+1}}\,\mathcal{D}_{11}^n \Psi_{13}, \qquad |\psi_<^{(n)}\rangle \simeq \frac{1}{n!\sqrt{n+1}}\,\mathcal{D}_{11}^n \dot\Psi_1^4. \tag{3.2}$$

The $\mathfrak{psu}(1,1|2)$ algebra has eight supersymmetry generators. We denote them collectively by $\mathfrak{Q}^{a\beta\mathfrak{c}}$ where a Greek index β can take the values '$+$', '$-$'. In gauge theory the supercharges translate to

$$\mathfrak{Q}^{a+>} = \mathfrak{Q}^a{}_1, \qquad \mathfrak{Q}^{a+<} = \varepsilon^{ab}\dot{\mathfrak{Q}}_{1b},$$
$$\mathfrak{Q}^{a->} = \dot{\mathfrak{S}}^{a1}, \qquad \mathfrak{Q}^{a-<} = \varepsilon^{ab}\mathfrak{S}^1{}_b. \tag{3.3}$$

[1] It may be noted that the product $T_{\rm fund}(x)\,T_{\overline{\rm fund}}(x)$ is again invariant under switching the v and $\dot v$. This is in agreement with the fact that the overall central charge for the two representations is zero.

[2] Their product does not have this degeneracy either.

At leading order they act on the states as follows,

$$\begin{aligned}
\mathfrak{Q}^{a+\mathfrak{b}}_{(0)}|\phi^{(n)}_c\rangle &= \sqrt{n+1}\,\delta^a_c\varepsilon^{\mathfrak{b}\mathfrak{d}}|\psi^{(n)}_{\mathfrak{d}}\rangle, &\qquad \mathfrak{Q}^{a+\mathfrak{b}}_{(0)}|\psi^{(n)}_{\mathfrak{c}}\rangle &= \sqrt{n+1}\,\delta^{\mathfrak{b}}_{\mathfrak{c}}\varepsilon^{ad}|\phi^{(n+1)}_d\rangle,\\
\mathfrak{Q}^{a-\mathfrak{b}}_{(0)}|\phi^{(n)}_c\rangle &= \sqrt{n}\,\delta^a_c\varepsilon^{\mathfrak{b}\mathfrak{d}}|\psi^{(n-1)}_{\mathfrak{d}}\rangle, &\qquad \mathfrak{Q}^{a-\mathfrak{b}}_{(0)}|\psi^{(n)}_{\mathfrak{c}}\rangle &= \sqrt{n+1}\,\delta^{\mathfrak{b}}_{\mathfrak{c}}\varepsilon^{ad}|\phi^{(n)}_d\rangle.
\end{aligned} \tag{3.4}$$

Furthermore, there are the $\mathfrak{su}(2)$ generators $\mathfrak{R}^{ab} = \mathfrak{R}^{ba}$, which translate to the notation of [4] as $\mathfrak{R}^{ab} = \varepsilon^{ac}\mathfrak{R}^b{}_c$. They act canonically on the bosonic doublet of states (to all orders)

$$\mathfrak{R}^{ab}|\phi^{(n)}_c\rangle = \delta^{\{a}_c\varepsilon^{b\}d}|\phi^{(n)}_d\rangle. \tag{3.5}$$

Finally, the $\mathfrak{su}(1,1)$ generators are denoted by $\mathfrak{J}^{\alpha\beta} = \mathfrak{J}^{\beta\alpha}$. They are related to the gauge theory notation as

$$\mathfrak{J}^{++} = \mathfrak{P}_{11}, \qquad \mathfrak{J}^{--} = \mathfrak{K}^{11}, \qquad \mathfrak{J}^{+-} = \tfrac{1}{2}\mathfrak{D} + \tfrac{1}{2}\mathfrak{L}^1{}_1 + \tfrac{1}{2}\dot{\mathfrak{L}}^1{}_1. \tag{3.6}$$

They act on the states by changing the index n by up to one unit

$$\begin{aligned}
\mathfrak{J}^{++}_{(0)}|\phi^{(n)}_a\rangle &= (n+1)|\phi^{(n+1)}_a\rangle, &\qquad \mathfrak{J}^{++}_{(0)}|\psi^{(n)}_{\mathfrak{a}}\rangle &= \sqrt{(n+1)(n+2)}|\psi^{(n+1)}_{\mathfrak{a}}\rangle,\\
\mathfrak{J}^{+-}_{(0)}|\phi^{(n)}_a\rangle &= (n+\tfrac{1}{2})|\phi^{(n)}_a\rangle, &\qquad \mathfrak{J}^{+-}_{(0)}|\psi^{(n)}_{\mathfrak{a}}\rangle &= (n+1)|\psi^{(n)}_{\mathfrak{a}}\rangle,\\
\mathfrak{J}^{--}_{(0)}|\phi^{(n)}_a\rangle &= n|\phi^{(n-1)}_a\rangle, &\qquad \mathfrak{J}^{--}_{(0)}|\psi^{(n)}_{\mathfrak{a}}\rangle &= \sqrt{n(n+1)}|\psi^{(n-1)}_{\mathfrak{a}}\rangle.
\end{aligned} \tag{3.7}$$

3.2 The Automorphism

In the above expressions, the Gothic indices $\mathfrak{a}, \mathfrak{b}, \ldots = <, >$ were introduced to handle the two fermionic states in a collective manner. The transformation rules (3.4,3.5,3.7) follow from $\mathfrak{psu}(1,1|2)$ symmetry alone. Curiously they can be written with the usual index contraction rules using only the auxiliary symbols $\delta^{\mathfrak{a}}_{\mathfrak{b}}$ and $\varepsilon^{\mathfrak{a}\mathfrak{b}}$. It is therefore obvious that the representation has an $\mathfrak{su}(2)$ automorphism, see e.g. [13], and that the Gothic indices label a doublet of this $\mathfrak{su}(2)$. We introduce the generators $\mathfrak{B}^{\mathfrak{a}\mathfrak{b}}$ of this $\mathfrak{su}(2)$, which rotate the fermions as

$$\mathfrak{B}^{\mathfrak{a}\mathfrak{b}}|\psi^{(n)}_{\mathfrak{c}}\rangle = \delta^{\{\mathfrak{a}}_{\mathfrak{c}}\varepsilon^{\mathfrak{b}\}\mathfrak{d}}|\psi^{(n)}_{\mathfrak{d}}\rangle. \tag{3.8}$$

The $\mathfrak{su}(2)$ automorphism can be viewed as an accidental symmetry in the $\mathfrak{psu}(1,1|2)$ sector of $\mathcal{N} = 4$ SYM: The generators $\mathfrak{B}^{<<}$ and $\mathfrak{B}^{>>}$ transform between fermions Ψ and conjugate fermions $\dot{\Psi}$ in gauge theory, cf. (3.2). However, none of the $\mathfrak{psu}(2,2|4)$ generators of the full theory acts in such a way. Only the Cartan generator $\mathfrak{B}^{<>}$ of the $\mathfrak{su}(2)$ automorphism is equivalent to a combination of the Lorentz generators: $\mathfrak{B}^{<>} = \mathfrak{L}^1{}_1 - \dot{\mathfrak{L}}^1{}_1$.

This means we have found an additional symmetry in this sector, which explains a higher degree of degeneracy in the spectrum. Indeed, in terms of the Cartan charges, the transformation of Bethe roots (2.3) has the same effect as the generators $\mathfrak{B}^{<<}$ and $\mathfrak{B}^{>>}$. The two flavors of auxiliary Bethe roots v and $\dot{v}$ effectively form a doublet of the $\mathfrak{su}(2)$ automorphism.[3] If there are M_0 auxiliary Bethe roots in total, the degeneracy is

[3]Note, however, that a pair of v and $\dot{v}$ taking the same value form a singlet because of Fermi statistics.

realized as the M_0-fold tensor product of $\mathfrak{su}(2)$ doublets. This tensor product is reducible, and $\mathfrak{su}(2)$ symmetry can only account for degeneracy within the irreducible components. Nevertheless, even the irreducible components turn out to be fully degenerate. Therefore, the $\mathfrak{su}(2)$ automorphism explains only part of the extended degeneracy, and there should be an even larger symmetry. This symmetry should have the full tensor product as one irreducible multiplet. This behavior is somewhat reminiscent of the Yangian symmetry in the Haldane-Shastry model [14,15], which also displays fully degenerate tensor products. We will return to this issue in Section 4, and consider only the $\mathfrak{su}(2)$ automorphism for the moment.

3.3 Zero-Momentum States

As discussed above, for zero-momentum states the symmetry is enhanced by two copies of $\mathfrak{psu}(1|1)$ with one central charge. We shall denote the fermionic generators by $\hat{\mathfrak{Q}}^{\mathfrak{a}}$ and $\hat{\mathfrak{S}}^{\mathfrak{a}}$ and the central charge by $\hat{\mathfrak{D}}$. In the gauge theory notation, they represent the supercharges

$$\hat{\mathfrak{Q}}^{<} = \dot{\mathfrak{Q}}_{23}, \qquad \hat{\mathfrak{Q}}^{>} = -\mathfrak{Q}^4{}_2,$$
$$\hat{\mathfrak{S}}^{<} = \mathfrak{S}^2{}_4, \qquad \hat{\mathfrak{S}}^{>} = \dot{\mathfrak{S}}^{32},$$

and the generator of anomalous dimensions

$$\hat{\mathfrak{D}} = \tfrac{1}{2}\mathfrak{D} + \mathfrak{L}^2{}_2 + \mathfrak{R}^4{}_4 = \tfrac{1}{2}\mathfrak{D} + \dot{\mathfrak{L}}^2{}_2 - \mathfrak{R}^3{}_3 = \tfrac{1}{2}\delta\mathfrak{D}. \tag{3.9}$$

The last two equalities are satisfied for states within the $\mathfrak{psu}(1,1|2)$ sector. The fermionic generators expand in odd powers of the coupling constant, and they act by increasing or decreasing the length of the spin chain by one unit. At the leading order $\mathcal{O}(g)$, the generators $\hat{\mathfrak{S}}^{\mathfrak{a}}_{(1)}$ act on two adjacent sites and turn them into a single site. Explicitly, the action takes the form [11]

$$\begin{aligned}
\hat{\mathfrak{S}}^{\mathfrak{a}}_{(1)}|\phi^{(m)}_b\psi^{(n)}_{\mathfrak{c}}\rangle &= -\frac{1}{\sqrt{n+1}}\,\delta^{\mathfrak{a}}_{\mathfrak{c}}\big|\phi^{(n+m+1)}_b\big\rangle,\\
\hat{\mathfrak{S}}^{\mathfrak{a}}_{(1)}|\psi^{(m)}_{\mathfrak{b}}\phi^{(n)}_c\rangle &= \frac{1}{\sqrt{m+1}}\,\delta^{\mathfrak{a}}_{\mathfrak{b}}\big|\phi^{(n+m+1)}_c\big\rangle,\\
\hat{\mathfrak{S}}^{\mathfrak{a}}_{(1)}|\psi^{(m)}_{\mathfrak{b}}\psi^{(n)}_{\mathfrak{c}}\rangle &= \frac{\sqrt{n+1}}{\sqrt{(m+1)(m+n+2)}}\,\delta^{\mathfrak{a}}_{\mathfrak{b}}\big|\psi^{(n+m+1)}_{\mathfrak{c}}\big\rangle\\
&\quad+\frac{\sqrt{m+1}}{\sqrt{(n+1)(m+n+2)}}\,\delta^{\mathfrak{a}}_{\mathfrak{c}}\big|\psi^{(n+m+1)}_{\mathfrak{b}}\big\rangle,\\
\hat{\mathfrak{S}}^{\mathfrak{a}}_{(1)}|\phi^{(m)}_b\phi^{(n)}_c\rangle &= \frac{1}{\sqrt{n+m+1}}\,\varepsilon^{bc}\varepsilon^{\mathfrak{a}\mathfrak{d}}\big|\psi^{(n+m)}_{\mathfrak{d}}\big\rangle.
\end{aligned}\tag{3.10}$$

Conversely, the generators $\hat{\mathfrak{Q}}^{\mathfrak{a}}_{(1)}$ act on a single site and turn it into two,

$$\hat{\mathfrak{Q}}^{\mathfrak{a}}_{(1)}|\phi^{(n)}_b\rangle = \sum_{k=0}^{n-1}\frac{1}{\sqrt{k+1}}\,\varepsilon^{\mathfrak{a}\mathfrak{c}}\big|\psi^{(k)}_{\mathfrak{c}}\phi^{(n-1-k)}_b\big\rangle - \sum_{k=0}^{n-1}\frac{1}{\sqrt{n-k}}\,\varepsilon^{\mathfrak{a}\mathfrak{c}}\big|\phi^{(k)}_b\psi^{(n-1-k)}_{\mathfrak{c}}\big\rangle,$$

$$
\begin{aligned}
\hat{\mathfrak{Q}}^{\mathfrak{a}}_{(1)}|\psi^{(n)}_{\mathfrak{b}}\rangle \;=\; & \sum_{k=0}^{n-1}\frac{\sqrt{n-k}}{\sqrt{(k+1)(n+1)}}\,\varepsilon^{\mathfrak{ac}}\big|\psi^{(k)}_{\mathfrak{c}}\psi^{(n-1-k)}_{\mathfrak{b}}\big\rangle \\
& +\sum_{k=0}^{n-1}\frac{\sqrt{k+1}}{\sqrt{(n-k)(n+1)}}\,\varepsilon^{\mathfrak{ac}}\big|\psi^{(k)}_{\mathfrak{b}}\psi^{(n-1-k)}_{\mathfrak{c}}\big\rangle \\
& -\sum_{k=0}^{n}\frac{1}{\sqrt{n+1}}\,\delta^{\mathfrak{a}}_{\mathfrak{b}}\varepsilon^{cd}\big|\phi^{(k)}_{c}\phi^{(n-k)}_{d}\big\rangle. \qquad (3.11)
\end{aligned}
$$

Again, by inspection the representations of $\mathfrak{psu}(1|1)^2$ turns out to have a manifest $\mathfrak{su}(2)$ automorphism. It is nice to see that the unified treatment of the two fermionic states as a doublet compresses the expressions found in [11] somewhat. Furthermore, when the construction of [11] is to be carried to higher perturbative orders one may expect the $\mathfrak{su}(2)$ symmetry to reduce the number of permitted terms and thus simplify the analysis. Finally, we should note that there is a unique lift of the action (3.10,3.11) to the nonplanar level. This means that the nonplanar $\mathfrak{psu}(1,1|2)$ sector of $\mathcal{N}=4$ SYM will also have the additional $\mathfrak{su}(2)$ symmetry.[4]

4 Some Degenerate States

Let us now consider the full observed degeneracy. We will try to get acquainted with it by constructing explicitly some degenerate states. Here and in the following sections we will work only at leading order in the coupling constant g. In other words, the $\mathfrak{psu}(1,1|2)$ generators $\mathfrak{Q},\mathfrak{J}$ are truncated at $\mathcal{O}(g^0)$, and for the $\mathfrak{psu}(1|1)^2$ generators $\hat{\mathfrak{Q}},\hat{\mathfrak{S}}$ we take only the $\mathcal{O}(g^1)$ contributions $\hat{\mathfrak{Q}}_{(1)},\hat{\mathfrak{S}}_{(1)}$ in (3.10,3.11).

4.1 Vacuum

The simplest state that is part of a nontrivial multiplet is

$$|0_L\rangle = |\psi^{(0)}_{<}\psi^{(0)}_{<}\psi^{(0)}_{<}\ldots\psi^{(0)}_{<}\rangle. \qquad (4.1)$$

We shall call it the vacuum state of length L. Note that it is not the ground state of the model, but it is a homogeneous eigenstate of the Hamiltonian, and we can place excitations on it by flipping some of the spins. In the above Bethe ansatz it is represented by $K=L$ main Bethe roots and $M=L$ auxiliary Bethe roots. The roots are the solutions to the algebraic equations (including $u=\infty$ and twice $v=\infty$)

$$(u+\tfrac{i}{2})^L=(u-\tfrac{i}{2})^L,\qquad (v+i)^L+(v-i)^L=2v^L. \qquad (4.2)$$

The equation for the main Bethe roots can be solved explicitly as $u_k=\frac{1}{2}\cot(\pi k/L)$. The momentum and energy of this state are given by (2.2)

$$P=\pi(L-1),\qquad E=4L. \qquad (4.3)$$

[4] It is likely, however, that the 2^{M_0} degeneracy will be lifted into the irreducible components of $\mathfrak{su}(2)$.

The eigenvalue of the transfer matrix in the spin representation reads (2.6)

$$T_{\text{spin}}(x) = 1 + \sum_{n=0}^{\infty} \frac{x^L\big(2x^L - (x+i)^L - (x-i)^L\big)}{(x-in)^L(x-in-i)^L}\,. \tag{4.4}$$

Note that for even L the overall momentum is maximal, $P \equiv \pi$, while for odd L the overall momentum is zero, $P \equiv 0$. Therefore only the states with odd L are physical states of AdS/CFT, and only for those the symmetry algebra enlarges by $\mathfrak{psu}(1|1)^2$.

The vacuum state is part of a $\mathfrak{su}(2)$ multiplet of $L+1$ *states*. The $L+1$ components are given by $(\mathfrak{B}^{<<})^{0,1,\ldots,L}|0_L\rangle$. Note also that it is part of a multiplet of $L-1$ *multiplets* of $\mathfrak{psu}(1,1|2)$.[5] The $L-1$ highest-weight components are obtained by acting with the cubic operator given in App. B.2; they read $((\mathfrak{J}^3)^{<<})^{0,1,\ldots,L-2}|0_L\rangle$.

4.2 Degenerate Eigenstates

Let us now consider the set of states where the flavor of one auxiliary Bethe root is flipped. One can convince oneself that a state is composed from basis states of the typical form

$$\mathfrak{Q}^{2-<}(k)\,\mathfrak{Q}^{1-<}(l)\,\mathfrak{J}^{++}(m)\,|0_L\rangle \sim |\ldots \overset{k}{\underset{\downarrow}{}}\phi_1^{(0)} \ldots \overset{l}{\underset{\downarrow}{}}\phi_2^{(0)} \ldots \overset{m}{\underset{\downarrow}{}}\psi_<^{(1)} \ldots\rangle. \tag{4.5}$$

The arguments of the generators correspond to the sites of the spin chain on which they should act. Here we have only displayed the excitations while the vacuum sites $\psi_<^{(0)}$ have been suppressed. The operators $\mathfrak{J}(k)$ act as the leading order generators in (3.4,3.7) on site k of the chain.[6] Note that if two or all of the three excitations coincide on a single site they will give rise to $\phi_1^{(1)}$, $\phi_2^{(1)}$ or $\psi_>^{(0)}$. We find precisely $L+1$ states of this form completely degenerate with the vacuum $|0_L\rangle$. Three of these states are descendants of $\mathfrak{psu}(1,1|2)$,

$$\varepsilon_{ab}\mathfrak{Q}^{a-<}\mathfrak{Q}^{b-<}\mathfrak{J}^{++}|0_L\rangle, \quad \varepsilon_{ab}\mathfrak{Q}^{a-<}\mathfrak{Q}^{b+<}|0_L\rangle, \quad \varepsilon_{ab}\mathfrak{Q}^{a+<}\mathfrak{Q}^{b-<}|0_L\rangle, \tag{4.6}$$

and one is the $\mathfrak{su}(2)$ descendant

$$\mathfrak{B}^{<<}|0_L\rangle. \tag{4.7}$$

However, since $\mathfrak{B}^{\mathfrak{ab}}$ does not commute with $\mathfrak{psu}(1,1|2)$, it is more convenient to use instead the cubic operator $(\mathfrak{J}^3)^{\mathfrak{ab}}$ presented in App. B.2 (built from cubic combinations of ordinary $\mathfrak{psu}(1,1|2)$ and $\mathfrak{su}(2)$ generators),

$$(\mathfrak{J}^3)^{<<}|0_L\rangle. \tag{4.8}$$

The generator $(\mathfrak{J}^3)^{\mathfrak{ab}}$ commutes with $\mathfrak{psu}(1,1|2)$ and therefore moves between $\mathfrak{psu}(1,1|2)$ highest weight states.

[5] Due to the $\mathfrak{su}(2)$ grading of the $\mathfrak{psu}(1,1|2)$ algebra these two numbers differ by two.

[6] The statistics of the fermionic generators $\mathfrak{Q}(k)$ is taken into account by first permuting it to its place of action. This may cause a sign flip.

For even L (and nonzero momentum) this exhausts the set of trivial descendants. There remain $L-3$ unexplained degenerate states. For odd L the vacuum is a zero-momentum state, and therefore the additional $\mathfrak{psu}(1|1)^2$ symmetry applies. It yields one further descendant,

$$\hat{\mathfrak{S}}^{<}\hat{\mathfrak{Q}}^{<}|0_L\rangle. \tag{4.9}$$

Consequently there are only $L-4$ unexplained degenerate states in this case.

Among the remaining degenerate states we find one state with the very simple form

$$\begin{aligned}|1_L\rangle = &\sum_{n,k=1}^{L}(-1)^k\mathfrak{J}^{++}(k+n)\,\varepsilon_{ab}\mathfrak{Q}^{a-<}(1+n)\,\mathfrak{Q}^{b-<}(L+n)\,|0_L\rangle\\ &-(1-(-1)^L)\mathfrak{B}^{<<}\,|0_L\rangle.\end{aligned} \tag{4.10}$$

One can confirm straightforwardly that it is a highest-weight state of $\mathfrak{psu}(1,1|2)$. For even length this state is indeed linearly independent of the above descendants. For odd length, however, the state is proportional to the $\mathfrak{psu}(1|1)^2$ descendant (4.9), $|1_L\rangle \sim \hat{\mathfrak{S}}^{<}\hat{\mathfrak{Q}}^{<}|0_L\rangle$. This turns out to be a special case because of the overall momentum being zero. We will return to this issue in the next section.

We have also found a second degenerate state with a slightly more complicated form,

$$\begin{aligned}|2_L\rangle = &\sum_{n,k=1}^{L}(-1)^k(2k-L-1+\delta_{k1}-\delta_{kL})\mathfrak{J}^{++}(k+n)\,\varepsilon_{ab}\mathfrak{Q}^{a-<}(1+n)\,\mathfrak{Q}^{b-<}(L+n)\,|0_L\rangle\\ &+\sum_{k=2}^{L}\sum_{n=1}^{L}(-1)^k\mathfrak{J}^{++}(k+n)\,\varepsilon_{ab}\mathfrak{Q}^{a-<}(2+n)\,\mathfrak{Q}^{b-<}(L+n)\,|0_L\rangle\\ &+(1+(-1)^L)(L-1)\mathfrak{B}^{<<}\,|0_L\rangle.\end{aligned} \tag{4.11}$$

This state is also a highest weight state of $\mathfrak{psu}(1,1|2)$, and for odd length is not a $\mathfrak{psu}(1|1)^2$ descendant of $|0_L\rangle$.

4.3 Parity

The degenerate states do not all have the same parity. For L even or odd we find $\frac{1}{2}(L-2)$ or $\frac{1}{2}(L-3)$ states, respectively, which have opposite parity than the vacuum.[7] Recalling the above results, this means that after removing the trivial descendants there is always one more degenerate state with opposite parity than with equal parity. More explicitly, we can say that $|1_L\rangle$ has the opposite parity as $|0_L\rangle$ for even L and the same parity as $|0_L\rangle$ for odd L. Conversely, the state $|2_L\rangle$ has the same parity as $|0_L\rangle$ for even L and the opposite parity as $|0_L\rangle$ for odd L.

[7] The definition of parity may include shifts $\mathcal{U}^k$ of the chain which act nontrivially on states with overall momentum. It is therefore more convenient to only specify the parity w.r.t. a reference state.

5 Nonlocal Symmetry

To account for the additional degeneracy, it is natural to seek new symmetry generators. We will take into account the findings regarding the Bethe ansatz and the form of the degenerate states found in the previous section to construct some nonlocal generators $\mathcal{Y}$. We will then investigate their algebra.

5.1 Bilocal Generators

First of all, an elementary step between two degenerate Bethe states consists in changing the flavor of one auxiliary Bethe root, as discussed in Section 2. The $\mathfrak{su}(2)$ generators $\mathfrak{B}^{\mathfrak{ab}}$ qualitatively act in the same way. This indicates that the new generators will be in the same representation, i.e. in the adjoint/spin-one/triplet representation of $\mathfrak{su}(2)$. We will thus denote them by $\mathcal{Y}^{\mathfrak{ab}} = \mathcal{Y}^{\mathfrak{ba}}$.

As the example degenerate states given in the previous section have multiple non-adjacent excitations, we should look for nonlocal generators. The simplest degenerate state $|1_L\rangle$ in (4.10) has a pair of adjacent excitations and a single excitation that is not near the pair. A generator that creates such a state from the vacuum $|0_L\rangle$ consequently has to be bilocal (at least). More complicated states with multilocal excitations such as $|2_L\rangle$ in (4.11) could in principle be generated by repeated application of these bilocal generators.

Furthermore, we know that the form of the example degenerate state $|1_L\rangle$ in (4.10) is qualitatively identical to the second order $\mathfrak{psu}(1|1)^2$ descendant $\hat{\mathfrak{S}}^{<}\hat{\mathfrak{Q}}^{<}|0_L\rangle$. Thus we expect $\mathcal{Y}^{\mathfrak{ab}}$ to act similarly to $\hat{\mathfrak{S}}^{\{\mathfrak{a}}\hat{\mathfrak{Q}}^{\mathfrak{b}\}}$.

Here we have to make a distinction between states with zero and states with nonzero momentum. For zero momentum the combination $\hat{\mathfrak{S}}^{\{\mathfrak{a}}\hat{\mathfrak{Q}}^{\mathfrak{b}\}}$ already explains the degenerate state $|1_L\rangle$. However, due to the $\mathfrak{psu}(1|1)^2$ algebra, it cannot explain any of the other degenerate states. Conversely, in the case of nonzero momentum the individual generators $\hat{\mathfrak{S}}^{\mathfrak{a}}$ and $\hat{\mathfrak{Q}}^{\mathfrak{b}}$ cannot be defined independently because it is not possible to change the length of the spin chain preserving the momentum.[8] It is nevertheless possible to consistently define the product $\hat{\mathfrak{S}}^{\{\mathfrak{a}}\hat{\mathfrak{Q}}^{\mathfrak{b}\}}$ for nonzero-momentum states because it preserves the length. This is the bilocal operator

$$\mathcal{Y}^{\mathfrak{ab}} = \sum_{j=0}^{L-1}\sum_{i=0}^{L+1}(1 - \tfrac{1}{2}\delta_{i,0} - \tfrac{1}{2}\delta_{i,L+1})\,\mathcal{U}^{j-i}\,\hat{\mathfrak{S}}^{\{\mathfrak{a}}(1,2)\,\mathcal{U}^{i}\,\hat{\mathfrak{Q}}^{\mathfrak{b}\}}(1)\,\mathcal{U}^{-j}. \tag{5.1}$$

Here, $\mathcal{U}$ is the operator that shifts the chain by one site to the right; it commutes with all of the local symmetry generators. The summation over j ensures that $\mathcal{Y}^{\mathfrak{ab}}$ acts homogeneously on the chain, and the symmetrization in the indices makes it a $\mathfrak{su}(2)$ triplet, as needed to explain the degeneracy. The generator $\hat{\mathfrak{Q}}(1)$ removes the first site of the chain and replaces it with two sites, while $\hat{\mathfrak{S}}(1,2)$ replaces the first two sites of the spin chain with one. So, the generator $\mathcal{Y}^{\mathfrak{ab}}$ consists of products of the $\hat{\mathfrak{Q}}$ and $\hat{\mathfrak{S}}$ generators acting all possible distances apart, with equal weight except for a symmetric

[8] The eigenvalues of a lattice momentum operator take the values $2\pi m/L$ (mod 2π). Changing the length L by one unit only preserves the eigenvalue zero.

regularization when a $\hat{\mathfrak{S}}$ interaction acts on both sites created by a $\hat{\mathfrak{Q}}$ interaction. The regularization resolves the one-site ambiguity in where to place newly created sites.

For zero-momentum states the action of $\mathcal{Y}^{\mathfrak{ab}}$ is equivalent to the action of $\hat{\mathfrak{S}}^{\{\mathfrak{a}}\hat{\mathfrak{Q}}^{\mathfrak{b}\}}$. Therefore, it cannot be used to immediately explain the additional degeneracy beyond the established $\mathfrak{psu}(1|1)^2$ symmetry in the zero-momentum sector. We will discuss this further in Section 5.5. However, $\mathcal{Y}^{\mathfrak{ab}}$ does commute exactly with $\mathfrak{psu}(1,1|2)$ and with the Hamiltonian even if the momentum is nonzero; a proof is given in Appendix C. Therefore the existence of $\mathcal{Y}^{\mathfrak{ab}}$ proves the additional degeneracies for all states with nonzero momentum.

The generators $\mathcal{Y}^{\mathfrak{ab}}$ immediately explain the form of the simplest degenerate state (4.10) found in the last section; it is related to the vacuum by applying $\mathcal{Y}^{<<}$ once,

$$|1_L\rangle \sim \mathcal{Y}^{<<}|0_L\rangle. \tag{5.2}$$

For even length $L \leq 10$ we have checked directly that the remaining descendants are given by

$$\mathcal{Y}^{<>}|1_L\rangle, \quad \ldots, \quad (\mathcal{Y}^{<>})^{(L-4)}|1_L\rangle. \tag{5.3}$$

Of course, further application of $\mathcal{Y}^{<>}$ generates no additional linearly independent states. Since the $(\mathcal{Y}^{<>})^m|1_L\rangle$ include all of the degenerate states, there is a linear combination of them that equals the degenerate state generated by the cubic invariant $(\mathfrak{J}^3)^{<<}|0_L\rangle$. Also, a short computation shows that the $\mathcal{Y}^{<>}$ transform under parity $\mathbf{p}$ as

$$\mathbf{p}\,\mathcal{Y}^{\mathfrak{ab}}\,\mathbf{p} = \mathcal{U}\mathcal{Y}^{\mathfrak{ab}}. \tag{5.4}$$

This is consistent with the counting of the parities of degenerate states done in Section 4.3. When acting on the even-length vacuum ($\mathcal{U}$ eigenvalue -1), the $\mathcal{Y}^{\mathfrak{ab}}$ are parity odd and generate a sequence of alternating parity degenerate states.

For the odd-length states, which have vanishing momentum, one can easily convince oneself using $\mathcal{Y}^{ab} \simeq \hat{\mathfrak{S}}^{\{\mathfrak{a}}\hat{\mathfrak{Q}}^{\mathfrak{b}\}}$ that the states (5.3) are all proportional to $|1_L\rangle$.

5.2 An Infinite-Dimensional Algebra

Let us first understand the algebra of $\mathcal{Y}^{\mathfrak{ab}}$ in the zero-momentum sector, where we have a representation in terms of $\mathfrak{psu}(1|1)^2$ generators. It is not difficult to convince oneself of the following relations,

$$\begin{aligned}\left[\mathfrak{B}^{\mathfrak{ab}}, \hat{\mathfrak{D}}^m\hat{\mathfrak{Q}}^{\mathfrak{c}}\hat{\mathfrak{S}}^{\mathfrak{d}}\right] &= \hat{\mathfrak{D}}^m\varepsilon^{\mathfrak{c}\{\mathfrak{b}}\hat{\mathfrak{Q}}^{\mathfrak{a}\}}\hat{\mathfrak{S}}^{\mathfrak{d}} - \hat{\mathfrak{D}}^m\hat{\mathfrak{Q}}^{\mathfrak{c}}\hat{\mathfrak{S}}^{\{\mathfrak{b}}\varepsilon^{\mathfrak{a}\}\mathfrak{d}},\\ \left[\hat{\mathfrak{D}}^m\hat{\mathfrak{Q}}^{\mathfrak{a}}\hat{\mathfrak{S}}^{\mathfrak{b}}, \hat{\mathfrak{D}}^n\hat{\mathfrak{Q}}^{\mathfrak{c}}\hat{\mathfrak{S}}^{\mathfrak{d}}\right] &= \hat{\mathfrak{D}}^{m+n+1}\varepsilon^{\mathfrak{cb}}\hat{\mathfrak{Q}}^{\mathfrak{a}}\hat{\mathfrak{S}}^{\mathfrak{d}} - \hat{\mathfrak{D}}^{m+n+1}\varepsilon^{\mathfrak{ad}}\hat{\mathfrak{Q}}^{\mathfrak{c}}\hat{\mathfrak{S}}^{\mathfrak{b}}.\end{aligned} \tag{5.5}$$

Denoting these combinations by $\mathcal{Y}_k^{\mathfrak{ab}}$, $k = 0, 1, 2, \ldots$, such that $\mathcal{Y}_0^{\mathfrak{ab}} = \mathfrak{B}^{\mathfrak{ab}}$ and $\mathcal{Y}_n^{\mathfrak{ab}} \simeq \hat{\mathfrak{D}}^{n-1}\mathcal{Y}^{\mathfrak{ab}}$, we obtain the infinite-dimensional algebra

$$[\mathcal{Y}_m^{\mathfrak{ab}}, \mathcal{Y}_n^{\mathfrak{cd}}] = \varepsilon^{\mathfrak{cb}}\mathcal{Y}_{m+n}^{\mathfrak{ad}} - \varepsilon^{\mathfrak{ad}}\mathcal{Y}_{m+n}^{\mathfrak{cb}}. \tag{5.6}$$

This algebra is a parabolic subalgebra of the loop algebra of $\mathfrak{su}(2)$.

We conjecture that the same algebra (5.6) holds not only for the zero-momentum sector, but for all states if we identify

$$\mathcal{Y}_0^{\mathfrak{ab}} = \mathfrak{B}^{\mathfrak{ab}}, \qquad \mathcal{Y}_1^{\mathfrak{ab}} = \mathcal{Y}^{\mathfrak{ab}}, \qquad \mathcal{Y}_{n+1}^{\mathfrak{ab}} = -\tfrac{1}{2}\varepsilon_{\mathfrak{cd}}[\mathcal{Y}^{\mathfrak{c}\{\mathfrak{a}}, \mathcal{Y}_n^{\mathfrak{b}\}\mathfrak{d}}]. \tag{5.7}$$

It is quite clear that the relations with $m = 0$ or $n = 0$ hold by $\mathfrak{su}(2)$ symmetry. Furthermore, the relation with $m = n = 1$ merely defines $\mathcal{Y}_2^{\mathfrak{ab}}$. The relations with $m + n \geq 3$ are nontrivial and have to be verified.

In fact, the relations with $m+n = 3$ are the Serre relations for the algebra, and they imply all the relations with $m + n > 3$. In the following we will prove this statement by induction. For convenience, we switch to an adjoint basis for $\mathcal{Y}_n^{\mathfrak{i}}$, $\mathfrak{i} = 1, 2, 3$ where the $\mathfrak{su}(2)$ structure constants are given by the totally antisymmetric tensor $\varepsilon^{\mathfrak{ijk}}$. The commutation relations can now be written for all nonnegative integer levels N as

$$[\mathcal{Y}_m^{\mathfrak{i}}, \mathcal{Y}_{N-m}^{\mathfrak{j}}] = \varepsilon^{\mathfrak{ijk}}\mathcal{Y}_N^{\mathfrak{k}}, \quad m = 0, \ldots N. \tag{5.8}$$

Assume (5.8) is satisfied at some level $N \geq 3$. Then we use five main steps to show that it is satisfied at level $N + 1$.

- Step 1. Using our inductive assumption, consider the equations for $m = 1, \ldots N - 2$ and their cyclic permutations,

$$\begin{aligned} 0 &= [\mathcal{Y}_1^2, \mathcal{Y}_{N-m}^1] + [\mathcal{Y}_1^1, \mathcal{Y}_{N-m}^2] \\ &= \big[\mathcal{Y}_m^3, [\mathcal{Y}_1^2, \mathcal{Y}_{N-m}^1]\big] + \big[\mathcal{Y}_m^3, [\mathcal{Y}_1^1, \mathcal{Y}_{N-m}^2]\big] \\ &= [\mathcal{Y}_1^2, \mathcal{Y}_N^2] - [\mathcal{Y}_{m+1}^1, \mathcal{Y}_{N-m}^1] + [\mathcal{Y}_{m+1}^2, \mathcal{Y}_{N-m}^2] - [\mathcal{Y}_1^1, \mathcal{Y}_N^1]. \end{aligned} \tag{5.9}$$

  Comparing the $m = M$ and $m = N - M - 1$ equations, we find that

$$[\mathcal{Y}_m^1, \mathcal{Y}_{N+1-m}^1] = [\mathcal{Y}_m^2, \mathcal{Y}_{N+1-m}^2] = [\mathcal{Y}_m^3, \mathcal{Y}_{N+1-m}^3], \quad m = 1, \ldots N. \tag{5.10}$$

- Step 2. We also have, for $m = 1, \ldots N$

$$\begin{aligned} [\mathcal{Y}_m^1, \mathcal{Y}_{N+1-m}^1] &= \big[\mathcal{Y}_m^1, [\mathcal{Y}_1^2, \mathcal{Y}_{N-m}^3]\big] \\ &= [\mathcal{Y}_{m+1}^3, \mathcal{Y}_{N-m}^3] - [\mathcal{Y}_1^2, \mathcal{Y}_N^2], \end{aligned} \tag{5.11}$$

  and cyclic permutations. Using the result from step 1, we find

$$[\mathcal{Y}_m^1, \mathcal{Y}_{N+1-m}^1] = m[\mathcal{Y}_1^1, \mathcal{Y}_N^1], \quad m = 1, \ldots N, \tag{5.12}$$

  and similarly for cyclic permutations. However, since

$$[\mathcal{Y}_1^1, \mathcal{Y}_N^1] = -[\mathcal{Y}_N^1, \mathcal{Y}_1^1]. \tag{5.13}$$

  we must have

$$0 = [\mathcal{Y}_m^1, \mathcal{Y}_{N+1-m}^1] = [\mathcal{Y}_m^2, \mathcal{Y}_{N+1-m}^2] = [\mathcal{Y}_m^3, \mathcal{Y}_{N+1-m}^3], \quad m = 1, \ldots N. \tag{5.14}$$

- Step 3. Commuting $\mathcal{Y}_0$ with $[\mathcal{Y}^1_m, \mathcal{Y}^1_{N+1-m}]$ (and cyclic permutations) yields

$$[\mathcal{Y}^{\mathfrak{i}}_m, \mathcal{Y}^{\mathfrak{j}}_{N+1-m}] = -[\mathcal{Y}^{\mathfrak{j}}_m, \mathcal{Y}^{\mathfrak{i}}_{N+1-m}], \quad m = 1, \dots N. \tag{5.15}$$

- Step 4. We can now show that there is a unique consistent way to define $\mathcal{Y}^{\mathfrak{k}}_{N+1}$. For instance, consider the following equations for $m = 1, \dots N-1$,

$$\begin{aligned}
[\mathcal{Y}^1_m, \mathcal{Y}^2_{N+1-m}] &= \big[\mathcal{Y}^1_m, [\mathcal{Y}^3_1, \mathcal{Y}^1_{N-m}]\big] = -[\mathcal{Y}^2_{m+1}, \mathcal{Y}^1_{N-m}] \\
&= [\mathcal{Y}^1_{m+1}, \mathcal{Y}^2_{N-m}] \\
&\dots \\
&= [\mathcal{Y}^1_N, \mathcal{Y}^2_1] \\
&= \mathcal{Y}^3_{N+1}.
\end{aligned} \tag{5.16}$$

- Step 5. It is now straightforward to use any of the equivalent expressions for $\mathcal{Y}^{\mathfrak{k}}_{N+1}$ to check that

$$[\mathcal{Y}^{\mathfrak{i}}_0, \mathcal{Y}^{\mathfrak{j}}_{N+1}] = \varepsilon^{\mathfrak{ijk}} \mathcal{Y}^{\mathfrak{k}}_{N+1}. \tag{5.17}$$

  This completes the set of equations at level $N+1$. Therefore, assuming the level-3 equations are satisfied, (5.8) is satisfied for all N.

At this time, a direct proof of the level-3 relations is beyond our technical capabilities. Note that to prove the level-3 relations, it is sufficient to check that (switching back to the previous $\mathfrak{su}(2)$ notation) $[\mathcal{Y}^{<>}_1, \mathcal{Y}^{<>}_2] = 0$, since commutators with the $\mathfrak{B}$ yield the remaining relations. This relation can also be written using only bilocal generators as

$$\big[\mathcal{Y}^{<>}, [\mathcal{Y}^{<<}, \mathcal{Y}^{>>}]\big] = 0. \tag{5.18}$$

Still, we have to gain confidence in the level-3 relations. As a start, using `Mathematica` we have checked that they are satisfied on many states of small excitation number, including all states of length 4 with 4 or fewer excitations (above the half-BPS vacuum) and all state of length 5 or 6 with 3 or fewer excitations. Also checked were states with larger lengths and excitation numbers, including a length-7, 7-excitation state. Checking much longer or higher excitation states rapidly becomes impractical because of combinatorics. However, we consider the evidence described above as persuasive. Hopefully, a complete proof will become possible in the future.

5.3 The Representation of the Loop Algebra

The observed degeneracies motivating this work should correspond to irreducible 2^M-dimensional representations of the above loop algebra. Finite-dimensional representations of loop algebras are typically tensor products of evaluation representations. In an evaluation representation, the level-n generator $\mathcal{Y}_n$ acts like the level-0 generator $\mathcal{Y}_0$ multiplied by the n-th power of the evaluation parameter x

$$\mathcal{Y}_n|x\rangle = x^n \mathcal{Y}_0|x\rangle. \tag{5.19}$$

L	p	u	v	$\dot{v}$
3	$\pm\frac{2\pi}{3}$	$1 \mp 4\sqrt{3}u - 12u^2 \pm 16\sqrt{3}u^3$	$1 \pm \sqrt{3}v - 6v^2$	$1 \mp \sqrt{3}\dot{v}$
4	$\pm\frac{\pi}{2}$	$1 \mp 16u - 40u^2 \pm 64u^3 + 80u^4$	$1 \pm 5v - 6v^2 \mp 10v^3$	$1 \pm \dot{v}$
4	π	$u - 4u^3$	$1 - 6v^2$	—
5	$\pm\frac{2\pi}{5}$	$-\frac{1}{2}\sqrt{1+\frac{2}{\sqrt{5}}}(1 - 40u^2 + 80u^4)$ $\pm 3u \mp 40x^3 \pm 48u^5$	$1 - 15v^2 + 15v^4$ $\pm 5\sqrt{1+\frac{2}{\sqrt{5}}}(v - 2v^3)$	$\sqrt{1+\frac{2}{\sqrt{5}}} \mp \dot{v}$
6	π	$-3u + 40u^3 - 48u^5$	$1 - 15v^2 + 15v^4$	—
8	π	$u - 28u^3 + 112u^5 - 64^7$	$1 - 28v^2 + 70v^4 - 28v^6$	—

Table 1: Eigenstates used for checking the relationship (5.24) between $\mathcal{Y}$ eigenvalues and Bethe roots. The first two columns give the length and momentum of the eigenstates. The last three columns give the polynomials whose zeros are the Bethe roots. Note that the contributions of equal auxiliary roots ($v_k = \dot{v}_j$) cancel in the expression for $\mathcal{Y}^{<>}_n$ (5.25).

Tensor products of evaluation representations $|x_k\rangle$ with distinct evaluation parameters x_k are generally irreducible. The basic reason is that the sum over $(x_k)^n$ is not proportional to the n-th power of the sum over x_k.

In our case the relevant evaluation module is two-dimensional and consists of the states

$$|{<}, x\rangle \quad \text{and} \quad |{>}, x\rangle. \tag{5.20}$$

Explicitly, the generators act on these states as (note that $\mathcal{Y}^{\mathfrak{ab}}_0 = \mathfrak{B}^{\mathfrak{ab}}$)

$$\begin{aligned}
&\mathcal{Y}^{<<}_n|{<},x\rangle = +x^n|{>},x\rangle, && \mathcal{Y}^{<<}_n|{>},x\rangle = 0,\\
&\mathcal{Y}^{>>}_n|{<},x\rangle = 0, && \mathcal{Y}^{>>}_n|{>},x\rangle = -x^n|{<},x\rangle,\\
&\mathcal{Y}^{<>}_n|{<},x\rangle = -\tfrac{1}{2}x^n|{<},x\rangle, && \mathcal{Y}^{<>}_n|{>},x\rangle = +\tfrac{1}{2}x^n|{<},x\rangle,
\end{aligned} \tag{5.21}$$

which is consistent with the algebra (5.6). Then, tensor products labeled by the highest-weight state

$$|\Psi\rangle = |{<},x_1\rangle \otimes |{<},x_2\rangle \otimes \ldots \otimes |{<},x_M\rangle \tag{5.22}$$

with distinct x_k form multiplets of dimension 2^M. These multiplets are characterized by the eigenvalues of the generator $\mathcal{Y}^{<>}_n$

$$\mathcal{Y}^{<>}_n|\Psi\rangle = -\frac{1}{2}\left(\sum_{i=1}^{M} x_k^n\right)|\Psi\rangle. \tag{5.23}$$

In fact, by examining some representative eigenstates listed in Table 1, we find that the x_k should be simply related to the auxiliary Bethe roots v_k and $\dot{v}_k$ as

$$x_k = \frac{i(1-e^{iP})}{v_k}\,, \tag{5.24}$$

where P is the overall momentum of the state. With this identification, the algebra implies that any nonzero momentum Bethe eigenstate $|\Psi\rangle$ characterized by auxiliary roots $\{v_1,\ldots,v_M\}$ and $\{\dot v_1,\ldots,\dot v_{\dot M}\}$ satisfies

$$\mathcal{Y}_n^{<>}|\Psi\rangle = -\tfrac{1}{2}\big(i(1-e^{iP})\big)^n \left(\sum_{k=1}^{M}\frac{1}{v_k^n} - \sum_{k=1}^{\dot M}\frac{1}{\dot v_k^n}\right)|\Psi\rangle. \tag{5.25}$$

This identification (5.24) is not surprising since the auxiliary Bethe roots are closely associated with the degeneracy. Furthermore, the inverse dependence on the v and on the $\dot v$ follows from (5.21). This is necessary for compatibility with invariance of $\mathcal{Y}^{\mathfrak{ab}}$ under the $\mathfrak{psu}(1,1|2)$ algebra. It is also consistent with the fact that a pair of equal auxiliary Bethe roots v and $\dot v$ leads to a singlet rather than a quadruplet.

It is curious that the overall momentum P appears in the definition of the evaluation parameter. It actually cancels the singularities that occur when there are auxiliary roots v or $\dot v$ at zero: As explained in Sec. 2.2 this can only happen for zero-momentum states, and in that case the factor in the numerator $(1-e^{iP})$ also goes to zero. The explicit evaluation of $\mathcal{Y}_n^{<>}$ for the odd-length vacuum state $|0_L\rangle$ in (4.1) gives the proper regularization (for $n>0$)

$$\mathcal{Y}_n^{<>}|0_L\rangle = -\tfrac{1}{2}\hat D^n|0_L\rangle. \tag{5.26}$$

Here $\hat D = \frac{1}{2}E = 2L$ is the eigenvalue of $\hat{\mathfrak{D}}$ which equals half the energy of the state. In other words, the state corresponds to the following tensor product of evaluation representations,

$$|0_L\rangle = |<,\hat D\rangle \otimes |<,0\rangle \otimes \ldots \otimes |<,0\rangle. \tag{5.27}$$

Therefore the generators $\mathcal{Y}_n$, $n>0$, transform effectively only the first doublet. This is fully consistent with our above findings that the generators $\mathcal{Y}_n$ cannot explain the degeneracy in the zero-momentum case and also with the algebra $\mathcal{Y}_n \simeq \hat{\mathfrak{D}}^{n-1}\mathcal{Y}$.

Finally, we should emphasize that we have not proven that the identification (5.24) is satisfied for all states, but we have given compelling evidence of its truth.

5.4 Relation to Yangian Symmetry

The apparent asymptotic integrability of the $\mathcal{N}=4$ SYM spin chain is equivalent to the existence of a Yangian symmetry [17], which is a nonlocal infinite-dimensional symmetry. The Yangian of the $\mathcal{N}=4$ SYM spin chain was constructed at leading order in [18], and its perturbative corrections in subsectors have been studied in [19–21]. The Yangian is a Hopf algebra whose structure is the subject of many recent investigations [22]. In general, for a Lie algebra with generators $\mathfrak{J}^A$, the Yangian is generated by the Lie generators $\mathfrak{J}_0^A = \mathfrak{J}^A$ combined with additional generators, $\mathfrak{J}_1^A$. In a spin chain description, the $\mathfrak{J}_1^A$ act as bilocal products of the $\mathfrak{J}_0^A$

$$\mathfrak{J}_1^A \simeq \sum_{k<n} f^A{}_{BC}\mathfrak{J}_0^B(k)\mathfrak{J}_0^C(n), \tag{5.28}$$

where $f^A{}_{BC}$ are the structure constants. From this action, it is clear that the $\mathfrak{J}_1^A$ transform in the adjoint of the Lie algebra. The Yangian generators must satisfy a Serre relation,[9]

$$\big[\mathfrak{J}_1^A,[\mathfrak{J}_1^B,\mathfrak{J}_0^C\}\big\} - \big[\mathfrak{J}_0^A,[\mathfrak{J}_1^B,\mathfrak{J}_1^C\}\big\} = \tfrac{1}{6}a^{ABC}{}_{DEF}\,\{\mathfrak{J}_0^D,\mathfrak{J}_0^E,\mathfrak{J}_0^F],$$
$$a^{ABC}{}_{DEF} = (-1)^{(EM)} f^{AK}{}_D f^B{}_E{}^L f^C{}_F{}^M f_{KLM}. \qquad (5.29)$$

The term on the right hand side implies that a Yangian is a deformation of the loop (sub)algebra of a Lie algebra. Also, combining this Serre relation with the adjoint transformation of the $\tilde{\mathfrak{J}}^A$ implies another relation,

$$\big[[\mathfrak{J}_1^A,\mathfrak{J}_1^B\},[\mathfrak{J}_0^P,\mathfrak{J}_1^Q\}\big\} + \big[[\mathfrak{J}_1^P,\mathfrak{J}_1^Q\},[\mathfrak{J}_0^A,\mathfrak{J}_1^B\}\big\} = \tfrac{1}{6}a^{ABC}{}_{DEF} f^{PQ}{}_C\{\mathfrak{J}_0^D,\mathfrak{J}_0^E,\mathfrak{J}_1^F]. \quad (5.30)$$

This second Serre relation is useful when considering a $\mathfrak{su}(2)$ algebra, since in that case the first Serre relation is trivial.

Let us now compare the action of $\mathcal{Y}^{\mathfrak{ab}}$ in (5.1) to the formal action of Yangian generators in (5.28). The former acts as a bilocal product of $\hat{\mathfrak{S}}^{\mathfrak{a}}$ and $\hat{\mathfrak{Q}}^{\mathfrak{b}}$. The same is true for the $\mathfrak{su}(2)$ automorphism $\hat{\mathfrak{B}}^{\mathfrak{ab}}$ of $\mathfrak{psu}(1|1)^2$ with vanishing central charges $\hat{\mathfrak{C}}^{\mathfrak{ab}} = 0$. In fact, the action is equal up to (not yet investigated) issues related to the length-changing nature of the $\mathfrak{psu}(1|1)^2$ generators $\hat{\mathfrak{S}}^{\mathfrak{a}}$ and $\hat{\mathfrak{Q}}^{\mathfrak{b}}$. Therefore it is natural to identify our generators $\mathcal{Y}^{\mathfrak{ab}}$ with the level-one generators of the $\mathfrak{su}(2)$ automorphism[10]

$$\mathcal{Y}^{\mathfrak{ab}} = \hat{\mathfrak{B}}_1^{\mathfrak{ab}}. \qquad (5.31)$$

The generators $\mathcal{Y}^{\mathfrak{ab}}$ would thus enlarge the $\mathfrak{psu}(1|1)^2$ Yangian (which is a part of the full $\mathfrak{psu}(2,2|4)$ Yangian [18]) by an automorphism in just the same way as the generators $\mathfrak{B}^{\mathfrak{ab}}$ enhance the the $\mathfrak{psu}(1|1)^2$ Lie algebra by the $\mathfrak{su}(2)$ automorphism. Consistently with this identification, the Serre relation (5.30) implies that the $\mathcal{Y}^{\mathfrak{ab}}$ generate an *undeformed* $\mathfrak{su}(2)$ loop (sub)algebra, since the relevant combinations of structure constants appearing on the right side vanishes for central charges $\mathfrak{C}^{\mathfrak{ab}} = 0$.[11]

Of course, for nonzero momentum states the $\mathfrak{psu}(1|1)^2$ symmetry no longer applies. However, it does apply for infinite-length states (which are typically required also for Yangian symmetry to be realized[12]). We can then view the loop algebra symmetry for nonzero-momentum states simply as the consequence of the extended $\mathfrak{psu}(1|1)^2$ Yangian for the infinite-length chain combined with the fact that the $\mathcal{Y} = \hat{\mathfrak{B}}_1$ are length-preserving. In contrast, the other Yangian generators, the $\hat{\mathfrak{Q}}_1$ or $\hat{\mathfrak{S}}_1$, clearly are not a symmetry for finite-length nonzero-momentum states since they also change the length of the chain. It is still unusual even for part of this Yangian symmetry to be realized

[9] The symmetric triple product is $\{x_1,x_2,x_3] = \sum_{i\neq j\neq k} x_i x_j x_k$, with appropriate additional signs for fermionic x.

[10] As in [20], the definition of the bilocal product here needs to be generalized naturally to allow for multisite (and even length-changing) symmetry generators. With appropriate modifications of the local terms, we could write the bilocal product also including terms with $\hat{\mathfrak{S}}$ acting first, in agreement with the bilocal action.

[11] This Yangian relation thus provides an efficient way to see that the level-3 relation suffices to guarantee that the loop subalgebra is satisfied. The level-3 relation is equivalent to the Serre relation for the $\mathfrak{su}(2)$ part of the Yangian.

[12] The bilocal Yangian generators usually cannot be defined consistently with periodic boundary conditions.

exactly by the Hamiltonian for finite-length states. It is closely tied to the fact that the generators $\hat{\mathfrak{S}}$ and $\hat{\mathfrak{Q}}$ change the length by a definite and opposite amount, so that a bilocal product consistent with periodic boundary conditions can be constructed.

While we identify the $\mathfrak{su}(2)$ automorphisms of the $\mathfrak{psu}(1,1|2)$ and $\mathfrak{psu}(1|1)^2$ algebras, $\mathfrak{B} = \hat{\mathfrak{B}}$, apparently the corresponding Yangians cannot be identified, $\mathfrak{B}_1 \neq \hat{\mathfrak{B}}_1$. For instance, the $\mathcal{Y} = \hat{\mathfrak{B}}_1$ commute with the $\mathfrak{psu}(1,1|2)$ algebra, while the Yangian generator $\mathfrak{B}_1$ should not commute. However, it would still be interesting to generalize [20] to obtain the $\mathcal{O}(g^2)$ corrections to the extended $\mathfrak{psu}(1,1|2)$ Yangian. It is possible still that there is some relation between the $\mathcal{O}(g^2)$ Yangian generators of the two automorphisms.

Finally, note that we can now expect contributions to other Yangian generators at $\mathcal{O}(g^2)$ that have similar spin chain structure as the $\mathcal{Y}^{\mathfrak{ab}}$. Suitable sectors that have generators that act nontrivially at $\mathcal{O}(g)$ include the $\mathfrak{su}(2|3)$ sector as well as the full $\mathfrak{psu}(2,2|4)$ spin chain.

5.5 Zero-Momentum Degeneracy and the Yangian Double

With our new understanding of the origin of the algebra generated by the $\mathcal{Y}$, there is a natural explanation for the remaining degeneracy of the zero-momentum sector. As noted above, any Bethe eigenstate in the zero-momentum sector has a root at $v = 0$ or $\dot{v} = 0$. The contribution from this root dominates the eigenvalue of the $\mathcal{Y}^{<>}$, so that these states only form doublets of the loop algebra. What is needed to explain the other degenerate states is a generator with the inverse eigenvalues, for which the nonzero roots would dominate. This is precisely what we would expect from the full $\mathfrak{su}(2)$ loop algebra, which would follow from the double Yangian [23] for the extended $\mathfrak{psu}(1|1)^2$ algebra.

The full $\mathfrak{su}(2)$ loop algebra takes the same form as in (5.6), except now the $\mathcal{Y}_m$ are defined for all integer values m. So the full algebra is generated by $\mathcal{Y}_0^{\mathfrak{ab}}$, $\mathcal{Y}_1^{\mathfrak{ab}}$, and $\mathcal{Y}_{-1}^{\mathfrak{ab}}$. The additional relations that would need to be checked to verify that the $\mathcal{Y}_{-1}^{\mathfrak{ab}}$ generate the rest of the algebra include the level-(-3) Serre relation and

$$[\mathcal{Y}_{-1}^{\mathfrak{ab}}, \mathcal{Y}_1^{\mathfrak{cd}}] = \varepsilon^{\mathfrak{cb}}\mathfrak{B}^{\mathfrak{ad}} - \varepsilon^{\mathfrak{ad}}\mathfrak{B}^{\mathfrak{cb}}. \tag{5.32}$$

It would be very interesting to find the spin chain representation for the $\mathcal{Y}_{-1}^{\mathfrak{ab}}$, which basically invert the $\mathcal{Y}_1^{\mathfrak{ab}}$. We leave this investigation for the future, but note that using the example states in Table 1 and (5.25) with $n = -1$ will provide significant information about these generators. However, unlike the $\mathcal{Y}_1^{\mathfrak{ab}}$ there does not appear to be as natural a representation in terms of ordinary symmetry generators. Finally, once one finds the $\mathcal{Y}_{-1}^{\mathfrak{ab}}$, one could immediately compute the $\hat{\mathfrak{Q}}_{-1}$ and $\hat{\mathfrak{S}}_{-1}$, which would not act just as products of ordinary symmetry generators.

5.6 A Singlet Bilocal Generator

It is curious to note that there exists a bilocal generator $\mathcal{X}$ very similar to the $\mathcal{Y}^{\mathfrak{ab}}$, which is a $\mathfrak{su}(2)$-singlet

$$\mathcal{X} = \sum_{j=0}^{L-1}\sum_{i=0}^{L+1}(1 - \tfrac{1}{2}\delta_{i,0} - \tfrac{1}{2}\delta_{i,L+1})\tfrac{1}{2}\varepsilon_{\mathfrak{ba}}\mathcal{U}^{j-i}\,\hat{\mathfrak{S}}^{\mathfrak{a}}(1,2)\,\mathcal{U}^{i}\,\hat{\mathfrak{Q}}^{\mathfrak{b}}(1)\,\mathcal{U}^{-j}. \tag{5.33}$$

Like the $\mathcal{Y}^{\mathfrak{ab}}$, it commutes with the $\mathfrak{psu}(1,1|2)$ algebra and the one-loop Hamiltonian, as discussed in Appendix C.

Similar to the reasoning used at the beginning of Section 5.2, we can use the zero-momentum reduction for $\mathcal{X}$ to conjecture that it commutes with the $\mathcal{Y}_n^{\mathfrak{ab}}$ for all n. Again, we have obtained very strong evidence using `Mathematica`. We have checked that these commutators vanish for the same set of states described in the last paragraph of Section 5.2. It is however presently not clear how to generalize $\mathcal{X}$ to -dimensional algebra of $\mathcal{X}_m$. For such an algebra, we would have $\mathcal{X}_0 = \hat{\mathfrak{A}}$, and for zero-momentum states the remaining generators would simply be $\mathcal{X}_n = \hat{\mathfrak{D}}^{n-1}\mathcal{X}$, yielding an abelian algebra that commutes with the $\mathcal{Y}_n^{\mathfrak{ab}}$.

Using the states in Table 1, we find that the eigenvalues of $\mathcal{X}$ only differ from those of $\mathcal{Y}_1^{<>}$ by the relative sign between the v and $\dot{v}$ contributions,

$$\mathcal{X}|\Psi\rangle = -\tfrac{i}{2}(1-e^{iP})\left(\sum_{k=1}^{M}\frac{1}{v_k}+\sum_{k=1}^{\dot{M}}\frac{1}{\dot{v}_k}\right)|\Psi\rangle. \tag{5.34}$$

From this and the fact that $\mathcal{X}$ reduces to a product of $\mathfrak{psu}(1|1)^2$ symmetry generators for zero-momentum states, we see that $\mathcal{X}$ does not map between different $\mathfrak{psu}(1,1|2)$ multiplets.

Similar arguments as in Section 5.4 imply that we can identify $\mathcal{X}$ as a bilocal Yangian generator, $\mathcal{X} = \hat{\mathfrak{A}}_1$, and the Serre relation (5.29) then implies that $\mathcal{X}$ commutes with the triplet $\mathcal{Y}$. Finally, $\mathcal{X}$ also should have a double, but the double would commute with $\mathcal{X}$.

6 Conclusions and Outlook

In this article we have investigated a curious 2^M-fold degeneracy of an integrable spin chain with $\mathfrak{psu}(1,1|2)$ symmetry. This degeneracy was observed at the level of Bethe equations in [3]. Here we have considered the symmetry algebra that explains the degeneracy. We have constructed two triplets of symmetry generators, $\mathfrak{B}$ and $\mathcal{Y}$, at the level of operators acting on spin chain states. The local generators $\mathfrak{B}$ form a $\mathfrak{su}(2)$ automorphism of $\mathfrak{psu}(1,1|2)$ while the bilocal generators $\mathcal{Y}$ commute with $\mathfrak{psu}(1,1|2)$. Together they apparently generate a subalgebra of the loop algebra of $\mathfrak{su}(2)$. This extended symmetry algebra commutes with the Hamiltonian and thus explains the degeneracy.

It remains an open problem to identify the spin chain operators that generate the 2^M degeneracy for zero-momentum states. As argued above, these operators are likely to generate the remaining part of the full $\mathfrak{su}(2)$ loop algebra. It is possible that these operators' spin chain representation is not simple. However, this still deserves further study especially because it is also possible that they would give new insight into the origin of the simple next-to-leading order corrections to the local symmetry generators obtained in [11].

While we have restricted our study to the one-loop Hamiltonian, it is clear that the symmetry enhancement persists at higher loops. The 2^M degeneracy of the Bethe ansatz is preserved by the higher-loop corrections [3]. Therefore, we expect the $\mathcal{Y}^{\mathfrak{ab}}$ to receive loop corrections so that they commute with the loop-corrected Hamiltonian. Note that the leading terms for the bilocal symmetry generators $\mathcal{Y}^{\mathfrak{ab}}_{(2)}$ discussed in this paper correspond to $\mathcal{O}(g^2)$. Given the Yangian origin of the $\mathcal{Y}^{\mathfrak{ab}}$, we expect the corrections for the bilocal generators to involve substituting the appropriate loop corrections for the $\mathfrak{psu}(1|1)^2$ generators appearing in the expression for the $\mathcal{Y}^{\mathfrak{ab}}$, similar to the quantum corrections to bilocal Yangian generators studied in [20]. That is, at $\mathcal{O}(g^{2\ell})$ the bilocal generators should take the form

$$\mathcal{Y}^{\mathfrak{ab}}_{(2\ell)} \simeq \sum_{m=1}^{\ell}\sum_{j=0}^{L-1}\sum_{i=0}^{L+1} \mathcal{U}^{j-i}\,\hat{\mathfrak{S}}^{\{\mathfrak{a}}_{(2m-1)}(1,\ldots,m+1)\,\mathcal{U}^{i}\,\hat{\mathfrak{Q}}^{\mathfrak{b}\}}_{(2\ell-2m+1)}(1,\ldots,\ell-m+1)\,\mathcal{U}^{-j}. \tag{6.1}$$

Explicit calculation will be required to find the regularization of the overlap between $\hat{\mathfrak{S}}$ and $\hat{\mathfrak{Q}}$. The study of these corrections may be very useful in constraining the higher-loop contributions to the local symmetry generators.

As at leading order, for cyclic states the $\mathcal{Y}^{\mathfrak{ab}}_{(2\ell)}$ will reduce to ordinary products of the (loop-corrected) $\mathfrak{psu}(1|1)^2$ generators. Therefore, we expect that the loop corrections will also preserve the algebra of the $\mathcal{Y}^{\mathfrak{ab}}_{n}$. This would be consistent with the loop algebra following from the extended $\mathfrak{psu}(1|1)^2$ Yangian of the infinite-length chain, as discussed in Section 5.4, which is expected also to all orders in perturbation theory.

The degeneracy was observed in the context of $AdS_5 \times S^5$ string theory. However, it might also be relevant for certain superstring models on $AdS_3 \times S^3$ or $AdS_2 \times S^2$ which also possess $\mathfrak{psu}(1,1|2)$ symmetry. Further suitable models include the principal chiral/WZW model on the group manifold $\widetilde{\mathrm{PSU}}(1,1|2)$ or some of it cosets. For instance, in some of these cases an additional su(2) and some even larger unexplained degeneracies were noticed in [24]. It is conceivable that they are of a similar origin as the ones discussed here.

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A Commutation Relations

In the following we shall list the commutation relations for the symmetry algebras.

A.1 Maximally Extended $\mathfrak{psu}(1,1|2)$ Algebra

Let us first consider the $\mathfrak{psu}(1,1|2)$ algebra. It consists of the three $\mathfrak{su}(2)$ generators $\mathfrak{R}^{ab}=\mathfrak{R}^{ba}$, the three $\mathfrak{su}(1,1)$ generators $\mathfrak{J}^{\alpha\beta}=\mathfrak{J}^{\beta\alpha}$, and the eight fermionic generators $\mathfrak{Q}^{a\beta\mathfrak{c}}$. All Latin, Greek and Gothic indices can take one out of two values. A summary of commutation relations reads

$$\begin{aligned}
[\mathfrak{R}^{ab},\mathfrak{R}^{cd}] &= \varepsilon^{cb}\mathfrak{R}^{ad}-\varepsilon^{ad}\mathfrak{R}^{cb},\\
[\mathfrak{J}^{\alpha\beta},\mathfrak{J}^{\gamma\delta}] &= \varepsilon^{\gamma\beta}\mathfrak{J}^{\alpha\delta}-\varepsilon^{\alpha\delta}\mathfrak{J}^{\gamma\beta},\\
[\mathfrak{R}^{ab},\mathfrak{Q}^{c\delta\mathfrak{e}}] &= \tfrac{1}{2}\varepsilon^{ca}\mathfrak{Q}^{b\delta\mathfrak{e}}+\tfrac{1}{2}\varepsilon^{cb}\mathfrak{Q}^{a\delta\mathfrak{e}},\\
[\mathfrak{J}^{\alpha\beta},\mathfrak{Q}^{c\delta\mathfrak{e}}] &= \tfrac{1}{2}\varepsilon^{\delta\alpha}\mathfrak{Q}^{c\beta\mathfrak{e}}+\tfrac{1}{2}\varepsilon^{\delta\beta}\mathfrak{Q}^{c\alpha\mathfrak{e}},\\
\{\mathfrak{Q}^{a\beta\mathfrak{c}},\mathfrak{Q}^{d\epsilon\mathfrak{f}}\} &= \varepsilon^{\beta\epsilon}\varepsilon^{\mathfrak{cf}}\mathfrak{R}^{da}-\varepsilon^{ad}\varepsilon^{\mathfrak{cf}}\mathfrak{J}^{\beta\epsilon}+\varepsilon^{ad}\varepsilon^{\beta\epsilon}\mathfrak{C}^{\mathfrak{cf}}.
\end{aligned}\tag{A.1}$$

For completeness, we have introduced a maximal set of three central charges $\mathfrak{C}^{\mathfrak{ab}}=\mathfrak{C}^{\mathfrak{ba}}$. In the case of the spin representation they act trivially. The algebra furthermore admits an $\mathfrak{su}(2)$ grading. The commutators with the generators $\mathfrak{B}^{\mathfrak{ab}}=\mathfrak{B}^{\mathfrak{ba}}$ of the automorphism are canonical,

$$\begin{aligned}
[\mathfrak{B}^{\mathfrak{ab}},\mathfrak{B}^{\mathfrak{cd}}] &= \varepsilon^{\mathfrak{cb}}\mathfrak{B}^{\mathfrak{ad}}-\varepsilon^{\mathfrak{ad}}\mathfrak{B}^{\mathfrak{cb}},\\
[\mathfrak{B}^{\mathfrak{ab}},\mathfrak{Q}^{c\delta\mathfrak{e}}] &= \tfrac{1}{2}\varepsilon^{\mathfrak{ea}}\mathfrak{Q}^{c\delta\mathfrak{b}}+\tfrac{1}{2}\varepsilon^{\mathfrak{eb}}\mathfrak{Q}^{c\delta\mathfrak{a}},\\
[\mathfrak{B}^{\mathfrak{ab}},\mathfrak{C}^{\mathfrak{cd}}] &= \varepsilon^{\mathfrak{cb}}\mathfrak{C}^{\mathfrak{ad}}-\varepsilon^{\mathfrak{ad}}\mathfrak{C}^{\mathfrak{cb}}.
\end{aligned}\tag{A.2}$$

Note that the "central charges" $\mathfrak{C}^{\mathfrak{ab}}$ now become a spin-1 triplet under this $\mathfrak{su}(2)$ automorphism, i.e. they are not central for the maximally extended algebra. All in all this algebra can be denoted as $\mathfrak{su}(2)\ltimes\mathfrak{psu}(1,1|2)\ltimes\mathbb{R}^3$.

A.2 Maximally Extended $\mathfrak{psu}(1|1)^2$ Algebra

The only nontrivial commutator of the $\mathfrak{psu}(1|1)^2$ algebra reads

$$\{\hat{\mathfrak{Q}}^{\mathfrak{a}},\hat{\mathfrak{S}}^{\mathfrak{b}}\}=\hat{\mathfrak{C}}^{\mathfrak{ab}}+\varepsilon^{ab}\hat{\mathfrak{D}}.\tag{A.3}$$

For completeness we have introduced a triplet $\hat{\mathfrak{C}}^{\mathfrak{ab}}$ of central charges to accompany the singlet $\hat{\mathfrak{D}}$. In our spin chain model the triplet acts trivially, $\hat{\mathfrak{C}}^{\mathfrak{ab}}=0$.

The algebra admits a $\mathfrak{u}(2)$ grading, which can be split up into $\mathfrak{su}(2)$ and $\mathfrak{u}(1)$ gradings. The $\mathfrak{su}(2)$ automorphism is defined by the commutation relations

$$\begin{aligned}
[\hat{\mathfrak{B}}^{\mathfrak{ab}},\hat{\mathfrak{B}}^{\mathfrak{cd}}] &= \varepsilon^{\mathfrak{cb}}\hat{\mathfrak{B}}^{\mathfrak{ad}}-\varepsilon^{\mathfrak{ad}}\hat{\mathfrak{B}}^{\mathfrak{cb}},\\
[\hat{\mathfrak{B}}^{\mathfrak{ab}},\hat{\mathfrak{Q}}^{\mathfrak{c}}] &= \tfrac{1}{2}\varepsilon^{\mathfrak{ca}}\hat{\mathfrak{Q}}^{\mathfrak{b}}+\tfrac{1}{2}\varepsilon^{\mathfrak{cb}}\hat{\mathfrak{Q}}^{\mathfrak{a}},\\
[\hat{\mathfrak{B}}^{\mathfrak{ab}},\hat{\mathfrak{S}}^{\mathfrak{c}}] &= \tfrac{1}{2}\varepsilon^{\mathfrak{ca}}\hat{\mathfrak{S}}^{\mathfrak{b}}+\tfrac{1}{2}\varepsilon^{\mathfrak{cb}}\hat{\mathfrak{S}}^{\mathfrak{a}},\\
[\hat{\mathfrak{B}}^{\mathfrak{ab}},\hat{\mathfrak{C}}^{\mathfrak{cd}}] &= \varepsilon^{\mathfrak{cb}}\hat{\mathfrak{C}}^{\mathfrak{ad}}-\varepsilon^{\mathfrak{ad}}\hat{\mathfrak{C}}^{\mathfrak{cb}},
\end{aligned}\tag{A.4}$$

while the $\mathfrak{u}(1)$ grading $\hat{\mathfrak{A}}$ distinguishes $\hat{\mathfrak{Q}}$ from $\hat{\mathfrak{S}}$,

$$\begin{aligned}
[\hat{\mathfrak{A}},\hat{\mathfrak{Q}}^{\mathfrak{a}}] &= +\hat{\mathfrak{Q}}^{\mathfrak{a}},\\
[\hat{\mathfrak{A}},\hat{\mathfrak{S}}^{\mathfrak{a}}] &= -\hat{\mathfrak{S}}^{\mathfrak{a}}.
\end{aligned}\tag{A.5}$$

Altogether the algebra can be denoted by $\mathfrak{u}(2) \ltimes \mathfrak{psu}(1|1)^2 \ltimes \mathbb{R}^4$.

A priori the $\mathfrak{su}(2)$ automorphisms $\mathfrak{B}$ and $\hat{\mathfrak{B}}$ of $\mathfrak{psu}(1,1|2)$ and $\mathfrak{psu}(1|1)^2$, respectively, are not identical, but they merely satisfy the same commutation relations. For the spin representation of the product of these algebras in perturbative gauge theory, they should however be identified $\mathfrak{B} = \hat{\mathfrak{B}}$.

The $\mathfrak{psu}(1|1)^2$ algebra can be embedded in another $\mathfrak{psu}(1,1|2)$ algebra, with the fermionic generators now written as $\hat{\mathfrak{Q}}^{\alpha\beta\mathfrak{c}}$. Then we have

$$\hat{\mathfrak{Q}}^{\mathfrak{a}} = \hat{\mathfrak{Q}}^{1+\mathfrak{a}}, \qquad \hat{\mathfrak{A}} = \hat{\mathfrak{R}}^{12},$$
$$\hat{\mathfrak{S}}^{\mathfrak{a}} = \hat{\mathfrak{Q}}^{2-\mathfrak{a}}, \qquad \hat{\mathfrak{D}} = -\hat{\mathfrak{J}}^{+-} + \hat{\mathfrak{R}}^{12}. \tag{A.6}$$

B Multilinear Operators

In this appendix we list some relevant multilinear operators for the symmetry algebra. These include the quadratic Casimir invariant, but also an interesting triplet of cubic operators. We then show that the cubic operators satisfy the same algebra as the $\mathcal{Y}^{\mathfrak{ab}}$ and can be used to deform the $\mathcal{Y}^{\mathfrak{ab}}$ while preserving this algebra.

B.1 Quadratic Invariants

It is straightforward to construct the quadratic Casimir for the maximally extended $\mathfrak{psu}(1,1|2)$ algebra introduced in Appendix A,

$$\mathfrak{J}^2 = 2\varepsilon_{\mathfrak{bc}}\varepsilon_{\mathfrak{da}}\mathfrak{B}^{\mathfrak{ab}}\mathfrak{C}^{\mathfrak{cd}} + \varepsilon_{bc}\varepsilon_{da}\mathfrak{R}^{ab}\mathfrak{R}^{cd} - \varepsilon_{\beta\gamma}\varepsilon_{\delta\alpha}\mathfrak{J}^{\alpha\beta}\mathfrak{J}^{\gamma\delta} - \varepsilon_{ad}\varepsilon_{\beta\epsilon}\varepsilon_{\mathfrak{cf}}\mathfrak{Q}^{a\beta\mathfrak{c}}\mathfrak{Q}^{d\epsilon\mathfrak{f}}. \tag{B.1}$$

For the algebra without central extensions, $\mathfrak{C}^{\mathfrak{ab}} = 0$, the first terms simply drops out. The centrally extended algebra without automorphism, on the other hand, does not have a quadratic invariant because the first term is important, but it requires $\mathfrak{B}^{\mathfrak{ab}}$.

For the maximally extended $\mathfrak{psu}(1|1)^2$ the quadratic Casimir operator reads

$$\hat{\mathfrak{J}}^2 = \varepsilon_{\mathfrak{bc}}\varepsilon_{\mathfrak{da}}\{\hat{\mathfrak{B}}^{\mathfrak{ab}}, \hat{\mathfrak{C}}^{\mathfrak{cd}}\} + \{\hat{\mathfrak{A}}, \hat{\mathfrak{D}}\} - \varepsilon_{\mathfrak{ab}}[\hat{\mathfrak{Q}}^{\mathfrak{a}}, \hat{\mathfrak{S}}^{\mathfrak{b}}]. \tag{B.2}$$

In the combined algebra of $\mathfrak{psu}(1,1|2)$ and $\mathfrak{psu}(1|1)^2$ with identified automorphisms $\mathfrak{B}^{\mathfrak{ab}} = \hat{\mathfrak{B}}^{\mathfrak{ab}}$ also the central charges have to be identified, $\mathfrak{C}^{\mathfrak{ab}} = \hat{\mathfrak{C}}^{\mathfrak{ab}}$, in order for a quadratic invariant to exist. This invariant is the sum of (B.1) and (B.2) but with the first term in both expressions appearing only once.

Some more invariant quadratic generators obviously include quadratic combinations of the central charges

$$\mathfrak{C}^2 = \varepsilon_{\mathfrak{bc}}\varepsilon_{\mathfrak{da}}\mathfrak{C}^{\mathfrak{ab}}\mathfrak{C}^{\mathfrak{cd}}, \qquad \hat{\mathfrak{C}}^2 = \varepsilon_{\mathfrak{bc}}\varepsilon_{\mathfrak{da}}\hat{\mathfrak{C}}^{\mathfrak{ab}}\hat{\mathfrak{C}}^{\mathfrak{cd}}, \qquad \hat{\mathfrak{D}}^2. \tag{B.3}$$

B.2 Triplet of Cubic $\mathfrak{psu}(1,1|2)$ Invariants

Curiously, there exist three cubic $\mathfrak{psu}(1,1|2)$ invariants $(\mathfrak{J}^3)^{\mathfrak{ab}} = (\mathfrak{J}^3)^{\mathfrak{ba}}$ for the algebra without central extensions, $\mathfrak{C}^{\mathfrak{ab}} = 0$,

$$\begin{aligned}(\mathfrak{J}^3)^{\mathfrak{ab}} = {} & \tfrac{1}{2}\varepsilon_{ce}\varepsilon_{dh}\varepsilon_{\zeta\iota}\mathfrak{R}^{cd}[\mathfrak{Q}^{e\zeta\mathfrak{a}}, \mathfrak{Q}^{h\iota\mathfrak{b}}] + \tfrac{1}{2}\varepsilon_{eh}\varepsilon_{\gamma\zeta}\varepsilon_{\delta\iota}\mathfrak{J}^{\gamma\delta}[\mathfrak{Q}^{e\zeta\mathfrak{a}}, \mathfrak{Q}^{h\iota\mathfrak{b}}] \\ & + \varepsilon_{de}\varepsilon_{fc}\mathfrak{B}^{\mathfrak{ab}}\mathfrak{R}^{cd}\mathfrak{R}^{ef} - \varepsilon_{\delta\epsilon}\varepsilon_{\zeta\gamma}\mathfrak{B}^{\mathfrak{ab}}\mathfrak{J}^{\gamma\delta}\mathfrak{J}^{\epsilon\zeta} - \varepsilon_{cf}\varepsilon_{\delta\eta}\varepsilon_{\mathfrak{eh}}\mathfrak{B}^{\mathfrak{ab}}\mathfrak{Q}^{c\delta\mathfrak{e}}\mathfrak{Q}^{f\eta\mathfrak{h}}.\end{aligned} \tag{B.4}$$

They transform as a triplet under $\mathfrak{B}$, and commute with the $\mathfrak{psu}(1,1|2)$ algebra. These cubic generators are important for the multiplet structure in the algebra with automorphism. For a multiplet of the extended algebra, the highest-weight states of $\mathfrak{psu}(1,1|2)$ form a multiplet of $\mathfrak{su}(2)$. To move about in this multiplet, one cannot simply use the $\mathfrak{su}(2)$ generators $\mathfrak{B}^{\mathfrak{ab}}$ because they do not commute with $\mathfrak{psu}(1,1|2)$. Instead, the cubic generators map between highest-weight states of $\mathfrak{psu}(1,1|2)$, i.e. they can be understood as $\mathfrak{su}(2)$ ladder generators.

B.3 Algebra of Cubic Invariants

The cubic operators $(\mathfrak{J}^3)^{\mathfrak{ab}}$ commute with all $\mathfrak{psu}(1,1|2)$ generators, and they transform as a triplet under the $\mathfrak{su}(2)$ automorphism

$$[\mathfrak{B}^{\mathfrak{ab}}, (\mathfrak{J}^3)^{\mathfrak{cd}}] = \varepsilon^{\mathfrak{cb}}(\mathfrak{J}^3)^{\mathfrak{ad}} - \varepsilon^{\mathfrak{ad}}(\mathfrak{J}^3)^{\mathfrak{cb}}. \tag{B.5}$$

It remains to be seen how they commute among themselves.

We first note that $(\mathfrak{J}^3)^{\mathfrak{ab}}$ in (B.4) contains the quadratic Casimir $\mathfrak{J}^2$ in (B.1) (with $\mathfrak{C}^{\mathfrak{ab}} = 0$) multiplied by the $\mathfrak{su}(2)$ generator $\mathfrak{B}^{\mathfrak{ab}}$. We can thus split it up into two parts

$$(\mathfrak{J}^3)^{\mathfrak{ab}} = (\tilde{\mathfrak{J}}^3)^{\mathfrak{ab}} + \mathfrak{J}^2\mathfrak{B}^{\mathfrak{ab}} \tag{B.6}$$

with the remainder

$$(\tilde{\mathfrak{J}}^3)^{\mathfrak{ab}} = \tfrac{1}{2}\varepsilon_{ce}\varepsilon_{dh}\varepsilon_{\zeta\iota}\mathfrak{R}^{cd}[\mathfrak{Q}^{e\zeta\mathfrak{a}}, \mathfrak{Q}^{h\iota\mathfrak{b}}] + \tfrac{1}{2}\varepsilon_{eh}\varepsilon_{\gamma\zeta}\varepsilon_{\delta\iota}\mathfrak{J}^{\gamma\delta}[\mathfrak{Q}^{e\zeta\mathfrak{a}}, \mathfrak{Q}^{h\iota\mathfrak{b}}]. \tag{B.7}$$

Now, $(\mathfrak{J}^3)^{\mathfrak{ab}}$ commutes with ordinary $\mathfrak{psu}(1,1|2)$ generators, and the $\tilde{\mathfrak{J}}^3$ are products of ordinary $\mathfrak{psu}(1,1|2)$ generators only. Therefore, the commutator of two nonidentical $\mathfrak{J}^3$ generators yields simply a product of the quadratic Casimir and a $\mathfrak{J}^3$,

$$[(\mathfrak{J}^3)^{\mathfrak{ab}}, (\mathfrak{J}^3)^{\mathfrak{cd}}] = \varepsilon^{\mathfrak{cb}}\mathfrak{J}^2(\mathfrak{J}^3)^{\mathfrak{ad}} - \varepsilon^{\mathfrak{ad}}\mathfrak{J}^2(\mathfrak{J}^3)^{\mathfrak{cb}}. \tag{B.8}$$

From this, it is straightforward to obtain the entire algebra generated by the cubic invariants. Define

$$(\mathfrak{J}^3_0)^{\mathfrak{ab}} = \mathfrak{B}^{\mathfrak{ab}} \qquad \text{and} \qquad (\mathfrak{J}^3_n)^{\mathfrak{ab}} = \left(\mathfrak{J}^2\right)^{n-1}(\mathfrak{J}^3)^{\mathfrak{ab}}, \quad n \geq 1. \tag{B.9}$$

It only takes a short computation to show that these $\mathfrak{J}^3_n$ satisfy a loop algebra (the same algebra as the $\mathcal{Y}_n$ in Sec. 5.2)

$$[(\mathfrak{J}^3_m)^{\mathfrak{ab}}, (\mathfrak{J}^3_n)^{\mathfrak{cd}}] = \varepsilon^{\mathfrak{cb}}(\mathfrak{J}^3_{m+n})^{\mathfrak{ad}} - \varepsilon^{\mathfrak{ad}}(\mathfrak{J}^3_{m+n})^{\mathfrak{cb}}. \tag{B.10}$$

For n or m equal to 0, this algebra is satisfied since the quadratic Casimir commutes even with $\mathfrak{B}^{\mathfrak{ab}}$. Assuming n and m are greater than 0, we substitute the definition (B.9) to obtain

$$\begin{aligned}
[(\mathfrak{J}^3_m)^{\mathfrak{ab}}, (\mathfrak{J}^3_n)^{\mathfrak{cd}}] &= [\left(\mathfrak{J}^2\right)^{m-1}(\mathfrak{J}^3)^{\mathfrak{ab}}, \left(\mathfrak{J}^2\right)^{n-1}(\mathfrak{J}^3)^{\mathfrak{cd}}] \\
&= \left(\mathfrak{J}^2\right)^{n+m-2}[(\mathfrak{J}^3)^{\mathfrak{ab}}, (\mathfrak{J}^3)^{\mathfrak{cd}}] \\
&= \varepsilon^{\mathfrak{cb}}\left(\mathfrak{J}^2\right)^{n+m-1}(\mathfrak{J}^3)^{\mathfrak{ad}} - \varepsilon^{\mathfrak{ad}}\left(\mathfrak{J}^2\right)^{n+m-1}(\mathfrak{J}^3)^{\mathfrak{cb}} \\
&= \varepsilon^{\mathfrak{cb}}(\mathfrak{J}^3_{n+m})^{\mathfrak{ad}} - \varepsilon^{\mathfrak{ad}}(\mathfrak{J}^3_{n+m})^{\mathfrak{cb}},
\end{aligned} \tag{B.11}$$

as required. We used the vanishing commutator between $\mathfrak{J}^2$ and $(\mathfrak{J}^3)^{\mathfrak{ab}}$, and (B.8). It is interesting that the role of the quadratic Casimir operator here resembles that of $\hat{\mathfrak{D}}$ in the $\mathcal{Y}$ algebra for cyclic states above (5.6).

B.4 Representation of the Algebra

Let us understand the representations of the above loop algebra generated by $(\mathfrak{J}^3)^{\mathfrak{ab}}$, cf. Sec. 5.3. We act with $(\mathfrak{J}^3_n)^{<>}$ on a $\mathfrak{su}(2) \ltimes \mathfrak{psu}(1,1|2)$ highest-weight state $|\Psi\rangle$ and find

$$\begin{aligned}(\mathfrak{J}^3_n)^{<>}|\Psi\rangle &= (\mathfrak{J}^2)^{n-1}(\tilde{\mathfrak{J}}^3)^{<>}|\Psi\rangle + (\mathfrak{J}^2)^n \mathfrak{B}^{<>}|\Psi\rangle \\ &= x^{n-1}(\tilde{\mathfrak{J}}^3)^{<>}|\Psi\rangle + x^n \mathfrak{B}^{<>}|\Psi\rangle,\end{aligned} \tag{B.12}$$

where x is the eigenvalue of the quadratic Casimir $\mathfrak{J}^2$ on $|\Psi\rangle$. Now it turns out that $(\tilde{\mathfrak{J}}^3)^{<>}|\Psi\rangle = 0$, and consequently

$$(\mathfrak{J}^3_n)^{<>}|\Psi\rangle = x^n \mathfrak{B}^{<>}|\Psi\rangle. \tag{B.13}$$

Therefore, the representation of the loop algebra of $\mathfrak{J}^3_n$ is an evaluation representation with evaluation parameter x. In the case of a $(m+1)$-dimensional $\mathfrak{su}(2)$ multiplet of $\mathfrak{psu}(1,1|2)$ representations, the highest weight is realized as a symmetric tensor product of m fundamental evaluation representations with equal evaluation parameters x

$$|{<},x\rangle \otimes |{<},x\rangle \otimes \ldots \otimes |{<},x\rangle. \tag{B.14}$$

B.5 A One-Parameter Deformation of the Loop Algebra

Assuming that the $\mathcal{Y}^{\mathfrak{ab}}$ satisfy a loop algebra as explained in Sec. 5.2, there is actually a one-parameter generalization of these generators using the $(\mathfrak{J}^3)^{\mathfrak{ab}}$. The same subalgebra of the $\mathfrak{su}(2)$ loop algebra is generated by

$$\hat{\mathcal{Y}}^{\mathfrak{ab}} = \mathcal{Y}^{\mathfrak{ab}} + \alpha\,(\mathfrak{J}^3)^{\mathfrak{ab}} \tag{B.15}$$

for any constant α.

Let us present the full deformation of the $\mathcal{Y}^{\mathfrak{ab}}_n$, which we label $\hat{\mathcal{Y}}^{\mathfrak{ab}}_n$. We parameterize the deformation with α. $\hat{\mathcal{Y}}^{\mathfrak{ab}}_0$ is still given by $\mathfrak{B}^{\mathfrak{ab}}$, but all other generators become

$$\hat{\mathcal{Y}}^{\mathfrak{ab}}_n = \alpha^n (\mathfrak{J}^3_n)^{\mathfrak{ab}} + \sum_{m=0}^{n-1} \alpha^m \binom{n}{m} \left(\mathfrak{J}^2\right)^m \mathcal{Y}^{\mathfrak{ab}}_{n-m}, \quad n \geq 1. \tag{B.16}$$

Again, the algebra relations take the same form,

$$[\hat{\mathcal{Y}}^{\mathfrak{ab}}_m, \hat{\mathcal{Y}}^{\mathfrak{cd}}_n] = \varepsilon^{\mathfrak{cb}} \hat{\mathcal{Y}}^{\mathfrak{ad}}_{m+n} - \varepsilon^{\mathfrak{ad}} \hat{\mathcal{Y}}^{\mathfrak{cb}}_{m+n}. \tag{B.17}$$

The commutators with n or m equal to zero are again satisfied because $\mathfrak{J}^2$ commutes with $\mathfrak{B}^{\mathfrak{ab}}$. In order to check the relations with $m = 1$, we need the commutators between the $(\mathfrak{J}^3_n)^{\mathfrak{ab}}$ and the $\mathcal{Y}^{\mathfrak{ab}}_1$. The vanishing commutator between the $\mathcal{Y}$ and the ordinary $\mathfrak{psu}(1,1|2)$ generators implies

$$\begin{aligned}&[\mathcal{Y}^{\mathfrak{ab}}_1, (\mathfrak{J}^3)^{\mathfrak{cd}}] = \mathfrak{J}^2 \left(\varepsilon^{\mathfrak{cb}} \mathcal{Y}^{\mathfrak{ad}}_1 - \varepsilon^{\mathfrak{ad}} \mathcal{Y}^{\mathfrak{cb}}_1\right), && [(\mathfrak{J}^3)^{\mathfrak{ab}}, \mathcal{Y}^{\mathfrak{cd}}_1] = \mathfrak{J}^2 \left(\varepsilon^{\mathfrak{cb}} \mathcal{Y}^{\mathfrak{ad}}_1 - \varepsilon^{\mathfrak{ad}} \mathcal{Y}^{\mathfrak{cb}}_1\right), \\ &[\mathcal{Y}^{\mathfrak{ab}}_1, (\mathfrak{J}^3_n)^{\mathfrak{cd}}] = \left(\mathfrak{J}^2\right)^n \left(\varepsilon^{\mathfrak{cb}} \mathcal{Y}^{\mathfrak{ad}}_1 - \varepsilon^{\mathfrak{ad}} \mathcal{Y}^{\mathfrak{cb}}_1\right), && [(\mathfrak{J}^3_n)^{\mathfrak{ab}}, \mathcal{Y}^{\mathfrak{cd}}_1] = \left(\mathfrak{J}^2\right)^n \left(\varepsilon^{\mathfrak{cb}} \mathcal{Y}^{\mathfrak{ad}}_1 - \varepsilon^{\mathfrak{ad}} \mathcal{Y}^{\mathfrak{cb}}_1\right).\end{aligned} \tag{B.18}$$

Now we expand the left side of the relations (B.17) with $m = 1$ using (B.16). We simplify using (B.18) and the algebras of the $(\mathfrak{J}^3_n)^{\mathfrak{ab}}$ and of the $\mathcal{Y}^{\mathfrak{ab}}_n$,

$$\begin{aligned}
[\hat{\mathcal{Y}}^{\mathfrak{ab}}_1, \hat{\mathcal{Y}}^{\mathfrak{cd}}_n] &= \alpha^n [\mathcal{Y}^{\mathfrak{ab}}_1, (\mathfrak{J}^3_n)^{\mathfrak{cd}}] + \alpha^{n+1}[(\mathfrak{J}^3)^{\mathfrak{ab}}, (\mathfrak{J}^3_n)^{\mathfrak{cd}}] \\
&\quad + \sum_{m=0}^{n-1} \alpha^m \binom{n}{m} \left([\mathcal{Y}^{\mathfrak{ab}}_1, \left(\mathfrak{J}^2\right)^m \mathcal{Y}^{\mathfrak{cd}}_{n-m}] + \alpha[(\mathfrak{J}^3)^{\mathfrak{ab}}, \left(\mathfrak{J}^2\right)^m \mathcal{Y}^{\mathfrak{cd}}_{n-m}]\right) \\
&= \alpha^n \varepsilon^{\mathfrak{cb}} \left(\mathfrak{J}^2\right)^n \mathcal{Y}^{\mathfrak{ad}}_1 - \alpha^n \varepsilon^{\mathfrak{ad}} \left(\mathfrak{J}^2\right)^n \mathcal{Y}^{\mathfrak{cb}}_1 + \alpha^{n+1}\varepsilon^{\mathfrak{cb}}(\mathfrak{J}^3_{n+1})^{\mathfrak{ad}} - \alpha^{n+1}\varepsilon^{\mathfrak{ad}}(\mathfrak{J}^3_{n+1})^{\mathfrak{cb}} \\
&\quad + \sum_{m=0}^{n-1} \alpha^m \binom{n}{m} \left(\mathfrak{J}^2\right)^m \left(\varepsilon^{\mathfrak{cb}} \mathcal{Y}^{\mathfrak{ad}}_{n+1-m} - \varepsilon^{\mathfrak{ad}} \mathcal{Y}^{\mathfrak{cb}}_{n+1-m}\right) \\
&\quad + \sum_{m=0}^{n-1} \alpha^{m+1} \binom{n}{m} \left(\mathfrak{J}^2\right)^{m+1} \left(\varepsilon^{\mathfrak{cb}} \mathcal{Y}^{\mathfrak{ad}}_{n-m} - \varepsilon^{\mathfrak{ad}} \mathcal{Y}^{\mathfrak{cb}}_{n-m}\right) \\
&= \alpha^{n+1}\varepsilon^{\mathfrak{cb}}(\mathfrak{J}^3_{n+1})^{\mathfrak{ad}} - \alpha^{n+1}\varepsilon^{\mathfrak{ad}}(\mathfrak{J}^3_{n+1})^{\mathfrak{cb}} \\
&\quad + \sum_{m=0}^{n} \alpha^m \binom{n+1}{m} \left(\mathfrak{J}^2\right)^m \left(\varepsilon^{\mathfrak{cb}} \mathcal{Y}^{\mathfrak{ad}}_{n+1-m} - \varepsilon^{\mathfrak{ad}} \mathcal{Y}^{\mathfrak{cb}}_{n+1-m}\right) \\
&= \varepsilon^{\mathfrak{cb}} \hat{\mathcal{Y}}^{\mathfrak{ad}}_{n+1} - \varepsilon^{\mathfrak{ad}} \hat{\mathcal{Y}}^{\mathfrak{cb}}_{n+1}.
\end{aligned} \tag{B.19}$$

We combined terms to reach the second-to-last expression, and substituted the definition (B.16) for the last line. The calculation proceeds in parallel for (B.17) with $n = 1$. Since this algebra's Serre relations are the level three equations, which have n or m equal to one, it follows that (B.17) is satisfied.

C Symmetries of the Bilocal Generators

In this appendix, we prove that the $\mathcal{Y}^{\mathfrak{ab}}$ commute with the $\mathfrak{psu}(1,1|2)$ generators, including the one-loop dilatation generator. The proofs can be modified straightforwardly to show the same for $\mathcal{X}$.

Again, since we work only at leading order the $\mathfrak{psu}(1,1|2)$ generators $\mathfrak{Q}, \mathfrak{J}$ are truncated at $\mathcal{O}(g^0)$, and the $\mathfrak{psu}(1|1)^2$ generators $\hat{\mathfrak{Q}}, \hat{\mathfrak{S}}$ only act with $\hat{\mathfrak{Q}}_{(1)}, \hat{\mathfrak{S}}_{(1)}$.

C.1 Invariance under $\mathfrak{psu}(1,1|2)$

We now prove that $\mathcal{Y}^{\mathfrak{ab}}$ commutes with the classical $\mathfrak{psu}(1,1|2)$ generators. It is sufficient to prove that the commutators with the $\mathfrak{Q}$ vanish since the $\mathfrak{Q}$ generate the complete algebra. Furthermore, using $\mathfrak{B}$ symmetry, it is sufficient to prove this for $\mathcal{Y}^{<<}$. Now, $\mathfrak{Q}^{a\beta<}$ commute exactly with the $\hat{\mathfrak{Q}}^{<}$ and $\hat{\mathfrak{S}}^{<}$, so it is clear that these commutators vanish. However, it is nontrivial to show that the $\mathfrak{Q}^{a+>}$ commute with $\mathcal{Y}^{<<}$, since they only commute with $\hat{\mathfrak{Q}}^{<}$ up to a gauge transformation

$$\{\mathfrak{Q}^{a+>}, \hat{\mathfrak{Q}}^{<}\}|X\rangle = \varepsilon^{ab}|X\phi^{(0)}_b\rangle - \varepsilon^{ab}|\phi^{(0)}_b X\rangle = \check{Z}^a(2) - \check{Z}^a(1). \tag{C.1}$$

Here we use the notation $\check{Z}^a(i)$ for the insertion of a bosonic field at a new site between the original sites i and $i+1$. It will be useful to note that we can use $\mathcal{U}$ to change the site indices of any generator that acts on site i and any number of following sites,

$$X(i+1\ldots) = \mathcal{U}X(i\ldots)\mathcal{U}^{-1}. \tag{C.2}$$

We are now ready to check the commutator directly. We use that the $\mathfrak{Q}^{a+>}$ still commute exactly with $\hat{\mathfrak{S}}^{<}$ and apply (C.1) and (C.2),

$$\begin{aligned}
[\mathfrak{Q}^{a+>}, \mathcal{Y}^{<<}] &= \sum_{j=0}^{L-1}\sum_{i=0}^{L+1}(1-\frac{\delta_{i,0}}{2}-\frac{\delta_{i,L+1}}{2})\mathcal{U}^{j-i}\,\hat{\mathfrak{S}}^{<}(1,2)\,\mathcal{U}^{i}\,(\check{Z}^a(1)-\check{Z}^a(2))\mathcal{U}^{-j} \\
&= \sum_{j=0}^{L-1}\sum_{i=0}^{L+1}(1-\frac{\delta_{i,0}}{2}-\frac{\delta_{i,L+1}}{2})\mathcal{U}^{j-i}\,\hat{\mathfrak{S}}^{<}(1,2)\,\mathcal{U}^{i}\,\check{Z}^a(1)\mathcal{U}^{-j} \\
&- \sum_{j=0}^{L-1}\sum_{i=0}^{L+1}(1-\frac{\delta_{i,0}}{2}-\frac{\delta_{i,L+1}}{2})\mathcal{U}^{(j+1)-(i+1)}\,\hat{\mathfrak{S}}^{<}(1,2)\,\mathcal{U}^{i+1}\,\check{Z}^a(1)\mathcal{U}^{-(j+1)} \\
&= \sum_{j=0}^{L-1}\sum_{i=0}^{L+1}(1-\frac{\delta_{i,0}}{2}-\frac{\delta_{i,L+1}}{2})\mathcal{U}^{j-i}\,\hat{\mathfrak{S}}^{<}(1,2)\,\mathcal{U}^{i}\,\check{Z}^a(1)\mathcal{U}^{-j} \\
&- \sum_{j=1}^{L}\sum_{i=1}^{L+2}(1-\frac{\delta_{i,1}}{2}-\frac{\delta_{i,L+2}}{2})\mathcal{U}^{j-i}\,\hat{\mathfrak{S}}^{<}(1,2)\,\mathcal{U}^{i}\,\check{Z}^a(1)\mathcal{U}^{-j}
\end{aligned} \tag{C.3}$$

We shifted summation variables to obtain the last line, $i \to (i+1)$ and $j \to (j+1)$. Since the chain is of length L initially and after the application of the commutator, $j = L$ is equivalent to $j = 0$. Now we can combine the two lines (being careful with the different ranges for i) and simplify,

$$\begin{aligned}
[\mathfrak{Q}^{a+>}, \mathcal{Y}^{<<}] &= \sum_{j=0}^{L-1}\sum_{i=0}^{L+2}\Big[\big((1-\frac{\delta_{i,0}}{2}-\frac{\delta_{i,L+1}}{2}-\delta_{i,L+2})-(1-\frac{\delta_{i,1}}{2}-\frac{\delta_{i,L+2}}{2}-\delta_{i,0})\big) \\
&\quad\times \mathcal{U}^{j-i}\,\hat{\mathfrak{S}}^{<}(1,2)\,\mathcal{U}^{i}\,\check{Z}^a(1)\mathcal{U}^{-j}\Big] \\
&= \frac{1}{2}\sum_{j=0}^{L-1}(\mathcal{U}^{j-1}\,\hat{\mathfrak{S}}^{<}(1,2)\,\mathcal{U}\,\check{Z}^a(1)\mathcal{U}^{-j}+\mathcal{U}^{j}\,\hat{\mathfrak{S}}^{<}(1,2)\,\check{Z}^a(1)\mathcal{U}^{-j} \\
&- \mathcal{U}^{j-2}\,\hat{\mathfrak{S}}^{<}(1,2)\,\mathcal{U}\,\check{Z}^a(1)\mathcal{U}^{-j}-\mathcal{U}^{j-1}\,\hat{\mathfrak{S}}^{<}(1,2)\,\check{Z}^a(1)\mathcal{U}^{-j}) \\
&= \frac{1}{2}(1-\mathcal{U}^{-1})\sum_{j=0}^{L-1}\mathcal{U}^{j}(\mathcal{U}^{-1}\,\hat{\mathfrak{S}}^{1}(1,2)\,\mathcal{U}\,\check{Z}^a(1)+\hat{\mathfrak{S}}^{<}(1,2)\,\check{Z}^a(1))\mathcal{U}^{-j}.
\end{aligned} \tag{C.4}$$

To reach the middle expressions, we used that the length of the chain is $L+1$ after $\check{Z}^a$ acts. The expression in parenthesis inside the sum in the last line gives a chain derivative by parity. To see this, we write the chain with site $0 = L$ first:

$$\begin{aligned}
(\mathcal{U}^{-1}\,\hat{\mathfrak{S}}^{<}(1,2)\,\mathcal{U}\,\check{Z}^a(1)+\hat{\mathfrak{S}}^{<}(1,2)\,\check{Z}^a(1))|Y_0\,Y_1\,Y_2\ldots\rangle &= \\
\frac{\varepsilon^{ab}}{2}(\mathfrak{S}^{<}(0,1)|Y_0\phi_b^{(0)}\,Y_1\,Y_2\ldots\rangle+\mathfrak{S}^{<}(1,2)|Y_0\,\phi_b^{(0)}\,Y_1\,Y_2\ldots\rangle) &= \\
\frac{\varepsilon^{ab}}{2}(-\mathfrak{S}^{<}(0,1)|\phi_b^{(0)}Y_0\,Y_1\,Y_2\ldots\rangle+\mathfrak{S}^{<}(1,2)|Y_0\,\phi_b^{(0)}\,Y_1\,Y_2\ldots\rangle).
\end{aligned} \tag{C.5}$$

We used parity to reach the last line. Since this term acts homogeneously on the chain, the first and second terms cancel.

The proof for the $\mathfrak{Q}^{a-2}$ is similar. They only commute with $\hat{\mathfrak{S}}^{<}$ up to the gauge transformations

$$\begin{aligned}\{\mathfrak{Q}^{a->},\hat{\mathfrak{S}}^{<}\}|X\phi_b^{(0)}\rangle &= -\delta_b^a|X\rangle = -\hat{Z}^a(2),\\ \{\mathfrak{Q}^{a->},\hat{\mathfrak{S}}^{<}\}|\phi_b^{(0)}X\rangle &= \delta_b^a|X\rangle = \hat{Z}^a(1).\end{aligned} \tag{C.6}$$

Here we have defined $\hat{Z}^a(i)$. Since the $\mathfrak{Q}^{a->}$ commute exactly with the $\hat{\mathfrak{Q}}^{<}$, again using (C.2) to shift site indices we find

$$\begin{aligned}[\mathfrak{Q}^{a->},\mathcal{Y}^{<<}] &= \sum_{j=0}^{L-1}\sum_{i=0}^{L+1}(1-\frac{\delta_{i,0}}{2}-\frac{\delta_{i,L+1}}{2})\mathcal{U}^{j-i}\,(\hat{Z}^a(1)-\hat{Z}^a(2))\,\mathcal{U}^i\,\hat{\mathfrak{Q}}^{<}(1)\mathcal{U}^{-j}\\ &= \frac{1}{2}(\mathcal{U}^{-1}-1)\sum_{j=0}^{L-1}\mathcal{U}^j(\mathcal{U}\,\hat{Z}^a(1)\mathcal{U}^{-1}\,\hat{\mathfrak{Q}}^{<}(1)+\hat{Z}^a(1)\,\hat{\mathfrak{Q}}^{<}(1))\mathcal{U}^{-j}.\end{aligned} \tag{C.7}$$

Again, the term in parenthesis is a chain derivative by parity. This completes the proof that the $\mathfrak{Q}$ commute with $\mathcal{Y}^{<<}$. It follows by $\mathfrak{B}$ and $\mathfrak{psu}(1,1|2)$ symmetry that the $\mathcal{Y}^{\mathfrak{ab}}$ commute with all of the classical $\mathfrak{psu}(1,1|2)$ generators.

It is clear from the above proof that $\mathcal{X}$ (5.33) also commutes with the classical $\mathfrak{psu}(1,1|2)$ generators, since the bilocal product of $\hat{\mathfrak{S}}^{<}$ and $\hat{\mathfrak{Q}}^{>}$ (or $\hat{\mathfrak{S}}^{>}$ and $\hat{\mathfrak{Q}}^{<}$) by itself commutes.

C.2 Conservation

To prove that the $\mathcal{Y}$ commute with the Hamiltonian, $\hat{\mathfrak{D}}$, we first need to consider how the $\mathfrak{psu}(1|1)^2$ generators commute with the Hamiltonian. Locally, we have

$$\begin{aligned}\hat{\mathfrak{D}} &= \{\hat{\mathfrak{Q}}^{<},\hat{\mathfrak{S}}^{>}\} + \text{chain derivative},\\ &= -\{\hat{\mathfrak{Q}}^{>},\hat{\mathfrak{S}}^{<}\} + \text{chain derivative},\\ &= \tfrac{1}{2}\delta\mathfrak{D}_2 + \text{chain derivative}.\end{aligned} \tag{C.8}$$

Here "locally" refers to the interactions that are summed over the length of the chain. For instance, the local expression for the one-loop commutators expand as one-site to one-site and two-site to two-site interactions,

$$\{\hat{\mathfrak{Q}}^{\mathfrak{a}},\hat{\mathfrak{S}}^{\mathfrak{b}}\} = (\hat{\mathfrak{S}}^{\mathfrak{b}}(1,2)\hat{\mathfrak{Q}}^{\mathfrak{a}}(1)) + (\hat{\mathfrak{Q}}^{\mathfrak{a}}(1)\hat{\mathfrak{S}}^{\mathfrak{b}}(1,2)+\hat{\mathfrak{S}}^{\mathfrak{b}}(2,3)\hat{\mathfrak{Q}}^{\mathfrak{a}}(1)+\hat{\mathfrak{S}}^{\mathfrak{b}}(1,2)\hat{\mathfrak{Q}}^{\mathfrak{a}}(2)). \tag{C.9}$$

The term inside the first parenthesis is one-site to one-site, and the remaining terms are two-site to two-site. A chain derivative summed over the length of a periodic chain gives zero, so when we commute $\mathcal{Y}^{\mathfrak{ab}}$ with the Hamiltonian, we can use any of the equivalent forms in (C.8) as long as each one acts homogeneously on the chain. We will use this freedom to always commute any $\mathfrak{psu}(1,1)^2$ generator with the commutator in (C.8) that involves the same generator. Therefore, it will be convenient to define

$$\begin{aligned}\mathfrak{D}_L &= \{\hat{\mathfrak{Q}}^{<},\hat{\mathfrak{S}}^{>}\}\\ \mathfrak{D}_R &= -\{\hat{\mathfrak{Q}}^{>},\hat{\mathfrak{S}}^{<}\}.\end{aligned} \tag{C.10}$$

Furthermore, the $\mathfrak{D}_L$ and $\mathfrak{D}_R$ split into local one-site to one-site and two-site to two-site interactions (C.9). Then we have the exact local equalities only involving the two-site to two-site interactions of $\mathfrak{D}_L$ and $\mathfrak{D}_R$,

$$\begin{aligned}
[\hat{\mathfrak{Q}}^<(i), \mathfrak{D}_L] &= \hat{\mathfrak{q}}^<(i-1,i) - \hat{\mathfrak{q}}^<(i,i+1),\\
[\hat{\mathfrak{Q}}^>(i), \mathfrak{D}_R] &= \hat{\mathfrak{q}}^>(i-1,i) - \hat{\mathfrak{q}}^>(i,i+1),\\
\hat{\mathfrak{q}}^<(i-1,i) &= \hat{\mathfrak{Q}}^<(i)\mathfrak{D}_L(i-1,i) - \mathfrak{D}_L(i-1,i)\hat{\mathfrak{Q}}^<(i)\\
\hat{\mathfrak{q}}^>(i-1,i) &= \hat{\mathfrak{Q}}^>(i)\mathfrak{D}_R(i-1,i) - \mathfrak{D}_R(i-1,i)\hat{\mathfrak{Q}}^>(i).
\end{aligned} \tag{C.11}$$

Note that the $\hat{\mathfrak{q}}(i,i{+}1)$ have two-site to three-site interaction, with final sites $(i,i{+}1,i{+}2)$, and that their explicit forms in terms of interactions are not needed. These equalities can be shown easily by expanding $\mathfrak{D}_L$ and $\mathfrak{D}_R$ and using the fact that $(\hat{\mathfrak{Q}}^\mathfrak{a})^2 = 0$ is even satisfied on a one-site chain:

$$(\hat{\mathfrak{Q}}^\mathfrak{a}(1) + \hat{\mathfrak{Q}}^\mathfrak{a}(2))\hat{\mathfrak{Q}}^\mathfrak{a}(1) = 0 \quad \text{(no sum)}. \tag{C.12}$$

Similarly, we have

$$\begin{aligned}
[\hat{\mathfrak{S}}^<(i,i+1), \mathfrak{D}_R] &= \hat{\mathfrak{s}}^<(i-1,i,i+1) - \hat{\mathfrak{s}}^<(i,i+1,i+2),\\
[\hat{\mathfrak{S}}^>(i,i+1), \mathfrak{D}_L] &= \hat{\mathfrak{s}}^>(i-1,i,i+1) - \hat{\mathfrak{s}}^>(i,i+1,i+2),\\
\hat{\mathfrak{S}}^<(i-1,i,i+1) &= \hat{\mathfrak{S}}^<(i,i+1)\mathfrak{D}_R(i-1,i) - \mathfrak{D}_R(i-1,i)\hat{\mathfrak{S}}^<(i,i+1)\\
\hat{\mathfrak{s}}^>(i-1,i,i+1) &= \hat{\mathfrak{S}}^>(i,i+1)\mathfrak{D}_L(i-1,i) - \mathfrak{D}_L(i-1,i)\hat{\mathfrak{S}}^>(i,i+1).
\end{aligned} \tag{C.13}$$

The $\hat{\mathfrak{s}}(i,i+1,i+2)$ have three-site to two-site interactions, with final sites $(i,i+1)$, and again we do not need their explicit forms. Now, using these commutation relations, and the identities that follow from (C.2)

$$\begin{aligned}
\hat{\mathfrak{q}}^\mathfrak{a}(i-1,i) &= \mathcal{U}^{-1}\,\hat{\mathfrak{q}}^\mathfrak{a}(i,i+1)\,\mathcal{U},\\
\hat{\mathfrak{s}}^\mathfrak{a}(i-1,i,i+1) &= \mathcal{U}^{-1}\,\hat{\mathfrak{s}}^\mathfrak{a}(i,i+1,i+2)\,\mathcal{U},
\end{aligned} \tag{C.14}$$

we find

$$\begin{aligned}
[\hat{\mathfrak{D}}, \mathcal{Y}^{\mathfrak{a}\mathfrak{b}}] &= \sum_{j=0}^{L-1}\sum_{i=0}^{L+1}(1 - \tfrac{1}{2}\delta_{i,0} - \tfrac{1}{2}\delta_{i,L+1})\mathcal{U}^{j-i}\,(\hat{\mathfrak{s}}^\mathfrak{a}(0,1,2) - \hat{\mathfrak{s}}^\mathfrak{a}(1,2,3))\,\mathcal{U}^i\,\hat{\mathfrak{Q}}^\mathfrak{b}(1)\mathcal{U}^{-j}\\
&\quad + \sum_{j=0}^{L-1}\sum_{i=0}^{L+1}(1 - \tfrac{1}{2}\delta_{i,0} - \tfrac{1}{2}\delta_{i,L+1})\mathcal{U}^{j-i}\,\hat{\mathfrak{S}}^\mathfrak{a}(1,2)\mathcal{U}^i\,(\hat{\mathfrak{q}}^\mathfrak{b}(0,1) - \hat{\mathfrak{q}}^\mathfrak{b}(1,2))\,\mathcal{U}^{-j}\\
&= -\tfrac{1}{2}(1 - \mathcal{U}^{-1})\sum_{j=0}^{L-1}\mathcal{U}^j\,(\hat{\mathfrak{s}}^\mathfrak{a}(1,2,3)\,\hat{\mathfrak{Q}}^\mathfrak{b}(1) + \mathcal{U}^{-1}\,\hat{\mathfrak{s}}^\mathfrak{a}(1,2,3)\,\mathcal{U}\,\hat{\mathfrak{Q}}^\mathfrak{b}(1))\mathcal{U}^{-j}\\
&\quad + \tfrac{1}{2}(1 - \mathcal{U}^{-1})\sum_{j=0}^{L-1}\mathcal{U}^j\,(\hat{\mathfrak{S}}^\mathfrak{a}(1,2)\,\hat{\mathfrak{q}}^\mathfrak{b}(1,2) + \mathcal{U}\,\hat{\mathfrak{S}}^\mathfrak{a}(1,2)\,\mathcal{U}^{-1}\,\hat{\mathfrak{q}}^\mathfrak{b}(1,2))\mathcal{U}^{-j}.
\end{aligned} \tag{C.15}$$

To complete the proof, we will now show that this vanishes since it is a homogeneous sum of a chain derivative. Equivalently,

$$\hat{\mathfrak{S}}^{\mathfrak{a}}(1,2)\,\hat{\mathfrak{q}}^{\mathfrak{b}}(1,2)+\mathcal{U}\,\hat{\mathfrak{S}}^{\mathfrak{a}}(1,2)\,\mathcal{U}^{-1}\,\hat{\mathfrak{q}}^{\mathfrak{b}}(1,2)-\hat{\mathfrak{s}}^{\mathfrak{a}}(1,2,3)\,\hat{\mathfrak{Q}}^{\mathfrak{b}}(1)-\mathcal{U}^{-1}\,\hat{\mathfrak{s}}^{\mathfrak{a}}(1,2,3)\,\mathcal{U}\,\hat{\mathfrak{Q}}^{\mathfrak{b}}(1), \tag{C.16}$$

acts as a chain derivative on sites 1 and 2.

First we simplify the first term. For simplicity, we consider the $(<<)$ component. By definition and using the two-site to two-site interactions of the defining commutator of $\mathfrak{D}_L$ (C.10), we have

$$\begin{aligned}\hat{\mathfrak{q}}^{<}(1,2) &= \hat{\mathfrak{Q}}^{<}(2)\mathfrak{D}_L(1,2)-\mathfrak{D}_L(1,2)\hat{\mathfrak{Q}}^{<}(2)\\ &= \hat{\mathfrak{Q}}^{<}(2)\hat{\mathfrak{Q}}^{<}(1)\hat{\mathfrak{S}}^{>}(1,2)+\hat{\mathfrak{Q}}^{<}(2)\hat{\mathfrak{S}}^{>}(1,2)\hat{\mathfrak{Q}}^{<}(2)+\hat{\mathfrak{Q}}^{<}(2)\hat{\mathfrak{S}}^{>}(2,3)\hat{\mathfrak{Q}}^{<}(1)\\ &\quad -\hat{\mathfrak{Q}}^{<}(1)\hat{\mathfrak{S}}^{>}(1,2)\hat{\mathfrak{Q}}^{<}(2)-\hat{\mathfrak{S}}^{>}(1,2)\hat{\mathfrak{Q}}^{<}(2)\hat{\mathfrak{Q}}^{<}(2)-\hat{\mathfrak{S}}^{>}(2,3)\hat{\mathfrak{Q}}^{<}(1)\hat{\mathfrak{Q}}^{<}(2).\end{aligned} \tag{C.17}$$

Now, in the second term of the last expression (on the second-to-last line), we can switch the order of $\hat{\mathfrak{Q}}^{<}(2)$ and $\hat{\mathfrak{S}}^{<}(1,2)$ (with a minus sign) since these two operators do not act on any shared sites, but being careful with site indices, we must use $\hat{\mathfrak{Q}}^{<}(3)$ instead. Then, by the identity (C.12) that $\hat{\mathfrak{Q}}^2=0$ even on one site, we find that the second term and the fifth term cancel, and we are left with the simpler expression

$$\begin{aligned}&\hat{\mathfrak{Q}}^{<}(2)\hat{\mathfrak{Q}}^{<}(1)\hat{\mathfrak{S}}^{>}(1,2)+\hat{\mathfrak{Q}}^{<}(2)\hat{\mathfrak{S}}^{>}(2,3)\hat{\mathfrak{Q}}^{<}(1)\\ &-\hat{\mathfrak{Q}}^{<}(1)\hat{\mathfrak{S}}^{>}(1,2)\hat{\mathfrak{Q}}^{<}(2)-\hat{\mathfrak{S}}^{>}(2,3)\hat{\mathfrak{Q}}^{<}(1)\hat{\mathfrak{Q}}^{<}(2).\end{aligned} \tag{C.18}$$

Now the first two terms of (C.16) can be written as

$$\hat{\mathfrak{S}}^{<}(1,2)\,\hat{\mathfrak{q}}^{<}(1,2)+\hat{\mathfrak{S}}^{<}(2,3)\,\hat{\mathfrak{q}}^{<}(1,2). \tag{C.19}$$

The contributions from the first term of (C.18) cancel using (C.12) and the identity[13]

$$(\hat{\mathfrak{S}}^{<}(1,2)\,\hat{\mathfrak{Q}}^{<}(1)-\hat{\mathfrak{S}}^{<}(2,3)\,\hat{\mathfrak{Q}}^{<}(2))\hat{\mathfrak{Q}}^{<}(1)=0 \tag{C.20}$$

So we are left with the following six terms for (C.19) (the first two terms of (C.16))

$$\begin{aligned}&\hat{\mathfrak{S}}^{<}(1,2)\hat{\mathfrak{Q}}^{<}(2)\hat{\mathfrak{S}}^{>}(2,3)\hat{\mathfrak{Q}}^{<}(1)-\hat{\mathfrak{S}}^{<}(1,2)\hat{\mathfrak{Q}}^{<}(1)\hat{\mathfrak{S}}^{>}(1,2)\hat{\mathfrak{Q}}^{<}(2)-\\ &\hat{\mathfrak{S}}^{<}(1,2)\hat{\mathfrak{S}}^{>}(2,3)\hat{\mathfrak{Q}}^{<}(1)\hat{\mathfrak{Q}}^{<}(2)+\hat{\mathfrak{S}}^{<}(2,3)\hat{\mathfrak{Q}}^{<}(2)\hat{\mathfrak{S}}^{>}(2,3)\hat{\mathfrak{Q}}^{<}(1)-\\ &\hat{\mathfrak{S}}^{<}(2,3)\hat{\mathfrak{Q}}^{<}(1)\hat{\mathfrak{S}}^{>}(1,2)\hat{\mathfrak{Q}}^{<}(2)-\hat{\mathfrak{S}}^{<}(2,3)\hat{\mathfrak{S}}^{>}(2,3)\hat{\mathfrak{Q}}^{<}(1)\hat{\mathfrak{Q}}^{<}(2).\end{aligned} \tag{C.21}$$

Similar steps can be used for the last two terms of (C.16). We find

$$\begin{aligned}&\hat{\mathfrak{S}}^{<}(2,3)\hat{\mathfrak{Q}}^{>}(1)\hat{\mathfrak{S}}^{<}(1,2)\hat{\mathfrak{Q}}^{<}(1)+\hat{\mathfrak{S}}^{<}(2,3)\hat{\mathfrak{S}}^{<}(1,2)\hat{\mathfrak{Q}}^{>}(2)\hat{\mathfrak{Q}}^{<}(1)-\\ &\hat{\mathfrak{S}}^{<}(1,2)\hat{\mathfrak{Q}}^{>}(2)\hat{\mathfrak{S}}^{<}(2,3)\hat{\mathfrak{Q}}^{<}(1)+\hat{\mathfrak{S}}^{<}(2,3)\hat{\mathfrak{Q}}^{>}(1)\hat{\mathfrak{S}}^{<}(1,2)\hat{\mathfrak{Q}}^{<}(2)+\\ &\hat{\mathfrak{S}}^{<}(2,3)\hat{\mathfrak{S}}^{<}(1,2)\hat{\mathfrak{Q}}^{>}(2)\hat{\mathfrak{Q}}^{<}(2)-\hat{\mathfrak{S}}^{<}(1,2)\hat{\mathfrak{Q}}^{>}(2)\hat{\mathfrak{S}}^{<}(2,3)\hat{\mathfrak{Q}}^{<}(2).\end{aligned} \tag{C.22}$$

[13]This identity can be proved without too much difficulty. The second term is minus the parity image of the first term, so one just needs to check that the first term is parity even. This can be done with a short computation because, by $\mathfrak{B}$ charge conservation, the only possible interactions are of the form (suppressing derivatives) $|\psi_{<}\rangle \to |\psi_{>}\psi_{>}\rangle$.

Recall that we need to show that (C.16) is a chain derivative. (C.16) is the sum of (C.21) and (C.22). At this point, it is necessary to explicitly expand these terms as a sum of interactions. However, we can use discrete symmetries to greatly reduce the amount of computation. Under the discrete transformation R that acts as

$$R|\psi_<^{(n)}\rangle = |\psi_>^{(n)}\rangle, \quad R|\psi_>^{(n)}\rangle = |\psi_<^{(n)}\rangle, \quad R|\phi_1^{(n)}\rangle = |\phi_1^{(n)}\rangle, \quad R|\phi_2^{(n)}\rangle = -|\phi_2^{(n)}\rangle, \tag{C.23}$$

the supercharges transform as

$$R\hat{\mathfrak{Q}}^< R^{-1} = -\hat{\mathfrak{Q}}^>, \quad R\hat{\mathfrak{S}}^< R^{-1} = \hat{\mathfrak{S}}^>, \tag{C.24}$$

as can be confirmed by examining the expressions for the $\mathfrak{psu}(1|1)^2$ generators (3.11). Then under the combined operation

$$X \to RX^\dagger R^{-1}, \tag{C.25}$$

(C.21) transforms into minus (C.22) (term by term). Also, (C.21) and (C.22) are both parity odd. Using these discrete symmetries, as well as $\mathfrak{R}$ symmetry and conservation of $\mathfrak{B}$ charge, one can infer the complete action of (C.16) by computing the following four types of interactions:

$$\begin{aligned}
|\phi_1^{(n)}\phi_2^{(m)}\rangle &\longrightarrow \sum_{k=0}^{n+m-1} f_1(n,m,k)|\psi_>^{(k)}\psi_>^{(n+m-k-1)}\rangle, \\
|\phi_1^{(n)}\psi_<^{(m)}\rangle &\longrightarrow \sum_{k=0}^{n+m} f_2(n,m,k)|\phi_1^{(k)}\psi_>^{(n+m-k)}\rangle + f_3(n,m,k)\sum_{k=0}^{n+m}|\psi_>^{(k)}\phi_1^{(n+m-k)}\rangle, \\
|\psi_<^{(n)}\psi_<^{(m)}\rangle &\longrightarrow \sum_{k=0}^{n+m} f_4(n,m,k)|\psi_<^{(k)}\psi_>^{(n+m-k)}\rangle.
\end{aligned} \tag{C.26}$$

Completing this still lengthy computation, and applying the known symmetries, we find that the $<<$ component of (C.16) is given by the chain derivative $X^{<<}(1) - X^{<<}(2)$, where the only nonvanishing action of $X^{<<}$ is

$$X^{<<}|\psi_<^{(n)}\rangle = \frac{2}{(n+1)^2}|\psi_>^{(n)}\rangle. \tag{C.27}$$

Therefore, the $<<$ component of the commutator with the Hamiltonian vanishes on periodic states, and by $\mathfrak{B}$ symmetry the $\mathcal{Y}^{\mathfrak{ab}}$ commute with $\hat{\mathfrak{D}}$.

Analogous steps to those above can be used to show that $\mathcal{X}$ also commutes with the Hamiltonian. However, we have only computed (via `Mathematica`) the two-site to two-site interactions in this case up to five excitations. That computation was consistent with the commutator being a chain derivative, but another lengthy computation is needed to complete the proof in this case (the five-excitation computation is extremely strong evidence).

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