

The octic E_8 invariant

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We give an explicit expression for the primitive E_8 -invariant tensor with eight symmetric indices. The result is presented in a manifestly $\text{spin}(16)/\mathbb{Z}_2$ -covariant notation. © 2007 American Institute of Physics. [DOI: 10.1063/1.2748615]

I. INTRODUCTION AND PRELIMINARIES

The largest of the finite-dimensional exceptional Lie groups, E_8 , with Lie algebra \mathfrak{e}_8 , is an interesting object, both from a mathematical and a physical point of view. It is an extraordinarily symmetric object, which, e.g., is reflected by the fact that its root lattice is the unique even self-dual lattice in eight dimensions (in Euclidean space, even self-dual lattices only exist in dimension $8n$). This property is essential for the existence of the $E_8 \times E_8$ heterotic string. Because of self-duality, there is only one conjugacy class of representations, the weight lattice equals the root lattice, and there is no “fundamental” representation smaller than the adjoint. As one in the E series of algebras, E_8 is relevant as a U-duality group of symmetries for compactification of M theory to three dimensions (see, e.g., Ref. 1).

In contrast to the large amount of elegance, calculations involving E_8 and representations of E_8 are generically very complicated. Anything resembling a tensor formalism is completely lacking. A basic ingredient in a tensor calculus is a set of invariant tensors, or “Clebsch-Gordan coefficients.” The only invariant tensors that are known explicitly for E_8 are the Killing metric and the structure constants (which by definition take analogous forms for any semisimple Lie algebra in a Cartan-Weyl basis). The goal of this paper is to take a first step towards a tensor formalism for E_8 by explicitly constructing an invariant tensor with eight symmetric adjoint indices. The motivation for our work is partly mathematical and partly physical. On the mathematical side, the disturbing absence of a concrete expression for this tensor is unique among the finite-dimensional Lie groups. Even for the smaller exceptional algebras \mathfrak{g}_2 , \mathfrak{f}_4 , \mathfrak{e}_6 , and \mathfrak{e}_7 , all invariant tensors are accessible in explicit forms, due to the existence of fundamental representations smaller than the adjoint and to the connections with octonions and Jordan algebras. On the physical side, we anticipate applications to U duality in the presence of higher-derivative terms.²

The orders of Casimir invariants are known for all finite-dimensional semisimple Lie algebras. They are polynomials in $U(\mathfrak{g})$, the universal enveloping algebra of \mathfrak{g} , of the form $t_{A_1, \dots, A_k} T^{A_1}, \dots, T^{A_k}$, where t is a symmetric invariant tensor and T are generators of the algebra, and they generate the center $U(\mathfrak{g})^{\mathfrak{g}}$ of $U(\mathfrak{g})$. The Harish-Chandra homomorphism is the restriction of an element in $U(\mathfrak{g})^{\mathfrak{g}}$ to a polynomial in the Cartan subalgebra \mathfrak{h} , which will be invariant under the Weyl group $W(\mathfrak{g})$ of \mathfrak{g} . Due to the fact that the Harish-Chandra homomorphism is an isomorphism from $U(\mathfrak{g})^{\mathfrak{g}}$ to $U(\mathfrak{h})^{W(\mathfrak{g})}$ one may equivalently consider finding a basis of generators for the latter, a much easier problem. The orders of the invariants follow more or less directly from a diagonalization of the Coxeter element, the product of the simple Weyl reflections (see, e.g., Refs.

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3 and 4). For infinite-dimensional algebras, one has to consider a completion of the universal enveloping algebra in order to find invariants beyond the Killing metric.⁵

In the case of \mathfrak{e}_8 , the center $U(\mathfrak{e}_8)^{\mathfrak{e}_8}$ of the universal enveloping subalgebra is generated by elements of orders 2, 8, 12, 14, 18, 20, 24, and 30. The quadratic and octic invariants correspond to primitive invariant tensors in terms of which the higher ones should be expressible. While the quadratic invariant is described by the Killing metric, the explicit form of the octic invariant is previously not known (see Ref. 6, p. 304). It is reasonable to assume that it will have an application in the construction of higher-derivative deformations of M theory compactified to three (and lower) dimensions.

Lifting an element in $U(\mathfrak{h})^{W(\mathfrak{g})}$ back to $U(\mathfrak{g})^{\mathfrak{g}}$ when \mathfrak{g} is not a matrix algebra and the corresponding invariant tensors are not known may be a tedious problem. We would like to spend a moment considering the choice of method. An obvious path to follow is to consider manifest symmetry only under a maximal subgroup $F \subset G = E_8$ and make an *Ansatz* for the invariant in terms of F invariants. E_8 has a number of maximal subgroups, but one of them, $\text{spin}(16)/\mathbb{Z}_2$, is natural for several reasons. Considering calculational complexity, this is the subgroup that leads to the smallest number of terms in the *Ansatz*. Considering the connection to the Harish-Chandra homomorphism, $K = \text{spin}(16)/\mathbb{Z}_2$ is the maximal compact subgroup of the split form $G = E_{8(8)}$. The Weyl group is a discrete subgroup of K , and the Cartan subalgebra \mathfrak{h} lies entirely in the coset directions $\mathfrak{g}/\mathfrak{k}$ (these statements apply in general). Finally, considering physical applications, G/K cosets, with K the maximal compact subgroup of the split form of G , are the ones occurring in sigma models for M-theory compactifications.

There is indeed a significant intermediary step in the Harish-Chandra homomorphism. Consider it as the decomposition $f \circ e$ of $e: U(\mathfrak{g})^{\mathfrak{g}} \rightarrow U(\mathfrak{g}/\mathfrak{k})^{\mathfrak{k}}$ and $f: U(\mathfrak{g}/\mathfrak{k})^{\mathfrak{k}} \rightarrow U(\mathfrak{h})^{W(\mathfrak{g})}$, where both e and f act as restrictions [the notation $U(\mathfrak{g}/\mathfrak{k})$ is of course not to be interpreted in the sense of a universal enveloping algebra, it is the space of polynomials on $\mathfrak{g}/\mathfrak{k}$]. The operator e obviously exists for any subalgebra, not only \mathfrak{k} , and f exists thanks to $\mathfrak{h} \subset \mathfrak{g}/\mathfrak{k}$ and $W(\mathfrak{g}) \subset K$. Since the Harish-Chandra homomorphism is an isomorphism, both e and f are isomorphisms as well. This is relevant for higher-derivative terms in sigma model actions, which (modulo multiplication by automorphic forms) then can equivalently be written as terms in $U(\mathfrak{g})^{\mathfrak{g}}$ with a \mathfrak{k} -valued Lagrange multiplier gauge connection, or as terms in $U(\mathfrak{g}/\mathfrak{k})^{\mathfrak{k}}$ with the gauge connection eliminated. One conceivable approach to find the full invariant would be to start from $U(\mathfrak{e}_8/\mathfrak{so}(16))^{\mathfrak{so}(16)}$ and use an E_8 group element formed by exponentiating the spinor generators to conjugate the spinor out in the full algebra. Our impression is that such a calculation would be at least as difficult as the direct check of invariance performed below.

II. THE INVARIANT

We thus consider the decomposition of the adjoint representation of E_8 into representations of the maximal subgroup $\text{spin}(16)/\mathbb{Z}_2$. The adjoint decomposes into the adjoint 120 and a chiral spinor 128. We often use Dynkin labels for highest weights to label representations; these have labels (01000000) and (00000010), respectively. Our convention for chirality is $\Gamma_{a_1, \dots, a_{16}} \phi = +\varepsilon_{a_1, \dots, a_{16}} \phi$. The \mathfrak{e}_8 algebra becomes

$$\begin{aligned}
 [T^{ab}, T^{cd}] &= 2\delta_{[c}^a T_{d]}^b, \\
 [T^{ab}, \phi^\alpha] &= \frac{1}{4}(\Gamma^{ab} \phi)^\alpha, \\
 [\phi^\alpha, \phi^\beta] &= \frac{1}{8}(\Gamma_{ab})^{\alpha\beta} T^{ab}.
 \end{aligned} \tag{2.1}$$

The coefficients in the first and second commutators are related by the $\mathfrak{so}(16)$ algebra. The normalization of the last commutator is free but is fixed by the choice for the quadratic invariant,

which for the case above is $X_2 = \frac{1}{2} T_{ab} T^{ab} + \phi_\alpha \phi^\alpha$. Spinor and vector indices are raised and lowered with δ . Equation (2.1) describes the compact real form $E_{8(-248)}$. By letting $\phi \rightarrow i\phi$ one gets $E_{8(8)}$, where the spinor generators are noncompact, which is the real form relevant as duality symmetry in three dimensions [other real forms contain a noncompact $\text{spin}(16)/\mathbb{Z}_2$ subgroup]. The Jacobi identities are satisfied thanks to the Fierz identity $(\Gamma_{ab})_{[\alpha\beta} (\Gamma^{ab})_{\gamma\delta]} = 0$, which is satisfied for $\mathfrak{so}(8)$ with chiral spinors, $\mathfrak{so}(9)$, and $\mathfrak{so}(16)$ with chiral spinors [in the former cases the algebras are $\mathfrak{so}(9)$, due to triality, and \mathfrak{f}_4].

The Harish-Chandra homomorphism tells us that the “heart” of the invariant lies in an octic Weyl invariant of the Cartan subalgebra. A first step may be to lift it to a unique $\text{spin}(16)/\mathbb{Z}_2$ invariant in the spinor, corresponding to applying the isomorphism f^{-1} above. It is gratifying to verify [using, e.g., LIE (Ref. 7)] that there is indeed an octic invariant [other than $(\phi\phi)^4$] and that no such invariant exists at lower order. Using Fierz identities (more below and in the Appendix), it is straightforward to show that the new invariant is proportional to

$$(\phi\Gamma_{ab}{}^{cd}\phi)(\phi\Gamma_{cd}{}^{ef}\phi)(\phi\Gamma_{ef}{}^{gh}\phi)(\phi\Gamma_{gh}{}^{ab}\phi) \quad (2.2)$$

or

$$\varepsilon^{a_1 \dots a_{16}} (\phi\Gamma_{a_1 a_2 a_3 a_4} \phi) (\phi\Gamma_{a_5 a_6 a_7 a_8} \phi) (\phi\Gamma_{a_9 a_{10} a_{11} a_{12}} \phi) (\phi\Gamma_{a_{13} a_{14} a_{15} a_{16}} \phi)$$

[the two expressions are proportional modulo $(\phi\phi)^4$]. This is the expression that would go into a deformation of the sigma model without an $\mathfrak{so}(16)$ gauge field. Making an *Ansatz* for the entire E_8 invariant, we need to include also the generators T^{ab} and write down the most general $\mathfrak{so}(16)$ invariant with terms of orders $T^8, T^6\phi^2, T^4\phi^4, T^2\phi^6, \phi^8$. The numbers of these are 6, 11, 12, 5, and 2, respectively. We then have to check invariance only under the action of the spinorial generators. Out of the 36 coefficients in the general *Ansatz*, we expect 34 to become determined in terms of the remaining two, giving a linear combination of the fourth power of the quadratic invariant and a traceless octic invariant.

The counting can be refined to determine the coinciding irreducible $\mathfrak{so}(16)$ representations in T^{8-2k} and ϕ^{2k} . This will give us a concrete guideline in writing down the *Ansatz*. At this stage, exact Fierz identities are not needed, just the knowledge that they exist to make the *Ansatz* complete. Let us take some examples. At order T^8 , the six independent terms are $\text{Pf}(T)$, $\text{tr } T^8$, $\text{tr } T^6 \text{tr } T^2$, $(\text{tr } T^4)^2$, $\text{tr } T^4 (\text{tr } T^2)^2$, and $(\text{tr } T^2)^4$. At order $T^6\phi^2$, ϕ^2 contains the representations $\otimes_s^2(00000010) = (00000000) \oplus (00000020) \oplus (00010000)$ (see the Appendix for Fierz identities). These have to be contracted to singlets with the same representations in $\otimes_s^6(01000000) = 3(00000000) \oplus 2(00000020) \oplus 6(00010000) \oplus \dots$. How these considerations go into the *Ansatz* is easily read off from the final expression for the invariant below. Going to higher order in ϕ and lower in T makes things more involved, although all one really has to take care of is to choose a linearly independent set of expressions in ϕ when a representation occurs with multiplicity greater than 1. We should mention that we do not actually work with irreducible representations. Forming an element of an irreducible representation containing a number of spinors involves symmetrizations and subtraction of traces, which can be rather complicated. This becomes even more pronounced when we are dealing with transformation of terms in our *Ansatz* under the spinor generators, which will transform as spinors. Then irreducibility also involves gamma-trace conditions. Instead we use simple expressions that we know contain the irreducible ones. To take an example at order ϕ^4 , just considering the structure of the vector indices in the expression $(\phi\Gamma_{ab}^{ij}\phi)(\phi\Gamma_{cdij}\phi)$ tells us that it may contain the representations (00000000), (00010000), (02000000), and (20000000). However, ϕ^4 contains no (02000000) and only one (00010000), which means that (02000000) vanishes and (00010000) can be represented by a “simpler” expression, $(\phi\phi)(\phi\Gamma_{abcd}\phi)$, as we will see in the Appendix. The (00000000) represents a trace that we do not subtract explicitly. We simply use the above expression to ensure that the representation (02000000) is present. So, our expressions in the *Ansatz*, and also in the equations, which we will not display in detail, will be related to irreducible representations by a (block-)triangular matrix. The results of all these considerations can be read off from the resulting invariant below.

The transformation of the *Ansatz* under the action of the spinorial generator is an $\mathfrak{so}(16)$ spinor. The vanishing of this spinor is equivalent to ϵ_8 invariance. The spinorial generator acts similar to a supersymmetry generator on a superfield, giving terms at order $T^{7-2k}\phi^{1+2k}$ from $T^{8-2k}\phi^{2k}$ and $T^{6-2k}\phi^{2+2k}$. Here, it is necessary to use the full machinery of Fierz identities, some of which are described in the Appendix. Even though all identities may be derived from the ones at ϕ^3 , it becomes increasingly difficult to do so by hand as the number of ϕ 's increases. We have used a combination of manual calculation and calculations in the MATHEMATICA package GAMMA,⁸ based on representation contents obtained with LIE.⁷ The number of equations is the number of spinors that can be formed as $T^{7-2k}\phi^{1+2k}$. For $k=0,1,2,3$ these numbers are 15, 37, 25, and 5, respectively. It is satisfying to see that of these 82 equations, only 34 are linearly independent, as anticipated above. In addition, the result is consistent with the form of the quadratic invariant, and it is possible to form a traceless octic invariant tensor. All of this is obviously seen from an E_8 perspective but acts as a (much needed) consistency check on the calculations. None of the coefficients in the *Ansatz* is determined by a single relation, most by several, so we are quite confident that our result is correct.

The final result for the octic invariant is up to an overall multiplicative constant:

$$\begin{aligned}
X_8 = & \frac{1}{3072} \varepsilon^{a_1 \dots a_{16}} T_{a_1 a_2 \dots a_{15} a_{16}} - 30 \operatorname{tr} T^8 + 14 \operatorname{tr} T^6 \operatorname{tr} T^2 + \frac{35}{4} (\operatorname{tr} T^4)^2 - \frac{35}{8} \operatorname{tr} T^4 (\operatorname{tr} T^2)^2 \\
& + \frac{15}{64} (\operatorname{tr} T^2)^4 + \left[2 \operatorname{tr} T^6 - \operatorname{tr} T^4 \operatorname{tr} T^2 + \frac{1}{8} (\operatorname{tr} T^2)^3 \right] (\phi\phi) + \left[\left(\frac{5}{4} \operatorname{tr} T^4 - \frac{1}{2} (\operatorname{tr} T^2)^2 \right) T^{ab} T^{cd} \right. \\
& + \frac{27}{4} \operatorname{tr} T^2 T^{ab} (T^3)^{cd} - 15 T^{ab} (T^5)^{cd} - 15 (T^3)^{ab} (T^3)^{cd} \left. \right] (\phi\Gamma_{abcd}\phi) + \left[\frac{1}{16} \operatorname{tr} T^2 T^{ab} T^{cd} T^{ef} T^{gh} \right. \\
& \left. - \frac{5}{8} T^{ab} T^{cd} T^{ef} (T^3)^{gh} \right] (\phi\Gamma_{abcdefgh}\phi) - \frac{1}{192} T^{ab} T^{cd} T^{ef} T^{gh} T^{ij} T^{kl} (\phi\Gamma_{abcdefghijkl}\phi) \\
& + \left[7 \operatorname{tr} T^4 - \frac{31}{8} (\operatorname{tr} T^2)^2 \right] (\phi\phi)^2 - \frac{3}{64} T^{ab} T^{cd} T^{ef} T^{gh} (\phi\phi) (\phi\Gamma_{abcdefgh}\phi) + \left[\frac{5}{64} T^{ab} T^{cd} T^{ef} T^{gh} \right. \\
& \left. - \frac{15}{16} T^{ab} T^{ce} T^{df} T^{gh} + \frac{5}{8} T^{ae} T^{bf} T^{cg} T^{dh} \right] (\phi\Gamma_{abcd}\phi) (\phi\Gamma_{efgh}\phi) + \left[\frac{3}{2} (T^3)^{ab} T^{cd} - \frac{1}{8} \operatorname{tr} T^2 T^{ab} T^{cd} \right] (\phi\phi) \\
& \times (\phi\Gamma_{abcd}\phi) + \left[\frac{15}{16} (T^3)^{ab} T^{cd} - \frac{3}{16} \operatorname{tr} T^2 T^{ab} T^{cd} + \frac{5}{4} (T^2)^{ac} (T^2)^{bd} \right] (\phi\Gamma_{ab}{}^{ij}\phi) (\phi\Gamma_{cdij}\phi) \\
& + \frac{15}{8} T^{ab} T^{cd} (T^2)^{ef} (\phi\Gamma_{abe}{}^i\phi) (\phi\Gamma_{cdfi}\phi) + \frac{1}{2} \operatorname{tr} T^2 (\phi\phi)^3 + \frac{55}{32} T^{ab} T^{cd} (\phi\phi)^2 (\phi\Gamma_{abcd}\phi) \\
& + \frac{1}{8} T^{ab} T^{cd} (\phi\phi) (\phi\Gamma_{ab}{}^{ij}\phi) (\phi\Gamma_{cdij}\phi) + \left[-\frac{1}{384} T^{ab} T^{cd} + \frac{7}{192} T^{ac} T^{bd} \right] (\phi\Gamma_{ab}{}^{ij}\phi) (\phi\Gamma_{cd}{}^{kl}\phi) \\
& \times (\phi\Gamma_{ijkl}\phi) - \frac{57}{32} (\phi\phi)^4 + \frac{1}{12 \cdot 288} (\phi\Gamma_{ab}{}^{cd}\phi) (\phi\Gamma_{cd}{}^{ef}\phi) (\phi\Gamma_{ef}{}^{gh}\phi) (\phi\Gamma_{gh}{}^{ab}\phi) \\
& + \beta \left[-\frac{1}{2} \operatorname{tr} T^2 + (\phi\phi) \right]^4. \tag{2.3}
\end{aligned}$$

Here, β is an arbitrary constant multiplying the fourth power of the quadratic invariant. The trace vanishes for $\beta=9/127$ (that such a value exists at all is nontrivial and provides a further check on the coefficients). The occurrence of the prime 127 is not incidental; taking the trace of $\delta^{AB} \delta^{CD} \delta^{EF} \delta^{GH}$ gives $\delta_{GH} \delta^{AB} \delta^{CD} \delta^{EF} \delta^{GH} = (\frac{1}{7} \times 248 + \frac{6}{7}) \delta^{AB} \delta^{CD} \delta^{EF} = [2(127)/7] \delta^{AB} \delta^{CD} \delta^{EF}$. The actual technique we use for calculating the trace is not to extract the eight-index tensor but to act on the invariant with $[(1/2)(\partial/\partial T_{ab})(\partial/\partial T^{ab})] + [(\partial/\partial \phi_a)(\partial/\partial \phi^a)]$. We remind that Eq. (2.3) gives the octic invariant for the compact form $E_{8(-248)}$. The corresponding expression for

the split form $E_{8(8)}$ is obtained by a sign change of the terms containing ϕ^{4k+2} .

It would of course be of great use if one could extend the present investigation to a tensor formalism for E_8 . Part of that project would be to identify all relations that the octic invariant tensor fulfills. For example, there is no new invariant at order 10. This means that $t^{(A_1 \dots A_5) BCD} t^{A_6 \dots A_{10} BCD} = a t^{(A_1 \dots A_8) \delta^{A_9 A_{10}}} + b \delta^{A_1 A_2} \dots \delta^{A_9 A_{10}}$. It is not within our power to check this to all orders. We have checked (using MATHEMATICA, due to the complexity of differentiating and tracing the expressions in T) that it, quite nontrivially from an $so(16)$ perspective happens at lowest order in ϕ , T^{10} . When t is traceless the coefficients are $a = 3 \times 5 \times 13 \times 23 \times 29 / (2 \times 7 \times 127)$, $b = 3^3 \times 5 \times 13 \times 19 \times 37 / (2^2 \times 127^2)$. We then have expressions also for the invariants at orders 12 and 14, namely, $t^{(A_1 \dots A_6) BC} t^{A_7 \dots A_{12} BC}$ and $t^{(A_1 \dots A_7) B} t^{A_8 \dots A_{14} B}$. In order to form higher invariants, one will need expressions with more than two t 's.

In conclusion, it is satisfactory that the octic invariant can be constructed. What one really would like to use it for is to derive identities for it, so that its explicit form in some basis can be dropped. In the present framework this task looks very difficult, unless one may find a way of automating the calculations. A possible refinement of the present formalism, inspired by the Harish-Chandra homomorphism, would be to derive higher Fierz identities in a specific $spin(16)/\mathbb{Z}_2$ frame, where the spinor lies entirely in the Cartan subalgebra. Such a formalism would presumably be straightforward to implement in MATHEMATICA and would lead to much less time-consuming calculations.

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APPENDIX: FIERZ IDENTITIES

In this appendix we will describe the Fierz identities that we have used to find the linear dependence between representations at order ϕ^n for $n \geq 3$. As explained above, for even n we only have to make sure that the basis elements in the *Ansatz* are linearly independent, while for odd n , we need to know the exact dependence for determining the equations that the coefficients in the *Ansatz* must satisfy.

Fierz identities relate different expressions with the same index structure. Our strategy is to go from lower to higher powers of ϕ and, for each order, with increasing number of indices. In this way, higher identities can be derived by hand from the lower ones, but the calculations will also generically be more and more complicated. Some of the identities have instead been obtained directly using the MATHEMATICA package GAMMA. We will not write down all the Fierz identities here but explain the method with some examples, starting from the bottom. The symmetric product of two spinor representations decomposes into irreducible representations as

$$\otimes_s^2(00000010) = (00000000) \oplus (00010000) \oplus (00000020), \quad (A1)$$

where the terms on the right hand side correspond to the basis elements $\phi\phi$, $\phi\Gamma_{abcd}\phi$ and $\phi\Gamma_{abcdefg}\phi$ at order ϕ^2 . We do not need any Fierz identities here but proceed to ϕ^3 where we have

$$\otimes_s^3(00000010) = (00000010) \oplus (01000010) \oplus (00010010) \oplus (00000030). \quad (A2)$$

Again all the irreducible representations come with multiplicity 1. The corresponding terms will have one spinor index each, and 0, 2, 4, and 8 antisymmetric vector indices, respectively. From the basis elements at ϕ^2 we can construct three expressions with no free vector indices by multiplying by a spinor and an antisymmetric product of gamma matrices. Since there is only one (00000010), any two of them must be linearly dependent and we make the *Ansatz*

$$\phi(\phi\phi) = \frac{A}{4!}\Gamma^{abcd}\phi(\phi\Gamma_{abcd}\phi) = \frac{B}{8!}\Gamma^{abcdefgh}\phi(\phi\Gamma_{abcdefgh}\phi), \quad (\text{A3})$$

or equivalently, writing out the spinor indices,

$$\delta_\alpha(\beta\delta_{\gamma\delta}) = \frac{A}{4!}(\Gamma^{abcd})_{\alpha(\beta}(\Gamma_{abcd})_{\gamma\delta)} = \frac{B}{8!}(\Gamma^{abcdefgh})_{\alpha(\beta}(\Gamma_{abcdefgh})_{\gamma\delta)}. \quad (\text{A4})$$

Contracting the equations with $\delta^{\beta\gamma}$ gives $A=1/28$ and $B=1/198$. We choose $\phi(\phi\phi)$ as a basis element corresponding to (00000010), but we could of course also choose $\Gamma^{abcd}\phi(\phi\Gamma_{abcd}\phi)$ or $\Gamma^{abcdefgh}\phi(\phi\Gamma_{abcdefgh}\phi)$. This method of contracting the spinor indices in an *Ansatz* to determine the coefficients is the one that we have implemented in GAMMA for direct calculations. In this example it is easily done by hand, but for more terms we have to contract not only with δ but also with Γ^{ijkl} or $\Gamma^{ijklmnpq}$, and for higher orders in ϕ we have to perform multiple contractions, since each contraction removes two spinor indices. This complicates the calculations considerably, but the principle is the same.

We return to the representations at order ϕ^3 . For expressions with free vector indices, we have to take into account that the irreducible representations are all gamma traceless. This means that, in order to obtain (01000010), we must combine any of the two expressions $\Gamma^{cd}\phi(\phi\Gamma_{abcd}\phi)$ and $\Gamma^{cdefgh}\phi(\phi\Gamma_{abcdefgh}\phi)$, constructed from the basis elements at ϕ^2 , with $\Gamma^{ab}\phi(\phi\phi)$, from the one that we already have at ϕ^3 . However, we do not need these gamma-traceless linear combinations, only the relation between them. Since (01000010) occurs with multiplicity 1, they must be proportional to each other, which means that the three expressions $\Gamma^{cd}\phi(\phi\Gamma_{abcd}\phi)$, $\Gamma^{cdefgh}\phi(\phi\Gamma_{abcdefgh}\phi)$, and $\Gamma^{ab}\phi(\phi\phi)$ are linearly dependent. This *Ansatz* leads to the Fierz identity

$$\frac{1}{6!}\Gamma^{cdefgh}\phi(\phi\Gamma_{abcdefgh}\phi) = -\Gamma^{cd}\phi(\phi\Gamma_{abcd}\phi) - 49\Gamma_{ab}\phi(\phi\phi), \quad (\text{A5})$$

and we choose $\Gamma^{cd}\phi(\phi\Gamma_{abcd}\phi)$ as a new basis element. This will in the same way give rise to terms corresponding to gamma traces in *Ansätze* for expressions with more than two indices. Since there is no (10000001) or (00100001), the expressions with one or three antisymmetric indices are pure gamma traces. We write only two of the four identities here:

$$\Gamma^{bcd}\phi(\phi\Gamma_{abcd}\phi) = 42\Gamma_a\phi(\phi\phi),$$

$$\Gamma^d\phi(\phi\Gamma_{abcd}\phi) = \frac{1}{4}\Gamma_{[a}\Gamma^{de}\phi(\phi\Gamma_{bc]de}\phi) + \frac{1}{2}\Gamma_{abc}\phi(\phi\phi). \quad (\text{A6})$$

Both of them can be used to obtain Fierz identities at ϕ^4 . The first one (multiplied by a gamma matrix and a spinor) shows that the (20000000) part of $(\phi\Gamma_{ab}^{ij}\phi)(\phi\Gamma_{cdij}\phi)$ vanishes. The second one is very useful for deriving higher Fierz identities in general. For example, we can apply it to the (00010000) part of the same expression

$$\begin{aligned} (\phi\Gamma_{[ab}^{ij}\phi)(\phi\Gamma_{cd]ij}\phi) &= (\phi\Gamma_{[ab}^i\Gamma^j\phi)(\phi\Gamma_{cd]ij}\phi) \\ &= \frac{1}{12}(\phi\Gamma_{[ab}^i\Gamma_c\Gamma^{kl}\phi)(\phi\Gamma_{d]ikl}\phi) + \frac{1}{6}(\phi\Gamma_{[ab}^i\Gamma_{|i}\Gamma^{kl}\phi)(\phi\Gamma_{cd]kl}\phi) \\ &\quad + \frac{1}{2}(\phi\Gamma_{[ab}^i\Gamma_{cd]i}\phi)(\phi\phi) \\ &= 2(\phi\Gamma_{[ab}^{kl}\phi)(\phi\Gamma_{cd]kl}\phi) + 4(\phi\Gamma_{abcd}\phi)(\phi\phi), \end{aligned} \quad (\text{A7})$$

and we see that the (00010000) at ϕ^4 is indeed represented by the simpler expression $(\phi\phi) \times (\phi\Gamma_{abcd}\phi)$.

We end with an example of a ϕ^5 identity, with a degree of complexity which is typical for the ones we use at this level, obtained by means of GAMMA. The identity, which relates seemingly different expressions for $(02000010) \oplus 2(00010000)$, reads

$$\begin{aligned} \Gamma^{ijkl} \phi(\phi \Gamma_{abij} \phi)(\phi \Gamma_{cdkl} \phi) &= -10 \phi(\phi \Gamma_{ab}{}^{ij} \phi)(\phi \Gamma_{cdij} \phi) + 24 \phi(\phi \phi)(\phi \Gamma_{abcd} \phi) - 4 \Gamma^{ij} \phi(\phi \Gamma_{[abc}{}^k \phi) \\ &\quad \times (\phi \Gamma_{d]ijk} \phi) - 6 \Gamma_{[a} \Gamma^i \phi(\phi \Gamma_{b]ijk} \phi)(\phi \Gamma_{cd}{}^{jk} \phi) - 6 \Gamma_{[c} \Gamma^i \phi(\phi \Gamma_{d]ijk} \phi) \\ &\quad \times (\phi \Gamma_{ab}{}^{jk} \phi) + \Gamma_{ab} \Gamma^{ij} \phi(\phi \Gamma_{cdij} \phi)(\phi \phi) + \Gamma_{cd} \Gamma^{ij} \phi(\phi \Gamma_{abij} \phi)(\phi \phi) \\ &\quad + 4 \Gamma_{[a][c} \Gamma^{ij} \phi(\phi \Gamma_{d][b]ij} \phi)(\phi \phi) + \frac{1}{2} \Gamma_{ab} \Gamma^{ij} \phi(\phi \Gamma_{cd}{}^{kl} \phi)(\phi \Gamma_{ijkl} \phi) \\ &\quad + \frac{1}{2} \Gamma_{cd} \Gamma^{ij} \phi(\phi \Gamma_{ab}{}^{kl} \phi)(\phi \Gamma_{ijkl} \phi) + 12 \Gamma_{abcd} \phi(\phi \phi)^2. \end{aligned} \quad (\text{A8})$$

Here, everything except the first three terms on the right hand side represents gamma traces, whose exact form and coefficients are still important. They are deduced from the representation content in $\oplus_s^5(00000010)$ with fewer than four vector indices, namely, (11000001) (line 3), $2(01000010)$ (lines 4–6), and (00000010) (line 7) [the full *Ansatz* contains another two terms with (01000010) and (00000010), whose coefficients turn out to vanish]. Rather than tracing four spinor indices in an *Ansatz* with these terms, already containing free vector indices, with products of symmetric elements in the Clifford algebra, we choose to form scalars by tracing and contracting with a suitable number of elements with the same tensor structure as the terms themselves. This method turns out to be much less time consuming. It may seem that Eq. (A8) gives rise to a ϕ^6 identity for $(\phi \Gamma^{ijkl} \phi)(\phi \Gamma_{abij} \phi)(\phi \Gamma_{cdkl} \phi)$ by multiplying by a spinor, which would then make our basis at ϕ^6 incomplete, but fortunately, this expression will be canceled by terms on the right hand side.

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