

Torsional Oscillations of Slowly Rotating Relativistic Stars

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ABSTRACT

We study the effects of rotation on the torsional modes of oscillating relativistic stars with a solid crust. Earlier works in Newtonian theory provided estimates of the rotational corrections for the torsional modes and suggested that they should become CFS unstable, even for quite low rotation rates. In this work, we study the effect of rotation in the context of general relativity using elasticity theory and in the slow-rotation approximation. We find that the Newtonian picture does not change considerably. The inclusion of relativistic effects leads only to quantitative corrections. The degeneracy of modes for different values of m is removed, and modes with $\ell = m$ are shifted towards zero frequencies and become secularly unstable at stellar rotational frequencies ~ 20 -30 Hz.

Key words: relativity – methods: numerical – stars: neutron – stars: oscillations – stars: rotation

1 INTRODUCTION

Neutron stars are objects of extremely rich internal structure. Although their interior structure is still very uncertain, it seems that observations and theoretical studies of neutron stars are quite in agreement concerning the structure of their exterior parts. More specifically, there is agreement that neutron stars, 1-2 minutes after their formation, are cold enough to solidify their exteriors and form a crystal crust thanks to Coulomb forces between the various atomic nuclei. The crust is covered by a very thin fluid ocean, while the interior is formed by a super-fluid mantle (up to 5km in size). The composition of the core is highly uncertain. The crystal crust extends from the neutron star's atmosphere 1km down where densities reach nuclear densities around $1.3 - 2.4 \times 10^{14}$ gr/cm³. The Coulomb forces of the crystal ions forming the crust can be described via the shear modulus μ which is inversely proportional to the 4th power of the ion spacing. Since the restoring force is the Coulomb force, the periods of the torsional modes will strongly depend on the shear modulus, and its values will characterize the spectrum. Up to now, there are only two detailed calculations of the crust equation of state (Negele & Vautherin 1973; Douchin & Haensel 2001). Both were based on an approximate formulation (Strohmayer et al. 1991) leading to a typical value for the shear modulus of $\mu = 1 \times 10^{30}$ erg cm⁻³ ρ_{14} (Schumaker & Thorne 1983).

In the Newtonian limit and in the absence of strong magnetic fields, Hansen & Cioffi (1980) found that the period of the fundamental torsional modes ${}_{\ell}P_0$ depends mainly on the radius of the star R , the speed of the shear waves v_s and the angular index ℓ via the following relation

$${}_{\ell}\sigma_0 \approx [\ell(\ell + 1)]^{1/2} v_s / R \quad (1)$$

where $v_s = (\mu/\rho)^{1/2}$, and μ , ρ are the shear modulus and the density, respectively. Torsional modes, which are axial-type oscillations, are thought to be more easily excited during a fracturing of the crust since they only involve oscillations of the velocity. The velocity field of torsional oscillations is actually divergence-free without a radial component. Torsional modes (t -modes) are labeled as ${}_{\ell}t_n$, where ℓ is the angular index, while the index n corresponds to the number of radial nodes in the eigenfunctions of the overtones for a specific ℓ .

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The shear and torsional modes are well studied in Newtonian theory, see e.g. Hansen & Cioffi (1980); McDermot et al. (1988); Carroll et al. (1986); Strohmayer (1991), while there are only a few studies in general relativity (Schumaker & Thorne 1983; Leins 1994; Messios et al. 2001; Sotani et al. 2007a; Samuelsson & Andersson 2007; Sotani et al. 2007b). Relativistic effects have been found to increase significantly the fundamental $\ell = 2$ torsional mode period by roughly 30% for a typical $M = 1.4M_{\odot}$, $R = 10\text{km}$ model.

The aim of this work is the study of the effect of rotation on torsional modes. Up to now this is studied only in Newtonian theory in a single study by Strohmayer (1991), see also Lee & Strohmayer (1996). That study suggested that the frequency of a torsional mode in a rotating star is given by

$$\sigma = \sigma_0 + \frac{m\Omega}{\ell(\ell+1)} \quad (2)$$

where σ_0 is the fundamental frequency of the torsional mode of a non-rotating star and Ω is its rotational frequency. Since the typical frequency of the torsional modes of non-rotating stars σ_0 is of the order of 25-40 Hz, it is obvious that they will be subject to the so called Chandrasekhar-Friedman-Schutz (CFS) instability, even for small rotational frequencies (Yoshida & Lee 2001). This is a quite interesting result for rotating neutron stars since the instability (together with that of the r-modes) might lead to further fracturing and/or melting of the crust.

Recently, there is increased interest in the study of torsional modes because of the belief that Soft Gamma Repeaters (SGRs) could be magnetars experiencing star-quakes that are connected (through the intense magnetic field) to gamma ray flare activity. Magnetar star-quakes may be driven by the evolving intense magnetic field which accumulates stress and eventually leads to crust fracturing. There are three SGR events detected up to now. The first event was detected already in 1979 from the source SGR 0526-66 (Mazets et al. 1979; Barat et al. 1983), the second in 1998 from SGR 1900+14 (Hurley et al. 1999), while the third and most energetic one was observed in December 2004 from the source SGR 1806-20 (Terasawa et al. 2005; Palmer et al. 2005). Analysis of the tail oscillations, for a full discussion see Sotani et al. (2007a); Watts & Strohmayer (2006), revealed the presence of oscillations from a few tenths of Hz up to about 2 kHz. In an attempt to fit the observed frequencies to the torsional modes of various EoS for the core and the crust, Sotani et al. (2007a) suggested that neutron star models should have stiff equations of state and a mass between 1.6 - 2 M_{\odot} . Still these results depend critically on the interpretation of the order of each mode. For this, see the discussion in (Samuelsson & Andersson 2007).

In the present work, we derive the perturbation equations for torsional oscillations of rotating stars in general relativity following the approach by Kojima (1992); Stavridis & Kokkotas (2005) in the Cowling approximation which neglects perturbations of the space-time. The article is organized as follows. In the next section, we describe the general-relativistic equations that have been used to describe the background stellar configuration, and we derive the perturbation equations for torsional oscillations of rotating stars in the Cowling approximation. In the third section, we describe the numerical techniques used to calculate the modes together with a toy problem. The last is meant to explain the numerical results. The article closes with a summary and discussion.

2 PERTURBATION EQUATIONS FOR TORSIONAL OSCILLATIONS

We consider a slowly rotating relativistic star described by the metric

$$ds^2 = -e^{2\nu} dt^2 + e^{2\lambda} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) - 2\omega r^2 \sin^2\theta dt d\phi \quad (3)$$

where ν , λ and ω (the frequency of the local frame) are functions of the radial coordinate r . Up to first order in Ω , the background 4-velocity of the star is given by

$$u^\mu = [e^{-\nu}, 0, 0, \Omega e^{-\nu}] \quad (4)$$

where Ω is the angular velocity of the star. The background stellar models are solutions of the TOV equations and an equation describing the dragging of inertial frames

$$\varpi'' - \left(\nu' + \lambda' - \frac{4}{r}\right)\varpi' - 16\pi e^{2\lambda}(p + \epsilon)\varpi = 0 \quad (5)$$

where $\varpi := \Omega - \omega$. We assume that the star consists of a perfect fluid described by the energy-momentum tensor

$$T_{\mu\nu} = (p + \epsilon)u_\mu u_\nu + pg_{\mu\nu} . \quad (6)$$

We also assume that the star is isotropic, therefore, the background shear tensor vanishes.

Due to the spherical symmetry of the background, the perturbations of the background configuration can be decomposed into spherical harmonics. This leads to a large system of partial differential equations Kojima (1992); Stavridis & Kokkotas (2005). Here the space-time perturbations are omitted (Cowling approximation) and also the coupling between spheroidal (polar) and toroidal (axial) perturbations since they only marginally affect the eigenfrequencies of the torsional modes.

Under these approximations, the radial component of the perturbed velocity field and the variations of pressure and density will remain unaffected since they are polar perturbations. The perturbation of the 4-velocity δu^α is related to the displacement vector ζ^a through the relation $\delta u^\alpha = \mathcal{L}_u \zeta^\alpha$. For an observer co-rotating with the star, this is translated to:

$$\delta u^\theta = -e^{-\nu} \dot{Z} \frac{1}{\sin \theta} \frac{\partial Y_{\ell m}}{\partial \phi}, \quad (7)$$

$$\delta u^\phi = e^{-\nu} \dot{Z} \frac{1}{\sin \theta} \frac{\partial Y_{\ell m}}{\partial \theta}, \quad (8)$$

$$\delta u^t = (\Omega - \omega) r^2 e^{-3\nu} \dot{Z} \sin \theta \frac{\partial Y_{\ell m}}{\partial \theta}. \quad (9)$$

In other words, the velocity perturbations are described by the time derivative a function $Z = Z(t, r)$ related to the displacement vector ζ^a , here the dot stands for the temporal derivative. The perturbed energy-momentum tensor, which includes the contribution from shear, is given by

$$\delta T_{\mu\nu} = (p + \epsilon)(\delta u_\mu u_\nu + u_\mu \delta u_\nu) - 2\mu \delta S_{\mu\nu} \quad (10)$$

where $S_{\mu\nu}$ is the shear tensor, $S_{\mu\nu}$ is defined by $\sigma_{\mu\nu} = \mathcal{L}_u S_{\mu\nu}$. Here $\sigma_{\mu\nu}$ is the rate of shear given by the Lie derivative of the shear along the world lines (Carter & Quintana 1972)

$$\sigma_{\mu\nu} = \frac{1}{2} (u_{\mu;\alpha} P^\alpha_\nu + u_{\nu;\alpha} P^\alpha_\mu) - \frac{1}{3} P_{\mu\nu} u^\beta_{;\beta}, \quad (11)$$

and $P_{\mu\nu}$ is the projection tensor

$$P_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu. \quad (12)$$

The speed of shear waves on the crust depends on the shear modulus μ , the density ϵ and the pressure p of the star according to $v_s^2 = \mu/(p + \epsilon)$. A typical value of the speed of shear waves is $v_s \approx 10^8$ cm/s. The perturbation equation for the energy-momentum tensor $\delta(T_{\mu\nu}{}^{;\mu}) = 0$ provides a single equation for the axial perturbations

$$\ddot{Z} = v_s^2 e^{2\nu-2\lambda} \left[Z'' + \left(\frac{4}{r} + \nu' - \lambda' + \frac{\mu'}{\mu} \right) Z' - e^{2\lambda} \frac{\Lambda - 2}{r^2} Z \right] + 2im\varpi \left[\frac{1}{\Lambda} + v_s^2 \left(1 - \frac{2}{\Lambda} \right) \right] \dot{Z} \quad (13)$$

where $\Lambda = \ell(\ell + 1)$. In the absence of shear ($\mu = 0, v_s^2 = 0$), an inertial observer obtains

$$\sigma = -m \left(\Omega - \frac{2\varpi}{\Lambda} \right). \quad (14)$$

This gives the well known relation for the r-mode frequency in the Newtonian limit ($\Omega = \varpi$), while in the relativistic case it leads to a continuous spectrum (Kojima 1998; Beyer & Kokkotas 1999). In the last case couplings between higher l and polar perturbations need to be included for a proper study of the spectrum.

3 ESTIMATES OF THE MODE FREQUENCIES

In this section, we present an approximate solution of the boundary value problem and describe the numerical techniques used in the calculation of the frequencies of torsional modes of rotating stars.

For the numerical estimation of the frequencies, we use two different techniques. The first approach assumes a harmonic time-dependence of the perturbations. This leads on a boundary value problem. The second approach is based on a direct time evolution of equation (13) followed by a Fourier transform in time of the obtained values at a fixed radial position. We will describe only the first approach. The second approach has only been used for the verification of the results.

The Fourier transform of equation (13), i.e., the assumption that $\dot{Z} = i\sigma Z$, leads to the following differential equation

$$Z'' + \left(\frac{4}{r} + \nu' - \lambda' + \frac{\mu'}{\mu} \right) Z' + e^{2\lambda-2\nu} \left\{ \frac{\sigma_r^2}{v_s^2} - 2m \frac{\varpi}{v_s} \frac{\sigma_r}{v_s} \left[\frac{1}{\Lambda} + v_s^2 \left(1 - \frac{2}{\Lambda} \right) \right] - e^{2\nu} \frac{\Lambda - 2}{r^2} \right\} Z = 0. \quad (15)$$

The boundary conditions in the center (or at the lower end of the crust) and on the stellar surface are

$$Z \sim r^{\ell-1} \quad \text{for } r \rightarrow 0 \quad \text{or} \quad Z' = 0 \quad \text{for } r \rightarrow R_c \quad \text{and} \quad Z' = 0 \quad \text{for } r \rightarrow R \quad (16)$$

where R_c is the distance of the lower end of the crust from the center.

The above system of equations defines an eigenvalue problem for the frequencies of the torsional modes σ . For its solution, we use two approaches. The first is an approximate analytic solution, and the second is a numerical solution.

3.1 Approximate solution

Here we provide an approximate analytic solution to the eigenvalue problem based on Bessel functions. In order to be able to use Bessel functions and to treat the problem semi-analytically, we need to simplify equation (15). For this we make the following simplifying assumptions, $e^\nu = e^\lambda \approx 1$, $\mu' = \nu' = \lambda' \approx 0$, $\varpi = \Omega$ and $v_s^2 \ll 1$. This reduces equation (15) to a Bessel equation

$$Z'' + \frac{4}{r}Z' + \left[\Sigma^2 - \frac{(\ell-1)(\ell+2)}{r^2} \right] Z = 0 \quad (17)$$

where

$$\Sigma^2 = \frac{\sigma_r^2}{v_s^2} - \frac{2m}{\Lambda} \frac{\Omega}{v_s} \frac{\sigma_r}{v_s}. \quad (18)$$

Together with the boundary conditions (16), this equation leads on a simpler boundary value problem. Actually, for high frequencies, the eigenvalues could easily be estimated using the WKB method. On the other hand, since the frequency of the fundamental torsional mode is relatively small, it is unclear whether the WKB approximation is applicable, in particular in the discussion of CFS instability. The last involves the investigation of the limit $\Sigma \rightarrow 0$.

The general solution of equation (17) can be given in the form of Bessel functions

$$Z(r) = c_1 r^{-3/2} J_{\ell+\frac{1}{2}}(\Sigma r) + c_2 r^{-3/2} Y_{\ell+\frac{1}{2}}(\Sigma r). \quad (19)$$

where c_1 and c_2 are arbitrary constants. Since the condition for regularity in the center demands that $Z \sim r^{l-1}$, the contribution of the Bessel functions $Y_n(r)$ is excluded since it is divergent for $r \rightarrow 0$. The second boundary condition demands that $Z'(R) = 0$ which leads to the following transcendental equation

$$(\ell-1)J_{\ell+\frac{1}{2}}(k) - k J_{\ell+\frac{3}{2}}(k) = 0, \quad (20)$$

where $k = \Sigma \cdot R$. The roots of this equation can be found by elementary numerical methods. This leads to the following values for k

$$\ell = 2 \quad : \quad {}_2k_n = 2.501, 7.136, 10.515, 13.772, \dots \quad (21)$$

$$\ell = 3 \quad : \quad {}_3k_n = 3.865, 8.445, 11.882, 15.175, \dots \quad (22)$$

$$\ell = 4 \quad : \quad {}_4k_n = 5.095, 9.713, 13.211, 16.544, \dots \quad (23)$$

...

The eigenfrequencies can be obtained from the roots of the equation

$${}_\ell \sigma_n^2 - \frac{2m\Omega}{\Lambda} {}_\ell \sigma_n = \left(\frac{{}_\ell k_n}{R} v_s \right)^2 \quad (24)$$

by setting ${}_\ell \sigma_n^{(0)} = {}_\ell k_n v_s / R$ (the frequency in the absence of rotation). In this way, we get an approximate form of the torsional mode frequency σ_r in the rotating frame

$${}_\ell \sigma_n = {}_\ell \sigma_n^{(0)} \sqrt{1 + \left[\frac{1}{{}_\ell \sigma_n^{(0)}} \frac{m\Omega}{\ell(\ell+1)} \right]^2} + \frac{m\Omega}{\ell(\ell+1)} \quad (25)$$

which for an inertial observer ($\sigma = \sigma_r - m\Omega$) has the form

$${}_\ell \sigma_n = {}_\ell \sigma_n^{(0)} \sqrt{1 + \left[\frac{1}{{}_\ell \sigma_n^{(0)}} \frac{m\Omega}{\ell(\ell+1)} \right]^2} - \frac{m\Omega(\ell^2 + \ell - 1)}{\ell(\ell+1)}. \quad (26)$$

Both relations agree with the Newtonian results of Strohmayer (1991), equation (25) and of Yoshida & Lee (2001) equation (26). Note that ${}_\ell k_0 \approx \sqrt{\ell(\ell+1)}$ leads to a well known form of the frequency of the fundamental torsional mode for non-rotating Newtonian stars, see equation (1).

3.2 Numerical solution

The numerical solution of the eigenvalue problem defined by equations (15) - (16) is solved by a shooting method. Its results have been tested by direct numerical evolution of the time dependent equation (13). The numerical results verify the suggestion by Yoshida & Lee (2001) that the torsional modes *are CFS unstable*.

In Figure 1, we plot the frequency of the fundamental torsional mode (${}_2 t_0$ for $m = -2 \dots 2$) as a function of the stellar rotation frequency Ω . We also show the results derived by the approximate analytic method described earlier. It is clear that

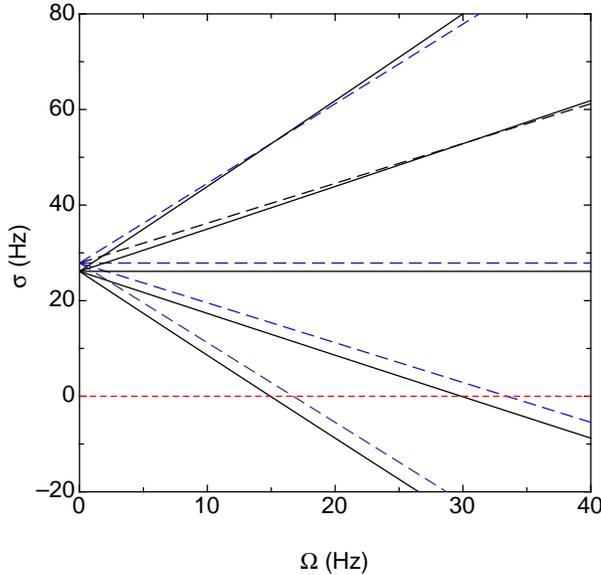


Figure 1. The frequency of the torsional mode ${}_2t_0$ as function of the rotation (both in Hz). The dashed line corresponds to the frequency of the toy model and the continuous line to the frequency of the relativistic model achieved by a direct numerical solution of the eigenvalue problem. The data are for a stellar model with a realistic EoS A (Pandharipande 1971) for the fluid core and an EoS for the crust given by Douchin & Haensel (2001) (model $A + DH_{14}$). This stellar model has radius $R = 9.49$ km $M = 1.4M_\odot$ and crust thickness $\Delta r/R \approx 5.15\%$.

there is a difference in the results of the order of 30%. Mainly, this is due to the fact that in one case we used the approximate Newtonian form of the equation, while in the other case we used its exact relativistic form.

It is obvious that the torsional modes become secularly (CFS) unstable even for very slowly rotating relativistic stars. In a Newtonian study (Strohmayer 1991) torsional mode frequencies, measured in a co-rotating reference frame, can be described approximately by the relation (1), where $\sigma = \sigma_r$. Moreover, an inertial observer will measure

$$\sigma_i = \sigma_r - m\Omega \approx \sigma_0 - m\Omega + \frac{m\Omega}{\ell(\ell+1)}. \quad (27)$$

The CFS instability sets in when the frequency of the torsional mode for the inertial observer $\sigma_i = 0$ or, equivalently, when the phase velocity of the mode is equal to the rotational frequency $\sigma_r/m = \Omega$, i.e., when the critical rotation frequency of the star is given by

$$\Omega_{\text{inst}} = \frac{\ell(\ell+1)}{m(\ell^2 + \ell - 1)}\sigma_0. \quad (28)$$

The approximate results derived earlier (25)-(26) verify the Newtonian results. Also the numerical results agree extremely well on the effect of rotation on the frequencies of the torsional modes, i.e.,

$$\left(\frac{\partial\sigma_i}{\partial\Omega}\right)_{\text{Numerical}} \approx \left(\frac{\partial\sigma_i}{\partial\Omega}\right)_{\text{Approximate}} \quad (29)$$

with a typical error of the order of 2-5%, depending on the compactness of the star.

The perturbation equations have been solved for a number of different equations of state for the fluid core and the crust which are listed in Table 1 of Sotani et al. (2007a). The overall picture is the same, i.e., shows a 20-30% difference of the frequencies of the relativistic and the Newtonian equations, and a 2-5% difference in the rate of the frequency as function of rotation ($\frac{\partial\sigma_i}{\partial\Omega}$). The frequencies plotted in Figure 1 are for a stellar model with a realistic EoS A (Pandharipande 1971) for the fluid core together with an EoS for the crust given by Douchin & Haensel (2001) (model $A + DH_{14}$). This stellar model has radius $R = 9.49$ km $M = 1.4M_\odot$ and crust thickness $\Delta r/R \approx 5.15\%$.

4 RESULTS AND DISCUSSION

In this article, we showed by means of numerical and semi-analytic methods that the torsional modes of rotating relativistic stars are subject to the CFS instability as suggested by Yoshida & Lee (2001) for Newtonian stars. This instability might be only of academic interest since viscosity works against it, and therefore it will probably never prevail.

We would like to emphasize that the CFS instability does not operate in the up to now observed SGRs because they are very slowly rotating stars with periods of the order of seconds. Still it is possible that the torsional modes of the newly born neutron stars (soon after they form a crust) will be CFS unstable because rotation periods of the order of tenths or hundreds of Hz are expected. The instability of torsional modes in the crust and the unstable r-modes in the fluid core might work together in the direction of breaking or melting the crust. If strong magnetic fields are present, then the accumulated stress might enhance the above scenario suggesting that the young neutron stars with strong magnetic fields will probably have a more frequent flare activity. CFS type rotational instabilities of magnetic field modes will also be present in rotating neutron stars, and their effect in the stellar flare activity is part of an extension of this work.

Finally, rotation (as well as the presence of strong magnetic fields) will produce frequency shifts towards both lower and higher frequencies which poses extra difficulties for the identification of the various observed frequencies from SGRs.

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REFERENCES

- Barat C. et al., 1983, *A&A*, 126, 400
 Beyer H. & Kokkotas K.D., 1999, *MNRAS*, 308, 745
 Carroll B.W., Zweibel E.G., Hansen C., McDermot P.N., Savedoff M.P., Thomas J.H., Van Horn H.M., 1986, *ApJ*, 305, 767
 Carter B. & Quintana H., 1972, *Proc. R. Soc. Lond. A*, 331, 57
 Douchin F. & Haensel P., 2001, *A&A*, 380, 151
 Hansen C. & Cioffi D., 1980, *ApJ*, 238, 740
 Hurley K. et al., 1999, *Nature*, 397, 41
 Kojima Y., 1992, *Phys. Rev. D*, 46, 4289
 Kojima Y., 1998, *MNRAS*, 293, 49
 Lee U. & Strohmayer T.E., 1996 *A&A*, 311, 155
 Leins M., PhD Thesis, 1994, University of Tübingen
 Mazets E.P. et al., 1979, *Nature*, 282, 587
 McDermott P.N., Van Horn H.M., Hansen C.J., 1988, *ApJ*, 325, 725
 Messios N., Papadopoulos D.B., Stergioulas N., 2001, *MNRAS*, 328, 1161
 Negele J.W. & Vautherin D., 1973, *Nucl. Phys. A*, 207, 298
 Palmer D.M. et al., 2005, *Nature*, 434, 1107
 Pandharipande V.R., 1971, *Nucl. Phys. A*, 178, 123
 Samuelsson L. & Andersson N., 2007, *MNRAS*, 374, 256
 Schumaker B.L. & Thorne K.S., 1983, *MNRAS*, 203, 457
 Sotani H., Kokkotas K.D., Stergioulas N., 2007, *MNRAS*, 375, 261
 Sotani H., Kokkotas K.D., Stergioulas N., Vavoulidis M., astro-ph/0611666
 Stavridis A. & Kokkotas K.D., 2005, *Int. J. Mod. Phys. D*, 14, 543
 Strohmayer T.E., 1991, *ApJ*, 372, 573
 Strohmayer T.E. et al., 1991, *ApJ*, 375, 679
 Terasawa T. et al., 2005, *Nature* 434, 1110
 Watts A.L., Strohmayer T.E., astro-ph/0612252
 Yoshida S. & Lee U., 2001, *ApJ*, 546, 1121