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Static Vacuum Solutions from Convergent Null Data Expansions at Space-Like Infinity

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Abstract. We study formal expansions of asymptotically flat solutions to the static vacuum field equations which are determined by minimal sets of freely specifyable data referred to as 'null data'. These are given by sequences of symmetric trace free tensors at space-like infinity of increasing order. They are 1 : 1 related to the sequences of Geroch multipoles. Necessary and sufficient growth estimates on the null data are obtained for the formal expansions to be absolutely convergent. This provides a complete characterization of all asymptotically flat solutions to the static vacuum field equations.

1. Introduction

In this article will be given a characterization of asymptotically flat, static solutions to Einstein's vacuum field equations $Ric[\tilde{g}] = 0$. We thus consider Lorentz metrics which take in coordinates suitably adapted to a hypersurface orthogonal, time-like Killing field K the form

$$\tilde{g} = v^2\, d\,t^2 + \tilde{h}\,, \qquad v = v(x^c) > 0\,, \qquad \tilde{h} = \tilde{h}_{ab}(x^c)\, dx^a\, dx^b\,, \tag{1.1}$$

where $\tilde{h}$ denotes a negative definite metric on the time slices $\tilde{S}_c = \{t = c = const.\}$ and the Killing field is given by $K = \partial_t$. In this representation Einstein's vacuum field equations reduce to the *static vacuum field equations*

$$R_{ab}[\tilde{h}] = \frac{1}{v}\, \tilde{D}_a\, \tilde{D}_b\, v\,, \qquad \Delta_{\tilde{h}}\, v = 0 \quad \text{on} \quad \tilde{S} \equiv \tilde{S}_0\,. \tag{1.2}$$

It will be assumed that $\tilde{S}$ is diffeomorphic to the complement of a closed ball $B_R(0)$ in $\mathbb{R}^3$ with a diffeomorphism whose components define coordinates x^a, $a = 1, 2, 3$, on $\tilde{S}$ in which the asymptotic flatness condition[1]

$$\tilde{h}_{ac} = \left(1 + \frac{2\,m}{|x|}\right) \delta_{ac} + O_k(|x|^{-(1+\epsilon)})\,, \quad v = 1 - \frac{m}{|x|} + O_k(|x|^{-(1+\epsilon)}) \;\text{ as }\; |x| \to \infty\,, \tag{1.3}$$

[1]The terms $O_k(|x|^{-(1+\epsilon)})$ behave like $O(|x|^{-(1+\epsilon+j)})$ under differentiations of order $j \le k$.

is realized with some $\epsilon > 0$ and $k \geq 2$, where $|\,.\,|$ denotes the standard Euclidean norm.

Solutions to (1.2) satisfying the fall-off conditions (1.3) have been characterized by Reula [23] and Miao [18] in terms of boundary value problems for the static field equations where the data are prescribed on the sphere $\partial\tilde{S}$, which encompasses the asymptotic end.

Our interest in static solutions comes, however, from the observation that for vacuum solutions arising from asymptotically flat, time symmetric initial data asymptotic smoothness at null infinity appears to be related to asymptotic staticity of the data at space-like infinity [14,25]. To analyse this situation we wish to control the static vacuum solutions in terms of quantities defined at space-like infinity.

Another reason for giving such a characterization results from the work by Corvino [5,6], Corvino and Schoen [7], and Chruściel and Delay [3,4]. These authors deform given asymptotically flat vacuum data outside prescribed compact sets to vacuum data which are *exactly static or stationary near* or *asymptotically static or stationary at* space-like infinity and use such data to discuss the existence of null geodesically complete solutions which have a smooth asymptotic structure at null infinity. To assess the scope of these results it is desirable to have a complete description of the asymptotically flat static vacuum solutions in terms of asymptotic quantities.

A characterization of this type has been suggested by Geroch by giving a definition of multipole moments for static solutions [16]. He assumes the metric $\tilde{h}$ to admit a smooth conformal extension in the following sense. With an additional point i, which is to represent space-like infinity, the set $S = \tilde{S} \cup \{i\}$ is assumed to acquire a smooth differential structure which induces on $\tilde{S}$ the given one, which makes S diffeomorphic to an open ball in $\mathbb{R}^3$ with the center representing i, and which admits a function $\Omega \in C^2(S) \cap C^\infty(\tilde{S})$ with the properties

$$\Omega > 0 \quad \text{on} \quad \tilde{S}, \tag{1.4}$$

$$h_{ab} = \Omega^2\, \tilde{h}_{ab} \quad \text{extends to a smooth negative definite metric on } S, \tag{1.5}$$

$$\Omega = 0\,, \quad D_a\Omega = 0\,, \quad D_a D_b \Omega = -2\, h_{ab} \quad \text{at} \quad i\,, \tag{1.6}$$

where D denotes the covariant derivative operator defined by h. We note that these conditions are preserved under rescalings $h \to \vartheta^4\, h$, $\Omega \to \vartheta^2\, \Omega$ with smooth positive functions ϑ satisfying $\vartheta(i) = 1$.

With these assumptions Geroch defines a sequence of tensor fields P, P_a, $P_{a_2 a_1}$, … near i by setting[2]

$$P = \Omega^{-1/2}\,(1 - v)\,, \quad P_a = D_a\, P\,, \quad P_{a_2 a_1} = \mathcal{C}\left(D_{a_2} P_{a_1} - \frac{1}{2}\, P\, R_{a_2 a_1}\right),$$

$$P_{a_{p+1}\dots a_1} = \mathcal{C}(D_{a_{p+1}} P_{a_p \dots a_1} - c_p\, P_{a_{p+1}\dots a_3}\, R_{a_2 a_1})\,,$$

$$\text{with} \quad c_p = \frac{p\,(2\,p-1)}{2}\,, \quad p = 2, 3, \dots\,,$$

[2]We depart from the convention of [16] by changing the sign of P.

where R_{ab} denotes the Ricci tensor of h_{ab} and $\mathcal{C}$ the projector onto the symmetric, trace free part of the respective tensor fields. The multipole moments are then defined as the tensors

$$\nu = P(i)\,, \qquad \nu_{a_p \dots a_1} = P_{a_p \dots a_1}(i)\,, \quad p = 1, 2, 3, \dots\,,$$

at i. Setting aside the monopole ν, we will denote the remaining series of multipoles by

$$\mathcal{D}_{mp} = \{\nu_{a_1}, \, \nu_{a_2 a_1}, \, \nu_{a_3 a_2 a_1}, \, \dots\}\,. \tag{1.7}$$

The problem of characterizing solutions to a quasi-linear, gauge-elliptic system of equations of the type (1.2) by a minimal set of data given at an ideal point representing space-like infinity is unusual and certainly quite different from a standard boundary value problem for (1.2). There are available some results which go into this direction but little has been done on the general question of existence.

Müller zum Hagen has shown that solutions v, $\tilde{h}_{ab}$ to (1.2) are real analytic in $\tilde{h}$-harmonic coordinates [20]. The question to what extent the multipoles introduced above determine the metric h_{ab} and the function v raises the question whether this metric is real analytic even at i in suitable coordinates and conformal scalings. Beig and Simon [2] have shown (under assumptions which have been relaxed later by Kennefick and O'Murchadha [17]) that the rescaled metric does indeed extend in a suitable gauge as a real analytic metric to i if it is assumed that the ADM mass satisfies

$$m \neq 0\,. \tag{1.8}$$

We shall assume this result in the following and shall not go through the argument again, though its structural basis will be pointed out in passing. Beig and Simon also provide an argument which essentially shows that a given sequence of multipoles determines a unique formal expansion of a 'formal solution' to the static vacuum field equations.

For axisymmetric static vacuum solutions, which are special in admitting explicit descriptions [26], the question under which assumptions a sequence of multipoles does indeed determine a converging expansion of a static solution has been studied by Bäckdahl and Herberthson [1]. For the general case, for which the freedom to prescribe data is much larger, this problem has never been analyzed. For this reason the results referred to above remained essentially of heuristic value.

It is the purpose of this article to derive, under the assumption (1.8), necessary and sufficient conditions for certain minimal sets of asymptotic data, denoted collectively by $\mathcal{D}_n$ and referred to as *null data*, to determine (unique) real analytic solutions and thus to provide a complete characterization of all possible asymptotically flat solutions to the static vacuum field equations. The behaviour of these solutions in the large will not be studied here. We shall only be interested in what could be called 'germs of static solutions at space-like infinity', for which S may comprise only a neighbourhood of the point i which is quite small in terms of h (in terms of $\tilde{h}$ they cover infinite domains extending to space-like infinity).

While the multipoles above are defined for any conformal gauge, it will be convenient for our analysis to remove the conformal gauge freedom. As shown below, the metric $h = \Omega^2\,\tilde{h}$ defined with the *preferred gauge*

$$\Omega = \left(\frac{1-v}{m}\right)^2,$$

on a suitable neighbourhood $\tilde{S}$ of space-like infinity, can be extended with (1.4)–(1.6) in suitable coordinates to a real analytic metric at i. The metric so obtained satisfies $R[h] = 0$ on S. In this gauge we get with the notation above

$$P = m\,, \quad P_a = 0\,, \quad P_{a_2a_1} = -\frac{m}{2}\,s_{a_2a_1}\,, \tag{1.9}$$

$$P_{a_{p+1}\ldots a_1} = \mathcal{C}(D_{a_{p+1}}P_{a_p\ldots a_1} - c_p\,P_{a_{p+1}\ldots a_3}\,s_{a_2a_1})\,, \quad p = 2, 3, \ldots\,, \tag{1.10}$$

where s_{ab} denotes the trace free part of the Ricci tensor of h. In the given gauge we consider now the set

$$\mathcal{D}_n = \big\{s_{a_2a_1}(i), \mathcal{C}(D_{a_3}s_{a_2a_1})(i), \mathcal{C}(D_{a_4}D_{a_3}s_{a_2a_1})(i), \\ \mathcal{C}(D_{a_5}D_{a_4}D_{a_3}s_{a_2a_1})(i),\ \ldots\ldots\ \big\}\,.$$

Given $m \neq 0$ and the sequence $\mathcal{D}_n$ associated with h, one calculate the multipoles $\mathcal{D}_{mp}$ of h and vice versa. The sets $\mathcal{D}_n$ and $\mathcal{D}_{mp}$ thus carry the same information, but $\mathcal{D}_n$ is easier to work with because the expressions are linear in the curvature.

Let now $c_{\mathbf{a}}$, $\mathbf{a} = 1, 2, 3$, be an h-orthonormal frame field near i which is h-parallelly propagated along the geodesics through i and denote the covariant derivative in the direction of $c_{\mathbf{a}}$ by $D_{\mathbf{a}}$. We express the tensors in $\mathcal{D}_n$ in terms of this frame and write

$$\mathcal{D}^*_n = \big\{s_{\mathbf{a}_2\mathbf{a}_1}(i), \mathcal{C}(D_{\mathbf{a}_3}s_{\mathbf{a}_2\mathbf{a}_1})(i), \mathcal{C}(D_{\mathbf{a}_4}D_{\mathbf{a}_3}s_{\mathbf{a}_2\mathbf{a}_1})(i), \\ \mathcal{C}(D_{\mathbf{a}_5}D_{\mathbf{a}_4}D_{\mathbf{a}_3}s_{\mathbf{a}_2\mathbf{a}_1})(i),\ \ldots\ \big\}\,. \tag{1.11}$$

We note that these tensors are defined uniquely up to a rigid rotation $c_{\mathbf{a}} \to s^{\mathbf{c}}{}_{\mathbf{a}}\,c_{\mathbf{c}}$ with $(s^{\mathbf{c}}{}_{\mathbf{a}}) \in O(3, \mathbb{R})$. These data will be referred to as the *null data of h in the frame $c_{\mathbf{a}}$*.

It will be shown that if these data are derived from a real analytic metric h near i there exist constants M, $r > 0$ so that the components of these tensors satisfy the *Cauchy estimates*

$$|\mathcal{C}(D_{\mathbf{a}_p}\ldots D_{\mathbf{a}_1}s_{\mathbf{b}\,\mathbf{c}})(i)| \leq \frac{M\,p!}{r^p}\,, \quad \mathbf{a}_p, \ldots, \mathbf{a}_1, \mathbf{b}, \mathbf{c} = 1, 2, 3\,, \quad p = 0, 1, 2, \ldots\,.$$

Conversely, we get the following existence result.

Theorem 1.1. *Suppose $m \neq 0$ and*

$$\hat{\mathcal{D}}_n = \{\psi_{\mathbf{a}_2\mathbf{a}_1},\ \psi_{\mathbf{a}_3\mathbf{a}_2\mathbf{a}_1},\ \psi_{\mathbf{a}_4\mathbf{a}_3\mathbf{a}_2\mathbf{a}_1},\ \ldots\ \}\,, \tag{1.12}$$

is a infinite sequence of symmetric, trace free tensors given in an orthonormal frame at the origin of a 3-dimensional Euclidean space. If there exist constants $M, r > 0$ *such that the components of these tensors satisfy the estimates*

$$|\psi_{\mathbf{a}_p\,\ldots\,\mathbf{a}_1\mathbf{b}\,\mathbf{c}}| \le \frac{M\,p!}{r^p}\,, \qquad \mathbf{a}_p, \ldots, \mathbf{a}_1, \mathbf{b}, \mathbf{c} = 1, 2, 3\,, \qquad p = 0, 1, 2, \ldots\,,$$

then there exists an analytic, asymptotically flat, static vacuum solution $(\tilde{h}, v)$ *with ADM mass* m*, unique up to isometries, so that the null data implied by* $h = \left(\frac{m}{1-v}\right)^4 \tilde{h}$ *in a suitable frame* $c_{\mathbf{a}}$ *as described above satisfy*

$$\mathcal{C}(D_{\mathbf{a}_q}\,\ldots\,D_{\mathbf{a}_3} s_{\mathbf{a}_2\mathbf{a}_1})(i) = \psi_{\mathbf{a}_q\,\ldots\,\mathbf{a}_1}\,, \qquad q = 2, 3, 4, \ldots\,.$$

A sequence of data of the form (1.12) (not necessarily satisfying any estimates) will in the following be referred to as *abstract null data*. The type of estimate imposed here on the abstract null data does not depend on the orthonormal frame in which they are given (cf. the discussion leading to (7.30)). Since these estimates are necessary as well as sufficient, all possible ends near space-like infinity of asymptotically flat static vacuum solutions are characterized by this result.

The proof of the result above will be given in terms of the conformal metric h_{ab}. For this purpose (1.2) are reexpressed in Chapter 2 as 'conformal static vacuum field equations' for h_{ab} and fields derived from h_{ab} and v. In Chapter 3 it is shown by a direct argument that in a certain setting a set of abstract null data defines the expansion coefficients of a formal expansion of a solution to these equations uniquely. Showing the convergence of the series so obtained appears difficult, however. Using the analyticity of the solutions to the conformal static vacuum field equations at the point i, we study in Chapter 4 their analytic extensions into the complex domain. Denote by $\mathcal{N}_i$ the 'cone' with vertex at i generated by the complex null geodesics through the point i. The null data are then represented by a function on $\mathcal{N}_i$, the component of the Ricci tensor obtained by contracting it with the null vector tangent to $\mathcal{N}_i$. In this setting the original problem assumes the form of a characteristic initial value problem with data prescribed on $\mathcal{N}_i$.

We wish to obtain the equations in a form which allows us to derive from prescribed estimates on the null data appropriate estimates on the expansion coefficients. This requires a choice of gauge which is suitably adapted to $\mathcal{N}_i$. Because of the vertex, any such gauge will necessarily be singular at a certain subset of the manifold. The manifold $\hat{S}$ considered in Chapter 4 organizes the singularity in a geometric way. In Chapter 5 the conformal static vaccum field equations are considered on $\hat{S}$, and it is shown how to determine a formal solution to the complete set of conformal field equations from a given set of abstract null data. The convergence of the series so obtained is shown in Chapter 6. Making use of the lemmas proven in the previous chapters, this result is translated in Chapter 7 into a gauge which is regular near i and allows us to prove Theorem 1.1. A translation of the estimates on the null data into equivalent estimates on the multipoles and a generalization of the present result to stationary solutions will be discussed elsewhere.

2. The static field equations in the conformal setting

The existence problem will be analyzed completely in terms of the conformally rescaled metric. We begin by describing the conformal gauge and then express the static field equations in terms of the conformal fields. This discussion follows essentially that of [12] and [14].

2.1. The choice of the conformal gauge

Consider a situation as described by conditions (1.4)–(1.6). If the metric $\tilde{h}$ is asymptotically flat and has vanishing Ricci scalar $R[\tilde{h}]$ on $\tilde{S}$ the function Ω satisfies (cf. [14])

$$\left(\Delta_h - \frac{1}{8}R[h]\right)(\Omega^{-1/2}) = 0 \quad \text{on} \quad \tilde{S} \quad \text{and} \quad r\,\Omega^{-1/2} \to 1 \quad \text{as} \quad r \to 0\,,$$

where r denotes the h-distance from i. Sufficiently close to i one obtains the representation

$$\Omega^{-1/2} = \zeta^{-1/2} + W\,,$$

with smooth functions ζ and W satisfying

$$\left(\Delta_h - \frac{1}{8}R[h]\right) W = 0\,, \tag{2.1}$$

and

$$\zeta(i) = 0\,, \quad D_a\zeta(i) = 0\,, \quad D_aD_b\zeta(i) = -2\,h_{ab}\,. \tag{2.2}$$

The functions ζ and W are real analytic if the metric h is real analytic. In [2] Beig and Simon consider static vacuum metrics of the form

$$\tilde{g} = e^{2\,U}\,dt^2 + e^{-2\,U}\,\hat{h}_{ab}\,dx^a\,dx^b\,,$$

related to (1.1) by $v = e^U$ and $\hat{h}_{ab} = v^2\,\tilde{h}_{ab}$, and show that the function $\omega = (U/m)^2$ and the metric

$$h'_{ab} = \omega^2\,\hat{h}_{ab} = \Omega'^2\,\tilde{h}_{ab} \quad \text{with} \quad \Omega' = \omega\,e^U\,, \tag{2.3}$$

extend in h'-harmonic coordinates near i to real analytic fields at i so that Ω' satisfies requirements (1.4)–(1.6) with the h'-covariant derivative operator D'.

It follows [12] that $\Omega'^{-1/2} = \zeta'^{-1/2} + W'$ with $\zeta' = \frac{\omega}{\cosh^2(U/2)}$ and $W' = \frac{m}{2}\,\frac{\sinh(U/2)}{U/2}$. Assume S to be chosen so that $U \neq 0$ on $\tilde{S}$. Rescaling with $\vartheta = W'/W'(i) > 0$ on S gives

$$h = \vartheta^4\,h' = \Omega^2\,\tilde{h} \quad \text{with} \quad \Omega = \vartheta^2\,\Omega'\,,$$

where the conformal factor can be written

$$\Omega = \left(\frac{1-v}{m}\right)^2 \quad \text{on} \quad S\,. \tag{2.4}$$

Because of (2.1) the metric h has then vanishing Ricci scalar

$$R[h] = 0 \quad \text{on} \quad S\,, \tag{2.5}$$

and it follows that

$$\Omega^{-1/2} = \zeta^{-1/2} + W\,, \tag{2.6}$$

where

$$W = \frac{m}{2}\,, \qquad \zeta = \frac{1}{\mu}\left(\frac{1-v}{1+v}\right)^2 \quad \text{with} \quad \mu = \frac{m^2}{4}\,. \tag{2.7}$$

The fields h and ζ are real analytic on S and the functions W and ζ satisfy (2.1), (2.2). In the following the gauge (2.4) and thus (2.5)–(2.7) will be assumed.

2.2. The conformal static vacuum field equations

The function ζ satisfies on S the equation

$$\Delta_h\,(\zeta^{-1/2}) = 4\,\pi\,\delta_i\,, \tag{2.8}$$

where δ_i denotes the Dirac distribution with weight 1 at i. This equation implies

$$2\,\zeta\,s = D_a\zeta\,D^a\zeta \quad \text{on} \quad S \quad \text{with} \quad s = \frac{1}{3}\,\Delta_h\,\zeta\,, \tag{2.9}$$

which, together with (2.2), implies in turn the equation above. The function $\zeta^{-1/2}$ can be characterized as a fundamental solution of Δ_h with pole at i so that ζ is real analytic on S and satisfies (2.2). It is uniquely determined by h because the expansion coefficients of ζ in h-normal coordinates centered at i are recursively determined by (2.2), (2.9).

We derive now a representation of the static vacuum field equations (1.2) in terms of the conformal metric h and fields derived from it. With (2.5) follows

$$R_{ab}[h] = s_{ab}\,, \tag{2.10}$$

where s_{ab} is a trace free symmetric tensor field. The first of (1.2) implies in the gauge (2.4)

$$0 = \Sigma_{ab} \equiv D_a\,D_b\,\zeta - s\,h_{ab} + \zeta\,(1-\mu\,\zeta)\,s_{ab}\,, \tag{2.11}$$

with s as in (2.9). With the Bianchi identity $D^a s_{ab} = 0$ the integrability conditions

$$0 = \frac{1}{2}\;D^c\,\Sigma_{ca}\,, \qquad 0 = \frac{1}{\zeta}\left(D_{[c}\,\Sigma_{a]b} + \frac{1}{2}\;D^d\,\Sigma_{d[c}\,h_{a]b}\right)$$

for the overdetermined system (2.11) take the form

$$0 = S_a \equiv D_a\,s + (1-\mu\,\zeta)\,s_{ab}\,D^b\,\zeta\,, \tag{2.12}$$

and

$$0 = H_{cab} \equiv (1-\mu\,\zeta)\,D_{[c}s_{a]b} - \mu\left(2\,D_{[c}\zeta\,s_{a]b} + D^d\,\zeta\,s_{d[c}\,h_{a]b}\right). \tag{2.13}$$

We note that this can be read as an expression of the Cotton tensor $B_{bca} = D_{[c}R_{a]b} - \frac{1}{4}\,D_{[c}R\,h_{a]b}$ in terms of the undifferentiated curvature. Its dualized version reads by (2.13)

$$B_{ab} = \frac{1}{2}\,B_{acd}\,\epsilon_b{}^{cd} = \frac{\mu}{1-\mu\,\zeta}\left(s_{da}\,\epsilon_b{}^{cd}D_c\zeta - \frac{1}{2}\,s_{de}\,\epsilon_{ba}{}^{d}D^e\zeta\right). \tag{2.14}$$

Equations (2.10), (2.11), (2.12), (2.13) together with conditions (2.2), which imply

$$s(i) = -2\,, \tag{2.15}$$

will be referred to as the *conformal static vacuum field equations* for the unknown fields

$$h\,,\ \ \zeta\,,\ \ s\,,\ \ s_{ab}\,. \tag{2.16}$$

The second of (1.2) implies that $R[\tilde{h}] = 0$ and can thus also be read as the conformally covariant Laplace equation for v. With the conformal covariance of the latter and (2.4), (2.5), (2.7), its conformal version reduces to (2.8). The identity

$$D_a(2\,\zeta\,s - D_c\,\zeta\,D^c\,\sigma) = 2\,\zeta\,S_a - 2\,\Sigma_{ac}\,D^c\zeta\,,$$

shows that (2.9), whence (2.8), is a consequence of equations (2.2) and (2.11). It follows that for given $m \neq 0$, which defines W and μ, a solution of the conformal static vacuum field equations provides a unique solution to the static vacuum field equations (1.2).

The system (2.10), (2.11), (2.12), (2.13) represents a quasi-linear, overdetermined system of PDE's which implies elliptic equations for all unknowns in a suitable gauge. The Ricci operator becomes elliptic in harmonic coordinates and the elliptic character of the remaining equations can be seen by taking the trace of (2.11), by contracting (2.12) with D^a, and by contracting (2.13) with D^c and using the Bianchi identity and (2.11) again so that in all three cases one obtains an equation with the Laplacian acting on the respective unknown. By deducing from the fall-off behaviour of the physical solution at space-like infinity a certain minimal smoothness of the conformal fields at i and invoking a general theorem of Morrey [19] on elliptic systems of this type, Beig and Simon [2] concluded that the solutions are in fact real analytic at i. To avoid introducing additional constraints by taking derivatives, we shall deal with the system of first order above.

3. The exact sets of equations argument

Constructing solutions from minimal sets of data prescribed at i poses quite an unusual problem for a system of the type of the static conformal field equations. To see how it might be done, we study expansions of the fields in normal coordinates.

For convenience assume in the following S to coincide with a convex h-normal neighbourhood of i. Let $c_{\mathbf{a}}$, $\mathbf{a} = 1, 2, 3$, be an h-orthonormal frame field on S which is parallelly transported along the h-geodesics through i and let x^a denote normal coordinates centered at i so that $c^b{}_{\mathbf{a}} \equiv \langle dx^b, c_{\mathbf{a}}\rangle = \delta^b{}_{\mathbf{a}}$ at i. We refer to such a frame as *normal frame centered at i*. Its dual frame will be denoted by $\chi^{\mathbf{c}} = \chi^{\mathbf{c}}{}_b\,dx^b$.

At the point with coordinates x^a the coefficients of the frame then satisfy

$$c^b{}_{\mathbf{a}}\,x^a = \delta^b{}_{\mathbf{a}}\,x^a\,, \quad x_b\,c^b{}_{\mathbf{a}} = x_b\,\delta^b{}_{\mathbf{a}}\,,$$

(where we set $x_a = x^b \delta_{ba}$ and assume, as in the following, that the summation rule does not distinguish between bold face and other indices). Equivalently, the coefficients of the dual frame satisfy

$$\chi^{\mathbf{a}}{}_b \, x^b = \delta^{\mathbf{a}}{}_b \, x^b \,, \quad x_a \, \chi^{\mathbf{a}}{}_b = x_a \, \delta^{\mathbf{a}}{}_b \,, \tag{3.1}$$

which implies with the coordinate expression $h_{ab} = -\delta_{\mathbf{ac}} \, \chi^{\mathbf{a}}{}_b \, \chi^{\mathbf{c}}{}_d$ of the metric the well known characterization $x^a \, h_{ab} = -x^a \, \delta_{ab}$ of the x^a as h-normal coordinates centered at i. In the following all tensor fields, except the frame field $c_{\mathbf{a}}$ and the coframe field $\chi^{\mathbf{c}}$, will be expressed in terms of this frame field, so that the metric is given by $h_{\mathbf{ab}} \equiv h(c_{\mathbf{a}}, c_{\mathbf{c}}) = -\delta_{\mathbf{ab}}$. With $D_{\mathbf{a}} \equiv D_{c_{\mathbf{a}}}$ the connection coefficients with respect to $c_{\mathbf{a}}$ are defined by $D_{\mathbf{a}} \, c_{\mathbf{c}} = \Gamma_{\mathbf{a}}{}^{\mathbf{b}}{}_{\mathbf{c}} \, c_{\mathbf{b}}$.

An analytic tensor field $T_{\mathbf{a}_1 \ldots \mathbf{a}_k}$ on S has in the normal coordinates x^a a *normal expansion* at i, which can be written (cf. [13])

$$T_{\mathbf{a}_1 \ldots \mathbf{a}_k}(x) = \sum_{p \geq 0} \frac{1}{p\,!} \, x^{c_p} \ldots x^{c_1} \, D_{\mathbf{c}_p} \ldots D_{\mathbf{c}_1} \, T_{\mathbf{a}_1 \ldots \mathbf{a}_k}(i) \,. \tag{3.2}$$

(This is a convenient short version of the correct expression; more precisely, the x^a should be replaced here by the components of the vector field X which has in normal coordinates the expansion $X(x) = x^b \, \delta^{\mathbf{a}}{}_b \, c_{\mathbf{a}}$ and which can be characterized as the non-identically vanishing vector field near i which satisfies $D_X \, X = X$, $X(i) = 0$.) In the following it will be shown how normal expansions can be obtained for solutions

$$h_{\mathbf{ab}} \,, \;\; \zeta \,, \;\; s \,, \;\; s_{\mathbf{ab}} \,, \tag{3.3}$$

to the conformal static vacuum field equations. In 3 dimensions the curvature tensor satisfies

$$R_{abcd}[h] = 2\big\{h_{a[c} L_{d]b} + h_{b[d} L_{c]a}\big\} \quad \text{with} \quad L_{ab}[h] = R_{ab}[h] - \frac{1}{4} \, R[h] \, h_{ab} \,,$$

and can be expressed because of (2.5) completely in terms of $s_{\mathbf{ab}}$. Once the latter is known, the connection coefficients $\Gamma_{\mathbf{a}}{}^{\mathbf{b}}{}_{\mathbf{c}}$ and the coefficients of the 1-forms $\chi^{\mathbf{a}}$ can be obtained, order by order, from the structural equations in polar coordinates cf. [8],

$$\frac{d}{d\,s}\big(s \, \chi^{\mathbf{a}}{}_b(s \, x^f)\big) = \delta^{\mathbf{a}}{}_b + \Gamma_{\mathbf{c}}{}^{\mathbf{a}}{}_{\mathbf{d}}(s \, x^f) \, s \, \chi^{\mathbf{c}}{}_b(s \, x^f) \; x^d \,,$$

$$\frac{d}{d\,s} \left(\Gamma_{\mathbf{a}}{}^{\mathbf{c}}{}_{\mathbf{e}}(s \, x^f) \, s \, \chi^{\mathbf{a}}{}_b(s \, x^f)\right) = R^{\mathbf{c}}{}_{\mathbf{eda}}(s \, x^f) \, x^d \;\, s \, \chi^{\mathbf{a}}{}_b\big(s \, x^f\big) \,,$$

where s denotes along the h-geodesics through i with unit tangent vectors an affine parameter which vanishes at i, so that $s^2 = \delta_{ab} \, x^a \, x^b$.

By formally taking covariant derivatives, the expansion coefficients of ζ and s up to order $m+2$ resp. $m+1$ can be obtained from equations (2.11) and (2.12) once $s_{\mathbf{ab}}$ is known up to order m. Calculating the expansion coefficients for $s_{\mathbf{ab}}$ by means of equation (2.13) leads, however, to some complicated algebra. It turns out that the latter simplifies considerably in the space spinor formalism.

To achieve the transition to the space-spinor formalism we introduce the constant van der Waerden symbols

$$\alpha^{AB}{}_a\,, \quad \alpha^a{}_{AB}\,, \quad a = 1,2,3\,, \quad A, B = 0,1\,,$$

which map one-index objects onto two-index objects which are symmetric in the two indices. If the latter are read as matrices, the symbols are given by

$$\xi^a \to \xi^{AB} = \alpha^{AB}{}_a\,\xi^a = \frac{1}{\sqrt{2}}\begin{pmatrix} -\xi^1 - i\xi^2 & \xi^3 \\ \xi^3 & \xi^1 - i\xi^2 \end{pmatrix},$$

$$\xi_a \to \xi_{AB} = \xi_a\,\alpha^a{}_{AB} = \frac{1}{\sqrt{2}}\begin{pmatrix} -\xi_1 + i\xi_2 & \xi_3 \\ \xi_3 & \xi_1 + i\xi_2 \end{pmatrix}.$$

With the summation rule also applying to capital indices one gets

$$\delta^c{}_a = \alpha^c{}_{AB}\,\alpha^{AB}{}_a\,, \qquad -\,\delta_{ab}\,\alpha^a{}_{AB}\,\alpha^b{}_{CD} = -\epsilon_{A(C}\,\epsilon_{D)B} \equiv h_{ABCD}\,,$$

$$a, b = 1,2,3\,, \quad A, B, C, D = 0,1\,,$$

where the constant ϵ-spinor is antisymmetric, $\epsilon_{AB} = -\epsilon_{BA}$, and satisfies $\epsilon_{01} = 1$. It is used to move indices according to the rules $\iota_B = \iota^A\,\epsilon_{AB}$, $\iota^A = \epsilon^{AB}\,\iota_B$, so that $\epsilon_A{}^B$ corresponds to the Kronecker delta. We shall denote the 'scalar product' $\kappa_A\,\iota^A$ of two spinors κ^A and ι^A occasionally also by $\epsilon(\kappa,\iota)$. It is important here to observe the order in which the spinors occur.

Given the van der Waerden symbols, we associate with a tensor field $T^{\mathbf{a}_1\ldots\mathbf{a}_p}{}_{\mathbf{b}_1\ldots\mathbf{b}_q}$ *given in the frame* $c_{\mathbf{a}}$ the space spinor field

$$T^{A_1B_1\ldots A_pB_p}{}_{C_1D_1\ldots C_qD_q} = T^{\mathbf{a}_1\ldots\mathbf{a}_p}{}_{\mathbf{b}_1\ldots\mathbf{b}_q}\alpha^{A_1B_1}{}_{a_1}\cdot\ldots\cdot\alpha^{b_q}{}_{C_qD_q}$$
$$= T^{(A_1B_1)\ldots(A_pB_p)}{}_{(C_1D_1)\ldots(C_qD_q)}\,.$$

In the following we shall employ tensor or spinor notation as it appears convenient. Consider the spinor field

$$\tau^{AA'} = \epsilon_0{}^A\,\epsilon_0{}^{A'} + \epsilon_1{}^A\,\epsilon_1{}^{A'}\,.$$

We assume that primed indices take values 0 and 1 and the summation rule applies, use a bar to denote complex conjugation, and take from $SL(2,\mathbb{C})$ two-index spinor theory the conventions that indices acquire a prime under complex conjugation and that the complex conjugate of ϵ_{AB} is denoted by $\epsilon_{A'B'}$. Setting

$$\xi^+_{AB\ldots H} = \tau_A{}^{A'}\,\tau_B{}^{B'}\,\ldots\tau_H{}^{H'}\,\bar{\xi}_{A'B'\ldots H'}\,,$$

one finds that a space spinor field

$$T_{A_1B_1\ldots A_pB_p} = T_{(A_1B_1)\ldots(A_pB_p)}\,,$$

arises from a real tensor field $T_{\mathbf{a}_1\ldots\mathbf{a}_p}$ if and only if it satisfies the reality condition

$$T_{A_1B_1\ldots A_pB_p} = (-1)^p\,T^+_{A_1B_1\ldots A_pB_p}\,. \tag{3.4}$$

It follows in particular

$$\xi_{AB}\,\xi^{AB} = 2\,(\xi_{00}\,\xi_{11} - \xi_{01}\,\xi_{01}) = 2\,\det(\xi_{AB}) = -\delta_{ab}\,\xi^a\,\xi^b\,,$$

and we can have $\xi_{AB}\,\xi^{AB} = 0$ for vectors $\xi^{AB} \neq 0$ only if ξ^a is complex. Since $\xi^{AB} = \xi^{(AB)}$, the relations $\xi_{AB}\,\xi^{AB} = 0$, $\xi^{AB} \neq 0$ imply by the equation above that $\xi^{AB} = \kappa^A\,\kappa^B$ for some $\kappa^A \neq 0$. This fact will allow us to interpret the data (1.11) as 'null data'.

Any spinor field $T_{ABC\ldots GH}$, symmetric or not, admits a decomposition into products of totally symmetric spinor fields and epsilon spinors which can be written schematically in the form (cf. [21])

$$T_{ABC\ldots GH} = T_{(ABC\ldots GH)} + \sum \epsilon' s \times \textit{symmetrized contractions of } T\,. \tag{3.5}$$

Later on it will be important for us that spinor fields $T_{A_1B_1\ldots A_pB_p}$ arising from tensor fields $T_{\mathbf{a}_1\ldots\mathbf{a}_p}$ satisfy

$$T_{(A_1B_1\ldots A_pB_p)} = \mathcal{C}(T_{\mathbf{a}_1\ldots\mathbf{a}_p})\,\alpha^{a_1}{}_{A_1B_1}\,\ldots\,\alpha^{a_p}{}_{A_pB_p}\,,$$

i.e., the projectors $\mathcal{C}$ onto the trace free symmetric part of tensors is represented in the space spinor notation simply by symmetrization. If convenient, we shall denote the latter also by the symbol *sym*.

To discuss vector analysis in terms of spinors, a complex frame field and its dual 1-form field are defined by

$$c_{AB} = \alpha^a{}_{AB}\,c_{\mathbf{a}}\,, \qquad \chi^{AB} = \alpha^{AB}{}_a\,\chi^{\mathbf{a}}\,,$$

so that $h(c_{AB}, c_{AB}) = h_{ABCD}$. If the derivative of a function f in the direction of c_{AB} is denoted by $c_{AB}(f) = f_{,a}\,c^a{}_{AB}$ and the spinor connection coefficients are defined by

$$\Gamma_{AB}{}^C{}_D = \frac{1}{2}\,\Gamma_{\mathbf{a}}{}^{\mathbf{b}}{}_{\mathbf{c}}\,\alpha^a{}_{AB}\,\alpha^{CH}{}_b\,\alpha^c{}_{DH}\,, \quad \text{so that} \quad \Gamma_{ABCD} = \Gamma_{(AB)(CD)}\,,$$

the covariant derivative of a spinor field ι^A is given by

$$D_{AB}\,\iota^C = e_{\,AB}(\iota^C) + \Gamma_{AB}{}^C{}_B\,\iota^B\,.$$

If it is required to satisfies the Leibniz rule with respect to tensor products, it follows that covariant derivatives in the $c_{\mathbf{a}}$-frame formalism translate under contractions with the van der Waerden symbols into spinor covariant derivatives and vice versa.

The commutator of covariant spinor derivatives satisfies

$$(D_{CD}\,D_{EF} - D_{EF}\,D_{CD})\,\iota^A = R^A{}_{BCDEF}\,\iota^B\,, \tag{3.6}$$

with the curvature spinor

$$R_{ABCDEF} = \frac{1}{2}\left\{\left(s_{ABCE} - \frac{R[h]}{6}\,h_{ABCE}\right)\epsilon_{DF} + \left(s_{ABDF} - \frac{R[h]}{6}\,h_{ABDF}\right)\epsilon_{CE}\right\},$$

where $R[h]$ is the Ricci scalar and $s_{ABCD} = s_{\mathbf{ab}}\,\alpha^a{}_{AB}\,\alpha^b{}_{CD}$ represents the trace free part of the Ricci tensor of h, which is completely symmetric, $s_{ABCD} = s_{(ABCD)}$. The gauge condition (2.5) implies

$$R_{ABCDEF} = \frac{1}{2}\,(s_{ABCE}\,\epsilon_{DF} + s_{ABDF}\,\epsilon_{CE})\,. \tag{3.7}$$

In the space-spinor formalism equations (2.13) acquire the concise form

$$D_A{}^E s_{BCDE} = \frac{2\,\mu}{1-\mu\,\zeta}\, s_{E(BCD}\, D_{A)}{}^E \zeta\,. \tag{3.8}$$

Applying to this equation and to the spinor versions of (2.11) and (2.12) the theory of 'exact sets of fields' discussed in [21], we get the following result.

Lemma 3.1. *Let there be given a sequence*

$$\hat{\mathcal{D}}_n = \{\psi_{A_2B_2A_1B_1},\ \psi_{A_3B_3A_2B_2A_1B_1},\ \psi_{A_4B_4A_3B_3A_2B_2A_1B_1},\ \ldots\},$$

of totally symmetric spinors satisfying the reality condition (3.4)*. Assume that there exists a solution h, ζ, s, s_{ABCD} to the conformal static field equations* (2.2)*,* (2.10)*,* (2.11)*,* (2.12)*,* (2.13) *so that the spinors given by $\hat{\mathcal{D}}_n$ coincide with the null data $\mathcal{D}_n^*$ given by* (1.11) *of the metric h in terms of an h-orthonormal normal frame centered at i, i.e.,*

$$\psi_{A_pB_p\ldots A_3B_3A_2B_2A_1B_1} = D_{(A_pB_p}\,\ldots\, D_{A_3B_3}\, s_{A_2B_2A_1B_1)}(i)\,, \quad p \geq 2\,. \tag{3.9}$$

Then the coefficients of the normal expansions (3.2) *of the fields* (2.16)*, in particular of*

$$s_{ABCD}(x) = \sum_{p\geq 0} \frac{1}{p!}\, x^{A_pB_p} \ldots x^{A_1B_1}\, D_{A_pB_p} \ldots D_{A_1B_1}\, s_{ABCD}(i)\,, \tag{3.10}$$

with $x^{AB} = \alpha^{AB}{}_a\, x^a$, are uniquely determined by the data $\hat{\mathcal{D}}_n$ and satisfy the reality conditions.

Proof. It holds $s_{ABCD}(i) = \psi_{ABCD}$ by assumption and the expansion coefficients for ζ, s of lowest order are given by (2.2), (2.15). The induction steps for ζ and s being obvious by (2.11) and (2.12), we only need to consider s_{ABCD} and (3.8). Assume $m \geq 0$. If spinors $D_{A_pB_p}\,\ldots\,D_{A_1B_1}\, s_{CDEF}(i)$, $p \leq m$, have been obtained which satisfy (3.9) and, up to that order, (3.8), the totally symmetric part of

$$D_{A_{m+1}B_{m+1}}\,\ldots\, D_{A_1B_1}\, s_{CDEF}(i)\,,$$

is given by the prescribed data while its contractions, which define the remaining terms in the decomposition corresponding to (3.5), are determined as follows. Observing the symmetries involved, essentially two cases can occur:

i) If one of the indices B_j is contracted with F, say, the operator $D_{A_jB_j}$ can be commuted with other covariant derivatives, generating by (3.6), (3.7) only terms of lower order, until it applies directly to s_{CDEF}. Equation (3.8) then shows how to express the resulting term by quantities of lower order.

ii) If the index B_j is contracted with B_k, $k \neq j$, the operators $D_{A_jB_j}$ and $D_{A_kB_k}$ can be commuted with other covariant derivatives, until the operator $D_{A_jH}\, D_{A_k}{}^H$ applies directly to s_{CDEF}. If the corresponding term is symmetrized in A_j and A_k the general identity

$$D_{H(A}\, D^H{}_{B)}\, s_{CDEF} = -2\, s_{H(CDE}\, s_{F)AB}{}^H,$$

implied by (3.6), (3.7) shows that this term is in fact of lower order. If a contraction of A_j and A_k is involved, the general identity

$$D_{AB}\, D^{AB}\, s_{CDEF} = -2\, D_F{}^G\, D_G{}^H\, s_{CDEH} + 3\, s_{GH(CD}\, s_{E)F}{}^{GH},$$

shows together with (3.8) that the corresponding term can again be expressed in terms of quantities of lower order, showing that $D_{A_{m+1}B_{m+1}} \ldots D_{A_1B_1}\, s_{CDEF}(i)$ is determined by our data and terms of order $\leq m$. That the expansion coefficients satisfy the reality condition is a consequence of the formalism and the fact that they are satisfied by the data $\hat{\mathcal{D}}_n$. □

To achieve our goal, we have to show the convergence of the formal series determined in Lemma 3.1. This requires us to impose estimates on the free coefficients given by $\mathcal{D}_n$. We get the following result.

Lemma 3.2. *A necessary condition for the formal series* (3.10) *determined in Lemma* 3.1 *to be absolutely convergent near the origin is that the data given by* $\hat{\mathcal{D}}_n$ *satisfy estimates of the type*

$$|\psi_{A_pB_p\ldots A_1B_1CDEF}| \leq \frac{p!\, M}{r^p}, \quad p = 0, 1, 2, \ldots, \tag{3.11}$$

with some constants $M, r > 0$.

Proof. If f is a real analytic function defined on some neighbourhood of the origin in $\mathbb{R}^n$, it can be analytically extended to a function which is defined, holomorphic, and bounded on a polydisc $P(0,r) = \{x \in \mathbb{C}^n \,|\, |x^j| < r,\ 1 \leq j \leq n\}$ with some $r > 0$. Its Taylor expansion $f = \sum_{|\alpha|\geq 0} \frac{1}{\alpha!}\, \partial^\alpha f(0)\, x^\alpha$ is absolutely convergent on $P(0,r)$ with $\sup_{x\in P(0,r)} |f(x)| \leq M < \infty$ so that its derivatives satisfy the estimates

$$|\partial^\alpha f(0)| \leq \frac{\alpha!\, M}{r^{|\alpha|}} \leq \frac{|\alpha|!\, M}{r^{|\alpha|}}. \tag{3.12}$$

The first of these estimates are known as Cauchy inequalities. Here $\alpha \in \mathbb{N}^n$ denotes a multi-index and we use the notation $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\alpha! = \alpha_1! \cdot \cdots \cdot \alpha_n!$, $\partial^\alpha = \partial_1^{\alpha_1} \cdot \cdots \cdot \partial_n^{\alpha_n}$, and $x^\alpha = (x^1)^{\alpha_1} \cdot \cdots \cdot (x^n)^{\alpha_n}$.

If the series (3.10) and thus

$$s_{\mathbf{ab}}(x) = \sum_{p\geq 0} \frac{1}{p!}\, x^{c_p} \ldots x^{c_1}\, D_{\mathbf{c}_p} \ldots D_{\mathbf{c}_1}\, s_{\mathbf{ab}}(i), \tag{3.13}$$

is absolutely convergent near the origin, there exist therefore by the second of the estimates (3.12) constants $M_*, r_* > 0$ with

$$|D_{\mathbf{c}_p} \ldots D_{\mathbf{c}_1}\, s_{\mathbf{ab}}(i)| \leq \frac{p!\, M_*}{r_*^p}, \qquad \mathbf{c}_p, \ldots, \mathbf{c}_1, \mathbf{a}, \mathbf{b} = 1, 2, 3, \qquad p = 0, 1, 2, \ldots .$$

Observing the transition rule from tensor to spinor quantities, one gets from this the estimates

$$|D_{A_pB_p} \ldots D_{A_1B_1}\, s_{CDEF}(i)| \leq \frac{p!\, M}{r^p}, \quad A_p, B_p, \ldots E, F = 0, 1, \quad p = 0, 1, 2, \ldots \tag{3.14}$$

with $M = 9\,c^2\,M_*$ and $r = r_*/3\,c$, where $c = \max_{a=1,2,3;\,A,B=0,1} |\alpha^a{}_{AB}|$. To derive from these estimates the estimates (3.11) we consider instead of (3.5) directly the symmetrization operator to get

$$\begin{aligned}|\psi_{A_pB_p\ldots A_1B_1CDEF}| &= |D_{(A_pB_p}\,\ldots\,D_{A_1B_1}\,s_{CDEF)}(i)|\\ &\leq \frac{1}{(2p+4)!}\sum_{\pi\in\mathcal{S}_{2p+4}} |D_{\pi(A_pB_p}\,\ldots\,D_{A_1B_1}\,s_{CDEF)}(i)|\\ &\leq \frac{p!\,M}{r^p}\,,\end{aligned}$$

where $\mathcal{S}_m$ denotes the group of permutations of m elements. □

We note for later use that if the derivatives of a smooth function f satisfy estimates of the type (3.12) with some constants $M, r > 0$ then the function f is real analytic near the origin because its Taylor series is majorized by

$$\sum_\alpha M\,r^{-|\alpha|}\,x^\alpha = \frac{M\,r^n}{(r-x^1)\cdot\ldots\cdot(r-x^n)}\,, \qquad |x^a| < 1\,, \tag{3.15}$$

and

$$\sum_\alpha \frac{|\alpha|\,!}{\alpha\,!}\,M\,r^{-|\alpha|}\,x^\alpha = \frac{M\,r}{(r-x^1-\cdots-x^n)}\,, \qquad \sum_{j=1}^n |x^j| < 1\,. \tag{3.16}$$

3.1. Relations between null data and multipoles

We express the relation between the sequences $\mathcal{D}^*_n$ of null data and the sequences $\mathcal{D}^*_{mp}$ of multipoles of h (in the same normal frame centered at i) in terms of space-spinor notation.

Lemma 3.3. *The spinor fields $P_{A_pB_p\ldots A_1B_1}$ near i, given by* (1.9), (1.10), *are of the form*

$$P_{A_pB_p\ldots A_1B_1} = -\frac{m}{2}\big\{D_{(A_pB_p}\,\ldots\,D_{A_3B_3}\,s_{A_2B_2A_1B_1)} + F_{A_pB_p\ldots A_1B_1}\big\}\,, \tag{3.17}$$

with symmetric spinor-valued functions

$$\begin{aligned}F_p &\equiv F_{A_pB_p\ldots A_1B_1}\\ &= F_{A_pB_p\ldots A_1B_1}\big[\{D_{(A_qB_q}\,\ldots\,D_{A_3B_3}\,s_{A_2B_2A_1B_1)}\}_{q\leq p-2}\big]\,, \quad p\geq 2\,,\end{aligned}$$

which satisfy

$$F_{A_2B_2A_1B_1} = 0\,, \quad F_{A_3B_3A_2B_2A_1B_1} = 0\,,$$

and which are real linear combinations of symmetrized tensor products of

$$s_{A_2B_2A_1B_1}\,,\; D_{(A_3B_3}\,s_{A_2B_2A_1B_1)}\,,\;\ldots\,,D_{(A_{p-2}B_{p-2}}\,\ldots\,D_{A_3B_3}\,s_{A_2B_2A_1B_1)}\,,$$

for $p \geq 4$.

Proof. The first two results on F follow by direct calculations from (1.9), (1.10). Inserting (3.17) into the recursion relation (1.10) gives for $p \geq 3$ the recursion relations

$$F_{A_{p+1}B_{p+1}\dots A_1B_1} = D_{(A_{p+1}B_{p+1}}\,F_{A_pB_p\dots A_1B_1)} - c_p\left\{s_{(A_{p+1}B_{p+1}A_pB_p}\,D_{A_{p-1}B_{p-1}}\,\cdots\,s_{A_2B_2A_1B_1)} + s_{(A_{p+1}B_{p+1}}\,F_{A_{p-1}B_{p-1}\dots A_1B_1)}\right\}. \tag{3.18}$$

With the induction hypothesis which assumes the properties of the F's stated above for $F_{A_qB_q\dots A_1B_1}$, $q \leq p$, the relations (3.18) imply these properties for $F_{A_{p+1}B_{p+1}\dots A_1B_1}$. □

A further calculation gives

$$F_4 = -c_3\,s_{(A_4B_4A_3B_3}\,s_{A_2B_2A_1B_1)}\,,$$
$$F_5 = -(2\,c_3 + c_4)\,s_{(A_5B_5A_4B_4}\,D_{A_3B_3}s_{A_2B_2A_1B_1)}\,,$$

and by induction the recursion law above implies the general expressions

$$F_{2p} = \alpha_{2p}\,sym(s \otimes D^{2p-4}s) + \cdots + \omega_{2p}\,sym(\otimes^p s)\,, \quad p \geq 3\,,$$
$$F_{2p+1} = \alpha_{2p+1}\,sym(s \otimes D^{2p-3}s) + \cdots + \omega_{2p+1}\,sym(\otimes^{p-1}s \otimes Ds)\,, \quad p \geq 3\,,$$

with real coefficients $\alpha_{2p}, \alpha_{2p+1}, \dots, \omega_{2p}, \omega_{2p+1}$. The first terms on the right hand sides denote the term with the highest power of D occurring in the respective expression. The sum of the powers of D occurring in each term is even in the case of F_{2p} and odd in the case of F_{2p+1}. The sum of the powers of D occurring in each of the terms indicated by dots lies between 2 and $2\,p-4$ in the case of F_{2p} and between 3 and $2\,p-3$ in the case of F_{2p+1}. The coefficients indicated above are determined by

$$\alpha_6 = -(2\,c_3 + c_4 + c_5)\,, \qquad \alpha_7 = -(2\,c_3 + c_4 + c_5 + c_6)\,,$$
$$\omega_5 = -(2\,c_3 + c_4)\,, \qquad \omega_6 = c_3\,c_5\,,$$

and, for $p \geq 3$, by

$$\alpha_{2p+1} = \alpha_{2p} - c_{2p}\,, \qquad \alpha_{2p+2} = \alpha_{2p+1} - c_{2p+1}\,,$$
$$\omega_{2p+1} = p\,\omega_{2p} - c_{2p}\,\omega_{2p-1}\,, \qquad \omega_{2p+2} = -c_{2p+1}\,\omega_{2p}\,,$$

which implies in particular

$$\omega_{2p} = (-1)^{p+1}\,\Pi_{l=1}^{p-1}\,c_{2l+1}\,, \quad p \geq 3\,. \tag{3.19}$$

Restricting the relation (3.17) to i defines with the identification (3.9) a non-linear map which can be read as a map

$$\Psi : \{\,\hat{\mathcal{D}}_n\,\} \to \{\,\hat{\mathcal{D}}_{mp}\,\}\,,$$

of the set of *abstract null data* into the set of *abstract multipoles* (i.e., sequences of symmetric spinors not necessarily derived from a metric) satisfying

$$\nu_{A_pB_p\,\dots\,A_1B_1} = -\frac{m}{2}\Big(\psi_{A_pB_p\,\dots\,A_1B_1} + F_{A_pB_p\,\dots\,A_1B_1}\big[\{\psi_{A_qB_q\,\dots\,A_1B_1}\}_{q\leq p-2}\big]\Big)\,, \qquad p\geq 2\,. \quad (3.20)$$

Corollary 3.4. *For given m the map Ψ which maps sequences $\hat{\mathcal{D}}_n$ of abstract null data onto sequences $\hat{\mathcal{D}}_{mp}$ of abstract multipoles is bijective.*

Proof. An inverse of Ψ can be constructed because $F_2 = 0$, $F_3 = 0$, and the F_p depend only on the $\psi_{A_qB_q\,\dots\,A_1B_1}$ with $q \leq p-2$. The relations (3.20) therefore determine for a given sequence $\hat{\mathcal{D}}_{mp}$ recursively a unique sequence $\hat{\mathcal{D}}_n$. □

It follows that for a given metric h the sequences of multipoles and the sequences of null data in a given standard frame carry the same information on h. The relation is not simple, however. It can happen that a sequence $\hat{\mathcal{D}}_n$ with only a finite number of non-vanishing members is mapped onto an sequence $\hat{\mathcal{D}}_{mp}$ with an infinite number of non-vanishing members and vice versa. For instance, the relations given above show that the sequence $\hat{\mathcal{D}}_n = \{\psi_2,\ 0,\ 0,\ 0,\dots\}$ with $\psi_2 \equiv \psi_{A_2B_2A_1B_1} \neq 0$ is mapped onto the sequence $\hat{\mathcal{D}}_{mp} = \{\nu_2,\ 0,\ \nu_4,\ 0,\ \nu_6,\ \dots\}$ with $\nu_q = \nu_{A_qB_q\,\dots\,A_1B_1}$, where

$$\nu_2 = \psi_2\,, \qquad \nu_{2p} = (-1)^{p+1}\,(\Pi_{l=1}^{p-1}\,c_{2l+1})\,sym(\otimes^p\psi_2) \neq 0\,, \quad p\geq 2\,.$$

4. The characteristic initial value problem

To complete the analysis one would have to show that the estimates (3.11) imply estimates of the type (3.14) for the coefficients of (3.10). The induction argument used in the proof of Lemma 3.1 leads, however, to complicated algebraic considerations. The commutation of covariant derivatives generates with the subsequent derivative operations more and more non-linear terms of lower order. Formalizing this procedure to derive estimates does not look very attractive. To arrive at a formulation of our question which looks more similar to a boundary value problem to which Cauchy–Kowalevskaya type arguments apply, we make use of the inherent geometric nature of the problem and the geometric meaning of the null data.

The fields h, ζ, s, s_{ABCD} are necessarily real analytic in the normal coordinates x^a and a standard frame c_{AB} centered at i. They can thus be extended near i by analyticity into the complex domain and considered as holomorphic fields on a complex analytic manifold S_c. Choosing S_c to be a sufficiently small neighbourhood of i, we can assume the extended coordinates, again denoted by x^a, to define a holomorphic coordinate system on S_c which identifies the latter with an open neighbourhood of the origin in $\mathbb{C}^3$. The original manifold S is then a real, 3-dimensional, real analytic submanifold of the real, 6-dimensional, real analytic manifold underlying S_c. If α^a, β^a, $a = 1,2,3$, define real local coordinates on the real 6-dimensional manifold underlying S_c so that the holomorphic coordinates x^a

can be written $x^a = \alpha^a + i\,\beta^a$, we use the standard notation $\partial_{x^a} = \frac{1}{2}(\partial_{\alpha^a} - i\,\partial_{\beta^a})$ and $\partial_{\bar{x}^a} = \frac{1}{2}(\partial_{\alpha^a} + i\,\partial_{\beta^a})$. The assumption that the complex-valued function $f = f(x^a)$ be holomorphic is then equivalent to the requirement that $\partial_{\bar{x}^a} f = 0$ so that we will only have to deal with the operators ∂_{x^a}. Under the analytic extension the main differential geometric concepts and formulas remain valid. The coordinates x^a and the extended frame, again denoted by c_{AB}, satisfy the same defining equations and the extended fields, denoted again by h, ζ, s, s_{ABCD}, satisfy the conformal static vacuum field equations as before.

The analytic function $\Gamma = \delta_{ab}\, x^a\, x^b$ on S extends to a holomorphic function on S_c which satisfies again the eikonal equation $h^{ab}\, D_a\Gamma\, D_b\Gamma = -4\,\Gamma$. On S it vanishes only at i, but the set

$$\mathcal{N}_i = \{p \in S_c |\ \Gamma(p) = 0\}\,,$$

is an irreducible analytical set (cf. [22]) such that $\mathcal{N}_i \setminus \{i\}$ is 2-dimensional complex submanifold of S_c. It is the cone swept out by the complex null geodesics through i and we will refer to it shortly as the *null cone at* i. While some of the following considerations may be reminiscent of considerations concerning cones swept out by real null geodesics through given points of 4-dimensional Lorentz spaces, there are basic differences. In the present case there do not exist splittings into future and past cones. The set $\mathcal{N}_i \setminus \{i\}$ is connected and its set of of complex null generators is diffeomorphic to $P^1(\mathbb{C}) \sim S^2$. If $\mathcal{N}_i \setminus \{i\}$ is considered as a 4-dimensional submanifold of the 6-dimensional real manifold underlying S_c, the set of real null generators is not simply connected but diffeomorphic to $SO(3,\mathbb{R})$.

The set $\mathcal{N}_i$ will be important for geometrizing our problem. Let $u \to x^a(u)$ be a null geodesic through i so that $x^a(0) = 0$. Its tangent vector is then of the form $\dot{x}^{AB} = \iota^A\,\iota^B$ with a spinor field $\iota^A = \iota^A(u)$ satisfying $D_{\dot{x}}\iota^A = 0$ along the geodesic. Then

$$s_0(u) = \dot{x}^a\,\dot{x}^b\, s_{ab}\big(x(u)\big) = \iota^A\,\iota^B\,\iota^C\,\iota^D\, s_{ABCD}\big(x(u)\big)\,, \tag{4.1}$$

is an analytic function of u with Taylor expansion

$$s_0 = \sum_{p=0}^{\infty} \frac{1}{p!}\, u^p\, \frac{d^p}{du^p}\, s_0(0)\,,$$

where

$$\begin{aligned}\frac{d^p}{du^p}\, s_0(0) &= \iota^{A_p}\,\iota^{B_p} \ldots \iota^C\,\iota^D\, D_{A_pB_p} \ldots D_{A_1B_1} s_{ABCD}(i)\\ &= \iota^{A_p}\,\iota^{B_p} \ldots \iota^C\,\iota^D\, D_{(A_pB_p} \ldots D_{A_1B_1} s_{ABCD)}(i)\,.\end{aligned}$$

Knowing these expansion coefficients for initial null vectors $\iota^A\,\iota^B$ covering an open subset of the null directions at i is equivalent to knowing the null data $\mathcal{D}^*_n$ of the metric h.

Our problem can thus be formulated as the boundary value problem for the conformal static vacuum equations with data given by the function (4.1) on $\mathcal{N}_i$, where the $\iota^A\iota^B$ are parallely propagated null vectors tangent to $\mathcal{N}_i$. The set $\mathcal{N}_i$ can be regarded as a (complex) characteristic of the (extended) operator Δ_h and also to the conformal static equations. Therefore we shall refer to this problem as the *characteristic initial value problem for the conformal static vacuum field equations with data on the null cone at space-like infinity.*

The conformal static vacuum field equations (2.10), (2.11), (2.12), (2.13) form a 3-dimensional analogue of the 4-dimensional conformal Einstein equations [9]. Characteristic initial value problems for these two type of systems are therefore quite similar in character.

The existence of analytic solutions to characteristic initial value problems for the conformal Einstein equations has been shown in [10] by using Cauchy–Kowalevskaya type arguments. In the present case we shall employ somewhat different techniques for the following reason.

The remaining and in fact the main difficulty in our problem arises from fact that $\mathcal{N}_i$ is not a smooth hypersurface but an analytic set with a vertex at the point i. A characteristic initial value problem for the conformal Einstein equations with data on a cone has been studied in [11] and some of the techniques introduced there and further developed in [13] will be used in the following. The method we use to derive estimates on the expansion coefficients has apparently not been used before in the context of Einstein's field equations.

4.1. The geometric gauge

To obtain a setting in which the mechanism of calculating the expansion coefficients allows one to derive estimates on the coefficients from the conditions imposed on the data, a gauge needs to be chosen which is suitably adapted to the singular set $\mathcal{N}_i$. The coordinates and the frame field will then necessarily be singular and the frame will no longer define a smooth lift to the bundle of frames but a subset which becomes tangent to the fibres over some points. The setting described in the following will organize this situation in a geometric way and provide control on the singularity and the smoothness of the fields.

Let $SU(2)$ be the group of complex 2×2 matrices $(s^A{}_B)_{A,B=0,1}$ satisfying

$$\epsilon_{AB}\, s^A{}_C\, s^B{}_D = \epsilon_{CD}\,, \quad \tau_{AB'}\, s^A{}_C\, \bar{s}^{B'}{}_{D'} = \tau_{CD'}\,, \tag{4.2}$$

where $s^B{}_D \to \bar{s}^{B'}{}_{D'}$ denotes complex conjugation. The map

$$SU(2) \ni s^A{}_B \to s^{(A}{}_{(C}\, s^{B)}{}_{D)} \to s^a{}_b = \alpha^a{}_{AB}\, s^A{}_C\, s^B{}_D\, \alpha^{CD}{}_b \in SO(3, R)\,, \tag{4.3}$$

realizes the $2:1$ covering homomorphism of $SU(2)$ onto the group $SO(3,\mathbb{R})$. Under holomorphic extension the map above extends to a $2:1$ covering homomorphism of the group $SL(2,\mathbb{C})$ onto the group $SO(3,\mathbb{C})$, where $SL(2,\mathbb{C})$ denotes the group of complex 2×2 matrices satisfying only the first of conditions (4.2).

We will make use of the principal bundle of normalized spin frames $SU(S) \stackrel{\pi}{\to} S$ with structure group $SU(2)$. A point $\delta \in SU(S)$ is given by a pair of spinors $\delta = (\delta^A_0, \delta^A_1)$ at a given point of S which satisfies

$$\epsilon(\delta_A,\, \delta_B) = \epsilon_{AB}\,, \quad \epsilon(\delta_A,\, \delta^+{}_{B'}) = \tau_{AB'}\,, \tag{4.4}$$

where the lower index, which labels the members of the spin frame, is assumed to acquire a prime under the "+"-operation. The action of the structure group is given for $s \in SU(2)$ by

$$\delta \to \delta \cdot s \quad \text{where} \quad (\delta \cdot s)_A = s^B{}_A \, \delta_B \,.$$

The projection π maps a frame δ onto its base point in S. The bundle of spin frames is mapped by a $2:1$ bundle morphism $SU(S) \xrightarrow{p} SO(S)$ onto the bundle $SO(S) \xrightarrow{\pi'} S$ of oriented, orthonormal frames on S so that $\pi' \circ p = \pi$. For any spin frame δ we can identify by (4.4) the matrix $(\delta^A_B)_{A,A,B=0,1}$ with an element of the group $SU(2)$. With this reading the map p will be assumed to be realized by

$$SU(S) \ni \delta \to p(\delta)_{AB} = \delta^E_A \, \delta^F_B \, c_{EF} \in SO(S) \,,$$

where c_{AB} denotes the normal frame field on S introduced before. We refer to $p(\delta)$ as the frame associated with the spin frame δ.

Under holomorphic extension the bundle $SU(S) \to S$ is extended to the principal bundle $SL(S_c) \xrightarrow{\pi} S_c$ of spin frames $\delta = (\delta^A_0, \delta^A_1)$ at given points of S_c which satisfy only the first of conditions (4.4). Its structure group is $SL(2,\mathbb{C})$. The bundle $SU(S) \xrightarrow{\pi} S$ is embedded into $SL(S_c) \xrightarrow{\pi} S_c$ as a real analytic subbundle. The bundle morphism p extends to a $2:1$ bundle morphism, again denoted by p, of $SL(S_c) \xrightarrow{\pi} S_c$ onto the bundle $S0(S_c) \xrightarrow{\pi'} S_c$ of oriented, normalized frames of S_c with structure group $SO(3,\mathbb{C})$. We shall make use of several structures on $SM(S_c)$.

With each $\alpha \in sl(2,\mathbb{C})$, i.e., $\alpha = (\alpha^A{}_B)$ with $\alpha_{AB} = \alpha_{BA}$, is associated a *vertical vector field* Z_α tangent to the fibres, which is given at $\delta \in SL(S_c)$ by $Z_\alpha(\delta) = \frac{d}{dv}(\delta \cdot \exp(v\,\alpha))|_{v=0}$, where $v \in \mathbb{C}$ and exp denotes the exponential map $sl(2,\mathbb{C}) \to SL(2,\mathbb{C})$.

The $\mathbb{C}^3$-valued *soldering form* $\sigma^{AB} = \sigma^{(AB)}$ maps a tangent vector $X \in T_\delta SL(S_c)$ onto the components of its projection $T_\delta(\pi)X \in T_{\pi(\delta)}S_c$ in the frame $p(\delta)$ associated with δ so that $T_\delta(\pi)X = \langle \sigma^{AB}, X \rangle p(\delta)_{AB}$. It follows that $\langle \sigma^{AB}, Z_\alpha \rangle = 0$ for any vertical vector field Z_α.

The $sl(2,\mathbb{C})$-valued *connection form* $\omega^A{}_B$ *on* $SL(S_c)$ transforms with the adjoint transformation under the action of $SL(2,\mathbb{C})$ and maps any vertical vector field Z_α onto its generator so that $\langle \omega^A{}_B, Z_\alpha \rangle = \alpha^A{}_B$.

With $x^{AB} = x^{(AB)} \in \mathbb{C}^3$ is associated the *horizontal vector field* H_x on $SL(S_c)$ which is horizontal in the sense that $\langle \omega^A{}_B, H_x \rangle = 0$ and which satisfies $\langle \sigma^{AB}, H_x \rangle = x^{AB}$. Denoting by H_{AB}, $A, B = 0, 1$, the horizontal vector fields satisfying $\langle \sigma^{AB}, H_{CD} \rangle = h^{AB}{}_{CD}$, it follows that $H_x = x^{AB} \, H_{AB}$. An integral curve of a horizontal vector field projects onto an h-geodesic and represents a spin frame field which is parallelly transported along this geodesic.

A holomorphic spinor field ψ on S_c is represented on $SL(S_c)$ by a holomorphic spinor-valued function $\psi_{A_1 \ldots A_j}(\delta)$ on $SL(S_c)$, given by the components of ψ in the frame δ. We shall use the notation $\psi_k = \psi_{(A_1 \ldots A_j)_k}$, $k = 0, \ldots, j$, where $(\ldots\ldots)_k$

denotes the operation *'symmetrize and set k indices equal to* 1 *the rest equal to* 0*'*. These functions completely specify ψ if ψ is symmetric. They are then referred to as the *essential components of* ψ.

4.2. The submanifold $\hat{S}$ of $SL(S_c)$

We combine the construction of a coordinate system and a frame field with the definition of an analytic submanifold M of $SL(S_c)$ which is obtained as follows. We choose a spin frame δ^* in the fibre of $SL(S_c)$ over i which is projected by π' onto the frame c_{AB} at considered i before. The curve

$$\mathbb{C} \ni v \to \delta(v) = \delta^* \cdot s(v) \in SL(S_c)\,,$$

with

$$s(v) = \exp(v\,\alpha) = \begin{pmatrix} 1 & 0 \\ v & 1 \end{pmatrix}, \quad \alpha = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in sl(2,\mathbb{C})\,, \tag{4.5}$$

in the fibre of $SL(S_c)$ over i defines a vertical, 1-dimensional, holomorphic submanifold I through δ^* on which v defines a coordinate. The associated family of frames $e_{AB} = e_{AB}(v)$ at i is given explicitly by

$$e_{00}(v) = c_{00} + 2\,v\,c_{01} + v^2\,c_{11}\,, \quad e_{01}(v) = c_{01} + v\,c_{11}\,, \quad e_{11}(v) = c_{11}\,.$$

The following construction is carried out in some neighbourhood of I. If the latter is chosen small enough all the following statements will be correct.

The set I is moved with the flow of H_{11} to obtain a holomorphic 2-manifold U_0 of $SL(S_c)$ containing I. The parameter on the integral curves of H_{11} which vanishes on I will be denoted by w and v is extended to U_0 by assuming it to be constant on the integral curves of H_{11}. All these integral curves are mapped by π onto the null geodesics $\gamma(w)$ with affine parameter w and tangent vector $\gamma'(0) = c_{11}$ at $\gamma(0) = i$. The parameter v specifies frame fields which are parallelly propagated along γ.

The set U_0 is moved with the flow of H_{00} to obtain a holomorphic 3-submanifold $\hat{S}$ of $SL(S_c)$ containing U_0. We denote by u the parameter on the integral curves of H_{00} which vanishes on U_0 and extend v and w to $\hat{S}$ by assuming them to be constant along the integral curves of H_{00}. The functions $z^1 = u$, $z^2 = v$, $z^3 = w$ define holomorphic coordinates on $\hat{S}$. The restriction the projection to $\hat{S}$ will be again denoted by π.

The projections of the integral curves of H_{00} with a fixed value of w sweep out, together with γ, the cone $\mathcal{N}_{\gamma(w)}$ near $\gamma(w)$ which is generated by the null geodesics through the point $\gamma(w)$. On the null geodesics u is an affine parameter which vanishes at $\gamma(w)$ while v parametrizes the different generators. In terms of the base space S_c our gauge is based on the nested family of cones $\mathcal{N}_{\gamma(w)}$ which share the generator γ. The set $W_0 = \{w = 0\}$, which projects onto $\mathcal{N}_i \setminus \gamma$, will define the initial data set for our problem. The map π induces a biholomorphic diffeomorphism of $\hat{S}' \equiv \hat{S} \setminus U_0$ onto $\pi(\hat{S}')$. The singularity of the gauge at points of U_0 (resp. over γ) consists in π dropping rank on U_0 because the curves $w = const.$ on U_0 are tangent to the fibres over $\gamma(w)$ where $\partial_v = Z_\alpha$. The null curve $\gamma(w)$ will be referred to as *the singular generator of* $\mathcal{N}_i$ *in the gauge determined by the spin frame* δ^* *resp. the corresponding frame* c_{AB} *at* i.

The soldering and the connection form pull back to holomorphic 1-forms on $\hat{S}$, which will be denoted again by σ^{AB} and $\omega^A{}_B$. Corresponding to the behaviour of π the 1-forms σ^{00}, σ^{01}, σ^{11} are linearly independent on $\hat{S}'$ while the rank of this system drops to 2 on U_0 because $\langle\sigma^{AB},\partial_v\rangle = \langle\sigma^{AB}, Z_\alpha\rangle = 0$. If the pull back of the curvature form $\Omega^A{}_B = \frac{1}{2}\, r^A{}_{BCDEF}\,\sigma^{CD}\wedge\sigma^{EF}$ to $\hat{S}$ is denoted again by $\Omega^A{}_B$, the soldering and the connection form satisfy the structural equations

$$d\,\sigma^{AB} = -\omega^A{}_C\wedge\sigma^{CB} - \omega^B{}_C\wedge\sigma^{AC}\,, \quad d\,\omega^A{}_B = -\omega^A{}_C\wedge\omega^C{}_B + \Omega^A{}_B\,.$$

By construction of $\hat{S}$ we have

$$\begin{aligned} &\langle\sigma^{AB},\partial_v\rangle = 0\,, \qquad \langle\sigma^{AB},\partial_w\rangle = \epsilon_1{}^A\,\epsilon_1{}^B \quad \text{on} \quad U_0\,,\\ &\langle\omega^A{}_B,\partial_w\rangle = 0\,, \qquad \langle\omega^A{}_B,\partial_v\rangle = \langle\omega^A{}_B, Z_\alpha\rangle = \epsilon_1{}^A\,\epsilon_B{}^0 \quad \text{on} \quad U_0\,,\\ &\langle\sigma^{AB},\partial_u\rangle = \epsilon_0{}^A\,\epsilon_0{}^B \quad \text{and} \quad \langle\omega^A{}_B,\partial_u\rangle = 0 \quad \text{on} \quad \hat{S}\\ &\qquad\qquad \text{while} \quad \langle\sigma^{AB},\partial_v\rangle \neq 0 \quad \text{on} \quad \hat{S}'\,. \end{aligned}$$

To obtain more precise information on σ^{AB} and $\omega^A{}_B$ we note the following general properties (cf. [11] and [13] for more details). If, for given $x^{AB}\in\mathbb{C}^3$, the Lie derivative with respect to H_x is denoted by $\mathcal{L}_x$, then

$$\mathcal{L}_x\sigma^{AB} = 2\,x^{C(A}\,\omega^{B)}{}_C\,, \quad \langle\mathcal{L}_x\omega^A{}_B,\,.\rangle = \langle\Omega^A{}_B, H_x\wedge\,.\rangle.$$

Since $0 = [\partial_u,\partial_v] = [H_{00},\partial_v]$ on $\hat{S}$ and $\Omega^A{}_B$ is horizontal, it follows that

$$\begin{aligned} \partial_u\langle\sigma^{AB},\partial_v\rangle &= 2\,\epsilon_0{}^{(A}\langle\omega^{B)}{}_0,\partial_v\rangle\,,\\ \partial_u\langle\omega^A{}_B,\partial_v\rangle|_{u=0} &= \langle\Omega^A{}_B, H_x\wedge Z_\alpha\rangle|_{u=0} = 0\,. \end{aligned}$$

This gives with the previous relations

$$\begin{aligned} &\langle\omega^A{}_B,\partial_v\rangle = \epsilon_1{}^A\,\epsilon_B{}^0 + O(u^2)\\ &\qquad\qquad \text{whence } \langle\omega^A{}_B,\partial_v\rangle = 2\,u\,\epsilon_0{}^{(A}\epsilon_1{}^{B)} + O(u^3) \quad \text{as} \quad u\to 0\,. \end{aligned}$$

Similarly we obtain with $0 = [\partial_u,\partial_w] = [H_{00},\partial_w]$ on $\hat{S}$

$$\partial_u\langle\sigma^{AB},\partial_w\rangle = 2\,\epsilon_0{}^{(A}\langle\omega^{B)}{}_0,\partial_w\rangle\,, \quad \partial_u\langle\omega^A{}_B,\partial_w\rangle|_{u=0} = \frac{1}{2}\,r^A{}_{B0011}\,.$$

In terms of the coordinates z^a we thus get $\sigma^{AB} = \sigma^{AB}{}_a\,dz^a$ on $\tilde{S}'$ with a co-frame matrix

$$(\sigma^{AB}{}_a) = \begin{pmatrix} 1 & \sigma^{00}{}_2 & \sigma^{00}{}_3\\ 0 & \sigma^{01}{}_2 & \sigma^{01}{}_3\\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & O(u^3) & O(u^2)\\ 0 & u + O(u^3) & O(u^2)\\ 0 & 0 & 1 \end{pmatrix} \quad \text{as} \quad u\to 0\,. \tag{4.6}$$

On $\hat{S}'$ there exist unique, holomorphic vector fields e_{AB} which satisfy

$$\langle\sigma^{AB}, e_{EF}\rangle = h^{AB}{}_{EF}\,.$$

If we write $e_{AB} = e^a{}_{AB}\,\partial_{z^a}$, the properties noted above imply for the frame coefficients

$$(e^a{}_{AB}) = \begin{pmatrix} 1 & e^1{}_{01} & e^1{}_{11} \\ 0 & e^2{}_{01} & e^2{}_{11} \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & O(u^2) & O(u^2) \\ 0 & \frac{1}{2\,u} + O(u) & O(u) \\ 0 & 0 & 1 \end{pmatrix} \quad \text{as} \quad u \to 0\,. \tag{4.7}$$

In the following we shall write

$$e^a{}_{AB} = e^{*a}{}_{AB} + \hat{e}^a{}_{AB}\,, \tag{4.8}$$

with singular part

$$e^{*a}{}_{AB} = \delta^a_1\,\epsilon_A{}^0\,\epsilon_B{}^0 + \delta^a_2\,\frac{1}{u}\,\epsilon_{(A}{}^0\,\epsilon_{B)}{}^1 + \delta^a_3\,\epsilon_A{}^1\,\epsilon_B{}^1\,, \tag{4.9}$$

and holomorphic functions $\hat{e}^a{}_{AB}$ on $\hat{S}$ which satisfy

$$\hat{e}^a{}_{AB} = O(u) \quad \text{as} \quad u \to 0\,. \tag{4.10}$$

We define connections coefficients on $\hat{S}'$ by writing $\omega^A{}_B = \Gamma_{CD}{}^A{}_B\,\sigma^{CD}$ with

$$\Gamma_{CD\,AB} \equiv \langle \omega_{AB}, e_{CD} \rangle\,,$$

so that $\Gamma_{CD\,AB} = \Gamma_{(CD)\,(AB)}$. The definition of the frame then implies

$$\Gamma_{00\,AB} = 0 \quad \text{on} \quad \hat{S} \quad \text{and} \quad \Gamma_{11\,AB} = 0 \quad \text{on} \quad U_0\,,$$

and it follows from the discussion above that

$$\Gamma_{ABCD} = \Gamma^*_{ABCD} + \hat{\Gamma}_{ABCD}\,, \tag{4.11}$$

with singular part

$$\Gamma^*_{AB\,CD} = -\frac{1}{u}\,\epsilon_{(A}{}^0\,\epsilon_{B)}{}^1\,\epsilon_C{}^0\,\epsilon_D{}^0\,, \tag{4.12}$$

and holomorphic functions $\hat{\Gamma}_{ABCD}$ on $\hat{S}$ which satisfy

$$\hat{\Gamma}_{ABCD} = O(u) \quad \text{as} \quad u \to 0\,. \tag{4.13}$$

The singular parts are 'universal' in the sense that their expressions only depend on the construction of $\hat{S}$ and not on properties of the metric. If the latter is flat the functions $\hat{e}^a{}_{AB}$ and $\hat{\Gamma}_{ABCD}$ vanish on $\hat{S}$. With the frame and the connection coefficients so defined we have the spin frame calculus in its standard form. The expressions above imply for any holomorphic spinor valued function $\psi_{A\ldots C}$ that $D_{00}\,\psi_{A\ldots C}$ and $D_{11}\,\psi_{A\ldots C}$ extend to $\hat{S}$ as holomorphic functions so that

$$D_{00}\,\psi_{A\ldots C} = \partial_u\,\psi_{A\ldots C} \;\text{ on }\; \hat{S} \quad \text{and} \quad D_{11}\,\psi_{A\ldots C} = \partial_w\,\psi_{A\ldots C} \;\text{ on }\; U_0\,.$$

4.3. Tensoriality and expansion type

A holomorphic function on $SL(S_c)$ induces a holomorphic function on $\hat{S}$ which can be considered as a holomorphic function of the coordinates z^a. While these coordinates are holomorphic on the submanifold $\hat{S}$ of $SL(S_c)$, the induced map π of $\hat{S}$ into S_c is singular on U_0. As a consequence, not every holomorphic function of the z^a can arise as a pull-back to $\hat{S}$ of a holomorphic function on $SL(S_c)$. The latter must have a special type of expansion in terms of the z^a which reflects the particular relation between the 'angular' coordinate v the 'radial' coordinate u. The following notion will be important for our discussion.

Definition. A holomorphic function f on $\hat{S}$ will be said to be of v-finite expansion type k_f, with k_f an integer, if it has in terms of the coordinates u, v, and w a Taylor expansion at the origin of the form

$$f = \sum_{p=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{2\,m+k_f} f_{m,n,p}\, u^m\, v^n\, w^p\,,$$

where it is assumed that $f_{m,n,p} = 0$ if $2\,m + k_f < 0$.

We note that the construction of $\hat{S}$ does not distinguish the set $I = \pi^{-1}(i)$ from the sets $\pi^{-1}(\gamma(w))$. Correspondingly, the Taylor expansions of the function f above at points $(0, 0, w_0)$ with w_0 close to 0 have the same structure with respect to u and v.

Lemma 4.1. *Let $\phi_{A_1 \ldots A_j}$ be a holomorphic, symmetric, spinor-valued function on $SL(S_c)$. Then the restrictions of its essential components $\phi_k = \phi_{(A_1 \ldots A_j)_k}$, $0 \leq k \leq j$, to $\hat{S}$ satisfy*

$$\partial_v\, \phi_k = (j-k)\, \phi_{k+1}\,, \quad k = 0, \ldots, j\,, \quad on \quad U_0\,, \tag{4.14}$$

(*where we set $\phi_{j+1} = 0$*) *and ϕ_k is of expansion type $j - k$.*

Proof. In the following we consider $\hat{S}$ as a submanifold of $SL(S_c)$. The tensorial transformation law of ϕ under the action of the 1-parameter subgroup (4.5) with generator $\alpha^A{}_B = \epsilon_1{}^A\, \epsilon_B{}^0$ implies

$$Z_\alpha\, \phi_k = (j-k)\, \phi_{k+1} \quad \text{for} \quad 0 \leq k \leq j \quad \text{on} \quad SL(S_c)\,,$$

and thus (4.14) because $Z_\alpha = \partial_v$ on U_0. From the relations above follows in particular that

$$Z_\alpha^{j-k+1}\, \phi_k = 0 \quad \text{on} \quad SL(S_c)\,. \tag{4.15}$$

A general horizontal vector field H_x has with Z_α the commutator

$$[Z_\alpha, H_x] = H_{\alpha \cdot x}\,,$$

where α acts on $x^{AB} = x^{(AB)}$ according to the induced action by

$$x^{AB} \rightarrow (\alpha \cdot x)^{AB} = \alpha^A{}_C\, x^{CB} + \alpha^B{}_C\, x^{AC} = 2\, \epsilon_1^{(A}\, x^{B)0}\,.$$

With $x^{AB} = \epsilon_0{}^A \epsilon_0{}^B$, so that $H_x = H_{00}$, it follows

$$[Z_\alpha, H_{00}] = 2\,H_{01}\,, \quad [Z_\alpha, H_{01}] = H_{11}\,, \quad [Z_\alpha, H_{11}] = 0\,.$$

By induction this gives the operator equations

$$Z_\alpha^n\,H_{00} = n\,(n-1)\,H_{11}\,Z_\alpha^{n-2} + 2\,n\,H_{01}\,Z_\alpha^{n-1} + H_{00}\,Z_\alpha^n\,, \quad n \geq 1\,,$$

and, more generally,

$$Z_\alpha^n\,H_{00}^m = a_{n,m}\,H_{11}^m\,Z_\alpha^{n-2m} + \sum_{l=0}^{2\,m-1} A_{n,m,l}\,Z_\alpha^{n-l} + H_{00}^m\,Z_\alpha^n\,, \quad m,n \geq 1\,,$$

where the $a_{n,m}$ are real coefficients, the $A_{n,m,l}$ denote operators which are sums of products of horizontal vector fields, and the terms in which Z_α formally appears with negative exponent are assumed to vanish. With (4.15) this implies

$$Z_\alpha^n\,H_{00}^m\,\phi_k = 0 \quad \text{for} \quad n > 2\,m + j - k \quad \text{on} \quad SL(S_c)\,.$$

The results follows because $Z_\alpha^n\,H_{00}^m\,\phi_k = \partial_v^n\,\partial_u^m\,\phi_k$ at points of U_0. □

4.4. The null data on W_0

We shall derive an expansion of the restriction of the essential component s_0 of the Ricci spinor to the hypersurface W_0, i.e.,

$$s_0(u,v) = s_{(ABCD)_0}|_{W_0}\,,$$

in terms of quantities on the base space S_c. Consider the normal frame c_{AB} on S_c near i which agrees at i with the frame associated with δ^* and denote by

$$\mathcal{D}_n^* \equiv \{D^*_{(A_1B_1} \ldots D^*_{A_pB_p} s^*_{ABCD)}(i)\,, \quad p = 0,1,2,\ldots\}\,,$$

the corresponding null data of h in the frame c_{AB}. Choose a fixed value of v and consider $s = s(v)$ as in (4.5). The vector $H_{00}(\delta^* \cdot s)$ then projects onto the null vector $s^A{}_0\,s^B{}_0\,c_{AB}$ at i. Since c_{AB} is a normal frame near i, the null vector field $s^A{}_0\,s^B{}_0\,c_{AB}$ is tangent to a null geodesic $\eta = \eta(u,v)$ on $\mathcal{N}_i$ with affine parameter u with $u = 0$ at i and the integral curve of H_{00} through $\delta^* \cdot s$ projects onto this null geodesic. It follows from this with the explicit expression for $s = s(v)$ that

$$\begin{aligned} s_0(u,v) &= s^A{}_0(v)\,s^B{}_0(v)\,s^C{}_0(v)\,s^D{}_0(v)\,s^*_{ABCD}|_{\eta(u,v)} \\ &= \sum_{m=0}^{\infty} \frac{1}{m\,!}\,u^m\,s^{A_1}{}_0(v)\,s^{B_1}{}_0(v)\ldots\,s^D{}_0(v)\,D^*_{(A_1B_1}\ldots D^*_{A_mB_m}\,s^*_{ABCD)}(i) \\ &= \sum_{m=0}^{\infty}\sum_{n=0}^{2\,m+4} \psi_{m,n}\,u^m\,v^n\,, \end{aligned} \tag{4.16}$$

with

$$\psi_{m,n} = \frac{1}{m\,!}\binom{2\,m+4}{n} D^*_{(A_1B_1}\ldots D^*_{A_mB_m}\,s^*_{ABCD)_n}(i)\,, \quad 0 \leq n \leq 2\,m+4\,.$$

This formula shows how to determine the function $s_0(u,v)$ from the null data $\mathcal{D}_n^*$ and vice versa. We note that the expansion above is consistent with s_0 being of

v-finite expansion type 4. We shall refer to (4.16) as the *null data on* W_0 in our gauge.

5. The conformal static vacuum field equations on $\hat{S}$

With the frame e_{AB} and the connection coefficients Γ_{ABCD} on $\hat{S}$ we have the standard frame calculus available. Given the fields ζ, s, s_{ABCD}, we define on $\hat{S}$ the quantities

$$t_{AB}{}^{EF}{}_{CD}\, e^a{}_{EF} \equiv 2\,\Gamma_{AB}{}^E{}_{(C}\, e^a{}_{D)E} - 2\,\Gamma_{CD}{}^E{}_{(A}\, e^a{}_{B)E} - e^a{}_{CD,b}\, e^b{}_{AB} + e^a{}_{AB,b}\, e^b{}_{CD}\,,$$

$$R_{ABCDEF} \equiv r_{ABCDEF} - \frac{1}{2}\,\{s_{ABCE}\,\epsilon_{DF} + s_{ABDF}\,\epsilon_{CE}\}\,,$$

with

$$\begin{aligned} r_{ABCDEF} \equiv{}& e_{CD}(\Gamma_{EFAB}) - e_{EF}(\Gamma_{CDAB}) \\ &+ \Gamma_{EF}{}^K{}_C\,\Gamma_{KDAB} + \Gamma_{EF}{}^K{}_D\,\Gamma_{CKAB} - \Gamma_{CD}{}^K{}_E\,\Gamma_{KFAB} \\ &- \Gamma_{CD}{}^K{}_F\,\Gamma_{EKAB} + \Gamma_{EF}{}^K{}_B\,\Gamma_{CDAK} - \Gamma_{CD}{}^K{}_B\,\Gamma_{EFAK} \\ &- t_{CD}{}^{GH}{}_{EF}\,\Gamma_{GHAB}\,, \end{aligned}$$

%endalign*$\Sigma_{AB} \equiv D_{AB}\,\zeta - \zeta_{AB}\,,$

$$\begin{aligned} \Sigma_{ABCD} &\equiv D_{AB}\,\zeta_{CD} - s\,h_{ABCD} + \zeta\,(1-\mu\,\zeta)\,s_{ABCD}\,, \\ S_{AB} &\equiv D_{AB}\,s + (1-\mu\,\zeta)\,s_{ABCD}\,\zeta^{CD}\,, \\ H_{ABCD} &\equiv D_A{}^E s_{BCDE} - \frac{2\,\mu}{1-\mu\,\zeta}\,s_{E(BCD}\,\zeta_{A)}{}^E\,. \end{aligned}$$

In terms of the tensor fields on the left hand side, which have been introduced as labels for the equations as well as for discussing the interdependencies of the equations, the conformal static vacuum equations read

$$t_{AB}{}^{EF}{}_{CD}\, e^a{}_{EF} = 0\,, \qquad R_{ABCDEF} = 0\,, \qquad \Sigma_{AB} = 0\,,$$
$$\Sigma_{ABCD} = 0\,, \qquad S_{AB} = 0\,, \qquad H_{ABCD} = 0\,.$$

The first equation is Cartan's first structural equation with the requirement that the (metric) connection be torsion free ($t_{AB}{}^{EF}{}_{CD}$ being the torsion tensor). The second equation is Cartan's second structural equation with the requirement that the Ricci tensor coincides with the trace free tensor s_{ab}. The third equation defines ζ_{AB}, the remaining equations have been considered before.

To discuss these equations in detail we need to write them out in our gauge, observing in particular the nature of the singularities in (4.8) and (4.11).

The equations $t_{AB}{}^{EF}{}_{00}\, e^a{}_{EF} = 0$:

$$\partial_u \hat{e}^1{}_{01} + \frac{1}{u}\,\hat{e}^1{}_{01} = -2\,\hat{\Gamma}_{0101} + 2\,\hat{\Gamma}_{0100}\,\hat{e}^1{}_{01}\,,$$
$$\partial_u \hat{e}^2{}_{01} + \frac{1}{u}\,\hat{e}^2{}_{01} = \frac{1}{u}\,\hat{\Gamma}_{0100} + 2\,\hat{\Gamma}_{0100}\,\hat{e}^2{}_{01}\,,$$
$$\partial_u \hat{e}^1{}_{11} = -2\,\hat{\Gamma}_{1101} + 2\,\hat{\Gamma}_{1100}\,\hat{e}^1{}_{01}\,,$$
$$\partial_u \hat{e}^2{}_{11} = \frac{1}{u}\hat{\Gamma}_{1100} + 2\,\hat{\Gamma}_{1100}\,\hat{e}^2{}_{01}\,.$$

The equations $R_{AB00EF} = 0$:

$$\partial_u \hat{\Gamma}_{0100} + \frac{2}{u}\,\hat{\Gamma}_{0100} - 2\,\hat{\Gamma}^2_{0100} = \frac{1}{2}\,s_0\,,$$
$$\partial_u \hat{\Gamma}_{0101} + \frac{1}{u}\,\hat{\Gamma}_{0101} - 2\,\hat{\Gamma}_{0100}\,\hat{\Gamma}_{0101} = \frac{1}{2}\,s_1\,,$$
$$\partial_u \hat{\Gamma}_{0111} + \frac{1}{u}\,\hat{\Gamma}_{0111} - 2\,\hat{\Gamma}_{0100}\,\hat{\Gamma}_{0111} = \frac{1}{2}\,s_2\,,$$
$$\partial_u \hat{\Gamma}_{1100} + \frac{1}{u}\,\hat{\Gamma}_{1100} - 2\,\hat{\Gamma}_{1100}\,\hat{\Gamma}_{0100} = s_1\,,$$
$$\partial_u \hat{\Gamma}_{1101} - 2\,\hat{\Gamma}_{1100}\,\hat{\Gamma}_{0101} = s_2\,,$$
$$\partial_u \hat{\Gamma}_{1111} - 2\,\hat{\Gamma}_{1100}\,\hat{\Gamma}_{0111} = s_3\,.$$

The equations $\Sigma_{00} = 0$, $\Sigma_{00CD} = 0$, $S_{00} = 0$:

$$0 = \partial_u \zeta - \zeta_{00}\,,$$
$$0 = \partial_u\,\zeta_{00} + \zeta\,(1 - \mu\,\zeta)\,s_0\,,$$
$$0 = \partial_u\,\zeta_{01} + \zeta\,(1 - \mu\,\zeta)\,s_1\,,$$
$$0 = \partial_u\,\zeta_{11} - s + \zeta\,(1 - \mu\,\zeta)\,s_2\,,$$
$$0 = \partial_u s + (1 - \mu\,\zeta)\,(s_0\,\zeta_{11} - 2\,s_1\,\zeta_{01} + s_2\,\zeta_{00})\,.$$

The equations $-H_{0(BCD)_k} = 0$ in the order $k = 0, 1, 2, 3$:

$$\partial_u\,s_1 - \frac{1}{2\,u}(\partial_v\,s_0 - 4\,s_1) - \hat{e}^1{}_{01}\partial_u\,s_0 - \hat{e}^2{}_{01}\partial_v\,s_0$$
$$= -4\,\hat{\Gamma}_{0101}\,s_0 + 4\,\hat{\Gamma}_{0100}\,s_1 - \frac{2\,\mu}{(1 - \mu\,\zeta)}\,\{s_0\,\zeta_{01} - s_1\,\zeta_{00}\}\,,$$
$$\partial_u\,s_2 - \frac{1}{2\,u}(\partial_v\,s_1 - 3\,s_2) - \hat{e}^1{}_{01}\partial_u\,s_1 - \hat{e}^2{}_{01}\partial_v\,s_1$$
$$= -\hat{\Gamma}_{0111}\,s_0 - 2\,\hat{\Gamma}_{0101}\,s_1 + 3\,\hat{\Gamma}_{0100}\,s_2 - \frac{\mu}{2\,(1 - \mu\,\zeta)}\,\{s_0\,\zeta_{11} + 2\,s_1\,\zeta_{01} + 3\,s_2\,\zeta_{00}\}\,,$$
$$\partial_u\,s_3 - \frac{1}{2\,u}(\partial_v\,s_2 - 2\,s_3) - \hat{e}^1{}_{01}\partial_u\,s_2 - \hat{e}^2{}_{01}\partial_v\,s_2$$
$$= -2\hat{\Gamma}_{0111}\,s_1 + 2\,\hat{\Gamma}_{0100}\,s_3 - \frac{\mu}{(1 - \mu\,\zeta)}\,\{s_1\,\zeta_{11} + s_3\,\zeta_{00}\}\,,$$

$$\partial_u\, s_4 - \frac{1}{2\,u}(\partial_v\, s_3 - s_4) - \hat{e}^1{}_{01}\partial_u\, s_3 - \hat{e}^2{}_{01}\partial_v\, s_3$$
$$= -3\,\hat{\Gamma}_{0111}\, s_2 + 2\,\hat{\Gamma}_{0101}\, s_3 + \hat{\Gamma}_{0100}\, s_4 - \frac{\mu}{2\,(1-\mu\,\zeta)}\,\{3\, s_2\,\zeta_{11} - 2\, s_3\,\zeta_{01} - s_4\,\zeta_{00}\}\,.$$

These equations, referred to as the ∂_u-*equations*, will be read as a system of PDE's for the set of functions

$$\hat{e}^1{}_{01}\,,\;\; \hat{e}^2{}_{01}\,,\;\; \hat{e}^1{}_{11}\,,\;\; \hat{e}^2{}_{11}\,,\;\; \hat{\Gamma}_{01AB}\,,\;\; \hat{\Gamma}_{11AB}\,,\;\; \zeta\,,\;\; \zeta_{AB}\,,\;\; s\,,\;\; s_1\,,\;\; s_2\,,\;\; s_3\,,\;\; s_4\,,$$

which comprises all the unknowns with the exception of s_0. The following features of them will be important.

All ∂_u-equations are *interior equations on the hypersurfaces* $\{w = w_0\}$ in the sense that only derivatives in the directions of u and v are involved.

The equations are singular with terms u^{-1} occurring in various places. It will be seen later that these terms come with the 'right' signs to possess (unique) solutions which are holomorphic in u, v and w. Remarkably, the equations for the s_k ensure regular solution to have the correct tensorial behaviour by the occurrence of terms u^{-1} with factors $\partial_v\, s_k - (4-k)\, s_{k+1}$. By Lemma 4.1 we know that they have to vanish U_0.

The system splits into a hierarchy of subsystems, with

$$t_{01}{}^{EF}{}_{00}\, e^2{}_{EF} = 0\,, \quad R_{000001} = 0\,,$$

being the first subsystem,

$$t_{01}{}^{EF}{}_{00}\, e^1{}_{EF} = 0\,, \qquad R_{010001} = 0\,, \qquad \Sigma_{00} = 0\,,$$
$$\Sigma_{0000} = 0\,, \qquad \Sigma_{0001} = 0\,, \qquad H_{0000} = 0\,,$$

being the second subsystem, and so on. The hierarchy has the following property. If s_0 is given on $\{w = w_0\}$, the first subsystem reduces to singular system of ODE's. Given its solution, the second subsystem also reduces to a system of ODE's (with coefficients which are calculated from the functions known so far by operation interior to $\{w = w_0\}$), and so on. Thus, given s_0 and the appropriate initial data on $U_0 \cap \{w = w_0\}$, all unknowns can be determined on $\{w = w_0\}$ by solving a sequence of systems of ODE's in the independent variable u.

The functions $\hat{e}^a{}_{AB}$ and $\hat{\Gamma}_{ABCD}$ vanish on U_0 by our gauge conditions. Therefore only initial data for ζ, ζ_{AB}, s, and s_k need to be determined on U_0 and the function s_0 needs to be provided on $\{w = w_0\}$. While s_0 will be prescribed on W_0 as our initial datum, an equation is needed to determine its evolution off W_0. For this purpose we will consider the following equations.

The equations $H_{1(BCD)_k} = 0$ in the order $k = 0, 1, 2, 3$:

$$\partial_w\, s_0 - \frac{1}{2\,u}(\partial_v\, s_1 - 3\, s_2) + \hat{e}^1{}_{11}\partial_u\, s_0 + \hat{e}^2{}_{11}\partial_v\, s_0 - \hat{e}^1{}_{01}\partial_u\, s_1 - \hat{e}^2{}_{01}\partial_v\, s_1$$
$$= -(\hat{\Gamma}_{0111} - 4\,\hat{\Gamma}_{1101})\, s_0 - (2\,\hat{\Gamma}_{0101} + 4\,\hat{\Gamma}_{1100})\, s_1 + 3\,\hat{\Gamma}_{0100}\, s_2$$
$$+ \frac{2\,\mu}{(1-\mu\,\zeta)}\,\frac{1}{4}\,\{s_0\,\zeta_{11} + 2\, s_1\,\zeta_{01} - 3\, s_2\,\zeta_{00}\}\,, \tag{5.1}$$

$$\partial_w s_1 - \frac{1}{2u}(\partial_v s_2 - 2 s_3) + \hat{e}^1{}_{11}\partial_u s_1 + \hat{e}^2{}_{11}\partial_v s_1 - \hat{e}^1{}_{01}\partial_u s_2 - \hat{e}^2{}_{01}\partial_v s_2$$
$$= \hat{\Gamma}_{1111} s_0 - (2\hat{\Gamma}_{0111} - 2\hat{\Gamma}_{1101}) s_1 - 3\hat{\Gamma}_{1100} s_2 + 2\hat{\Gamma}_{0100} s_3$$
$$+ \frac{2\mu}{(1-\mu\zeta)}\frac{1}{2}\{s_1\zeta_{11} - s_3\zeta_{00}\},$$
$$\partial_w s_2 - \frac{1}{2u}(\partial_v s_3 - 2 s_4) + \hat{e}^1{}_{11}\partial_u s_2 + \hat{e}^2{}_{11}\partial_v s_2 - \hat{e}^1{}_{01}\partial_u s_3 - \hat{e}^2{}_{01}\partial_v s_3$$
$$= 2\hat{\Gamma}_{1111} s_1 - 3\hat{\Gamma}_{0111} s_2 - (2\hat{\Gamma}_{1100} - 2\hat{\Gamma}_{0101}) s_3 + \hat{\Gamma}_{0100} s_4$$
$$+ \frac{2\mu}{(1-\mu\zeta)}\frac{1}{4}\{3 s_2\zeta_{11} - 2 s_3\zeta_{01} - s_4\zeta_{00}\},$$
$$\partial_w s_3 - \frac{1}{2u}\partial_v s_4 + \hat{e}^1{}_{11}\partial_u s_3 + \hat{e}^2{}_{11}\partial_v s_3 - \hat{e}^1{}_{01}\partial_u s_4 - \hat{e}^2{}_{01}\partial_v s_4$$
$$= 3\hat{\Gamma}_{1111} s_2 - (4\hat{\Gamma}_{0111} + 2\hat{\Gamma}_{1101}) s_3 - (\hat{\Gamma}_{1100} - 4\hat{\Gamma}_{0101}) s_4$$
$$+ \frac{2\mu}{(1-\mu\zeta)}\{s_3\zeta_{11} - s_4\zeta_{01}\}.$$

All singular terms cancel in the equations $0 = H_{0(BCD)_{k+1}} + H_{1(BCD)_k}$, which are given in the order $k = 0, 1, 2$ by

$$\partial_w s_0 - \partial_u s_2 + \hat{e}^1{}_{11}\partial_u s_0 + \hat{e}^2{}_{11}\partial_v s_0 \tag{5.2}$$
$$= 4\hat{\Gamma}_{1101} s_0 - 4\hat{\Gamma}_{1100} s_1 + \frac{\mu}{(1-\mu\zeta)}\{s_0\zeta_{11} + 2 s_1\zeta_{01} - 3 s_2\zeta_{00}\},$$
$$\partial_w s_1 - \partial_u s_3 + \hat{e}^1{}_{11}\partial_u s_1 + \hat{e}^2{}_{11}\partial_v s_1$$
$$= \hat{\Gamma}_{1111} s_0 + 2\hat{\Gamma}_{1101} s_1 - 3\hat{\Gamma}_{1100} s_2 - \frac{2\mu}{(1-\mu\zeta)}\{s_1\zeta_{11} - s_3\zeta_{00}\},$$
$$\partial_w s_2 - \partial_u s_4 + \hat{e}^1{}_{11}\partial_u s_2 + \hat{e}^2{}_{11}\partial_v s_2$$
$$= 2\hat{\Gamma}_{1111} s_1 - 2\hat{\Gamma}_{1100} s_3 + \frac{\mu}{(1-\mu\zeta)}\{3 s_2\zeta_{11} - 2 s_3\zeta_{01} - s_4\zeta_{00}\}.$$

We can consider (5.1) or (5.2) as equation prescribing the propagation of s_0 transverse to the hypersurfaces $\{w = const.\}$.

Because $\Gamma_{11CD} = 0$ on U_0, the equations $\Sigma_{11} = 0$, $\Sigma_{11CD} = 0$, $S_{11} = 0$ reduce on U_0 to the ODE's

$$\partial_w \zeta = \zeta_{11}, \quad \partial_w \zeta_{CD} = s\, h_{11CD} - \zeta(1-\mu\zeta)\, s_{11CD}, \quad \partial_w s = -(1-\mu\zeta)\, s_{11CD}\,\zeta^{CD}.$$

By (2.2), (2.15) we must impose

$$\zeta = 0, \quad \zeta_{AB} = 0, \quad s(i) = -2 \quad \text{on} \quad I = \{u = 0, w = 0\}.$$

This implies with the equations above

$$\zeta = 0, \quad \zeta_{01} = 0, \quad \zeta_{11} = 0 \quad \text{on} \quad U_0 = \{u = 0\}. \tag{5.3}$$

To determine ζ, ζ_{AB}, and s on U_0 it remains to solve on U_0 the equations

$$\partial_w \zeta_{00} = s, \quad \partial_w s = -s_4\,\zeta_{00}. \tag{5.4}$$

The tensorial properties of ζ_{AB} and s imply with (5.3) that

$$\partial_v^n \zeta_{00} = 0\,, \quad \partial_v^n \hat{s} = 0 \quad \text{on} \quad U_0 \quad \text{for} \quad n \geq 1\,. \tag{5.5}$$

Later it will be important that these relations can in fact be deduced from (5.3), (5.4), (5.6), and the initial conditions on I.

To ensure the tensor relations for the s_k and thus the existence of regular solutions to the equation for the s_k, we determine the initial data for $s_1, \ldots, s_4$ on U_0 by imposing the conditions

$$\partial_v s_k = (4-k)\, s_{k+1}\,, \quad k = 0\,, \ldots, 3\,, \quad \text{on} \quad U_0\,. \tag{5.6}$$

They imply recursively the expressions

$$\partial_v^n \partial_w^p s_k = \frac{(4-k)!}{4!} \partial_v^{k+n} \partial_w^p s_0\,, \qquad k = 0\,, \ldots 4\,, \quad n,\, p \geq 0 \quad \text{at} \quad \{u=0, v=0, w=0\}\,.$$

5.1. Calculating the formal expansion

The system of equations is overdetermined. We choose from it a subset of equations to define a systematic way of calculating a formal expansion of the solution. It will then follow from Lemma 5.5 that the expansion obtained by this procedure will lead to a formal solution of the full system of equations. A solution obtained by any other procedure will thus have to coincide with the present one.

It will be convenient to replace s by the unknown

$$\hat{s} = 2 + s\,,$$

and it will also be useful to write

$$s_k = s_k^* + \hat{s}_k \quad \text{with} \quad \partial_u s_k^* = 0 \quad \text{and} \quad s_k^*|_{u=0} = s_k|_{u=0} \qquad \text{so that} \quad \hat{s}_k = O(u) \quad \text{as} \quad u \to 0\,.$$

By (5.6) we can then assume that

$$\partial_v s_k^* = (4-k)\, s_{k+1}^*\,,$$

and the ∂_u-equations for the $\hat{s}_k$ can be written in the form

$$0 = -H_{0\,(BCD)_k} = \partial_u \hat{s}_{k+1} + \frac{4-k}{2\,u}\, \hat{s}_{k+1} - \frac{1}{2\,u}\, \partial_v \hat{s}_k + \hat{e}^a{}_{01}\, \partial_a(s_k^* + \hat{s}_k) + \textit{terms of zeroth order}\,,$$

so that the coefficient $(4-k)/2$ of the singular term $u^{-1}\, \hat{s}_{k+1}$ is positive and the term $u^{-1}\, \partial_v \hat{s}_k$, which involves the unknown $\hat{s}_k$ determined in an earlier step of the integration procedure, creates no problem because $\hat{s}_k = 0$ on U_0. Writing

$$x = \left(\hat{e}^a{}_{AB},\, \hat{\Gamma}_{ABCD},\, \zeta,\, \zeta_{AB},\, \hat{s},\, s_1,\, s_2,\, s_3,\, s_4\right),$$

so that the full set of unknowns are given by x and s_0, we proceed as follows.

On W_0 we prescribe s_0 as given in (4.16) with the null data $\mathcal{D}_n^*$ satisfying the reality conditions and the estimates (3.11). By (5.6) all components of x can be determined on I.

We successively integrate the subsystems in the hierarchy of ∂_u-equations to determine all components of x on W_0. These will be holomorphic in u and v and unique, because the relevant operators in the singular equations are of the form $\partial_u f + c\,u^{-1} f$ with non-negative constants c (a proof of this statement follows from the derivation of the estimates discussed below).

The equation $H_{0100} + H_{1000} = 0$ is used to determine $\partial_w s_0$ from the fields x and s_0 on W_0 as a holomorphic function of u and v.

Applying the operator ∂_w formally to the ∂_u-equations, one obtains equations for $\partial_w x$ on W_0 which can be solved with the initial data on $\{w = 0, u = 0\}$ which are obtained by using (5.4) and by applying ∂_w to (5.6). Applying ∂_w to the equation $H_{0100} + H_{1000} = 0$, one obtains $\partial^2 s_0$ on W_0.

Repeating these steps by applying successively the operator ∂_w^p, $p = 2, 3, \ldots$, one gets an sequence of functions $\partial_w^p x$, $\partial_w^p s_0$ on W_0, which are holomorphic in u and v.

Expanding the functions so obtained at $u = 0$, $v = 0$ we get the following result.

Lemma 5.1. *The procedure described above determines at the point $O = (u = 0, v = 0, w = 0)$ from the data s_0, given on W_0 according to* (4.16)*, a unique sequence of expansion coefficients*

$$\partial_u^m \, \partial_v^n \, \partial_w^p \, f(O)\,, \quad m, n, p = 0, 1, 2, \ldots\,,$$

where f stands for any of the functions

$$\hat{e}^a{}_{AB}\,,\ \hat{\Gamma}_{ABCD}\,,\ \zeta\,,\ \zeta_{AB}\,,\ \hat{s}\,,\ s_j\,.$$

If the corresponding Taylor series are absolutely convergent in some neighbourhood P of O, they define a solution to the ∂_u-equations and to the equation $H_{1000} = 0$ on P which satisfies on $P \cap U_0$ equations (5.6) *and* $\Sigma_{11} = 0$, $\Sigma_{11CD} = 0$, $S_{11} = 0$.

By Lemma 4.1 all spinor-valued functions should have a specific v-finite expansion type. The following result will be important for our convergence proof.

Lemma 5.2. *If the data s_0 are given on W_0 as in* (4.16)*, the formal expansions of the fields obtained in Lemma* 5.1 *correspond to ones of functions of v-finite expansion types given by*

$$k_{s_j} = 4 - j\,, \qquad k_{\zeta_i} = 2 - i\,, \qquad k_\zeta = 0\,, \qquad k_s = k_{\hat{s}} \le 2\,,$$
$$k_{\hat{e}^1_{AB}} = -A - B\,, \qquad k_{\hat{e}^2_{AB}} = 3 - A - B \quad \textit{for} \quad AB = 01,\, 10 \quad \textit{or} \quad 11\,.$$
$$k_{\hat{\Gamma}_{01AB}} = 2 - A - B\,, \quad k_{\hat{\Gamma}_{11AB}} = 1 - A - B \quad \textit{for} \quad A, B = 0 \quad \textit{or} \quad 1\,.$$

Remark 5.3. The scalar functions s, $\hat{s}$ must have expansion type $k_s = k_{\hat{s}} = 0$. As pointed out below, this does not follow with the simple arguments used here. Since it will not be important for the following discussions, we shall make no effort to retrieve this information from the equations.

Proof. We note the following properties of v-finite expansion types:

For given integer k the functions of expansion type k form a complex vector space which comprises the functions of expansion type $\leq k$.

If the functions f and g have expansion type k_f and k_g respectively, their product $f\,g$ has expansion type $k_{fg} = k_f + k_g$.

If f has expansion type k_f, the function $\partial_u f$ has expansion type $k_f + 2$. Conversely, if $\partial_u f$ has expansion type $k_f + 2$ and if the function independent of u which agrees on U_0 with f has expansion type k_f (for instance if $f|_{u=0} = 0$), then f has expansion type k_f.

If f has expansion type k_f and $f|_{u=0} = 0$ then $\frac{1}{u}\,f$ has expansion type $k_f + 2$.

If f has expansion type k_f, the function $\partial_v f$ has expansion type $k_f - 1$.

If f has an expansion type, the function $\partial_w f$ has the same expansion type.

Applying these rules one can check that the expansion types listed above are consistent with the ∂_u-equations, the equation $H_{1000} + H_{0100} = 0$ and the equations $S_{11} = 0$, $\Sigma_{1100} = 0$ used on U_0 in the sense that all terms in the equations have the same expansion types.

Assuming the given expansion types for the s_k, the ∂_u-equations for the $\hat{\Gamma}_{ABCD}$ imply at lowest order in u that in general the $k_{\hat{\Gamma}_{ABCD}}$ must take the values given above. It follows then from the ∂_u-equations for the $\hat{e}^a_{AB}$ at lowest order in u that the $k_{\hat{e}^a_{AB}}$ must take in general the values above. The remaining ∂_u-equations then imply at lowest order the other expansion types.

With these observations the Lemma follows from our procedure by a straightforward though lengthy induction argument. We do not write out the details. □

The equation

$$0 = S_{00} = \partial_u\, s + (1 - \mu\,\zeta)\, s_{00CD}\, \zeta^{CD}\,,$$

should imply more precisely $k_s = 0$, because the expansion type of the tensorial component $s_{00CD}\,\zeta^{CD}$ should be 2. The contraction of the spinor fields on the right hand side implies cancellations which lower the expansion types of the contracted quantities on the right hand side. These cancellations cannot be controlled in the explicit expression

$$0 = \partial_u s + (1 - \mu\,\zeta)\,(s_0\,\zeta_{11} - 2\,s_1\,\zeta_{01} + s_2\,\zeta_{00})\,,$$

by the simple rules given above, they only suggest an expansion type $k_s \leq 2$. Fortunately, this does not prevent us from determining the other expansion types. In the equation

$$0 = \Sigma_{0011} = \partial_u\,\zeta_{11} - s + \zeta\,(1 - \mu\,\zeta)\, s_{0011}\,,$$

s is added to a field of expansion type 2 and the equation

$$0 = S_{11} = \partial_w\, s + s_{11CD}\,\zeta^{CD} = \partial_w\, s + s_{1111}\,\zeta_{00} \quad \text{on} \quad U_0\,,$$

is consistent with $k_s \leq 2$. No further equation involving s is needed in the convergence proof.

5.2. The complete set of equations on $\hat{S}$

Because only a certain subset of the system of equations has been used to determine the formal expansions of the fields, it remains to be shown that the latter define in fact a formal solution to the complete system of conformal static vacuum field equations. To simplify stating the following result it will be assumed in this subsection that the formal expansions for

$$\hat{e}^a{}_{AB}\,,\ \hat{\Gamma}_{ABCD}\,,\ \zeta\,,\ \zeta_{AB}\,,\ \hat{s}\,,\ s_j\,,$$

determined in Lemma 5.1 define in fact absolutely convergent series on an open neighbourhood of the point O, which we assume to coincide with $\hat{S}$. There will arise no problem from this assumption because the following two lemmas will not be used in the derivation of the estimates in the next section.

Lemma 5.4. *With the assumptions above the corresponding functions*

$$e^a{}_{AB}\,,\ \Gamma_{ABCD}\,,\ \zeta\,,\ \zeta_{AB}\,,\ s\,,\ s_j\,,$$

satisfy the complete set of the conformal vacuum field equations on the set U_0 in the sense that the fields

$$t_{AB}{}^{EF}{}_{CD}\,,\ \ R_{ABCDEF}\,,\ \ \Sigma_{AB}\,,\ \ \Sigma_{ABCD}\,,\ \ S_{AB}\,,\ \ H_{ABCD}\,,$$

calculated from these functions on $\hat{S}\setminus U_0$ have vanishing limit as $u\to 0$.

Proof. Because of the equations solved already and the symmetries involved, we only need to examine the behaviour of the fields

$$t_{11}{}^{EF}{}_{01}\,,\ \ R_{AB0111}\,,\ \ \Sigma_{01}\,,\ \ \Sigma_{01CD}\,,\ \ S_{01}\,,\ \ H_{1(BCD)_k}\,,\ \ k=1,2,3\,,$$

near U_0.

With (4.8), (4.9), (4.11), (4.12) the ∂_u-equations imply for the frame and the dual frame coefficients the slightly stronger results (4.6), (4.7). A direct calculation gives then

$$\begin{aligned} t_{01}{}^{EF}{}_{11} = {} & 2\,\Gamma_{01}{}^{(E}{}_{1}\,\epsilon_1{}^{F)} - \Gamma_{11}{}^{(E}{}_{0}\,\epsilon_1{}^{F)} - \Gamma_{11}{}^{(E}{}_{1}\,\epsilon_0{}^{F)} \\ & - \sigma^{EF}{}_a\,(e^a{}_{11,c}\,e^c{}_{01} - e^a{}_{01,c}\,e^c{}_{11}) = O(u)\,, \end{aligned}$$

as $u\to 0$.

With the particular result

$$t_{01}{}^{01}{}_{11} = \Gamma_{0111} - \frac{1}{2}\,e^2{}_{11,2} - \frac{1}{2\,u}\,e^1{}_{11} + O(u^2) = O(u)\,,$$

follows

$$\begin{aligned} R_{000111} = {} & \Gamma_{1100,1}\,e^1{}_{01} + \Gamma_{1100,2}\,e^2{}_{01} - \Gamma_{0100,1}\,e^1{}_{11} - \Gamma_{0100,2}\,e^2{}_{11} - \Gamma_{0100,3} \\ & - \Gamma_{1100}\,\Gamma_{1100} + 2\,\Gamma_{0100}\,(\Gamma_{1101} - \Gamma_{0111}) - t_{01}{}^{01}{}_{11}\,\Gamma_{0100} \\ & - t_{01}{}^{11}{}_{11}\,\Gamma_{1100} - \frac{1}{2}\,s_{0011} \\ = {} & \frac{1}{2\,u}\left(\Gamma_{1100,2} - 2\,\Gamma_{1101} + 3\,\Gamma_{0111} - \frac{1}{2}\,e^2{}_{11,2} - \frac{3}{2\,u}\,e^1{}_{11}\right) - \frac{1}{2}\,s_{0011} + O(u) \end{aligned}$$

$$\to \frac{1}{2}\Big(\partial_v\partial_u\,\Gamma_{1100} - 2\,\partial_u\,\Gamma_{1101} + 3\,\partial_u\,\Gamma_{0111} - \frac{1}{2}\partial_v\,\partial_u\,e^2{}_{11} - \frac{3}{4}\partial_u^2\,e^1{}_{11} - s_{0011}\Big)$$
$$= 0 \quad \text{as} \quad u \to 0\,,$$

where the ∂_u-equations and the relation $\partial_v\,s_1 = 3\,s_2$ on U_0 are used to calculate the limit. Similarly,

$$R_{010111} = \frac{1}{2\,u}\,\Gamma_{1101,2} - \frac{1}{2\,u}\,\Gamma_{1111} - \frac{1}{2}\,s_{0111} + O(u)$$
$$\to \frac{1}{2}\,(\partial_v\,\partial_u\,\Gamma_{1101} - \partial_u\,\Gamma_{1111} - s_{0111}) = 0 \quad \text{as} \quad u \to 0\,,$$

where the ∂_u-equations and the relation $\partial_v\,s_2 = 2\,s_3$ on U_0 are used,

$$R_{110111} = \frac{1}{2\,u}\,\Gamma_{1111,2} - \frac{1}{2}\,s_{1111} + O(u) \to \frac{1}{2}\,(\partial_v\,\partial_u\,\Gamma_{1111} - s_{1111}) = 0 \quad \text{as} \quad u \to 0\,,$$

where the ∂_u-equations and the relation $\partial_v\,s_3 = s_4$ on U_0 are used.

By (5.3) and the remark following (5.5) we know that $\zeta = 0$, $\zeta_{01} = 0$, $\zeta_{11} = 0$, $\partial_v\,\zeta_{00} = 0$, $\partial_v\,s = 0$ on U_0. The ∂_u-equations and (5.6) imply

$$\Sigma_{01} = \frac{1}{2\,u}\partial_v\,\zeta - \zeta_{01} + O(u) \to \frac{1}{2}\partial_v\,\zeta_{00} - \zeta_{01} = 0\,,$$
$$\Sigma_{0100} = \frac{1}{2\,u}\,(\partial_v\,\zeta_{00} - 2\,\zeta_{01}) + O(u) \to \frac{1}{2}\,(\partial_v\,\partial_u\,\zeta_{00} - 2\,\partial_u\,\zeta_{01}) = 0\,,$$
$$\Sigma_{0101} = \frac{1}{2\,u}\,(\partial_v\,\zeta_{01} - \zeta_{11}) + \frac{1}{2}\,s + O(u) \to \frac{1}{2}\,(\partial_v\,\partial_u\,\zeta_{01} - \partial_u\,\zeta_{11} + s) = 0\,,$$
$$\Sigma_{0111} = \frac{1}{2\,u}\,\partial_v\,\zeta_{11} + O(u) \to \frac{1}{2}\,\partial_v\,\partial_u\,\zeta_{11} = 0\,.$$
$$S_{01} = \frac{1}{2\,u}\,\partial_v\;s + s_{0111}\,\zeta_{00} + O(u) \to \frac{1}{2}\,(\partial_v\,\partial_u\,s + 2\,s_{0111}\,\zeta_{00})$$
$$= \frac{1}{2}\,\partial_v\,(\partial_u\,s + s_{0011}\,\zeta_{00}) = 0\,, \quad \text{as} \quad u \to 0\,.$$

With our assumptions (and formally setting $s_5 = 0$) we get for $k = 0, \ldots, 3$

$$\gamma_k \equiv \lim_{u\to 0}\,(-2\,H_{0(ABC)_k}) = (6-k)\,\partial_u\,s_{k+1} - \partial_v\,\partial_u\,s_k - (4-k)\,\mu\,s_{k+1}\,\zeta_{00}\,,$$
$$\beta_k \equiv \lim_{u\to 0}\,(-2\,H_{1(ABC)_k}) = 2\,\partial_w\,s_k - \partial_v\,\partial_u\,s_{k+1} + (3-k)\;\partial_u\,s_{k+2}$$
$$- (3-k)\,\mu\,s_{k+2}\,\zeta_{00}\,.$$

The expected tensorial nature of s_{ABCD} and H_{ABCD} (cf. Lemma 4.1) would imply

$$4\,\beta_1 = \partial_v\,\beta_0 - \partial_v\,\gamma_1 + 2\,\gamma_2\,,$$
$$12\,\beta_2 = \partial_v^2\,\beta_0 - \partial_v^2\,\gamma_1 - 2\,\partial_v\,\gamma_2 + 4\,\gamma_3\,,$$
$$24\,\beta_3 = \partial_v^3\,\beta_0 - \partial_v^3\,\gamma_1 - 2\,\partial_v^2\,\gamma_2 - 8\,\partial_v\,\gamma_3 \quad \text{on} \quad U_0\,.$$

It turns out that these relations can in fact be verified by a direct calculation with the expressions for γ_k, β_k obtained above. Because the equations used to establish

Lemma 5.1 imply $\gamma_k = 0$, $\beta_0 = 0$, it follows that $\beta_1 = \beta_2 = \beta_3 = 0$ so that in fact $H_{ABCD} \to 0$ as $u \to 0$. □

We can now prove the desired result.

Lemma 5.5. *The functions*

$$e^a{}_{AB}\,, \quad \Gamma_{ABCD}\,, \quad \zeta\,, \quad \zeta_{AB}\,, \quad s\,, \quad s_j\,,$$

corresponding to the expansions determined in Lemma 5.1 *satisfy the complete set of conformal vacuum field equations on the set* $\hat{S}$.

Proof. It needs to be shown that the *zero quantities*

$$t_{01}{}^{EF}{}_{11}\,, \quad R_{AB0111}\,, \quad \Sigma_{01}\,, \quad \Sigma_{11}\,, \quad \Sigma_{01CD}\,, \quad \Sigma_{11CD}\,, \quad S_{01}\,, \quad S_{11}\,, \quad H_{1ABCD}\,,$$

vanish on $\hat{S}$. For this purpose we shall derive a system of *subsidiary equations* for these fields.

Given the fields

$$e^a{}_{AB}\,, \quad \Gamma_{ABCD}\,, \quad \zeta\,, \quad \zeta_{AB}\,, \quad s\,, \quad s_{ABCD}\,,$$

we have the 1-forms σ^{AB} dual to e_{AB} and the connection form $\omega^A{}_B = \Gamma_{CD}{}^A{}_B \sigma^{CD}$.

To derive the subsidiary system we consider the torsion form

$$\Theta^{AB} = \frac{1}{2}\, t_{CD}{}^{AB}{}_{EF}\, \sigma^{CD} \wedge \sigma^{EF}\,,$$

and the form

$$\Omega^{*A}{}_B \equiv \Omega^A{}_B - \hat{\Omega}^A{}_B = \frac{1}{2}\, R^A{}_{BCDEF}\, \sigma^{CD} \wedge \sigma^{EF}\,,$$

obtained as difference of the curvature form

$$\Omega^A{}_B = \frac{1}{2}\, r^A{}_{BCDEF}\, \sigma^{CD} \wedge \sigma^{EF}\,,$$

and the form

$$\hat{\Omega}^A{}_B = \frac{1}{2}\, s^A{}_{BCE}\, \sigma^C{}_F \wedge \sigma^{EF}\,.$$

The following general relations will be used: The identity $\sigma^a \wedge \sigma^b \wedge \sigma^c = \epsilon^{abc}\,\nu$ with $\nu = \frac{1}{3!}\,\epsilon_{def}\,\sigma^d \wedge \sigma^e \wedge \sigma^f$, which holds in 3-dimensional spaces. In space spinor form it takes the form

$$\sigma^{AB} \wedge \sigma^{CD} \wedge \sigma^{EF} = \epsilon^{AB\,CD\,EF}\,\nu$$

$$\text{with} \quad \epsilon^{AB\,CD\,EF} = \frac{i}{\sqrt{2}}\left(\epsilon^{AC}\,\epsilon^{BF}\,\epsilon^{DE} - \epsilon^{AE}\,\epsilon^{BD}\,\epsilon^{FC}\right),$$

which implies

$$\sigma^{AB} \wedge \sigma^C{}_D \wedge \sigma^{ED} = -i\,\sqrt{2}\,\epsilon^{A(C}\,\epsilon^{E)B}\,\nu = i\,\sqrt{2}\,h^{AB\,CE}\,\nu\,,$$

and thus

$$\hat{\Omega}^A{}_B \wedge \sigma^{BD} = \frac{1}{2}\, s^A{}_{BCE}\, \sigma^{BD} \wedge \sigma^C{}_F \wedge \sigma^{EF} = 0\,.$$

The equations

$$i_H\,(\alpha\wedge\beta)=i_H\,\alpha\wedge\beta+(-1)^k\alpha\wedge i_H\,\beta\,,\qquad \mathcal{L}_H\,\alpha=(d\circ i_H+i_H\circ d)\,\alpha\,,$$

which holds for arbitrary vector field H, k-form α, and j-form β. Finally, we note that in the presence of torsion the Ricci identity for a spinor field $\iota_{E\ldots H}$ of degree m reads

$$\begin{aligned}(D_{AB}\,D_{CD}-D_{CD}\,D_{AB})\,\iota_{EF\ldots H}=&-\,\iota_{LF\ldots H}\;r^L{}_{E\,AB\,CD}-\,\iota_{EL\ldots H}\;r^L{}_{F\,AB\,CD}\\&-\cdots-\,\iota_{EF\ldots L}\;r^L{}_{H\,AB\,CD}\\&-t_{AB}{}^{KL}{}_{CD}\,D_{KL}\,\iota_{EF\ldots H}\,.\end{aligned}$$

We shall derive now the subsidiary equations. The fields Θ^{AB} and $\Omega^A{}_B$ satisfy the first structural equation

$$d\,\sigma^{AB}=-\omega^A{}_C\wedge\sigma^{CB}-\omega^B{}_C\wedge\sigma^{AC}+\Theta^{AB}\,,$$

and the second structural equation

$$d\,\omega^A{}_B=-\omega^A{}_C\wedge\omega^C{}_B+\Omega^A{}_B\,,$$

respectively. These equations imply

$$d\,\Theta^{AB}=2\,\Omega^{(A}{}_C\wedge\sigma^{B)C}-2\,\omega^{(A}{}_C\wedge\Theta^{B)C}=2\,\Omega^{*(A}{}_C\wedge\sigma^{B)C}-2\,\omega^{(A}{}_C\wedge\Theta^{B)C}\,.$$

We set $H=e_{00}$ and observe that the gauge conditions and the ∂_u-equations imply

$$i_H\,\sigma^{AB}=\epsilon_0{}^A\,\epsilon_0{}^B=h_{00}{}^{AB}\,,\quad i_H\,\omega^A{}_B=0\,,\quad i_H\,\Theta^{AB}=0\,,\quad i_H\,\Omega^{*A}{}_B=0\,.$$

It follows that

$$\mathcal{L}_H\,\Theta^{AB}=(d\circ i_H+i_H\circ d)\,\Theta^{AB}=2\,\Omega^{*(A}{}_0\,\epsilon_0{}^{B)}\,,$$

and thus

$$\begin{aligned}\mathcal{L}_H\langle\Theta^{AB},e_{01}\wedge e_{11}\rangle=&\;2\langle\Omega^{*(A}{}_0,e_{01}\wedge e_{11}\rangle\epsilon_0{}^{B)}\\&+\big\langle\Theta^{AB},[H,e_{01}]\wedge e_{11}\big\rangle+\big\langle\Theta^{AB},e_{01}\wedge[H,e_{11}]\big\rangle\,.\end{aligned}$$

The first structural equation, the gauge conditions, and the ∂_u-equations imply

$$0=\langle\Theta^{EF},H\wedge e_{CD}\rangle e_{EF}=-\Gamma_{CD}{}^{EF}{}_{00}\,e_{EF}-[H,e_{CD}]\,,$$

whence

$$[H,e_{CD}]=-2\,\Gamma_{CD01}\,e_{00}+2\,\Gamma_{CD00}\,e_{01}\,.$$

This implies

$$\mathcal{L}_H\langle\Theta^{AB},e_{01}\wedge e_{11}\rangle=2\,\Gamma_{0100}\langle\Theta^{AB},e_{01}\wedge e_{11}\rangle+\,2\big\langle\Omega^{*(A}{}_0,e_{01}\wedge e_{11}\big\rangle\epsilon_0{}^{B)}\,,$$

i.e.,

$$\left(\partial_u+\frac{1}{u}\right)\,t_{01}{}^{AB}{}_{11}=2\,\hat{\Gamma}_{0100}\;t_{01}{}^{AB}{}_{11}+2\,R^{(A}{}_{0\,01\,11}\,\epsilon_0{}^{B)}\,.\qquad(5.7)$$

With the first structural equation we obtain

$$d\,\hat{\Omega}_{AB} - \omega^H{}_A \wedge \hat{\Omega}_{HB} - \omega^H{}_B \wedge \hat{\Omega}_{AH} = \frac{1}{2}\,D_{GH}\,s_{ABCD}\,\sigma^{GH} \wedge \sigma^C{}_F \wedge \sigma^{DF} = \frac{i}{\sqrt{2}}\,H^E{}_{ABE}\,\nu\,,$$

and from the second structural equation we get

$$d\,\Omega_{AB} - \omega^H{}_A \wedge \Omega_{HB} - \omega^H{}_B \wedge \Omega_{AH} = 0\,,$$

which give together

$$d\,\Omega^*_{AB} - \omega^H{}_A \wedge \Omega^*_{HB} - \omega^H{}_B \wedge \Omega^*_{AH} = -\frac{i}{\sqrt{2}}\,H^E{}_{ABE}\,\nu\,,$$

and thus, with the equations above,

$$\left(\partial_u + \frac{1}{u}\right) R_{AB\,01\,11} = 2\,\hat{\Gamma}_{0100}\,R_{AB\,01\,11} + \frac{1}{2}\,H_{1AB0}\,. \tag{5.8}$$

The identity

$$D_{AB}\,\Sigma_{CD} - D_{CD}\,\Sigma_{AB} = t_{AB}{}^{EF}{}_{CD}\,D_{EF}\zeta + \Sigma_{CDAB} - \Sigma_{ABCD}\,,$$

gives with the gauge conditions and the ∂_u-equations

$$\partial_u\,\Sigma_{CD} + \frac{2}{u}\,\epsilon_{(C}{}^0\,\epsilon_{D)}{}^1\,\Sigma_{01} = 2\,\hat{\Gamma}_{CD00}\,\Sigma_{01} + \Sigma_{CD00}\,. \tag{5.9}$$

The identity

$$\begin{aligned} D_{AB}\,\Sigma_{CDEF} - D_{CD}\,\Sigma_{ABEF} = {} & -2\,\zeta_{K(E}\,R^K{}_{F)ABCD} + t_{AB}{}^{GH}{}_{CD}\,D_{GH}\zeta_{EF} \\ & + S_{CD}\,h_{ABEF} - S_{AB}\,h_{CDEF} \\ & + (1 - 2\,\mu\,\zeta)(\Sigma_{AB}\,s_{CDEF} - \Sigma_{CD}\,s_{ABEF}) \\ & + \zeta\,(1 - \mu\,\zeta)\,(\epsilon_{CA}\,H_{BDEF} + \epsilon_{DB}\,H_{CAEF})\,, \end{aligned}$$

implies with the gauge conditions and the ∂_u-equations

$$\begin{aligned} \partial_u\,\Sigma_{CDEF} + \frac{2}{u}\,\epsilon_{(C}{}^0\,\epsilon_{D)}{}^1\,\Sigma_{01EF} = {} & 2\,\hat{\Gamma}_{CD00}\,\Sigma_{01EF} + S_{CD}\,h_{00EF} \\ & - (1 - 2\,\mu\,\zeta)\,\Sigma_{CD}\,s_{00EF} \\ & + \zeta\,(1 - \mu\,\zeta)\,\epsilon_{D0}\,H_{C0EF}\,. \end{aligned} \tag{5.10}$$

The identity

$$\begin{aligned} D_{AB}\,S_{CD} - D_{CD}\,S_{AB} = {} & t_{AB}{}^{EF}{}_{CD}\,D_{EF}s \\ & - \mu\,\{\Sigma_{AB}\,s_{CDEF} - \Sigma_{CD}\,s_{ABEF}\}\,\zeta^{EF}(1 - \mu\,\zeta) \\ & \times \big\{\Sigma_{AB}{}^{EF}\,s_{CDEF} - \Sigma_{CD}{}^{EF}\,s_{ABEF} \\ & + (\epsilon_{CA}\,H_{BDEF} + \epsilon_{DB}\,H_{CAEF})\,\zeta^{EF}\big\}\,, \end{aligned}$$

implies with the gauge conditions and the ∂_u-equations

$$\partial_u S_{CD} + \frac{2}{u}\,\epsilon_{(C}{}^{0}\,\epsilon_{D)}{}^{1}\,S_{01} = 2\,\hat{\Gamma}_{CD00}\,S_{01} + \mu\,\Sigma_{CD}\,s_{00EF}\,\zeta^{EF} - (1-\mu\,\zeta)\left\{\Sigma_{CD}{}^{EF}\;s_{00EF} - \epsilon_{D0}\,H_{C0EF}\,\zeta^{EF}\right\}. \tag{5.11}$$

Finally we have the identity

$$2\,D^{EF}\,H_{EFAB} = -4\,s_{K(BGH}\,R^{K}{}_{A)}{}^{EG}{}_{E}{}^{H} + t^{E}{}_{F}{}^{KL}{}_{EH}\,D_{KL}\,s_{AB}{}^{FH} - \frac{4\,\mu}{1-\mu\,\zeta}\,s_{H(ABF}\,\Sigma^{EF}{}_{G)}{}^{H} - \frac{2\,\mu^2}{(1-2\,\mu)^2}\,\Sigma^{EF}\,s_{H(ABF}\,\zeta_{E)}{}^{H} + \frac{\mu}{1-\mu\,\zeta}\left\{2\,H_{EHAB}\,\zeta^{EH} - 2\,H^{E}{}_{EH(A}\,\zeta_{B)}{}^{H}\right\}, \tag{5.12}$$

where the right hand side is a linear function of the zero quantities. The gauge conditions and the equations $H_{0ABC} = 0$, $H_{1000} = 0$ imply for the left hand side

$$D^{EF}\,H_{EFAB} = \partial_u\,H_{11AB} + \frac{1}{u}\left\{H_{11AB} + H_{110(A}\,\epsilon_{B)}{}^{0}\right\} - \left(\frac{1}{2\,u}\,\partial_v + \hat{e}^{a}{}_{01}\partial_{z^a}\right)H_{10AB} - 2\,\hat{\Gamma}_{0100}\,H_{11AB} - \hat{\Gamma}_{010A}\,H_{110B} - \hat{\Gamma}_{010B}\,H_{110A} + \hat{\Gamma}_{011A}\,H_{100B} + \hat{\Gamma}_{011B}\,H_{100A} + \hat{\Gamma}_{1100}\,H_{10AB}\,. \tag{5.13}$$

Equations (5.7), (5.8), (5.9), (5.10), (5.11), and equation (5.12) with (5.13) observed on the left hand side provide the system of subsidiary equations. Note that the right hand side of this system is a linear function of the zero quantities. It implies with Lemma 5.4 that all zero quantities vanish on $\hat{S}$. □

If the series considered in Lemma 5.1 are absolutely convergent it thus follows from Lemma 5.5 that they define in fact a solution to the complete set of static conformal vacuum field equations on $\hat{S}$.

6. Convergence of the formal expansion

Let there be given a sequence

$$\hat{\mathcal{D}}_n = \{\psi_{A_2B_2A_1B_1},\ \psi_{A_3B_3A_2B_2A_1B_1},\ \psi_{A_4B_4A_3B_3A_2B_2A_1B_1},\ \ldots\},$$

of totally symmetric spinors as in Lemma 3.1 and set in the expansion (4.16) of $s_0(u,v)$

$$D^*_{(A_1B_1}\,\ldots\,D^*_{A_mB_m}\,s^*_{ABCD)}(i) = \psi_{A_1B_1\ldots A_mB_mABCD}\,, \quad m \geq 0\,.$$

Observing the estimates (3.11), one finds as a necessary condition for the function s_0 on W_0 to determine an analytic solution to the conformal static vacuum field equations that its non-vanishing Taylor coefficients at the point O satisfy estimates of the form

$$|\partial_u^m \partial_v^n s_0(O)| = m!\, n!\, |\psi_{m,n}| \leq \binom{2m+4}{n} m!\, n!\, M\, r_1^{-m}\,, \qquad m \geq 0\,, \quad 0 \leq n \leq 2\,m+4\,. \quad (6.1)$$

A slightly different type of estimate will be more convenient for us.

Lemma 6.1. *Let e denote the Euler number. For given $\rho_0 \in \mathbb{R}$, $0 < \rho_0 \leq e^2$, there exist positive constants r_0, c_0 so that* (6.1) *implies estimates of the form*

$$|\partial_u^m \partial_v^n s_0(O)| \leq c_0 \frac{m!\, n!\, r_0^m\, \rho_0^n}{(1+m)^2\,(1+n)^2}\,, \quad m \geq 0\,, \quad 0 \leq n \leq 2\,m+4\,. \qquad (6.2)$$

Proof. With $r_0 = 4\,e^6\, r_1^{-1}\, \rho_0^{-2}$ and $c_0 = 16\, M\, e^8\, \rho_0^{-4}$, the estimate $1 \leq \binom{2\,m+4}{n} \leq 2^{2m+4}$, which follows from the binomial law $(1+x)^{2m+4} = \sum_{n=0}^{2m+4} \binom{2\,m+4}{n} x^n$, and the estimate $e^x \geq 1 + x$, which holds for $x \geq 0$, we get

$$\begin{aligned}\binom{2m+4}{n} m!\, n!\, M\, r_1^{-m} &\leq 16\, M\, m!\, n!\, (4\, r_1^{-1})^m \\ &= c_0\, m!\, n!\, \frac{r_0^m}{(e^m)^2}\, \frac{\rho_0^n}{(e^n)^2} \left(\frac{\rho_0}{e^2}\right)^{2\,m+4-n} \\ &\leq c_0\, m!\, n!\, \frac{r_0^m}{(1+m)^2}\, \frac{\rho_0^n}{(1+n)^2}\,, \\ &\qquad m \geq 0\,, \quad 0 \leq n \leq 2\,m+4\,. \end{aligned}$$

□

The following lemma provides our main estimates.

Lemma 6.2. *Suppose $s_0 = s_0(u, v)$ is a holomorphic function defined on some open neighbourhood U of $O = \{u = 0, v = 0, w = 0\}$ in $W_0 = \{w = 0\}$ which has an expansion of the form*

$$s_0(u, v) = \sum_{m=0}^{\infty} \sum_{n=0}^{2m+4} \psi_{m,n}\, u^m\, v^n\,,$$

so that its Taylor coefficients at the point O satisfy estimates of the type (6.2) *with some positive constants c_0^*, r_0, and $\rho_0 < 1/2$. Then there exist positive constants $r \geq r_0$, ρ, $c_{\hat{e}^a_{AB}}$, $c_{\hat{\Gamma}_{ABCD}}$, c_ζ, c_{ζ_i}, $c_{\hat{s}}$, c_k so that the expansion coefficients determined from s_0 in Lemma* 5.1 *satisfy for $m, n, p = 0, 1, 2, \ldots$*

$$|\partial_u^m \partial_v^n \partial_w^p s_k(O)| \leq c_k \frac{r^{m+p}\,(m+p)!\, \rho^n\, n!}{(m+1)^2\,(n+1)^2\,(p+1)^2}\,, \qquad (6.3)$$

and

$$|\partial_u^m \partial_v^n \partial_w^p f(O)| \leq c_f \frac{r^{m+p-1}\,(m+p)!\, \rho^n\, n!}{(m+1)^2\,(n+1)^2\,(p+1)^2}\,, \qquad (6.4)$$

where f stands for any of the functions $\hat{e}^a_{AB}$, $\hat{\Gamma}_{ABCD}$, ζ, ζ_i, $\hat{s}$.

Remark 6.3. Observing the v-finite expansion types discussed in Lemma 5.2, we can replace the right hand sides in the estimates above by zero if n is large enough relative to m. This will not be pointed out at each step and for convenience the estimates will be written as above. The expansion types obtained in Lemma 5.2 will become important and will be observed, however, when we derive the estimates.

We shall make use of arguments discussed in [24]. The following four lemmas are essentially given in that article.

Lemma 6.4. *For any non-negative integer n there is a positive constant C independent of n so that*

$$\sum_{k=0}^{n} \frac{1}{(k+1)^2(n-k+1)^2} \le C\,\frac{1}{(n+1)^2}\,.$$

Proof. Denoting by $[n/2]$ the largest integer $\le n/2$, we get with $C = \sum_{k=0}^{\infty} \frac{8}{(k+1)^2}$

$$\sum_{k=0}^{n} \frac{1}{(k+1)^2(n-k+1)^2} \le \sum_{k=0}^{[n/2]} \frac{2}{(k+1)^2(n-k+1)^2}$$
$$\le \sum_{k=0}^{[n/2]} \frac{2}{(k+1)^2\big([n/2]+1\big)^2} \le C\,\frac{1}{(n+1)^2}\,. \qquad \square$$

In the following C will always denote the constant above.

Lemma 6.5. *For any integers m, n, k, j with $0 \le k \le m$, and $0 \le j \le n$ resp. $0 \le j \le n-1$ holds*

$$\binom{m}{k}\binom{n}{j} \le \binom{m+n}{k+j} \qquad \textit{resp.} \qquad \binom{m}{k}\binom{n-1}{j} \le \binom{m+n}{k+j}.$$

Proof. This follows by induction, using the general formula $\binom{n+1}{j} = \binom{n}{j} + \binom{n}{j-1}$, or by expanding $(x+y)^{m+n} = (x+y)^m\,(x+y)^n$, using the binomial law $(x+y)^p = \sum_{j=0}^{p} \binom{p}{j}\, x^j\, y^{p-j}$. $\square$

If f is holomorphic on the polydisk $P = \{(u,v,w,) \in \mathbb{C}^3 |\ |u| \le 1/r_1, |v| \le 1/r_2, |w| \le 1/r_3\}$, with some $r_1, r_2, r_3 > 0$, one has the Cauchy estimates

$$|\partial_u^m\, \partial_v^n\, \partial_w^p\, f(O)| \le r_1^m\ r_2^n\ r_3^p\ m!\ n!\ p!\ \sup_P |f|\,, \qquad m,n,p = 0,1,2,\ldots \tag{6.5}$$

where O denotes the origin $u=0$, $v=0$, $w=0$. We need a slight modification of this.

Lemma 6.6. *If f is holomorphic near O, there exist positive constants c, r_0, ρ_0 so that*

$$|\partial_u^m\, \partial_v^n\, \partial_w^p\, f(O)| \le c\,\frac{r^{m+p}\,(m+p)!\,\rho^n\, n!}{(m+1)^2\,(n+1)^2\,(p+1)^2}\,, \qquad m,n,p = 0,1,2,\ldots$$

for any $r \geq r_0$, $\rho \geq \rho_0$. *If in addition* $f(0,v,0) = 0$, *the constants can be chosen such that*

$$|\partial_u^m \, \partial_v^n \, \partial_w^p \, f(O)| \leq c \, \frac{r^{m+p-1} \, (m+p)! \, \rho^n \, n!}{(m+1)^2 \, (n+1)^2 \, (p+1)^2} \, , \qquad m,n,p = 0,1,2,\dots$$

for any $r \geq r_0$, $\rho \geq \rho_0$.

Proof. Let α be a positive number for which precise values will be considered below. Choosing an estimate of the type (6.5) with $r_1 = r_3$ and setting $c = \alpha \sup_P |f|$, $r_0 = e^2 \, r_1 = e^2 \, r_3$, $\rho_0 = e^2 \, r_2$, one gets from (6.5)

$$\begin{aligned} |\partial_u^m \, \partial_v^n \, \partial_w^p \, f(O)| &\leq c \, \alpha^{-1} r_0^{m+p} \, (m+p)! \, \rho_0^n \, n! \, e^{-2(m+n+p)} \\ &\leq c \, \alpha^{-1} \frac{r_0^{m+p} \, (m+p)! \, \rho_0^n \, n!}{(m+1)^2 \, (n+1)^2 \, (p+1)^2} \, . \end{aligned}$$

With $\alpha = 1$ the monotonicity of $x \to x^q$, $q \geq 0$, $x > 0$ implies the first estimate. With $\alpha = r_0$ the estimate above implies

$$|\partial_u^m \, \partial_v^n \, \partial_w^p \, f(O)| \leq c \, \frac{r_0^{m+p-1} \, (m+p)! \, \rho_0^n \, n!}{(m+1)^2 \, (n+1)^2 \, (p+1)^2} \, .$$

If $f(0,v,0) = 0$, then $\partial_u^0 \, \partial_v^n \, \partial_w^0 \, f(O) = 0$ for $n \in N_0$ and the last relation remains true for $m + p = 0$, i.e., $m = 0$ and $p = 0$, if r_0 and ρ_0 are replaced by $r \geq r_0$ and $\rho \geq \rho_0$. If $m + p > 0$ the result follows as above. □

Lemma 6.7. *Let* m, n, p *be non-negative integers and* f_i, $i = 1, \dots, N$, *be smooth complex valued functions of* u, v, w *on some neighbourhood* U *of* O *whose derivatives satisfy on* U (*resp. at a given point* $p \in U$) *estimates of the form*

$$|\partial_u^j \, \partial_v^k \, \partial_w^l \, f_i| \leq c_i \, \frac{r^{j+l+q_i} \, (j+l)! \, \rho^k \, k!}{(j+1)^2 \, (k+1)^2 \, (l+1)^2} \qquad \text{for} \quad 0 \leq j \leq m \, , \quad 0 \leq k \leq n \, , \quad 0 \leq l \leq p \, ,$$

with some positive constants c_i, r, ρ *and some fixed integers* q_i (*independent of* j, k, l). *Then one has on* U (*resp. at* p) *the estimates*

$$|\partial_u^m \, \partial_v^n \, \partial_w^p \, (f_1 \cdot \ldots \cdot f_N)| \leq C^{3\,(N-1)} \; c_1 \cdot \ldots \cdot c_N \; \frac{r^{m+p+q_1+\ldots+q_N} \, (m+p)! \, \rho^n \, n!}{(m+1)^2 \, (n+1)^2 \, (p+1)^2} \, . \tag{6.6}$$

Remark 6.8.

(i) Lemma 6.7 remains obviously true if m, n, p are replaced in (6.6) by integers m', n', p' with $0 \leq m' \leq m$, $0 \leq n' \leq n$, $0 \leq p' \leq p$.

(ii) By the argument given below the factor $C^{3\,(N-1)}$ in (6.6) can be replaced by $C^{(3-r)\,(N-1)}$ if r of the integers m, n, p vanish.

Proof. We prove the case $N = 2$. The general case then follows with the first of Remarks 6.8 by an induction argument. With the estimates above and Lemmas 6.4 and 6.5 we get on U (resp. at p)

$$\begin{aligned}
|\partial_u^m \partial_v^n \partial_w^p (f_1 f_2)| &\le \sum_{j=0}^{m} \sum_{k=0}^{n} \sum_{l=0}^{p} \binom{m}{j} \binom{n}{k} \binom{p}{l} |\partial_u^j \partial_v^k \partial_w^l f_1| |\partial_u^{m-j} \partial_v^{n-k} \partial_w^{p-l} f_2| \\
&\le \sum_{j=0}^{m} \sum_{k=0}^{n} \sum_{l=0}^{p} \binom{m}{j} \binom{n}{k} \binom{p}{l} \frac{c_1 r^{j+l+q_1} (j+l)!\, \rho^k k!}{(j+1)^2 (k+1)^2 (l+1)^2} \\
&\quad \times \frac{c_2 r^{m-j+p-l+q_2} (m-j+p-l)!\, \rho^{n-k} (n-k)!}{(m-j+1)^2 (n-k+1)^2 (p-l+1)^2} \\
&\le \sum_{j=0}^{m} \sum_{k=0}^{n} \sum_{l=0}^{p} \frac{\binom{m}{j}\binom{p}{l}}{\binom{m+p}{j+l}} \\
&\quad \times \frac{c_1 c_2 r^{m+p+q_1+q_2} (m+p)!\, \rho^n n!}{(j+1)^2 (k+1)^2 (l+1)^2 (m-j+1)^2 (n-k+1)^2 (p-l+1)^2} \\
&\le C^3 c_1 c_2 \frac{r^{m+p+q_1+q_2} (m+p)!\, \rho^n n!}{(m+1)^2 (n+1)^2 (p+1)^2} .
\end{aligned}$$

□

We are now able to prove our main estimates.

Proof of Lemma 6.2. Following the procedure which led to Lemma 5.1, the proof will be given by induction with respect to m and p. It is easy to see that the constants can be chosen to satisfy the estimates at lowest order. Leaving the choice of the constants open, we will derive from the induction hypothesis for the derivatives of the next order estimates of the form

$$|\partial_u^m \partial_v^n \partial_w^p s_k(O)| \le c_k \frac{r^{m+p} (m+p)!\, \rho^n n!}{(m+1)^2 (n+1)^2 (p+1)^2} A_{s_k} ,$$

$$|\partial_u^m \partial_v^n \partial_w^p f(O)| \le c_f \frac{r^{m+p-1} (m+p)!\, \rho^n n!}{(m+1)^2 (n+1)^2 (p+1)^2} A_f ,$$

with certain constants A_{s_k}, A_f which depend on m, n, p and the constants c_k, c_f, r, and ρ. Sometimes superscripts will indicate to which order of differentiability particular constants A_{s_k}, A_f refer. Occasionally we will have to make assumptions on r to proceed with the induction step. We shall collect these conditions and the constants A_{s_k}, A_f, or estimates for them, and at the end it will be shown that the constants c_k, c_f, r, and ρ can be adjusted so that all conditions are satisfied and $A_{s_k} \le 1$, $A_f \le 1$. This will complete the induction proof.

In the following it is understood that, as above, a function in a modulus sign is evaluated at the origin O. The symbol x will stand for any of the fields

$$\hat{e}^a{}_{AB} ,\ \hat{\Gamma}_{ABCD} ,\ \zeta ,\ \zeta_0 ,\ \zeta_1 ,\ \zeta_2 ,\ \hat{s} ,\ s_1 ,\ s_2 ,\ s_3 ,\ s_4 .$$

For the quantities which are known to vanish at I the estimates are correct for $m = 0$, $p = 0$. Since we consider $\hat{s}$ as an unknown and $s(0) = -2$ as part of the equations, we thus only need to discuss the s_k. They are given on I by

$$s_k = \frac{(4-k)!}{4!}\,\partial_v^k\, s_0\,.$$

It thus follows by our assumptions

$$|\partial_u^0\,\partial_v^n\,\partial_w^0\, s_k| = \left|\frac{(4-k)!}{4!}\partial_v^{k+n}\, s_0\right| \leq \left\{\begin{array}{c} \frac{(4-k)!}{4!}\, c_0\, \frac{\rho^{n+k}\,(n+k)!}{(n+k+1)^2} \quad \text{for} \quad n \leq 4-k \\ 0 \quad \text{for} \quad n > 4-k \end{array}\right\}$$
$$= c_k\, \frac{\rho^n\, n!}{(n+1)^2}\, A_{s_k}^{m=0,p=0}\,,$$

with

$$A_{s_k}^{m=0,p=0} = \frac{c_0}{c_k}\,\rho^k\, h_{k,n} \leq \frac{c_0}{c_k}\,\rho^k\,,$$

because

$$h_{k,n} \equiv \left\{\begin{array}{c} \frac{(4-k)!}{4!}\,\frac{(n+k)!}{n!}\,\frac{(n+1)^2}{(n+k+1)^2} \quad \text{for} \quad n \leq 4-k \\ 0 \quad \text{for} \quad n > 4-k \end{array}\right\} \leq 1\,.$$

We should study now under which conditions on the constants it can be shown by induction with respect to m that the quantities $|\partial_u^m\,\partial_v^n\,\partial_w^0\, x|$, $n \in N_0$, satisfy the estimates given in the lemma. We shall skip the details of this step, because the arguments used here are similar to those used to discuss the quantities $|\partial_u^m\,\partial_v^n\,\partial_w^p\, x|$ for general p and the requirements obtained in that case are in fact stronger that those obtained for $p = 0$.

It will be assumed now that $p \geq 1$, that the estimates for $|\partial_u^m\,\partial_v^n\,\partial_w^l\, x|$ given in the lemma hold true for $m, n \in N_0$, $0 \leq l \leq p-1$, and try to determine conditions so that the induction step $p-1 \to p$ can be performed.

By taking formal derivatives of the equation

$$0 = H_{0100} + H_{1000}\,,$$

we get with our assumptions

$$\begin{aligned} |\partial_u^m\,\partial_v^n\,\partial_w^p\, s_0| \leq{} & |\partial_u^{m+1}\,\partial_v^n\,\partial_w^{p-1}\, s_2| + |\partial_u^m\,\partial_v^n\,\partial_w^{p-1}\,(\hat{e}^1{}_{11}\,\partial_u\, s_0)| \\ & + |\partial_u^m\,\partial_v^n\,\partial_w^{p-1}\,(\hat{e}^2{}_{11}\,\partial_v\, s_0)| + 4\,|\partial_u^m\,\partial_v^n\,\partial_w^{p-1}\,(\hat{\Gamma}_{1101}\, s_0 + \hat{\Gamma}_{1100}\, s_1)| \\ & + \mu\left|\partial_u^m\,\partial_v^n\,\partial_w^{p-1}\left(\frac{1}{1-\mu\,\zeta}\left\{s_0\,\zeta_2 + 2\, s_1\,\zeta_1 - 3\, s_2\,\zeta_0\right\}\right)\right|. \end{aligned}$$

For the first term on the right hand side follows immediately

$$|\partial_u^{m+1}\,\partial_v^n\,\partial_w^{p-1}\, s_2| \leq c_2\,\frac{r^{m+p}\,(m+p)!\,\rho^n\, n!}{(m+2)^2(n+1)^2 p^2}\,.$$

A slight variation of the calculations in the proof Lemma 6.7 gives

$$
\begin{aligned}
&|\partial_u^m\,\partial_v^n\,\partial_w^{p-1}\,(\hat{e}^1{}_{11}\,\partial_u\,s_0)|\\
&\quad\leq \sum_{j=0}^{m}\sum_{k=0}^{n}\sum_{l=0}^{p-1}\binom{m}{j}\binom{n}{k}\binom{p-1}{l}|\partial_u^j\,\partial_v^k\,\partial_w^l\,\hat{e}^1{}_{11}||\partial_u^{m-j+1}\,\partial_v^{n-k}\,\partial_w^{p-l-1}\,s_0|\\
&\quad\leq \sum_{j=0}^{m}\sum_{k=0}^{n}\sum_{l=0}^{p-1}\frac{\binom{m}{j}\binom{p-1}{l}}{\binom{m+p}{j+l}}\\
&\qquad\times\ \frac{c_{\hat{e}^1{}_{11}}\,c_0\,r^{m+p-1}\,(m+p)\,!\,\rho^n\,n\,!}{(j+1)^2\,(k+1)^2\,(l+1)^2\,(m-j+2)^2\,(n-k+1)^2\,(p-l)^2}\\
&\quad\leq C^3\,c_{\hat{e}^1{}_{11}}\,c_0\,\frac{r^{m+p-1}\,(m+p)\,!\,\rho^n\,n\,!}{(m+2)^2\,(n+1)^2\,p^2}\,,
\end{aligned}
$$

where the sum over j has been extended in the last step to $m+1$.

Similarly one gets

$$
\begin{aligned}
&|\partial_u^m\,\partial_v^n\,\partial_w^{p-1}\,(\hat{e}^2{}_{11}\,\partial_v\,s_0)|\\
&\quad\leq \sum_{j=0}^{m}\sum_{k=0}^{n}\sum_{l=0}^{p-1}\binom{m}{j}\binom{n}{k}\binom{p-1}{l}|\partial_u^j\,\partial_v^k\,\partial_w^l\,\hat{e}^2{}_{11}||\partial_u^{m-j}\,\partial_v^{n-k+1}\,\partial_w^{p-l-1}\,s_0|\\
&\quad\leq \sum_{j=0}^{m}\sum_{k=0}^{n}\sum_{l=0}^{p-1}\frac{\binom{m}{j}\binom{n}{k}\binom{p-1}{l}}{\binom{m+p-1}{j+l}\binom{n+1}{k}}\\
&\qquad\times\ \frac{c_{\hat{e}^2{}_{11}}\,c_0\,r^{m+p-2}\,(m+p-1)\,!\,\rho^{n+1}\,(n+1)\,!}{(j+1)^2\,(k+1)^2\,(l+1)^2\,(m-j+1)^2\,(n-k+2)^2\,(p-l)^2}\\
&\quad\leq C^3\,c_{\hat{e}^2{}_{11}}\,c_0\,\frac{r^{m+p-2}\,(m+p-1)\,!\,\rho^{n+1}\,(n+1)\,!}{(m+1)^2\,(n+2)^2\,p^2}\,,
\end{aligned}
$$

where the sum over k has been extended in the last step to $n+1$.

We emphasize here again an observation which is important for us. By Lemma 5.2 the terms $\partial_u^j\,\partial_v^k\,\partial_w^l\,\hat{e}^2{}_{11}$ and $\partial_u^{m-j}\,\partial_v^{n-k+1}\,\partial_w^{p-l-1}\,s_0$ in the second line vanish if $k>2\,j+1$ and $n-k+1>2\,(m-j)+4$ respectively. This implies that the term on the left hand side vanishes if $n>2\,m+4$, consistently with Lemma 5.2. When we estimate the expression in the last line above we can thus assume that $n\leq 2\,m+4$.

Lemma 6.7 implies immediately

$$
\begin{aligned}
&4\,|\partial_u^m\,\partial_v^n\,\partial_w^{p-1}\,(\hat{\Gamma}_{1101}\,s_0+\hat{\Gamma}_{1100}\,s_1)|\\
&\qquad\qquad\leq 4\,C^3\left(c_0\,c_{\hat{\Gamma}_{1101}}+c_1\,c_{\hat{\Gamma}_{1100}}\right)\frac{r^{m+p-2}\,(m+p-1)\,!\,\rho^n\,n\,!}{(m+1)^2\,(n+1)^2\,p^2}\,,
\end{aligned}
$$

and, observing that $\zeta(O) = 0$,

$$\mu \left| \partial_u^m \, \partial_v^n \, \partial_w^{p-1} \left(\frac{1}{1 - \mu\,\zeta} \left\{ s_0 \, \zeta_2 + 2\, s_1 \, \zeta_1 - 3\, s_2 \, \zeta_0 \right\} \right) \right|$$
$$\leq \mu \sum_{l=0}^{\infty} \left| \partial_u^m \, \partial_v^n \, \partial_w^{p-1} \left((\mu\,\zeta)^l \left\{ s_0 \, \zeta_2 + 2\, s_1 \, \zeta_1 - 3\, s_2 \, \zeta_0 \right\} \right) \right|$$
$$\leq \mu \sum_{l=0}^{\infty} \mu^l \, c_\zeta^l \, C^{3\,(l+1)} \, (c_0 \, c_{\zeta_2} + 2\, c_1 \, c_{\zeta_1} + 3\, c_2 \, c_{\zeta_0}) \frac{r^{m+p-l-2} \, (m+p-1)\,! \, \rho^n \, n\,!}{(m+1)^2 \, (n+1)^2 \, p^2}$$
$$= \frac{\mu}{1 - \frac{\mu\, c_\zeta \, C^3}{r}} \, C^3 \, (c_0 \, c_{\zeta_2} + 2\, c_1 \, c_{\zeta_1} + 3\, c_2 \, c_{\zeta_0}) \frac{r^{m+p-2} \, (m+p-1)\,! \, \rho^n \, n\,!}{(m+1)^2 \, (n+1)^2 \, p^2} \,,$$

where it is assumed that

$$r > \mu \, c_\zeta \, C^3 \,.$$

Together this gives

$$|\partial_u^m \, \partial_v^n \, \partial_w^p \, s_0| \leq c_2 \, \frac{r^{m+p} \, (m+p)! \, \rho^n \, n!}{(m+2)^2 (n+1)^2 p^2}$$
$$+ C^3 \, c_{\hat{e}^1{}_{11}} \; c_0 \, \frac{r^{m+p-1} \, (m+p)\,! \, \rho^n \, n\,!}{(m+2)^2 \, (n+1)^2 \, p^2}$$
$$+ C^3 \, c_{\hat{e}^2{}_{11}} \; c_0 \, \frac{r^{m+p-2} \, (m+p-1)\,! \, \rho^{n+1} \, (n+1)\,!}{(m+1)^2 \, (n+2)^2 \, p^2}$$
$$+ 4\, C^3 \, (c_0 \, c_{\hat{\Gamma}_{1101}} + c_1 \, c_{\hat{\Gamma}_{1100}}) \, \frac{r^{m+p-2} \, (m+p-1)\,! \, \rho^n \, n\,!}{(m+1)^2 \, (n+1)^2 \, p^2}$$
$$+ \frac{\mu}{1 - \frac{\mu\, c_\zeta \, C^3}{r}} \, C^3 \, (c_0 \, c_{\zeta_2} + 2\, c_1 \, c_{\zeta_1} + 3\, c_2 \, c_{\zeta_0}) \frac{r^{m+p-2} \, (m+p-1)\,! \, \rho^n \, n\,!}{(m+1)^2 \, (n+1)^2 \, p^2}$$
$$\leq c_0 \, \frac{r^{m+p} \, (m+p)! \, \rho^n \, n!}{(m+1)^2 (n+1)^2 (p+1)^2} \, A^*_{s_0} \,,$$

with a factor

$$A^*_{s_0} = \frac{c_2}{c_0} \, \frac{(m+1)^2 \, (p+1)^2}{(m+2)^2 \; p^2} + \frac{1}{r} \, C^3 \, c_{\hat{e}^1{}_{11}} \, \frac{(m+1)^2 \, (p+1)^2}{(m+2)^2 \; p^2}$$
$$+ \frac{1}{r^2} \, C^3 \, c_{\hat{e}^2{}_{11}} \, \frac{\rho \, (n+1)^3 \, (p+1)^2}{(n+2)^2 \, p^2 \, (m+p)}$$
$$+ \frac{4}{r^2} \; C^3 \, \left(c_{\hat{\Gamma}_{1101}} + \frac{c_1}{c_0} \, c_{\hat{\Gamma}_{1100}} \right) \, \frac{(p+1)^2}{p^2 \, (m+p)}$$
$$+ \frac{1}{r^2} \, \frac{\mu}{1 - \frac{\mu\, c_\zeta \, C^3}{r}} \, C^3 \, \left(c_{\zeta_2} + 2\, \frac{c_1}{c_0} \, c_{\zeta_1} + 3\, \frac{c_2}{c_0} \, c_{\zeta_0} \right) \, \frac{(p+1)^2}{p^2 \, (m+p)} \,.$$

Recalling that we can assume $n \leq 2\,m + 4$ in the third term on the right hand side, this finally gives

$$A^*_{s_0} \leq 4\,\frac{c_2}{c_0} + \frac{4}{r}\,C^3\,c_{\hat{e}^1{}_{11}} + \frac{20\,\rho}{r^2}\,C^3\,c_{\hat{e}^2{}_{11}} + \frac{16}{r^2}\,C^3\left(c_{\hat{\Gamma}_{1101}} + \frac{c_1}{c_0}\,c_{\hat{\Gamma}_{1100}}\right) + \frac{1}{r^2}\,\frac{4\,\mu}{1 - \frac{\mu\,c_\zeta\,C^3}{r}}\,C^3\left(c_{\zeta_2} + 2\,\frac{c_1}{c_0}\,c_{\zeta_1} + 3\,\frac{c_2}{c_0}\,c_{\zeta_0}\right).$$

We have the relations

$$s_k = \frac{(4-k)!}{4\,!}\,\partial_v^k\,s_0 \quad \text{on} \quad U_0\,,$$

the equation $0 = H_{0100} + H_{1000}$ reduces to

$$\partial_w\,s_0 = \partial_u\,s_2 + 3\,\mu\,s_2\,\zeta_0 \quad \text{on} \quad U_0\,,$$

and we have seen that

$$\partial_v\,\zeta_0 = 0 \quad \text{on} \quad U_0\,.$$

This implies for $p \geq 1$ the estimates

$$\begin{aligned}
|\partial_u^0\,\partial_v^n\,\partial_w^p\,s_k| &\leq \frac{(4-k)!}{4\,!}\,(|\partial_u^1\,\partial_v^{n+k}\,\partial_w^{p-1}\,s_2| + 3\,\mu\,|\partial_u^0\,\partial_v^{n+k}\,\partial_w^{p-1}\,(s_2\,\zeta_0)|) \\
&\leq \begin{cases} \frac{(4-k)!}{4!}\,c_2\,\frac{r^p\,p!\,\rho^{n+k}\,(n+k)!}{4\,p^2\,(n+k+1)^2} & \text{for} \quad n \leq 4-k \\ 0 & \text{for} \quad n > 4-k \end{cases} \\
&\quad + \begin{cases} 3\,\mu\,\frac{(4-k)!}{4!}\,\sum_{l=0}^{p-1}\binom{p-1}{l}|\partial_v^{n+k}\,\partial_w^l\,s_2|\,|\partial_w^{p-1-l}\,\zeta_0| & \text{for} \quad n \leq 2-k \\ 0 & \text{for} \quad n > 2-k \end{cases} \\
&\leq c_k\,\frac{r^p\,p!\,\rho^n\,n!}{(n+1)^2(p+1)^2}\,A_{s_k}^{m=0,p\geq 1},
\end{aligned}$$

with

$$A_{s_k}^{m=0,p\geq 1} = \frac{c_2}{c_k}\,\rho^k\,f_{k,n} + \frac{3}{r}\,\mu\,C\,\frac{c_2\,c_{\zeta_0}}{c_k}\,\rho^k\,g_{k,n} \leq \frac{c_2}{c_k}\,\rho^k + \frac{12}{r}\,\mu\,C\,\frac{c_2\,c_{\zeta_0}}{c_k}\,\rho^k\,,$$

because

$$f_{k,n} \equiv \begin{cases} \frac{(4-k)!}{4!}\,\frac{(n+k)!\,(n+1)^2\,(p+1)^2}{n!\,(n+k+1)^2\,4\,p^2} & \text{for} \quad n \leq 4-k \\ 0 & \text{for} \quad n > 4-k \end{cases} \leq 1\,,$$

$$g_{k,n} \equiv \begin{cases} \frac{(4-k)!}{4!}\,\frac{(n+k)!\,(n+1)^2\,(p+1)^2}{n!\,(n+k+1)^2\,p^3} & \text{for} \quad n \leq 2-k \\ 0 & \text{for} \quad n > 2-k \end{cases} \leq 4\,.$$

From the equation $\Sigma_{1100} = 0$, which reads

$$\partial_w\,\zeta_0 = -2 + \hat{s} \quad \text{on} \quad U_0\,,$$

it follows

$$|\partial_u^0\,\partial_v^n\,\partial_w\,\zeta_0| = |\partial_v^n\,(-2 + \hat{s})| = 2\,\delta_0^n \leq c_{\zeta_0}\,\frac{\rho^n\,n!}{(n+1)^2}\,A_{\zeta_0}^{m=0,p=1}\,,$$

with

$$A_{\zeta_0}^{m=0,p=1} = \frac{2}{c_{\zeta_0}} .$$

Furthermore, for $p \geq 2$,

$$|\partial_u^0 \, \partial_v^n \, \partial_w^p \, \zeta_0| = |\partial_v^n \, \partial_w^{p-1} \, \hat{s}| = c_{\hat{s}} \, \frac{r^{p-1} \, (p-1)! \, \rho^n \, n!}{(n+1)^2 \, p^2}$$
$$\leq c_{\zeta_0} \, \frac{r^p \, p! \, \rho^n \, n!}{(n+1)^2 \, (p+1)^2} \, A_{\zeta_0}^{m=0,p\geq 2} ,$$

with

$$A_{\zeta_0}^{m=0,p\geq 2} = \frac{1}{r} \, \frac{c_{\hat{s}}}{c_{\zeta_0}} \, \frac{(p+1)^2}{p^3} \leq \frac{2}{r} \, \frac{c_{\hat{s}}}{c_{\zeta_0}} .$$

The equation $S_{11} = 0$, which reads

$$\partial_w \, \hat{s} = -s_4 \, \zeta_0 \quad \text{on} \quad U_0 ,$$

implies

$$|\partial_u^0 \, \partial_v^n \, \partial_w \, \hat{s}| = 0 \leq c_{\hat{s}} \, \frac{\rho^n \, n!}{(n+1)^2} ,$$

and for $p \geq 2$

$$|\partial_u^0 \, \partial_v^n \, \partial_w^p \, \hat{s}| = |\partial_u^0 \, \partial_v^n \, \partial_w^{p-1} \, (s_4 \, \zeta_0)| \leq C^2 \, c_4 \, c_{\zeta_0} \, \frac{r^{p-2} \, (p-1)! \, \rho^n \, n!}{(n+1)^2 \, p^2}$$
$$\leq c_{\hat{s}} \, \frac{r^{p-1} \, p! \, \rho^n \, n!}{(n+1)^2 \, (p+1)^2} \, A_{\hat{s}}^{m=0,p\geq 2} ,$$

with

$$A_{\hat{s}}^{m=0,p\geq 2} = \frac{1}{r} \, C^2 \, \frac{c_4 \, c_{\zeta_0}}{c_{\hat{s}}} \, \frac{(p+1)^2}{p^3} \leq \frac{2}{r} \, C^2 \, \frac{c_4 \, c_{\zeta_0}}{c_{\hat{s}}} .$$

Having studied the quantities $|\partial_u^m \, \partial_v^n \, \partial_w^p \, x|$ for $m = 0$, we shall now derive the conditions which arise from the requirement that we can obtain the desired estimates for these quantities inductively for all positive integers m. We shall provide detailed arguments only for some representative ∂_u-equations and just state the analogues results for the remaining equations.

Multiplication of the equation

$$\partial_u \hat{e}^2{}_{01} + \frac{1}{u} \, \hat{e}^2{}_{01} = \frac{1}{u} \, \hat{\Gamma}_{0100} + 2 \, \hat{\Gamma}_{0100} \, \hat{e}^2{}_{01} ,$$

with u and formal differentiation gives with Lemma 6.6 for $m \geq 1$

$$\begin{aligned}
|\partial_u^m \partial_v^n \partial_w^p \hat{e}^2{}_{01}| &\leq \frac{1}{m+1} \left(|\partial_u^m \partial_v^n \partial_w^p \hat{\Gamma}_{0100}| + 2\,m\,|\partial_u^{m-1} \partial_v^n \partial_w^p (\hat{\Gamma}_{0100}\,\hat{e}^2{}_{01})|\right) \\
&\leq \frac{1}{m+1} \left(c_{\hat{\Gamma}_{0100}} \frac{r^{m+p-1}\,(m+p)!\,\rho^n\,n!}{(m+1)^2\,(n+1)^2\,(p+1)^2} \right. \\
&\quad \left. + 2\,m\,C^3\,c_{\hat{e}^2{}_{01}}\,c_{\hat{\Gamma}_{0100}} \frac{r^{m+p-3}\,(m+p-1)!\,\rho^n\,n!}{m^2\,(n+1)^2\,(p+1)^2} \right) \\
&= c_{\hat{e}^2{}_{01}} \frac{r^{m+p-1}\,(m+p)!\,\rho^n\,n!}{(m+1)^2\,(n+1)^2\,(p+1)^2} A^{m\geq 1}_{\hat{e}^2{}_{01}} ,
\end{aligned}$$

with

$$A^{m\geq 1}_{\hat{e}^2{}_{01}} = \frac{c_{\hat{\Gamma}_{0100}}}{c_{\hat{e}^2{}_{01}}} \frac{1}{m+1} + \frac{1}{r^2}\,C^3\,c_{\hat{\Gamma}_{0100}} \frac{2\,(m+1)}{m\,(m+p)} .$$

Proceeding in a similar way with the equations for the other frame coefficients one gets for the factors which need to be controlled the estimates

$$\begin{aligned}
A^{m\geq 1}_{\hat{e}^2_{01}} &\leq \frac{c_{\hat{\Gamma}_{0100}}}{2\,c_{\hat{e}^2_{01}}} + \frac{4}{r^2}\,C^3\,c_{\hat{\Gamma}_{0100}} , &\qquad A^{m\geq 1}_{\hat{e}^2_{11}} &\leq \frac{c_{\hat{\Gamma}_{1100}}}{c_{\hat{e}^2_{11}}} + \frac{8}{r^2}\,C^3 \frac{c_{\hat{\Gamma}_{1100}}\,c_{\hat{e}^2_{01}}}{c_{\hat{e}^2_{11}}} , \\
A^{m\geq 1}_{\hat{e}^1_{01}} &\leq \frac{4}{r} \frac{c_{\hat{\Gamma}_{0101}}}{c_{\hat{e}^1_{01}}} + \frac{4}{r^2}\,C^3\,c_{\hat{\Gamma}_{0100}} , &\qquad A^{m\geq 1}_{\hat{e}^1_{11}} &\leq \frac{8}{r} \frac{c_{\hat{\Gamma}_{1101}}}{c_{\hat{e}^1_{11}}} + \frac{8}{r^2}\,C^3 \frac{c_{\hat{\Gamma}_{1100}}\,c_{\hat{e}^1_{01}}}{c_{\hat{e}^1_{11}}} .
\end{aligned}$$

The same inequalities, with C^3 replaced by C^2, are obtained in the case $p = 0$. In the last two inequalities the occurrence of $1/r$ in both terms reflects the fact that $\hat{e}^1_{01}$ and $\hat{e}^1_{11}$ are both of the order $O(u^2)$ near O.

Multiplication of the equation

$$\partial_u \hat{\Gamma}_{0100} + \frac{2}{u}\,\hat{\Gamma}_{0100} = 2\,(\hat{\Gamma}_{0100})^2 + \frac{1}{2}\,s_0 ,$$

with u and formal differentiation gives for $m \geq 1$

$$\begin{aligned}
|\partial_u^m \partial_v^n \partial_w^p \hat{\Gamma}_{0100}| &\leq \frac{m}{m+2} \left(2\,|\partial_u^{m-1} \partial_v^n \partial_w^p \hat{\Gamma}_{0100}| + \frac{1}{2}\,|\partial_u^{m-1} \partial_v^n \partial_w^p s_0|\right) \\
&\leq \frac{2\,m}{m+2}\,C^3\,c^2_{\hat{\Gamma}_{0100}} \frac{r^{m+p-3}\,(m+p-1)!\,\rho^n\,n!}{m^2\,(n+1)^2\,(p+1)^2} \\
&\quad + \frac{m}{2\,(m+2)}\,c_0 \frac{r^{m+p-1}\,(m+p-1)!\,\rho^n\,n!}{m^2\,(n+1)^2\,(p+1)^2} \\
&\leq c_{\hat{\Gamma}_{0100}} \frac{r^{m+p-1}\,(m+p)!\,\rho^n\,n!}{(m+1)^2\,(n+1)^2\,(p+1)^2} A^{m\geq 1}_{\hat{\Gamma}_{0100}} ,
\end{aligned}$$

with

$$A^{m\geq 1}_{\hat{\Gamma}_{0100}} = \frac{1}{r^2}\,C^3\,c_{\hat{\Gamma}_{0100}} \frac{2\,(m+1)^2}{m\,(m+2)\,(m+p)} + \frac{c_0}{c_{\hat{\Gamma}_{0100}}} \frac{(m+1)^2}{2\,m\,(m+2)\,(m+p)} .$$

Proceeding in a similar way with the equations for the other connection coefficients one gets for the factors which need to be controlled the estimates

$$A^{m\geq 1}_{\hat{\Gamma}_{0100}} \leq \frac{c_0}{c_{\hat{\Gamma}_{0100}}} + \frac{4}{r^2}\,C^3\,c_{\hat{\Gamma}_{0100}}\,, \qquad A^{m\geq 1}_{\hat{\Gamma}_{0101}} \leq \frac{c_1}{c_{\hat{\Gamma}_{0101}}} + \frac{4}{r^2}\,C^3\,c_{\hat{\Gamma}_{0100}}\,,$$

$$A^{m\geq 1}_{\hat{\Gamma}_{0111}} \leq \frac{c_2}{c_{\hat{\Gamma}_{0111}}} + \frac{4}{r^2}\,C^3\,c_{\hat{\Gamma}_{0100}}\,, \qquad A^{m\geq 1}_{\hat{\Gamma}_{1100}} \leq \frac{2\,c_1}{c_{\hat{\Gamma}_{1100}}} + \frac{4}{r^2}\,C^3\,c_{\hat{\Gamma}_{0100}}\,,$$

$$A^{m\geq 1}_{\hat{\Gamma}_{1101}} \leq \frac{4\,c_2}{c_{\hat{\Gamma}_{1101}}} + \frac{8}{r^2}\,C^3\,\frac{c_{\hat{\Gamma}_{1100}}c_{\hat{\Gamma}_{0101}}}{c_{\hat{\Gamma}_{1101}}}\,, \quad A^{m\geq 1}_{\hat{\Gamma}_{1111}} \leq \frac{4\,c_3}{c_{\hat{\Gamma}_{1111}}} + \frac{8}{r^2}\,C^3\,\frac{c_{\hat{\Gamma}_{1100}}c_{\hat{\Gamma}_{0111}}}{c_{\hat{\Gamma}_{1111}}}\,,$$

The same inequalities, with C^3 replaced by C^2, are obtained in the case $p = 0$. Being slightly more generous, one gets inequalities which can be written in the concise form

$$\begin{aligned} A^{m\geq 1}_{\hat{\Gamma}_{01AB}} &\leq \frac{c_{A+B}}{c_{\hat{\Gamma}_{01AB}}} + \frac{4}{r^2}\,C^3\,c_{\hat{\Gamma}_{0100}}\,, \\ A^{m\geq 1}_{\hat{\Gamma}_{11AB}} &\leq \frac{4\,c_{A+B+1}}{c_{\hat{\Gamma}_{11AB}}} + \frac{8}{r^2}\,C^3\,\frac{c_{\hat{\Gamma}_{1100}}c_{\hat{\Gamma}_{01AB}}}{c_{\hat{\Gamma}_{11AB}}}\,, \quad A, B = 0, 1\,, \end{aligned}$$

where the c_{A+B}, c_{A+B+1} denote for suitable numerical values of the indices A, B the constants $c_0, \ldots, c_4$.

An analogous discussion of the equations

$$\begin{aligned} \partial_u \zeta &= \zeta_0\,, \\ \partial_u\,\zeta_0 &= -\zeta\,(1-\mu\,\zeta)\,s_0\,, \\ \partial_u\,\zeta_1 &= -\zeta\,(1-\mu\,\zeta)\,s_1\,, \\ \partial_u\,\zeta_2 &= -2 + \hat{s} - \zeta\,(1-\mu\,\zeta)\,s_2\,, \\ \partial_u\,\hat{s} &- (1-\mu\,\zeta)\,(s_0\,\zeta_{11} - 2\,s_1\,\zeta_{01} + s_2\,\zeta_{00})\,, \end{aligned}$$

does not require new considerations. For the factors which need to be controlled we get the estimates

$$A^{m\geq 1, p\geq 0}_{\zeta} \leq \frac{4}{r}\,\frac{c_{\zeta_0}}{c_\zeta}\,,$$

$$A^{m\geq 1, p\geq 0}_{\zeta_0} \leq \frac{4}{r}\,C^3\,\frac{c_0\,c_\zeta}{c_{\zeta_0}} + \frac{4}{r^2}\,\mu\,C^6\,\frac{c_0\,c_\zeta^2}{c_{\zeta_0}}\,,$$

$$A^{m\geq 1, p\geq 0}_{\zeta_1} \leq \frac{4}{r}\,C^3\,\frac{c_1\,c_\zeta}{c_{\zeta_1}} + \frac{4}{r^2}\,\mu\,C^6\,\frac{c_0\,c_\zeta^2}{c_{\zeta_1}}\,,$$

$$A^{m\geq 1, p\geq 0}_{\zeta_2} \leq \begin{cases} \frac{8}{c_{\zeta_2}} + \frac{4}{r}\left(\frac{c_{\hat{s}}}{c_{\zeta_2}} + C^3\,\frac{c_2\,c_\zeta}{c_{\zeta_2}}\right) + \frac{4}{r^2}\,\mu\,C^6\,\frac{c_2\,c_\zeta^2}{c_{\zeta_2}} & \text{for } m = 1\,, n = 0\,, p = 0\,, \\ \frac{4}{r}\left(\frac{c_{\hat{s}}}{c_{\zeta_2}} + C^3\,\frac{c_2\,c_\zeta}{c_{\zeta_2}}\right) + \frac{4}{r^2}\,\mu\,C^6\,\frac{c_2\,c_\zeta^2}{c_{\zeta_2}} & \text{otherwise} \end{cases}\,,$$

$$A^{m\geq 1}_{\hat{s}} \leq \left(\frac{4}{r}\,C^3 + \frac{4}{r^2}\,\mu\,C^6\,c_\zeta\right)\left(\frac{c_0\,c_{\zeta_2}}{c_{\hat{s}}} + 2\,\frac{c_1\,c_{\zeta_1}}{c_{\hat{s}}} + \frac{c_2\,c_{\zeta_0}}{c_{\hat{s}}}\right).$$

We consider the ∂_u-equations for the curvature component s_1. Multiplication with $2\,u$ gives

$$2\,u\,\partial_u\,s_1 + 4\,s_1 = \partial_v\,s_0 + 2\,u\,\hat{e}^1{}_{01}\partial_u s_0 + 2\,u\,\hat{e}^2{}_{01}\partial_v s_0 - 8\,u\,(\hat{\Gamma}_{0101}\,s_0 - \hat{\Gamma}_{0100}\,s_1) - u\,\frac{4\,\mu}{(1-\mu\,\zeta)}\left\{s_0\,\zeta_1 - s_1\,\zeta_0\right\},$$

which implies for $m \geq 1$

$$\begin{aligned}|\partial_u^m\,\partial_v^n\,\partial_w^p\,s_1| \leq{}& \frac{1}{2\,m+4}\,|\partial_u^m\,\partial_v^{n+1}\,\partial_w^p\,s_0| \\ &+ \frac{2\,m}{2\,m+4}\,\left(|\partial_u^{m-1}\,\partial_v^n\,\partial_w^p\,(\hat{e}^1{}_{01}\,\partial_u\,s_0)| + |\partial_u^{m-1}\,\partial_v^n\,\partial_w^p\,(\hat{e}^2{}_{01}\,\partial_v\,s_0)|\right) \\ &+ \frac{4\,m}{2\,m+4}\left(2\,|\partial_u^{m-1}\,\partial_v^n\,\partial_w^p\,(\hat{\Gamma}_{0101}\,s_0 - \hat{\Gamma}_{0100}\,s_1)|\right. \\ &\left.+ \mu\left|\partial_u^{m-1}\,\partial_v^n\,\partial_w^p\left\{\frac{1}{1-\mu\,\zeta}\,(s_0\,\zeta_1 - s_1\,\zeta_0)\right\}\right|\right).\end{aligned}$$

The terms arising here are estimated in a similar way as the terms in the curvature equation above. Again the expansion types allows one to assume that $0 \leq n \leq 2\,m + 4 - k$. Again r is restricted to values with

$$r > \mu\,c_\zeta\,C^3\,.$$

Proceeding similarly with the other ∂_u-equations for the curvature, the following estimates are obtained for the factors which need to be controlled.

$$\begin{aligned}A^{m\geq 1}_{s_1} \leq{}& \frac{c_0}{c_1}\,\rho + \frac{1}{r}\,C^3\,\frac{c_0}{c_1}\,c_{\hat{e}^1_{01}} + \frac{8\,\rho}{r^2}\,C^3\,\frac{c_0}{c_1}\,c_{\hat{e}^2_{01}} + \frac{8}{r^2}\,C^3\,\left(\frac{c_0}{c_1}\,c_{\hat{\Gamma}_{0101}} + c_{\hat{\Gamma}_{0100}}\right) \\ &+ \frac{1}{r^2}\,C^3\,\frac{4\,\mu}{1-\frac{\mu\,c_\zeta\,C^3}{r}}\left(\frac{c_0}{c_1}\,c_{\zeta_1} + c_{\zeta_0}\right), \\ A^{m\geq 1}_{s_2} \leq{}& \frac{c_1}{c_2}\,\rho + \frac{1}{r}\,C^3\,\frac{c_1}{c_2}\,c_{\hat{e}^1_{01}} + \frac{8\,\rho}{r^2}\,C^3\,\frac{c_1}{c_2}\,c_{\hat{e}^2_{01}} \\ &+ \frac{4}{r^2}\,C^3\,\left(\frac{c_0}{c_2}\,c_{\hat{\Gamma}_{0111}} + 2\,\frac{c_1}{c_2}\,c_{\hat{\Gamma}_{0101}} + 3\,c_{\hat{\Gamma}_{0100}}\right) \\ &+ \frac{1}{r^2}\,C^3\,\frac{2\,\mu}{1-\frac{\mu\,c_\zeta\,C^3}{r}}\left(\frac{c_0}{c_2}\,c_{\zeta_2} + 2\,\frac{c_1}{c_2}\,c_{\zeta_1} + 3\,c_{\zeta_0}\right), \\ A^{m\geq 1}_{s_3} \leq{}& \frac{c_2}{c_3}\,\rho + \frac{1}{r}\,C^3\,\frac{c_2}{c_3}\,c_{\hat{e}^1_{01}} + \frac{8\,\rho}{r^2}\,C^3\,\frac{c_2}{c_3}\,c_{\hat{e}^2_{01}} \\ &+ \frac{8}{r^2}\,C^3\,\left(\frac{c_1}{c_3}\,c_{\hat{\Gamma}_{0111}} + c_{\hat{\Gamma}_{0100}}\right) + \frac{1}{r^2}\,C^3\,\frac{4\,\mu}{1-\frac{\mu\,c_\zeta\,C^3}{r}}\left(\frac{c_1}{c_3}\,c_{\zeta_2} + c_{\zeta_0}\right),\end{aligned}$$

$$A^{m\geq 1}_{s_3} \leq \frac{c_3}{c_4}\,\rho + \frac{1}{r}\,C^3\,\frac{c_3}{c_4}\,c_{\hat{e}^1_{01}} + \frac{8\,\rho}{r^2}\,C^3\,\frac{c_3}{c_4}\,c_{\hat{e}^2_{01}} + \frac{4}{r^2}C^3\left(3\,\frac{c_2}{c_4}\,c_{\hat{\Gamma}_{0111}} + 2\,\frac{c_3}{c_4}\,c_{\hat{\Gamma}_{0101}} + c_{\hat{\Gamma}_{0100}}\right) + \frac{1}{r^2}\,C^3\,\frac{2\,\mu}{1-\frac{\mu\,c_\zeta\,C^3}{r}}\left(3\,\frac{c_2}{c_4}\,c_{\zeta_2} + 2\,\frac{c_3}{c_4}\,c_{\zeta_1} + c_{\zeta_0}\right).$$

This gives all the needed information.

To arrange now the constants so that the induction argument can successfully be carried out, we proceed as follows. The estimates for the decisive factors which have been obtained above are of the general form

$$A \leq \alpha + \frac{1}{r}\,\beta + \frac{1}{r^2}\,\gamma\,,$$

with α, β, and γ depending on all the constants except r. If $\beta = 0$ and $\gamma = 0$ it suffices to ensure $\alpha \leq 1$. In the other cases we require $\alpha \leq a$ where a is a given constant, $a < 1$, and then choose r large enough so that $A \leq 1$. A first set of conditions arising this way reads

$$\frac{c_k}{c_{k+1}}\,\rho \leq a\,, \quad \frac{c_0}{c_k}\,\rho^k \leq 1\,, \quad \frac{c_2}{c_k}\,\rho^k \leq a\,, \quad 4\,\frac{c_2}{c_0} \leq a\,.$$

These conditions can be satisfied simultaneously. The first equation implies $c_k \geq (\rho/a)^k\,c_0$. With

$$c_k = \left(\frac{\rho}{a}\right)^k c^*_0\,,$$

where $0 < \rho$, $a < 1$, the first two relations hold true, the fourth relation implies $\rho^2 \leq a^3/4$ and with this restriction the third relation holds as well. We choose

$$\rho = \rho_0\,, \quad a = \left(4\,\rho_0^2\right)^{1/3}.$$

The conditions

$$\frac{2}{c_{\zeta_0}} \leq 1\,, \quad \frac{8}{c_{\zeta_2}} \leq a\,,$$

are met by setting

$$c_{\zeta_0} \equiv 2\,, \quad c_{\zeta_2} \equiv \frac{8}{a}\,.$$

The conditions

$$\frac{c_{A+B}}{c_{\hat{\Gamma}_{01AB}}} \leq a\,, \quad \frac{4\,c_{1+A+B}}{c_{\hat{\Gamma}_{11AB}}} \leq a\,, \qquad A, B = 0, 1\,,$$

are then dealt with by setting

$$c_{\hat{\Gamma}_{01AB}} \equiv \frac{1}{a}\,c_{A+B}\,, \quad c_{\hat{\Gamma}_{11AB}} \equiv \frac{1}{a}\,c_{1+A+B}\,.$$

The conditions

$$\frac{c_{\hat{\Gamma}_{0100}}}{c_{\hat{e}^2_{01}}} \leq a\,, \quad \frac{c_{\hat{\Gamma}_{1100}}}{c_{\hat{e}^2_{11}}} \leq a\,,$$

are satisfied by setting

$$c_{\hat{e}^2_{01}} \equiv \frac{1}{a}\, c_{\hat{\Gamma}_{0100}}\,, \quad c_{\hat{e}^2_{11}} \equiv \frac{1}{a}\, c_{\hat{\Gamma}_{1100}}\,.$$

After this we choose some positive constants

$$\hat{e}^1_{01}\,, \;\; \hat{e}^1_{11}\,, \;\; c_{\zeta}\,, \;\; c_{\zeta_1}\,, \;\; c_{\hat{s}}\,.$$

That these constants are not further restricted by the procedure reflects the fact that the corresponding functions vanish to higher order at O. Their choice affects, however, the value of the constant r. After all constants except r have been fixed we can choose r so large that

$$r > \max\left\{r_0,\; \mu\, c_{\zeta}\, C^3\right\},$$

and that all the A's are ≤ 1. This finishes the induction proof. □

The following statement of the convergence result, obtained by using the v-finite expansion types of the various functions, emphasizes the role of v as an angular coordinate.

Lemma 6.9. *The estimates* (6.3) *and* (6.4) *for the derivatives of the functions* s_k *and* f *and the expansion types given in Lemma* 5.2 *imply that the associated Taylor series are absolutely convergent in the domain* $|v| < \frac{1}{\alpha\,\rho}$, $|u| + |w| < \frac{\alpha^2}{r}$, *for any real number* α, $0 < \alpha \leq 1$. *It follows that the formal expansion determined in Lemma* 5.1 *defines indeed a* (*unique*) *holomorphic solution to the conformal static vacuum field equations which induces the datum* s_0 *on* W_0.

Proof. The estimates (6.3) and (6.4) imply

$$\begin{aligned}
|\partial_u^m\, \partial_v^n\, \partial_w^p\, s_k(O)| &\leq \frac{c_k}{\alpha^{4-k}}\, \frac{(r/\alpha^2)^{m+p}\,(m+p)!\,(\alpha\,\rho)^n\, n!}{(m+1)^2\,(n+1)^2\,(p+1)^2}\, \alpha^{4-k+2m+2p-n}\\
&\leq \frac{c_k}{\alpha^{4-k}}\, \frac{(r/\alpha^2)^{m+p}\,(m+p)!\,(\alpha\,\rho)^n\, n!}{(m+1)^2\,(n+1)^2\,(p+1)^2}\\
&\qquad\qquad \text{for} \quad n \leq 2\,m + 4 - k, \quad m, p = 0, 1, 2, \ldots\\
|\partial_u^m\, \partial_v^n\, \partial_w^p\, f(O)| &\leq \frac{c_f}{\alpha^{k_f-2}}\, \frac{(r/\alpha^2)^{m+p-1}\,(m+p)!\,(\alpha\,\rho)^n\, n!}{(m+1)^2\,(n+1)^2\,(p+1)^2}\, \alpha^{k_f+2m+2p-n}\\
&\leq \frac{c_f}{\alpha^{k_f-2}}\, \frac{(r/\alpha^2)^{m+p-1}\,(m+p)!\,(\alpha\,\rho)^n\, n!}{(m+1)^2\,(n+1)^2\,(p+1)^2}\\
&\qquad\qquad \text{for} \quad n \leq 2\,m + k_f, \quad m, p = 0, 1, 2, \ldots\,.
\end{aligned}$$

Since the other derivatives vanish because of the respective expansion types, the first assertion is an immediate consequence of the majorizations (3.15), (3.16). The second assertion then follows with Lemma 5.5. □

7. Analyticity at space-like infinity

Due to our singular gauge the holomorphic solution of the conformal static field equations obtained in Lemma 6.9 does not cover a full neighbourhood of the point i. To analyse the situation we study the part of the solution which we have obtained by the convergence proof in terms of a normal frame based on the frame c_{AB} at i and associated normal coordinates. We write the geodesic equation $D_{\dot z}\dot z = 0$ for $z^a(s) = (u(s), v, (s), w(s))$ in the form

$$\begin{aligned}\dot z^a &= m^{AB}\, e^a_{AB} = m^{AB}\,(e^{*a}_{AB} + \hat e^a_{AB})\,,\\ \dot m^{AB} &= -2\, m^{CD}\Gamma_{CD}{}^{(A}{}_B\, m^{B)E}\\ &= -2\, m^{CD}\Gamma^*_{CD}{}^{(A}{}_B\, m^{B)E} - 2\, m^{CD}\hat\Gamma_{CD}{}^{(A}{}_B\, m^{B)E}\,,\end{aligned}$$

With the explicit expressions for the singular parts, the system takes the form

$$\begin{aligned}&\dot u = m^{00} + m^{AB}\,\hat e^1_{AB}\,, && \dot m^{00} = -2\, m^{CD}\hat\Gamma_{CD}{}^0{}_B\, m^{0B}\,,\\ &\dot v = \frac{1}{u}\, m^{01} + m^{AB}\,\hat e^2_{AB}\,, && \dot m^{01} = -\frac{1}{u}\, m^{01}\, m^{00} - 2\, m^{CD}\hat\Gamma_{CD}{}^{(0}{}_B\, m^{1)B}\,,\\ &\dot w = m^{11}\,, && \dot m^{11} = -\frac{2}{u}\, m^{01}\, m^{01} - 2\, m^{CD}\hat\Gamma_{CD}{}^1{}_B\, m^{1B}\,.\end{aligned}$$

These equations have to be solved with the initial conditions

$$u|_{s=0} = 0\,, \quad w|_{s=0} = 0\,, \tag{7.1}$$

for the curves to start at i. An arbitrary value

$$v_0 = v|_{s=0}\,, \tag{7.2}$$

can be prescribed to determine the ∂_u-∂_w-plane over i in which the tangent vector is lying, and an arbitrary choice of

$$m^{AB}|_{s=0} = m^{AB}_0 = m^{AB}_0\,\epsilon_0{}^A\,\epsilon_0{}^B + m^{AB}_0\,\epsilon_1{}^A\,\epsilon_1{}^B\,, \quad \dot u_0 \neq 0\,,$$

can be prescribed to specify the tangent vector in the ∂_u-∂_w-plane. Regularity and the equations require

$$m^{00}_0 = \dot u|_{s=0} \equiv \dot u_0\,, \quad m^{01}_0 = 0\,, \quad m^{11}_0 = \dot w|_{s=0} \equiv \dot w_0\,. \tag{7.3}$$

If the frame e_{AB} at a point of I is identified with its projection into T_iS_c, then

$$m^{AB}_0\, e_{AB} = m^{AB}_0\, s^C{}_A(v_0)\, s^D{}_B(v_0)\, c_{CD} = m^{*AB}\, c_{AB}\,,$$

holds at i with

$$m^{*00} = \dot u_0\,, \quad m^{*01} = \dot u_0\, v_0\,, \quad m^{*11} = \dot u_0\, v_0^2 + \dot w_0\,, \quad \dot u_0 \neq 0\,.$$

For arbitrarily given $m^{*AB} \in \mathbb{C}^3$ with $m^{*00} \neq 0$ this relation determines $\dot u_0$, v_0, $\dot w_0$ uniquely. Using $c_{AB} = \alpha^a{}_{AB}\, c_{\mathbf a}$, the tangent vectors can be written $m^{*AB}\, c_{AB} =$

$x^a\, c_{\mathbf{a}}$ with

$$x^1 = \frac{1}{\sqrt{2}}\left(\dot{w}_0 + (v_0^2 - 1)\,\dot{u}_0\right), \quad x^2 = \frac{i}{\sqrt{2}}\left(\dot{w}_0 + (v_0^2 + 1)\,\dot{u}_0\right), \quad x^3 = \sqrt{2}\,v_0\,\dot{u}_0$$
$$\dot{u}_0 \neq 0\,, \quad (7.4)$$

or, equivalently,

$$\dot{u}_0(x^a) = -\frac{x^1 + i\,x^2}{\sqrt{2}}\,, \quad v_0(x^a) = -\frac{x^3}{x^1 + i\,x^2}\,, \quad \dot{w}_0(x^a) = \frac{\delta_{ab}\,x^a\,x^b}{\sqrt{2}(x^1 + i\,x^2)}\,,$$
$$x^1 + i\,x^2 \neq 0\,. \quad (7.5)$$

The vectors $x^a c_{\mathbf{a}}$ cover all directions at i except those tangent to the complex null hyperplane $(c_{\mathbf{1}} + i\,c_{\mathbf{2}})^{\perp} = \{a(c_{\mathbf{1}} + i\,c_{\mathbf{2}}) + b\,c_{\mathbf{3}}|\ a, b \in \mathbb{C}\}$.

To determine the normal frame centered at i and based on the frame c_{AB} at i, we write the equation $D_{\dot{x}} c_{AB} = 0$ for the normal frame as an equation for the transformation $t^A{}_B \in SL(2,\mathbb{C})$, which relates the frame e_{AB} to the normal frame $c_{AB} = t^C{}_A\, t^D{}_B\, e_{CD}$. The resulting equation

$$0 = \frac{d}{ds}(t^C{}_A\, t^D{}_B) + m^{GH}\,\Gamma_{GH}{}^{CD}{}_{EF}\, t^E{}_A\, t^F{}_B\,,$$

can be written in the form $\dot{t}^A{}_B = -m^{DE}\Gamma_{DE}{}^A{}_C\, t^C{}_B$. Taking into account the structure of the connection coefficients, this gives

$$\dot{t}^A{}_B = -\frac{1}{u}\, m^{01}\, \epsilon_1{}^A\, t^0{}_B - m^{DE}\, \hat{\Gamma}_{DE}{}^A{}_C\, t^C{}_B\,. \quad (7.6)$$

This equation has to be solved along $z(s)$ with the initial condition

$$t^A{}_B|_{s=0} = s^A{}_B(-v_0)\,. \quad (7.7)$$

The initial value problems above make sense because the functions $\hat{e}^a{}_{AB}$ and $\hat{\Gamma}_{ABCD}$ are, by Lemma 6.9, holomorphic near the point $u = 0$, $v = v_0$, $w = 0$ for any prescribed value of v_0. The singularity of the system at that particular point requires, however, some attention.

We prepare the statement and the proof of the existence result, to be given in Lemma 7.2, by casting the system of ODE's into a suitable form. It will be convenient to make use of the *replacements resp. change of notation*

$$v \to v_0 + v\,, \quad m^{AB} \to m_0^{AB} + m^{AB}\,, \quad (7.8)$$

so that all unknowns vanish at $s = 0$. Furthermore, by setting

$$\tilde{e}^a_{AB}(u,v,w) = \hat{e}^a_{AB}(u, v_0 + v, w)\,, \quad \tilde{\Gamma}_{ABCD}(u,v,w) = \hat{\Gamma}_{ABCD}(u, v_0 + v, w)\,,$$

we define functions $\tilde{e}^a_{AB}$, $\tilde{\Gamma}_{ABCD}$ of the new unknowns which are holomorphic near $u = v = w = 0$. The regular equations read with this notation

$$\begin{aligned}
\dot{u} &= \dot{u}_0 + m^{00} + \dot{w}_0\,\tilde{e}^1_{11} + 2\,\tilde{e}^1_{01}\,m^{01} + \tilde{e}^1_{11}\,m^{11}\,,\\
\dot{w} &= \dot{w}_0 + m^{11}\,,\\
\dot{m}^{00} &= -2\Big\{\dot{u}_0\,\dot{w}_0\,\tilde{\Gamma}_{1101} + \dot{u}_0\,(2\,\tilde{\Gamma}_{0101}\,m^{01} + \tilde{\Gamma}_{1101}\,m^{11})\\
&\quad + \dot{w}_0\,(\tilde{\Gamma}_{1101}\,m^{00} + \tilde{\Gamma}_{1111}\,m^{01}) + 2\tilde{\Gamma}_{0101}\,m^{00}\,m^{01} + 2\tilde{\Gamma}_{0111}\,m^{01}\,m^{01}\\
&\quad + \tilde{\Gamma}_{1101}\,m^{00}\,m^{11} + \tilde{\Gamma}_{1111}\,m^{01}\,m^{11}\Big\}
\end{aligned}$$

The singular equations take the form

$$\begin{aligned}
u\,\dot{v} &= m^{01} + u\,(\dot{w}_0\,\tilde{e}^2_{AB} + 2\,\tilde{e}^2_{01}\,m^{01} + \tilde{e}^2_{11}\,m^{11})\\
u\,\dot{m}^{01} &= -\dot{u}_0\,m^{01} - m^{00}\,m^{01} + u\Big\{\dot{u}_0\,\dot{w}_0\,\tilde{\Gamma}_{1100} - \dot{w}_0^2\,\tilde{\Gamma}_{1111}\\
&\quad + \dot{u}_0\,(2\,\tilde{\Gamma}_{0100}\,m^{01} + \tilde{\Gamma}_{1100}\,m^{11})\\
&\quad + \dot{w}_0\,(\tilde{\Gamma}_{1100}\,m^{00} - 2\,\tilde{\Gamma}_{0111}\,m^{01} - 2\,\tilde{\Gamma}_{1111}\,m^{11})\\
&\quad + 2\,\tilde{\Gamma}_{0100}\,m^{00}\,m^{01} - 2\,\tilde{\Gamma}_{0111}\,m^{01}\,m^{11} + \tilde{\Gamma}_{1100}\,m^{00}\,m^{11} - \tilde{\Gamma}_{1111}\,m^{11}\,m^{11}\Big\}\,,\\
u\,\dot{m}^{11} &= -2\,m^{01}\,m^{01}\\
&\quad + 2\,u\Big\{\dot{w}_0^2\,\tilde{\Gamma}_{1101} + \dot{w}_0\,(2\,\tilde{\Gamma}_{0101}\,m^{01} + \tilde{\Gamma}_{1100}\,m^{01} + 2\,\tilde{\Gamma}_{1101}\,m^{11})\\
&\quad + 2\,\tilde{\Gamma}_{0100}\,m^{01}\,m^{01} + 2\,\tilde{\Gamma}_{0101}\,m^{01}\,m^{11} + \tilde{\Gamma}_{0100}\,m^{01}\,m^{11} + \tilde{\Gamma}_{1101}\,m^{11}\,m^{11}\Big\}\,.
\end{aligned}$$

Finally, (7.6) reads

$$\dot{t}^A{}_B = -\frac{1}{u}\,m^{01}\,\epsilon_1{}^A\,t^0{}_B - (2\,m^{01}\,\hat{\Gamma}_{01}{}^A{}_C + \dot{w}_0\,\hat{\Gamma}_{11}{}^A{}_C + m^{11}\,\hat{\Gamma}_{11}{}^A{}_C)\,t^C{}_B\,. \tag{7.9}$$

After applying ∂_s resp. ∂_s^2 to the geodesic equations and restricting all equations to $s = 0$ one obtains with the initial conditions (7.1), (7.2), (7.3) the relations

$$\dot{v}|_{s=0} = 0\,,\quad \dot{m}^{AB}|_{s=0} = 0\,,\quad \ddot{u}|_{s=0} = 0\,, \tag{7.10}$$

and, by taking a further derivative,

$$\partial_s^3 u(0) = \dot{u}_0^2\,\dot{w}_0\,\Big\{\partial_u^2\,\hat{e}^1{}_{11} - 2\,\partial_u\,\hat{\Gamma}_{1101}\Big\}_{u=0,v=v_0,w=0}\,.$$

This gives with the ∂_u-equations

$$\partial_s^3 u(0) = -4\,\dot{u}_0^2\,\dot{w}_0\,(s_2)_{u=0,v=v_0,w=0} = -\frac{1}{3}\dot{u}_0^2\,\dot{w}_0\,\left(\partial_v^2 s_0\right)_{u=0,v=v_0,w=0}\,, \tag{7.11}$$

which can be determined from the null data.

Because of Lemma 6.9 and the behaviour (4.7), (4.13) of the metric and the connection coefficients, which follows from the ∂_u-equations, there exist functions

f, g, h, k, l which are holomorphic on a polycylinder $P_{\epsilon'} = \{x \in \mathbb{C}^6 |\, |x_j| < \epsilon'\}$ with some $\epsilon' > 0$ so that the equations above can be written

$$\dot{u} = \dot{u}_0 + m^{00} + u^2 f\,, \tag{7.12}$$

$$u\,\dot{v} = m^{01} + u^2 g\,, \tag{7.13}$$

$$\dot{w} = \dot{w}_0 + m^{11}\,, \tag{7.14}$$

$$\dot{m}^{00} = u\,h\,, \tag{7.15}$$

$$u\,\dot{m}^{01} = -\dot{u}_0\, m^{01} - m^{00}\, m^{01} + u^2\, k\,, \tag{7.16}$$

$$u\,\dot{m}^{11} = -2\, m^{01}\, m^{01} + u^2\, l\,, \tag{7.17}$$

with f, g, h, k, l depending on the $\mathbb{C}^6$-valued function $z(s)$ comprising our unknowns in the form

$$z(s) = \left(z^j(s)\right)_{j=1,\dots,6} = \left(u(s),\, v(s),\, w(s),\, m^{00}(s),\, m^{01}(s),\, m^{11}(s)\right),$$

(which agrees after the replacement $v \to v - v_0$ in the first 3 components with the notation introduced earlier).

If F stands for any of the functions f, g, h, k, l, then it has on $P_{\epsilon'}$ an absolutely convergent expansion

$$F = \sum_{\alpha \in N^6} F_\alpha\, z^\alpha\,,$$

at $z^j = 0$, where again the multi-index notation is used. If $0 < \epsilon < \epsilon'$, there exists thus an $M > 0$ so that

$$\sup_{x \in P_\epsilon} \sum_\alpha |F_\alpha|\, |z^\alpha| \le M\,.$$

Lemma 7.1. *Let $p \ge 0$ be an integer and c and t real numbers which satisfy with the constant C of Lemma 6.4*

$$c \ge \frac{M}{C}\,, \quad t \ge \max\left\{1, \frac{c\,C}{\epsilon}\right\}. \tag{7.18}$$

If the derivatives of the functions $z^j(s)$ at $s = 0$ exist and satisfy the estimates

$$|\partial_s^k z^j| \le c\,\frac{t^{k-1}\,k\,!}{(k+1)^2}\,, \quad k = 1, \dots, 6\,, \quad k \le p\,,$$

then

$$\left|\partial_s^p F\big(z(s)\big)\right|_{s=0} \le c\,\frac{t^p\,p\,!}{(p+1)^2}\,.$$

If, in addition, u satisfies $u(0) = 0$, $\dot{u}(0) = \dot{u}_0$ and

$$|\partial_s^k u(s)|_{s=0} \le c\,\frac{t^{k-2}\,k\,!}{(k+1)^2}\,, \qquad 2 \le k \le p\,,$$

then

$$\left|\partial_s^p\Big(u\,F\big(z(s)\big)\Big)\right|_{s=0} \le |\dot{u}_0|\,c\,\frac{t^{p-1}\,p\,!}{p^2} + c^2\,C\,\frac{t^{p-2}\,p\,!}{(p+1)^2}\,,$$

for $p \geq 1$, where the second term on the right hand side is to be dropped if $p < 2$, and

$$\left|\partial_s^p\Big(u^2\,F\big(z(s)\big)\Big)\right|_{s=0} \leq 2\,|\dot u_0|^2\,c\,\frac{t^{p-2}\,p\,!}{(p-1)^2} + 4\,|\dot u_0|\,c^2\,C\,\frac{t^{p-3}\,p\,!}{(p+1)^2} + c^3\,C^2\,\frac{t^{p-4}\,p\,!}{(p+1)^2}\,,$$

for $p \geq 2$, where the second term on the right hand side is to be dropped if $p < 3$ and the third term is to be dropped if $p < 4$.

On the left hand sides of the following equations will be considered the modulus of the values of the functions at the point $s = 0$.

Proof. Observing Lemma 6.7 and the subsequent remark, one gets

$$\begin{aligned}|\partial_s^p F(z)| &\leq \sum_{|\alpha|\leq p} |F_\alpha||\partial_s^p z^\alpha| \leq \sum_{|\alpha|\leq p} |F_\alpha|\,C^{|\alpha|-1}\,c^{|\alpha|}\frac{t^{p-|\alpha|}\,p\,!}{(p+1)^2}\\ &\leq \frac{1}{c\,C}\sum_{|\alpha|\leq p}|F_\alpha|\left(\frac{c\,C}{t}\right)^{|\alpha|} c\,\frac{t^p\,p\,!}{(p+1)^2} \leq \frac{M}{c\,C}\,c\,\frac{t^p\,p\,!}{(p+1)^2} \leq c\,\frac{t^p\,p\,!}{(p+1)^2}\,,\end{aligned}$$

by the choice of c and t. With Lemma 6.4 this gives

$$\begin{aligned}\big|\partial_s^p\big(u\,F(z)\big)\big| &\leq p\,|\dot u_0|\,|\partial_s^{p-1}F(z)| + \sum_{j=2}^{p}\binom{p}{j}|\partial_s^j u|\,|\partial_s^{p-j}F(z)|\\ &\leq p\,|\dot u_0|\,c\,\frac{t^{p-1}\,(p-1)!}{p^2} + \sum_{j=2}^{p}\binom{p}{j}c\,\frac{t^{j-2}\,(j)!}{(j+1)^2}\,c\,\frac{t^{p-j}\,(p-j)!}{(p-j+1)^2}\\ &\leq |\dot u_0|\,c\,\frac{t^{p-1}\,p\,!}{p^2} + c^2\,C\,\frac{t^{p-2}\,p\,!}{(p+1)^2}\,,\end{aligned}$$

and similarly

$$\begin{aligned}\big|\partial_s^p\big(u^2\,F(z)\big)\big| &\leq \sum_{l=0}^{p}\binom{p}{l}\sum_{j=0}^{l}\binom{l}{j}\,|\partial_s^j u|\,|\partial_s^{l-j}u|\,|\partial_s^{p-l}F(z)|\\ &= 4\binom{p}{2}\,|\dot u_0|^2\,|\partial_s^{p-2}F(z)| + \sum_{l=3}^{p}\binom{p}{l}\,2\,l\,|\dot u_0|\,|\partial_s^{l-1}u|\,|\partial_s^{p-l}F(z)|\\ &\quad + \sum_{l=2}^{p}\binom{p}{l}\sum_{j=2}^{l-2}\binom{l}{j}\,|\partial_s^j u|\,|\partial_s^{l-j}u|\,|\partial_s^{p-l}F(z)|\\ &\leq 2\,|\dot u_0|^2\,c\,\frac{t^{p-2}\,p\,!}{(p-1)^2} + 4\,|\dot u_0|\,c^2\,C\,\frac{t^{p-3}\,p\,!}{(p+1)^2} + c^3\,C^2\,\frac{t^{p-4}\,p\,!}{(p+1)^2}\,.\end{aligned}$$

□

Lemma 7.2. *The requirement that $z(s)$ be a holomorphic solution of equations (7.12)–(7.17) near $s = 0$ satisfying $x(0) = 0$ and $\partial_s u(0) = \dot u_0 \neq 0$ determines* *a unique formal expansion of $z(s)$ at $s = 0$. There exist real constants c and t satisfying*

$$c \geq \max\left\{4\,|\dot{u}_0|, 4\,|\dot{w}_0|,\, |\dot{u}_0|^2\,|\,\dot{w}_0|\,|(\partial_v^2 s_0)_{u=0,v=v_0,w=0}|,\, \frac{M}{C}\right\}, \quad t \geq \max\left\{1, \frac{c\,C}{\epsilon}\right\}, \tag{7.19}$$

with C the constant of Lemma 6.4, so that the Taylor coefficients of $z(s)$ at $s = 0$ satisfy the estimates

$$|\partial_s^q z^j| \leq c\,\frac{t^{q-1}\,q\,!}{(q+1)^2}\,, \quad q = 0, 1, 2, \dots\,, \tag{7.20}$$

and the Taylor coefficients of $u(s)$ at $s = 0$ satisfy in addition the estimates

$$|\partial_s^{q+2} u| \leq c\,\frac{t^q\,(q+2)!}{(q+3)^2}\,, \quad q = 0, 1, 2, \dots\,. \tag{7.21}$$

It follows that for any given initial data $\dot{u}_0$, v_0, $\dot{w}_0$ with $\dot{u}_0 \neq 0$ there exists a number $t = t(\dot{u}_0, v_0, \dot{w}_0)$ and a unique holomorphic solutions $z^j(s) = z^j(s, \dot{u}_0, v_0, \dot{w}_0)$ of the initial value problem for the geodesic equations with initial data as described above which is defined for $|s| \leq 1/t$. The functions $z^j(s, \dot{u}_0, v_0, \dot{w}_0)$ are in fact holomorphic functions of all four variables in a certain domain.

Proof. The existence of a unique formal expansion follows immediately by applying ∂_s^p for $p = 1, 2, 3, \dots$ formally to equations (7.12)–(7.17), restricting to $s = 0$, and observing $\dot{u}_0 \neq 0$ and the initial data.

That the estimates (7.20) hold for $q = 0, 1$ follows from the initial condition $x(0) = 0$, the equations at $s = 0$ and our conditions on c and t. That the estimates (7.21) hold for $q = 0, 1$ follows from (7.10), (7.11), and our conditions on c and t.

Let $p \geq 1$ be an integer. We show that c and t can be chosen such that the estimates (7.20), (7.21) for $q \leq p$ imply with the equations the corresponding estimates for $p+1$. From (7.15) and Lemma 7.1 (with the provisos given there not repeated here) follows

$$|\partial_s^{p+1}\,m^{00}| = |\partial_s^p\,(u\,h)| \leq |\dot{u}_0|\,c\,\frac{t^{p-1}\,p\,!}{p^2} + c^2\,C\,\frac{t^{p-2}\,p\,!}{(p+1)^2} \leq A_{00}\,c\,\frac{t^p\,(p+1)\,!}{(p+2)^2}\,,$$

with

$$A_{00} = \frac{1}{t}\,|\dot{u}_0|\,\frac{p\,!}{p^2}\,\frac{(p+2)^2}{(p+1)\,!} + \frac{1}{t^2}\,c\,C\,\frac{p\,!}{(p+1)^2}\,\frac{(p+2)^2}{(p+1)\,!} \leq \frac{5}{t}\,|\dot{u}_0| + \frac{2}{t^2}\,c\,C\,.$$

Similarly one gets from (7.12)

$$\begin{aligned}|\partial_s^{p+2}\,u| &\leq |\partial_s^{p+1}\,m^{00}| + |\partial_s^{p+1}\,(u^2\,f)|\\ &\leq A_{m^{00}}\;c\,\frac{t^p\,(p+1)\,!}{(p+2)^2} + 2\,|\dot{u}_0|^2\,c\,\frac{t^{p-1}\,(p+1)\,!}{p^2} + 4\,|\dot{u}_0|\,c^2\,C\,\frac{t^{p-2}\,(p+1)\,!}{(p+2)^2}\\ &\quad + c^3\,C^2\,\frac{t^{p-3}\,(p+1)\,!}{(p+2)^2} \leq A_u\,c\,\frac{t^p\,(p+2)!}{(p+3)^2}\,,\end{aligned}$$

with

$$A_u = A_{00}\,\frac{(p+1)\,!}{(p+2)^2}\,\frac{(p+3)^2}{(p+2)\,!} + \frac{2}{t}\,|\dot{u}_0|^2\,\frac{(p+3)^2}{p^2(p+2)} + \frac{4}{t^2}\,|\dot{u}_0|\,c\,C\,\frac{(p+3)^2}{(p+2)^3}$$
$$+ \frac{1}{t^3}\,c^2\,C^2\,\frac{(p+3)^2}{(p+2)^3}$$
$$\leq \frac{3}{t}\,|\dot{u}_0|(1+4\,|\dot{u}_0|) + \frac{1}{t^2}\,c\,C\,(1+4\,|\dot{u}_0|) + \frac{1}{t^3}\,c^2\,C^2\,,$$

and from (7.14)

$$|\partial_s^{p+1}\,w| = |\partial_s^p\,m^{11}| \leq c\,\frac{t^{p-1}\,p\,!}{(p+1)^2} \leq A_w\,c\,\frac{t^p\,(p+1)\,!}{(p+2)^2}\,,$$

with

$$A_w = \frac{1}{t}\,\frac{(p+2)^2}{(p+1)^3} \leq \frac{2}{t}\,.$$

Applying ∂_s^{p+1} to (7.16) and observing the initial conditions, gives at $s = 0$ for $p \geq 1$

$$(p+2)\,\dot{u}_0\,\partial_s^{p+1} m^{01} = -\sum_{j=2}^{p+1}\binom{p+1}{j}\,\partial_s^j\,u\,\partial_s^{p+2-j} m^{01}$$
$$-\sum_{j=1}^{p}\binom{p+1}{j}\,\partial_s^j\,m^{00}\,\partial_s^{p+1-j} m^{01} + \partial_s^{p+1}(u^2\,k)\,,$$

whence

$$|\partial_s^{p+1} m^{01}| \leq \frac{1}{(p+2)\,|\dot{u}_0|}\left\{\sum_{j=2}^{p+1}\binom{p+1}{j}\,c^2\,\frac{t^{j-2}\,j\,!}{(j+1)^2}\,\frac{t^{p+1-j}\,(p+2-j)\,!}{(p+3-j)^2}\right.$$
$$\left.+\sum_{j=1}^{p}\binom{p+1}{j}\,c^2\,\frac{t^{j-1}\,j\,!}{(j+1)^2}\,\frac{t^{p-j}\,(p+1-j)\,!}{(p+2-j)^2} + |\partial_s^{p+1}(u^2\,k)|\right\}$$
$$\leq \frac{1}{|\dot{u}_0|} c^2\,C\,t^{p-1}\,(p+1)\,!\left\{\frac{1}{(p+3)^2} + \frac{1}{(p+2)^2}\right\} + 2\,|\dot{u}_0|\,c\,\frac{t^{p-1}\,(p+1)\,!}{p^2\,(p+2)}$$
$$+ 4\,c^2\,C\,\frac{t^{p-2}\,(p+1)\,!}{(p+2)^3} + \frac{1}{|\dot{u}_0|}\,c^3\,C^2\,\frac{t^{p-3}\,(p+1)\,!}{(p+2)^3}$$
$$= A_{01}\;c\,\frac{t^p\,(p+1)\,!}{(p+2)^2}\,,$$

with

$$A_{01} = \frac{1}{t}\,\left\{\frac{c\,C}{|\dot{u}_0|}(1+\frac{(p+2)^2}{(p+3)^2}) + 2\,|\dot{u}_0|\,\frac{(p+2)}{p^2}\right\} + \frac{4\,c\,C}{t^2}\,\frac{1}{p+2} + \frac{c^2\,C^2}{t^3\,|\dot{u}_0|}\,\frac{1}{p+2}$$
$$\leq \frac{1}{t}\,\left\{\frac{2\,c\,C}{|\dot{u}_0|} + 4\,|\dot{u}_0|\right\} + \frac{2\,c\,C}{t^2} + \frac{c^2\,C^2}{t^3\,|\dot{u}_0|}\,.$$

Similarly we get from (7.13)

$$|\partial_s^{p+1} v| \leq \frac{1}{(p+1)\,|\dot{u}_0|} \left\{ \sum_{j=2}^{p+1} \binom{p+1}{j} |\partial_s^j u|\, |\partial_s^{p+2-j} v| + |\partial_s^{p+1} m^{01}| + |\partial_s^{p+1}(u^2\, h)| \right\}$$

$$\leq \frac{1}{(p+1)\,|\dot{u}_0|} \left\{ \sum_{j=2}^{p+1} \binom{p+1}{j} c^2\, \frac{t^{j-2}\, j\,!}{(j+1)^2}\, \frac{t^{p+1-j}\,(p+2-j)\,!}{(p+3-j)^2} \right.$$

$$\left. + |\partial_s^{p+1} m^{01}| + |\partial_s^{p+1}(u^2\, h)| \right\}$$

$$\leq A_v\ c\, \frac{t^p\,(p+1)\,!}{(p+2)^2}\,,$$

with

$$A_v = \frac{A_{01}}{(p+1)\,|\dot{u}_0|} + \frac{1}{t}\,\frac{2\,c\,C}{|\dot{u}_0|}\,\frac{(p+2)^2}{(p+3)^2} + \frac{2|\dot{u}_0|}{t}\,\frac{(p+2)^2}{p(p+1)} + \frac{4\,c\,C}{t^2}\,\frac{1}{p+1} + \frac{c^2\,C^2}{t^3\,|\dot{u}_0|}\,\frac{1}{p+1}$$

$$\leq \frac{1}{t}\left\{9\,|\dot{u}_0| + 2 + \frac{2\,c\,C}{|\dot{u}_0|} + \frac{c\,C}{|\dot{u}_0|^2}\right\} + \frac{c\,C}{t^2}\left\{2 + \frac{1}{|\dot{u}_0|}\right\} + \frac{c^2\,C^2}{t^3}\left\{\frac{1}{|\dot{u}_0|} + \frac{1}{|\dot{u}_0|^2}\right\},$$

and finally from (7.17)

$$|\partial_s^{p+1} m^{11}| \leq \frac{1}{(p+1)\,|\dot{u}_0|} \left\{ \sum_{j=2}^{p+1} \binom{p+1}{j} c^2\, \frac{t^{j-2}\, j\,!}{(j+1)^2}\, \frac{t^{p+1-j}\,(p+2-j)!}{(p+3-j)^2} \right.$$

$$\left. + \sum_{j=1}^{p} \binom{p+1}{j} c^2\, \frac{t^{j-1}\, j\,!}{(j+1)^2}\, \frac{t^{p-j}\,(p+1-j)!}{(p+2-j)^2} + |\partial_s^{p+1}(u^2\, l)| \right\}$$

$$\leq A_{11} c \frac{t^p\,(p+1)!}{(p+2)^2}\,,$$

with

$$A_{11} \leq \frac{1}{t}\left\{18\,|\dot{u}_0| + \frac{2\,c\,C}{|\dot{u}_0|}\right\} + \frac{2\,c\,C}{t^2} + \frac{c^2\,C^2}{t^3\,|\dot{u}_0|}\,.$$

From the estimates for the A's it follows that given a choice of c which satisfies the first of the estimates (7.19), we can determine t large enough such that the second of the estimates (7.19) and the conditions

$$A_u, A_v, A_w, A_{00}, A_{01}, A_{11} \leq 1\,,$$

are satisfied. With this choice the induction step can be carried out.

It follows immediately from estimates (7.20) that the Taylor expansions of the functions z^j at $s = 0$, $z^j(s) = \sum_{p=0}^{\infty} z_p^j\, s^p$ with $z_p^j = \frac{1}{p!}\,\partial_s^p z^j(0)$, are absolutely convergent for $|s| < 1/t$.

The coefficients $z_p^j = z_p^j(\dot{u}_0, v_0, \dot{w}_0)$ depend on v_0 via the expansion coefficients of the functions $\tilde{e}^a_{AB}$, $\tilde{\Gamma}_{ABCD}$. This implies a polynomial dependence of the z^j_p on v_0 due to the v-finite expansion types of the functions $\hat{e}^a_{AB}$, $\hat{\Gamma}_{ABCD}$. The explicit dependence of the right hand sides of equations (7.12)–(7.17) on $\dot{u}_0$ and $\dot{w}_0$ alone would lead to a polynomial dependence of the z^j_p on $\dot{u}_0$ and $\dot{w}_0$. The occurrence of the factors u on the left hand sides of equations (7.15)–(7.17) implies, however, that the z^j_p are polynomials in $\dot{u}_0$, v_0, $\dot{w}_0$ divided by certain powers of $\dot{u}_0$.

The number t which restricts the domain of convergence ensured by our argument depends via ϵ and M on v_0, and via c and the A's on $\dot{u}_0$, $1/\dot{u}_0$ and $\dot{w}_0$ with the effect that $t \to \infty$ as $\dot{u}_0 \to 0$. It follows, however, from the form of the estimates (7.20) and the way they have been obtained that for $(\dot{u}_0, v_0, \dot{w}_0)$ in a compactly embedded subset U of $(\mathbb{C} \setminus \{0\}) \times \mathbb{C} \times \mathbb{C}$ a common number t can be determined so that the Taylor series will be absolutely convergent for $(s, \dot{u}_0, v_0, \dot{w}_0) \in P_{1/t}(0) \times U$.

If K is compact in $P_{1/t}(0) \times U$, there exists $t' > t$ with $K \subset P_{1/t'}(0) \times U$ and it follows from (7.20) that the sequence of holomorphic functions $f^j_n = \sum_{p=0}^n z^j_p\, s^p$ on $P_{1/t}(0) \times U$ satisfies

$$\sup_K |f^j_n - z^j| \leq \sum_{p=n+1}^{\infty} c\, \frac{t^{p-1}}{(p+1)^2} \left(\frac{1}{t'}\right)^p \leq \frac{c}{t'}\, \frac{(t/t')^n}{1 - t/t'} \to 0 \quad \text{as} \quad n \to \infty\,,$$

so that the f^j_n converge uniformly to z^j on K. Standard results on compactly converging sequences of holomorphic functions [22] then imply that the $z^j = z^j(s, \dot{u}_0, v_0, \dot{w}_0)$ are holomorphic function of all four variables on $P_{1/t}(0) \times U$. □

Lemma 7.3. *Along the geodesic corresponding to* $s \to z^j(s, \dot{u}_0, v_0, \dot{w}_0)$ *equations* (7.9) *have a unique holomorphic solution* $t^A{}_B(s) = t^A{}_B(s, \dot{u}_0, v_0, \dot{w}_0)$ *satisfying the initial conditions* (7.7). *The functions* $t^A{}_B(s, \dot{u}_0, v_0, \dot{w}_0)$ *are holomorphic in all four variables in the domain where the* $z^j(s, \dot{u}_0, v_0, \dot{w}_0)$ *are holomorphic.*

Proof. By the previous discussion we have $m^{01} = O(s^2)$, $u = O(s)$ with $\dot{u}_0 \neq 0$ so that $m^{01}/u = O(s)$ as $s \to 0$. It follows that (7.9) is in fact a linear ODE with holomorphic coefficients and the lemma follows from standard ODE theory. □

For later use we note that (7.7), (7.9) imply as an immediate consequence that

$$t^{-1A}{}_B(s) = s^A{}_B(v_0) + O\big(|s|^2\big) \quad \text{as} \quad s \to 0\,. \tag{7.22}$$

To discuss the transformation to normal coordinates the notation employed before the transition (7.8) will be used again, so that

$$s \to z^a\big(\exp(s\, x^a c_a)\big) = z^a(s, \dot{u}_0, v_0, \dot{w}_0)\,,$$

denotes in the coordinates $z^1 = u$, $z^2 = v$, $z^3 = w$ the geodesic which has at $s = 0$ the tangent vector $x^a c_{\mathbf{a}}$ with $x^a = x^a(\dot{u}_0, v_0, \dot{w}_0)$ at i. We note that by the discussion above

$$u(s,\dot u_0,v_0,\dot w_0) = \dot u_0\, s + O\big(|s|^3\big)\,,$$
$$v(s,\dot u_0,v_0,\dot w_0) = v_0 + O\big(|s|^2\big)\,,$$
$$w(s,\dot u_0,v_0,\dot w_0) = \dot w_0\, s + O\big(|s|^3\big)\,. \tag{7.23}$$

In terms of the map (7.5) the transformation of the normal coordinates x^c centered at i and based on the frame $c_{\mathbf{a}}$ at i into the coordinates z^a is the given by

$$x^a \to z^a(x^c) = z^a\big(1,\dot u_0(x^c), v_0(x^c), \dot w_0(x^c)\big)\,, \tag{7.24}$$

for small enough $|x^a|$ with $x^1 + i\,x^2 \neq 0$. The geodesics being given in normal coordinates by the curves $s \to s\,x^a$, this implies

$$s\,x^a \to z^a\big(1,\dot u_0(s\,x^c), v_0(s\,x^c), \dot w_0(s\,x^c)\big) = z^a\big(s,\dot u_0(x^c), v_0(x^c), \dot w_0(x^c)\big)\,.$$

We use the relation on the right hand side to derive a convenient expression for the map (7.24). Observing that

$$\dot u_0(s\,x^c) = s\,\dot u_0(x^c)\,,\quad v_0(s\,x^c) = v_0(x^c)\,,\quad \dot w_0(s\,x^c) = s\,\dot w_0(x^c)\,,\quad s\in C\,,$$

by (7.5), we write $x^a = s\,x^a_*$ with s chosen so that $\dot u_0(x^c_*) = 1$, whence $\dot u_0(x^c) = s$, and get with the relation above the map (7.24) in the form

$$\begin{aligned} z^a(x^c) = z^a\big(1,\dot u_0(x^c), v_0(x^c), \dot w_0(x^c)\big) &= z^a\big(s,\dot u_0(x^c_*), v_0(x^c_*), \dot w_0(x^c_*)\big)\\ &= z^a\left(\dot u_0(x^c), 1, v_0(x^c), \frac{\dot w_0(x^c)}{\dot u_0(x^c)}\right).\end{aligned}$$

With (7.23) this gives, as $|x| \equiv \sqrt{\delta_{ab}\,\bar x^a\,x^b} \to 0$, $x^1 + i\,x^2 \neq 0$,

$$u(x^c) = -\frac{x^1 + i\,x^2}{\sqrt2} + O\big(|x|^3\big)\,,\quad v(x^c) = -\frac{x^3}{x^1+i\,x^2} + O\big(|x|^2\big)\,, \tag{7.25}$$

$$\begin{aligned} w(x^c) &= \frac{1}{\sqrt2}\left(x^1 - i\,x^2 + \frac{(x^3)^2}{x^1+i\,x^2}\right) + O\big(|x|^3\big)\\ &= \frac{\delta_{ab}\,x^a\,x^b}{\sqrt2\,(x^1+i\,x^2)} + O\big(|x|^3\big)\,. \end{aligned} \tag{7.26}$$

In the flat case the order symbols must be omitted in these expressions.

With (4.6), (7.22) and

$$du = -\frac{1}{\sqrt2}\big(dx^1 + i\,dx^2\big) + O\big(|x|^2\big)\,,$$
$$dv = \frac{dx^3}{\sqrt2\;u} + \frac{v}{\sqrt2\;u}\big(dx^1 + i\,dx^2\big) + O\big(|x|\big)\,,$$
$$dw = \frac{1}{\sqrt2}\left(dx^1 - i\,dx^2 - 2\,v\,dx^3 - v^2\left(dx^1 + i\,dx^3\right)\right) + O\big(|x|^2\big)\,,$$

one gets for the forms $\chi^{AB} = \chi^{AB}{}_c \, dx^c$ dual to the normal frame c_{AB} indeed

$$\begin{aligned} \chi^{AB}(x^c) &= t^{-1A}{}_C \, t^{-1B}{}_D \left(\sigma^{CD}{}_1 \, du + \sigma^{CD}{}_2 \, dv + \sigma^{CD}{}_3 \, dw \right) \\ &= \left(\alpha^{AB}{}_a + \hat{\chi}^{AB}{}_a \right) dx^a \, , \end{aligned}$$

with some functions $\hat{\chi}^{AB}{}_a(x^c)$ which satisfy $\hat{\chi}^{AB}{}_a = O(|x|^2)$ as $|x| \to 0$. Correspondingly, the coefficients $c^a_{AB} = \langle dx^a , \, c_{AB} \rangle$ of the normal frame in the normal coordinates satisfy

$$c^a{}_{AB}(x^c) = \alpha^a{}_{AB} + \hat{c}^a{}_{AB}(x^c) \, ,$$

with holomorphic functions $\hat{c}^a{}_{AB}(x^c)$ which satisfy $\hat{c}^a{}_{AB}(x^c) = O(|x|^2)$ as $|x| \to 0$.

Since the three 1-forms $\alpha^{AB}{}_a \, dx^a$ are linearly independent this shows that for small $|x^c|$ the coordinate transformation $x^a \to z^a(x^c)$, where defined, is non-degenerate and the forms χ^{AB} behave as required by normal forms in normal coordinates. The relations (3.1), which characterize coefficients of normal forms in normal coordinates, are a consequence of the equations satisfied by $z^a(s)$ and $t^A{}_B(s)$. All the tensor fields which enter the conformal static vacuum field equations can now be expressed in term of the coordinates x^c and the frame field c_{AB}.

All ingredients are now available to derive our main result.

Proof of Theorem 1.1. The coordinates x^c cover a domain (i.e., a connected open set) U in $\mathbb{C}^3$ on which the frame vector fields $c^a{}_{AB} \, \partial / \partial_{x^c}$ exist, are linearly independent and holomorphic and where the other tensor fields expressed in terms of the x^a and c_{AB} are holomorphic. It follows from Lemmas 6.9, 7.2, and 7.3 that given any initial data $\dot{u}_0, v_0, \dot{w}_0$ with $\dot{u}_0 \neq 0$, there exists a solution $z^a(s, \dot{u}_0, v_0, \dot{w}_0)$ of the corresponding geodesic equations which is defined for $|s| \leq 1/t$ with some $t > 0$. The dicussion above shows, however, that t will become large if $|v_0|$ becomes large or $|\dot{u}_0|$ becomes very small. This implies that the U will not contain the hypersurface $x^1 + i \, x^2 = 0$ but the boundary of U will become tangent to this hypersurface at $x^a = 0$. From the estimates obtained so far it cannot be concluded that the coordinates extend holomorphically to a domain containing an open neighbourhood of the origin.

To analyse this question, we make use of the remaining gauge freedom to perform with some $t^A{}_B \in SU(2)$ a rotation $\delta^* \to \delta^* \cdot t$ of the spin frame and the associated rotation

$$c_{AB} \to c^t_{AB} = t^C{}_A \, t^D{}_B \, c_{CD}$$

of the frame c_{AB} at i on which the construction of the submanifold $\hat{S}$ and the related gauge is based. Starting with these frames at i all the previous constructions and derivations can be repeated.

Let u', v', w' and e^t_{AB} denote the analogues in the new gauge of the coordinates u, v, w and the frame e_{AB}. The sets $\{w = 0\}$ and $\{w' = 0\}$ are then both to be thought of as lift of the set $\mathcal{N}_i$ to the bundle of spin frames, the coordinates u and u' can both be interpreted as affine parameters on the null generators of $\mathcal{N}_i$ which vanish at i, the coordinates v, v' both label these null generators, and the frame vectors e_{00} and e^t_{00} can be identified with auto-parallel vector fields tangent to the null generators.

If v and v' then label the same generator η of $\mathcal{N}_i$, a relation

$$s^C{}_0(v')\, s^D{}_0(v')\, t^E{}_C\, t^F{}_D\, c_{EF} = e^t_{00} = f^2\, e_{00} = f^2\, s^C{}_0(v)\, s^D{}_0(v)\, c_{CD}\,,$$

must hold at i with some $f \neq 0$ and $e^t_{00} = f^2\, e_{00}$ must hold in fact along η, with f constant along η because e^t_{00} and e_{00} are auto-parallel. Absorbing the undetermined sign in f, this leads to

$$t^E{}_C\, s^C{}_0(v') = f\, s^E{}_0(v)\,.$$

With

$$(t^A{}_B) = \begin{pmatrix} a & -\bar{c} \\ c & \bar{a} \end{pmatrix} \quad \text{where} \quad a, c \in C, \quad |a|^2 = |c|^2 = 1\,, \tag{7.27}$$

this gives

$$v' = \frac{-c + a\, v}{\bar{a} + \bar{c}\, v}\,, \quad f = \frac{1}{\bar{a} + \bar{c}\, v}\,, \quad \text{resp.} \quad v = \frac{c + \bar{a}\, v'}{a - \bar{c}\, v'}\,, \quad f = a - \bar{c}\, v'\,.$$

Moreover, the relations

$$\langle du, e_{00}\rangle = 1 = \langle du', e^t_{00}\rangle = \langle du', f^2\, e_{00}\rangle\,,$$

imply for the affine parameters along η

$$u = f^2\, u'\,,$$

so that $\eta(u', v') = \eta(u, v)$ holds with these relations. We note that choices of $t^A{}_B$ with $c \neq 0$ can supply new information, because then $v \to \infty$ as $v' \to a/\bar{c}$ so that the singular generator of the c_{AB}-gauge, about whose neighbourhood we need information, is then contained in the regular domain of the c^t_{AB}-gauge.

For the null datum in the new gauge one gets with (4.16)

$$s^t_0(u', v') = s^A{}_0(v')\, \ldots\, s^C{}_0(v')\, t^E{}_A\, \ldots\, t^H{}_D\, s^*_{E\ldots H}|_{\eta(u',v')} = f^4\, s_0(u, v)$$

$$= \sum_{m=0}^{\infty} \frac{1}{m!}\, u'^{\,m}\, f^{2m+4}\, s^{A_1}{}_0(v)\, s^{B_1}{}_0(v) \ldots s^D{}_0(v)\, D^*_{(A_1 B_1} \ldots D^*_{A_m B_m}\, s^*_{ABCD)}(i)$$

$$= \sum_{m=0}^{\infty} \frac{1}{m!}\, u'^{\,m}\, f^{2m+4}\, s^{A_1}{}_0(v')\, s^{B_1}{}_0(v') \ldots s^D{}_0(v')\, D^t_{(A_1 B_1} \ldots D^t_{A_m B_m}\, s^t_{ABCD)}(i)\,,$$

and thus

$$s^t_0(u', v') = \sum_{m=0}^{\infty} \sum_{n=0}^{2m+4} \psi^t_{m,n}\, u'^{\,m}\, v'^{\,n}\,, \tag{7.28}$$

with

$$D^t_{(A_1 B_1} \ldots D^t_{A_m B_m}\, s^t_{ABCD)}(i) \equiv t^{G_1}{}_{A_1}\, t^{H_1}{}_{B_1}\, \ldots\, t^N{}_D\, D^*_{(G_1 H_1} \ldots D^*_{G_m H_m}\, s^*_{LKMN)}(i)\,,$$

and

$$\psi^t_{m,n} = \frac{1}{m\,!}\binom{2\,m+4}{n} D^t_{(A_1B_1}\dots D^t_{A_mB_m}\, s^t_{ABCD)_n}(i)$$
$$= \frac{1}{m\,!}\binom{2\,m+4}{n}\sum_{j=0}^{2\,m+4}\binom{2\,m+4}{j} t^{(G_1}{}_{(A_1}\, t^{H_1}{}_{B_1}\,\dots\, t^{N)_j}{}_{D)_n}$$
$$\times\, D^*_{(G_1H_1}\dots D^*_{G_mH_m}\, s^*_{LKMN)_j}(i)$$
$$= \binom{2\,m+4}{n}\sum_{j=0}^{2\,m+4} t^{(G_1}{}_{(A_1}\, t^{H_1}{}_{B_1}\,\dots\, t^{N)_j}{}_{D)_n}\,\psi_{m,j}\,.$$

It is convenient to write this in the form

$$\psi^t_{m,n} = \sum_{j=0}^{2\,m+4}\binom{2\,m+4}{n}^{1/2}\binom{2\,m+4}{j}^{-1/2} T_{2\,m+4}{}^{j}{}_{n}(t)\,\psi_{m,j}\,, \tag{7.29}$$

where the numbers

$$T_{2\,m+4}{}^{j}{}_{n}(t) = \binom{2\,m+4}{n}^{1/2}\binom{2\,m+4}{j}^{1/2} t^{(G_1}{}_{(A_1}\, t^{H_1}{}_{B_1}\,\dots\, t^{N)_j}{}_{D)_n}\,,$$

are so defined [11] that they represent the matrix elements of a unitary representation of $SU(2)$ and thus satisfy

$$|T_{2\,m+4}{}^{j}{}_{n}(t)| \le 1\,, \quad m = 0,1,2,\dots\,, \quad 0 \le j\,, \quad n \le 2\,m+4\,.$$

With the expressions above it is easy to see that the type of the estimate (3.11) and the type of the resulting estimate (6.1) are preserved under the gauge transformation. With (7.28) and (7.29) follows from (6.1) at the point $O' = (u' = 0, v' = 0)$

$$|\partial^m_{u'}\,\partial^n_{v'}\, s^t_0(O')| = m!\,n!\,|\psi^t_{m,n}| \le m!\,n!\sum_{j=0}^{2\,m+4}\binom{2\,m+4}{n}^{1/2}\binom{2\,m+4}{j}^{-1/2}$$
$$\times\ |T_{2\,m+4}{}^{j}{}_{n}(t)|\,|\psi_{m,j}|$$
$$\le m!\,n!\sum_{j=0}^{2\,m+4}\binom{2\,m+4}{n}^{1/2}\binom{2\,m+4}{j}^{1/2} M\,r_1^{-m}$$
$$\le m!\,n!\binom{2\,m+4}{n}\sum_{j=0}^{2\,m+4}\binom{2\,m+4}{j} M\,r_1^{-m}$$
$$= m!\,n!\binom{2\,m+4}{n} M'\,r_t^{-m}\,, \tag{7.30}$$

with $M' = 16\,M$ and $r_t = r_1/4$.

Assuming now that $c \neq 0$ in (7.27), the resulting c^t_{AB}-*gauge* can be studied from two different points of view:

i) The singular generator of $\mathcal{N}_i$ in the c^t_{AB}-gauge will coincide with the regular generator of $\mathcal{N}_i$ on which $v = -\bar{a}/\bar{c}$ in the c_{AB}-gauge. By starting from the solution in the c_{AB}-gauge, we are thus able to directly determine near that generator the transformation into the c^t_{AB}-gauge and to determine the expansion of the solution in the c_{AB}-gauge in terms of the coordinates u', v', w' and the frame field e^t_{AB}.
ii) Alternatively, with the null data $s^t_0(u', v')$ at hand, one can go through the discussions of the previous sections to show the existence of a solution to the conformal static vacuum equations in the coordinates u', v', w' pertaining to the c^t_{AB}-gauge. All the observations made above, in particular statements about domains of convergence, apply to this solution as well. Important for us is that this solution covers the generator $v' = a/\bar{c}$ near $u' = 0$ and $w' = 0$, which corresponds to the singular generator in the c_{AB}-gauge.

Because the formal expansions of the fields in terms of u', v', w' are uniquely determined by the data $s^t_0(u', v')$, the solutions obtained by the two methods are holomorphically related to each other on certain domains by the gauge transformation obtained in (i). The solution obtained in (ii) can be expressed in terms of the normal coordinates x^a_t and the normal frame field c^t_{AB} so that the x^a_t cover an certain domain $U_t \subset \mathbb{C}^3$ and the frame field c^t_{AB} is non-degenerate and all our tensor fields expressed in terms of x^a_t and c^t_{AB} are holomorphic on U_t as discussed above. It follows then that the solution in the c_{AB}-gauge and the solution in the c^t_{AB}-gauge are related on certain domains by the simple transformation (cf. (4.3))

$$x^a_t = t^{-1\,a}{}_b\, x^b\,, \quad c^t_{AB} = t^C{}_A\, t^D{}_B\, c_{CD}\,.$$

Extending this as a coordinate and frame transformation to the solution obtained in (ii) to express all field in terms x^a and c_{AB} so that they are defined and holomorphic on $t^{-1}\, U_t$, one finds that the solution obtained in (ii) and our original solution define in fact genuine holomorphic extensions of each other because each one covers the singular generator of the other one away from the origin in a regular way.

By letting $t^A{}_B$ go through $SU(2)$ and observing the corresponding extensions, one obtains in fact a holomorphic solution to the conformal static vacuum field equations in the normal coordinates x^a centered at i associated with the frame δ^* resp. c_{AB} at i on a domain which covers a full neighbourhood of space-like infinity. Consider again the solution we obtained in the c_{AB}-gauge. From the discussion above it follows that the domain U in $\mathbb{C}^3$ on which the solution is holomorphic in the coordinates x^a covers a connected open subset U' of the hypersurface $\{x^3 = 0\}$ of $\mathbb{C}^3$ which has empty intersection with the line $\{x^1 + i\, x^2 = 0,\, x^3 = 0\}$ (corresponding to the singular generator of the c_{AB}-gauge) and whose boundary becomes tangent to this line at the origin $x^a = 0$. Under the transition

$$\dot{u}_0 \to \dot{u}_0\,, \quad v_0 \to e^{i\,\theta/2}\, v_0\,, \quad \dot{w}_0 \to e^{i\,\theta}\, \dot{w}_0\,, \quad \theta \in R\,,$$

which leaves the quantities $|\dot{u}_0|$, $|v_0|$, $|\dot{w}_0|$ entering the estimates above invariant, one gets by (7.4)

$$x^1 + i\,x^2 \to x^1 + i\,x^2\,, \quad x^1 - i\,x^2 \to e^{i\,\theta}\left(x^1 - i\,x^2\right), \quad x^3 \to e^{i\,\theta}\,x^3\,.$$

Thus the set U' can be assumed to be invariant under this transformation.

Consider now the $c^{t^*}_{AB}$-gauge where the special transformation $t^{*\,A}{}_B$ is given by (7.27) with $a = 0$, $c = 1$. Let U'_{t^*} denote a subset of the hypersurface $\{x^3_{t^*} = 0\}$ in $\mathbb{C}^3$ analogous to U'. It has empty insection with the line $\{x^1_{t^*} + i\,x^2_{t^*} = 0,\, x^3_{t^*} = 0\}$ but its boundary becomes tangent to it at $x^a_{t^*} = 0$. It holds

$$c^{t^*}_{00} = c_{11}\,, \quad c^{t^*}_{01} = -c_{01}\,, \quad c^{t^*}_{11} = c_{00} \quad \text{at} \quad i\,,$$

and the corresponding normal coordinates are related by

$$x^1_{t^*} = -x^1\,, \quad x^2_{t^*} = x^2\,, \quad x^3_{t^*} = -x^3\,.$$

The holomorphic transformation $\{x^3_{t^*} = 0\} \ni (x^1_{t^*}, x^2_{t^*}) \to (-x^1, x^2) \in \{x^3 = 0\}$ maps U'_{t^*} onto a subset of $\mathbb{C}^2 \sim \mathbb{C}^2 \times \{0\}$, denoted by $t^{*\,-1}\,U'_{t^*}$, which has non-empty intersection with U'. After the transformation above the two solutions coincide on $t^{*\,-1}\,U'_{t^*} \cap U'$.

On the other hand, the image of the $c^{t^*}_{AB}$-regular line $\{x^1_{t^*} - i\,x^2_{t^*} = 0,\, x^3_{t^*} = 0\} \cap U'_{t^*}$ under this transformation contains the intersection of a neighbourhood of the origin with the singular line $\{x^1 - i\,x^2 = 0,\, x^3 = 0,\, x^a \neq 0\}$ of the c_{AB}-gauge. In fact, the set $t^{*\,-1}\,U'_{t^*} \cup U'$, which admits a holomorphic extension of our solution in the coordinates x^a and the frame c_{AB}, contains a punctured neighbourhood of the origin. As we have seen above, the field c_{AB} on this neighbourhood extends continously to the origin.

Let now $x^a_* \neq 0$ be an arbitrary point in $\mathbb{C}^3$. We want to show that the solution extends in the coordinates x^a to a domain which covers the set $s\,x^a_*$ for $0 < |s| < \epsilon$ for some $\epsilon > 0$. Since $x^a_* = y^a + i\,z^a$ with y^a, $z^a \in \mathbb{R}^3$ there is a vector $u^a \in \mathbb{R}^3$ of unit length and orthogonal to x^a with respect to the standard product $u \cdot x = \delta_{ab}\,u^a\,x^b$. Consider the c^t_{AB}-gauges with $t^A{}_B \in SU(2)$ so that $u^a{}_t = t^{-1\,a}{}_b\,u^b = \delta^a{}_3$. It follows then that $x^a_{*t} = t^{-1\,a}{}_b\,x^b_* \in \{x^3_t = 0\}$ and by the preceeding observation $t^A{}_B$ can in fact be chosen such that there exist an $\epsilon > 0$ so that the points $s\,x^a_{*t}$ with $0 < |s| < \epsilon$ are covered by U'_t. Transforming back we find that the set $U \in \mathbb{C}^3$ covered by the coordinates x^a can be extended so that the points $s\,x^a_*$ with $0 < |s| < \epsilon$ are covered by U and all field are holomorphic on U in the coordinates x^a. It follows that we can assume U to contain a punctured neighbourhood of the origin in which the solution is holomorphic in the normal coordinates x^a and the normal frame c_{AB}. Since holomorphic functions in more than one dimension cannot have isolated singularities [15] the solution is then in fact holomorphic on a full neighbourhood of the origin $x^a = 0$, which represents the point i.

By Lemma 3.1 the exact sets of equations argument determines from null data satisfying the reality conditions a formal expansion of the solution with expansion coefficients satisfying the reality conditions. By the various uniqueness statements obtained in the lemmas this expansion must coincide with the expansion in normal coordinates of the solution obtained above. This implies the existence of a 3-dimensional real slice on which the tensor fields satisfy the reality conditions. It is obtained by requiring the coordinates x^a to assume values in $\mathbb{R}^3$. □

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