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Algebraic quantum gravity (AQG): I. Conceptual setup

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Abstract

We introduce a new top down approach to canonical quantum gravity, called algebraic quantum gravity (AQG). The quantum kinematics of AQG is determined by an abstract *-algebra generated by a countable set of elementary operators labelled by an algebraic graph. The quantum dynamics of AQG is governed by a single master constraint operator. While AQG is inspired by loop quantum gravity (LQG), it differs drastically from it because in AQG there is fundamentally no topology or differential structure. A natural Hilbert space representation acquires the structure of an infinite tensor product (ITP) whose separable strong equivalence class Hilbert subspaces (sectors) are left invariant by the quantum dynamics. The missing information about the topology and differential structure of the spacetime manifold as well as about the background metric to be approximated is supplied by coherent states. Given such data, the corresponding coherent state defines a sector in the ITP which can be identified with a usual QFT on the given manifold and background. Thus, AQG contains QFT on all curved spacetimes at once, possibly has something to say about topology change and provides the contact with the familiar low energy physics. In particular, in two companion papers we develop semiclassical perturbation theory for AQG and LQG and thereby show that the theory admits a semiclassical limit whose infinitesimal gauge symmetry agrees with that of general relativity. In AQG everything is computable with sufficient precision and no UV divergences arise due to the background independence of the fundamental combinatorial structure. Hence, in contrast to lattice gauge theory on a background metric, no continuum limit has to be taken. There simply is no lattice regulator that must be sent to zero.

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1. Introduction

The present paper introduces a new conceptual framework for canonical quantum gravity resulting in a novel top-to-bottom approach. To justify it, a rather complex reasoning is required based on the current status of the quantum dynamics of loop quantum gravity (LQG). Therefore we will devote quite some space in this introduction to make the motivations, concepts and techniques clear and in order to show how this theory differs from the more traditional framework.

1.1. Anomalies and the semiclassical analysis of LQG

Loop quantum gravity (LQG) has advanced in recent years to one of the major candidates for a theory of quantum gravity. See [1] for books and [2] for recent reviews on the subject. The theory has a mathematically rigorous basis of the quantum kinematics [3–5] and there is a mathematically well-defined formulation of the quantum dynamics [6]. However, one problem has remained unsettled so far within LQG: the demonstration that the theory has general relativity as its semiclassical limit. Related to this, so far it has not been revealed that the algebra of the quantum constraints, while free of anomalies, mimics the algebra of the classical constraints.

The reason for this so far elusive evidence has a complicated but clear technical reason and in what follows we will try to explain it in some detail.

In the current setup, LQG is formulated in terms of gauge field variables, that is, non-Abelian electric fluxes and magnetic holonomies, just as in lattice gauge theory. The corresponding surfaces and curves are embedded into a spatial manifold σ of given topology. These define an abstract *-Poisson algebra. Using the physically well-motivated condition of spatial diffeomorphism invariance, one can show that there is only one unitary equivalence class of cyclic representations of this holonomy-flux algebra [7, 8]. Thus, the *kinematical* framework of LQG is rather tight and well under control.

The unique (up to unitary equivalence) Hilbert space $\mathcal{H}$ can be realized as the closure of the finite linear combinations of cylindrical functions. A cylindrical function is a complex valued, square integrable (with respect to a certain measure) function of holonomies along the edges of some finite graph and all finite graphs embedded into σ are allowed. Thus, in contrast to lattice gauge theory, the lattice is not fixed; rather, all lattices (or graphs) are considered simultaneously which is why LQG is a continuum rather than a lattice theory.

The problem with establishing the semiclassical limit of LQG has to do with the quantum *dynamics*.

There is a natural action of the spatial diffeomorphism group Diff(σ) on this Hilbert space which simply consists in mapping graphs to their images under the given diffeomorphism. This action is a unitary representation of Diff(σ) and therefore the spatial diffeomorphisms are represented without anomalies. However, the action is not weakly continuous. This means that the infinitesimal generators of Diff(σ), that is, the Lie algebra diff(σ), cannot be defined on $\mathcal{H}$. In contrast, the infinite number of Hamiltonian constraints can be defined on $\mathcal{H}$ [6]. However, since the classical Poisson algebra of constraints involves diff(σ), it should come at no surprise that the part of the quantum algebra that involves the Hamiltonian constraints does not manifestly mimic the classical algebra because in the quantum theory we can only define finite diffeomorphisms. In fact, there is a finite diffeomorphism analogue for the commutator between diff(σ) and the Hamiltonian constraints and that part of the algebra is realized without anomalies [6]. However, the commutator between two Hamiltonian constraints classically is a linear combination, with phase space dependent coefficients, of elements of diff(σ) and it is this commutator which is problematic in LQG.

The philosophy that has been adopted in [6] is that the quantization of the Hamiltonian constraints should be anomaly free in the sense that the (dual of the) commutator between two Hamiltonian constraints should annihilate the space of spatially diffeomorphism invariant states constructed in [5]. This is indeed possible to achieve and one can show that this requires that the Hamiltonian constraints, which are densely defined on cylindrical functions, *necessarily change (enlarge)* the graph that underlies a given cylindrical function. This is also natural to happen because the natural regularization of the constraint involves small loops that are attached to the vertices of a given graph which shrink towards the vertex as the regulator is removed. However, the shrinking process can be compensated for by a spatial diffeomorphism and since the limit is taken in an operator topology which involves spatially diffeomorphism invariant states, the loops actually do not completely shrink to the vertex. See [6] or the second book in [1] for details.

While the commutator of two Hamiltonian constraints then is anomaly free in the sense explained, in addition one would like to check that the classical limit of the commutator between quantum Hamiltonian constraints is precisely the corresponding Poisson bracket between the classical constraints. Here again we are faced with an obstacle: for graph changing operators such as the Hamiltonian constraints it turns out to be extremely difficult to define coherent (or semiclassical) states, that is, states labelled by points in the classical phase space with respect to which the operator assumes an expectation value which reproduces the value of the corresponding classical function at that point in phase space and with respect to which the (relative) fluctuations are small. The reason why this happens is that the existing coherent states for LQG [9] are defined over a finite collection of finite graphs and these suppress very effectively the fluctuations of those degrees of freedom that are labelled by the given graph. However, the Hamiltonian constraints add degrees of freedom to the state on which they act and the fluctuations of those are therefore no longer suppressed. Indeed, the semiclassical behaviour of the Hamiltonian constraints with respect to these coherent states is rather bad.

Hence we see that the problems of investigating the classical limit of LQG and verifying the quantum algebra of constraints are very much interlinked:

(1) spatial diffeomorphism invariance enforces a weakly discontinuous representation of spatial diffeomorphisms;
(2) anomaly freeness in the presence of only finite diffeomorphisms enforces graph changing Hamiltonian constraints;
(3) graph changing Hamiltonians seem to prohibit appropriate semiclassical states.

1.2. The master constraint programme for LQG

The purpose of the master constraint programme [10, 11] for LQG is to overcome those problems. The classical master constraint for a given (infinite) set of classical constraints is essentially the weighted sum of the squares of the individual constraints. The resulting master constraint carries the same information about the reduced phase space as the original set of individual constraints. Since the infinite set of constraints was replaced by a single one, there are trivially no quantum anomalies no matter whether operators act in a graph changing or non-graph changing fashion. However, whether or not the original quantum constraints that enter the construction of the master constraint are anomalous manifests itself in the spectrum of the master constraint [12]: if the original algebra is anomalous then it is expected that zero is not contained in the spectrum of the master constraint. This can be cured by subtracting from the master constraint the minimum of the spectrum provided of course that it is finite and vanishes as $\hbar \to 0$ so that the modified constraint still has the same classical limit as the original one. One then defines the physical Hilbert space as the (generalized) kernel of the master constraint; see the first reference of [12] for the mathematical details.

The master constraint for GR involves the weighted sum of squares of the Hamiltonian constraints such that the resulting expression is spatially diffeomorphism invariant. In [11] the master constraint has been quantized in two different ways: in the first version one used the graph changing operators defined in [6]. Since the operator must be spatially diffeomorphism invariant, from the results of [5] this operator must be defined directly on the spatially diffeomorphism invariant Hilbert space whose states are labelled by (generalized) knot classes. The semiclassical analysis of this first operator is again difficult because it changes knot classes and because so far no semiclassical spatially diffeomorphism invariant states have been defined in LQG. In the second version one used a non-graph changing operator which therefore can be defined directly on the kinematical Hilbert space. The Hamiltonian constraints that enter this operator would be anomalous. However, as we said, the master constraint does not care about this; moreover, the second master constraint is manifestly spatially diffeomorphism invariant. The second operator therefore can in principle be analysed by existing semiclassical tools.

1.3. Removing the graph dependence of semiclassical states for LQG: algebraic graphs

However, there is still one caveat. As already mentioned, the semiclassical tools for LQG developed so far are based on pure states over single graphs or mixed states based on a certain class of graphs. None of these states involves all the graphs that are allowed in LQG and therefore those states cannot be semiclassical for all degrees of freedom of LQG. See, e.g., the discussion in [13]. One cannot sum over all graphs because the sum is over uncountably many states; hence the state is not normalizable. Rather than taking an uncountable sum one could try to consider an uncountable tensor product which gives normalizable states [14]. The problem here is that there is no such thing as a maximal graph in LQG of which all other graphs are subgraphs.

Therefore, the existing semiclassical tools of LQG are heavily *graph dependent*.

It is at this point where we depart in a crucial way from LQG: we discard the notion of embedded graphs and consider algebraic graphs instead. An algebraic graph is simply a labelling set consisting of abstract points (vertices) together with information about how many abstract arrows (edges) point between points. There is no information about the knotting and braiding of those edges or about the location of the points. All that an algebraic graph knows about is the number of points and their oriented valence (that is, how many arrows point between different vertices with in or outgoing orientation). Hence we lose information about topology and differential structure of the spatial manifold underlying LQG. We call the theory based on algebraic graphs algebraic quantum gravity (AQG) in order to distinguish it from LQG by which it is inspired.

The point of introducing the notion of an algebraic graph is that it can be embedded in all possible ways into a given spatial manifold. Thus, at least all embedded graphs with the same valence structure as the underlying algebraic graph can be obtained in this way and we will see that this is enough in order to do semiclassical physics because all physical (gauge invariant) operators, such as the master constraint, can be defined in an embedding independent (algebraic) fashion. One just has to lift the action of a given LQG operator on embedded graphs to the algebraic graph. What we have achieved by this is that our theory has lost its graph dependence, the chosen algebraic graph is *fundamental* or maximal. It turns out that the algebraic graph necessarily must have a countably infinite number of edges; see below.

1.4. The extended master constraint

There is a problem with this idea which has to do with spatial diffeomorphism invariance. Since there is no σ to begin with, there cannot be any Diff(σ). Therefore, the natural action of the diffeomorphism group on the Hilbert space of LQG is not available in AQG. One could try to argue that the Hilbert space of AQG in some sense is already a space of spatially diffeomorphism invariant states. However, as shown in [15], this would make the physical Hilbert space of gauge invariant states too large. Therefore one somehow must also perform a gauge reduction with respect to the spatial diffeomorphism constraints. Even if we forget about the fact that in AQG there is no Diff(σ) and embed the algebraic graph into some σ and thus consider a fixed embedded lattice, there are problems in defining lattice analogues of the generators of spatial diffeomorphisms, their algebra does not close for finite lattice length (see, e.g., [16]).

It is at this point at which we invoke the *extended* master constraint introduced in [10]. The classical extended master constraint also involves the weighted sum of squares of the spatial diffeomorphism constraints such that the resulting expression is spatially diffeomorphism invariant. It can be quantized in LQG in a graph non-changing and spatially diffeomorphism invariant fashion similar to the simple master constraint. This may come as a surprise because the infinitesimal generator of spatial diffeomorphisms cannot be defined in LQG. The solution of the puzzle is that the weight function that enters the sum over squares becomes an operator which mildens the UV behaviour of the formally singular quantum generators of spatial diffeomorphisms. The point is now that this extended master constraint also naturally lifts to algebraic graphs. This way we have also achieved the implemention of spatial diffeomorphism invariance on the algebraic level without running into anomalies.

Note that many aspects of this idea to work at the embedding independent level had been spelled out already in [17]. However, the programme could then not be pushed to its logical frontiers because it was unclear how to deal with spatial diffeomorphism invariance, that is, the (extended) master constraint programme was not yet developed. Also, there are certain operators in LQG such as the volume operator [18–20] crucial for the quantum dynamics which do carry embedding dependent information and therefore cannot be immediately lifted to the algebraic level. The way we deal with this here is that we choose a fixed algebraic graph once and for all and choose a *generic* embedding (this will be made precise later). We then lift the volume operator of LQG for those embeddings. This will mean that the semiclassical limit of this operator will come out right only if the semiclassical states are defined using a generic embedding but again this turns out to be sufficient for semiclassical purposes.

1.5. The structure of AQG and semiclassical states

As already mentioned, an algebraic graph does not contain any information about the braiding of its edges and is not embedded into any 3-manifold. On such an algebraic graph one can define an abstract *- or C^*-algebra of elementary algebra elements out of which the master constraint is constructed as a composite operator. We use a specific representation of this algebra on a Hilbert space which is motivated from LQG and in this representation the master constraint is a positive, self-adjoint operator. In order to derive the classical limit of the theory we must give the following data: (1) a 3-manifold σ, (2) initial data m on σ (equivalently: a point in the classical phase space, for gravity essentially a 3-metric and its extrinsic curvature) and (3) an embedding of the algebraic graph (and a graph dual to it) into σ. Out of these data one can then construct a coherent state along the lines of [9].

In order that we can define a semiclassical limit for all σ we must necessarily work with (countably) infinite algebraic graphs in order to be able to deal with asymptotically flat topologies. If σ is compact, the embedding of the algebraic graph will contain accumulation points but this is no obstacle for our formulation because we can leave all but finitely many of the edges (and dual faces) of the embedded graph unexcited thus effectively avoiding accumulation points. This leads us naturally to von Neumann's infinite tensor product (ITP) which was applied in the context of LQG in [14]. Moreover, the ITP enables us to embed the algebraic graph as densely as we wish, thus making the semiclassical approximation as good as we like[3].

As an aside we should mention that while the (extended) master constraint can also be defined in LQG in a non-graph changing fashion, such an operator is there rather *ad hoc* because one has to define it also on rather coarse graphs. On those graphs the expression for the operator proposed in [10] cannot be obtained by a regularization process from the classical expression because the loops and edges involved might be 'large'. In contrast, in AQG there is a single graph to be considered and it is typically embedded in such a way that all loops and edges are small, thus being close in appearance to the classical continuum expression.

There is a crucial difference between the semiclassical states of LQG and of AQG. In both theories the coherent states are labelled by embedded graphs. However, in LQG these states are linear combinations of spin network functions[4] over the embedded graph with certain coefficients which carry the above data. In AQG the coherent states are linear combinations of spin network functions only if σ is compact and even then these spin network functions are labelled by the unique abstract graph while the coefficients are labelled by the embedded graph. This tiny difference has, e.g., the consequence that in LQG coherent states over different graphs are automatically orthogonal while in AQG this is not necessarily the case.

Since we can accommodate any σ in our formulation, AQG can presumably deal with topology change. Moreover, as was pointed out in [14], the nonseparable ITP is a direct sum of separable Hilbert spaces (sectors), some of which can be identified with excitations of our semiclassical states just discussed which could make contact with Fock spaces and low energy physics, as sketched in [21].

Note that in AQG, in contrast to the embedded graphs of LQG, the infinite algebraic graph is fixed. AQG theories defined on different infinite algebraic graphs are unitarily equivalent if and only if there is a permutation of the vertices such that the algebraic graphs can be transformed into each other. Hence, in AQG the algebraic graph is a fundamental object. An interesting question is whether one could extend AQG in such a way as to accommodate all algebraic graphs. This seems neither necessary nor meaningful to us because one would need to relate the edges of different algebraic graphs to each other; however, without an embedding this is not possible[5].

[3] This does not work for all operators of the theory but only for those which classically would come from volume integrals. Classical functions of this type separate the points of the classical phase space; see [13] for a discussion.

[4] Spin network functions (SNF) provide an orthonormal basis in LQG, in particular, SNFs labelled by different graphs are orthogonal.

[5] We could declare the Hilbert spaces labelled by different algebraic graphs as orthogonal to each other where different means that there is no permutation transformation between the corresponding adjacency matrices (see section 2). The elementary operators of the theory would then also be labelled by the algebraic graph in addition to edges and vertices and one would embed different algebraic graphs in such a way that they are disjoint in order to be consistent with LQG where such states would be orthogonal. However, there seems to be no physical justification for such a choice at present.

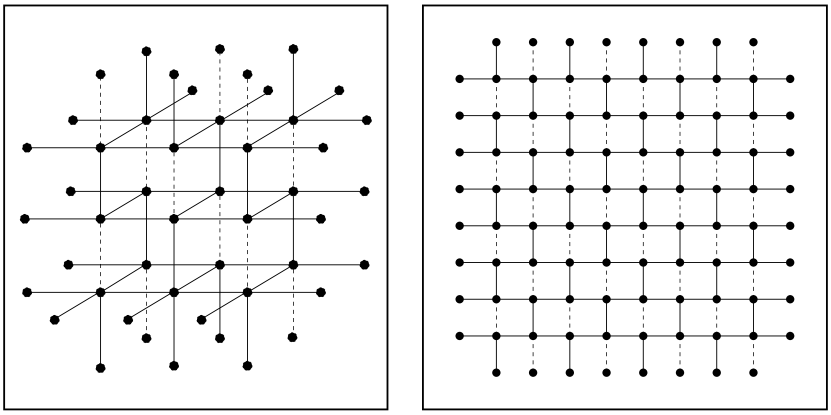

Figure 1. Illustrative example in three (two) dimensions how a five (three)-valent graph can be constructed from a cubic graph by assigning the trivial representation (dashed lines) at one of the edges at each vertex of the graph.

A different viewpoint is to use the maximal (countable) algebraic graph. This is simply the algebraic graph which consists of a countably infinite number of vertices such that each vertex has countably infinite connectivity with every other vertex. In other words, this is the algebraic graph the entries of whose adjacency matrix are all equal to countable infinity $\aleph$. Obviously, all algebraic graphs with finite entries in its adjacency matrix are algebraic subgraphs of this maximal algebraic graph. For an example how a five-valent graph can be constructed from a cubic one by deletion of edges, see figure 1. These subgraphs are the ones that are relevant in practical calculations because one uses states which are only excited on finitely many edges between any two vertices and it turns out that for suitable operators (such as the volume operator) the contribution from unexcited edges drops out[6].

Yet another viewpoint, possibly sufficient in order to accommodate most or at least a big class of embedded graphs at least in semiclassical approximations, is the observation that, given an algebraic graph with finite, given entries in its adjacency matrix, we can choose to embed some of its edges in such a way that they are arbitrarily short with respect to the spatial metric to be approximated in the semiclassical calculations. This edge is then almost equivalent to a virtual edge, it is effectively only an intertwiner and its endpoints almost merge into a single vertex (see also figure 2). This way we can generate, effectively, almost any valence of vertices and we can effectively avoid usage of the maximal algebraic graphs or algebraic graphs with large entries in their adjacency matrix.

Therefore, in order to keep the discussion in this paper and the companion papers simple, we will avoid the maximal algebraic graph but rather choose a sufficiently complex algebraic graph as our fundamental graph. We will indicate where modifications would be necessary in order to work with the maximal algebraic graph. Concretely, we will focus on cubic algebraic graphs (all vertices have valence six and two vertices have at most one edge in common) which will simplify our calculations and turns out to be sufficient in order to perform semiclassical calculations.

Note that no continuum limit has to be performed on the algebraic graph. None of the operators of the theory depends on a lattice length. This is not possible because the theory is

[6] If it does not, then in order to avoid infinities one has to invoke projection operators which again reduce the infinite sums over edges to those which are excited (see [21].)

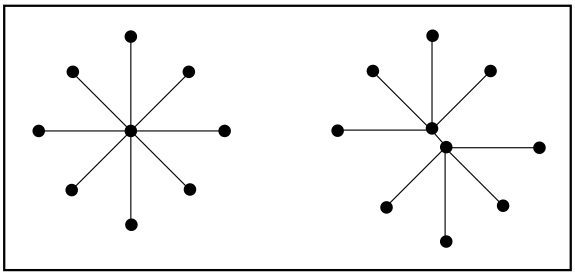

Figure 2. Idea how an eight-valent algebraic graph can be expressed in terms of two five-valent graphs with an connecting (inner) edge which is very short with respect to the background metric approximated by the coherent states.

manifestly background independent. There are no scales to be sent to zero, everything is UV finite. The precision with which the semiclassical limit is reached depends on the choice of the embedding (its 'fineness' with respect to the background metric to be approximated) which is a feature of the state but the fundamental quantum algebra does not know about this. This is in strong contrast to, say, lattice QCD on Minkowski space, where the Hamiltonian depends explicitly on the lattice length. One could interpret AQG as saying that lattice calculations are correct and that the lattice is actually fundamental if it is thought of as the concrete embedding of the algebraic graph. Lattice refinements are then to be thought of as different choices of embeddings of the fundamental algebraic graph. This also sheds new light on Wilson's notion of the renormalization group.

1.6. The semiclassical limit

With this setup, in two companion papers [22, 23] we will establish that the semiclassical limit of the extended master constraint is correct. More precisely, in [22] we carry out an exact computation using a simplification which consists in replacing the non-Abelian group $SU(2)$ by the Abelian group $U(1)^3$. This computation reproduces the classical $U(1)^3$ analogue of the master constraint to zeroth order in $\hbar$. The point of this approximation is that the $U(1)^3$ analogue of the volume operator, which enters the master constraint in a pivotal way, is analytically diagonizable. This is not the case for $SU(2)$ and prohibits exact semiclassical calculations. In [23] we develop semiclassical perturbation theory for AQG and LQG *with error control* which allows us to analytically calculate coherent state matrix elements of positive fractional powers of the $SU(2)$ volume operator up to any order in $\hbar$. The resulting semiclassical $SU(2)$ calculation is then exactly analogous to the $U(1)^3$ and reproduces the same classical limit as follows from the results of [9]. Hence [22, 23] together imply that the infinitesimal gauge generators of AQG have the correct classical limit. This is what is so far missing in LQG.

The coherent states chosen may be further improved, for instance, by statistical averaging over a certain class of embedded graphs so as to produce a density matrix (see, e.g., [24]). Note that here again there is a crucial difference between LQG and AQG. In LQG the statistical average of coherent states, which are linear combinations of spin network states, affects both the spin network states and their coefficients. In AQG it affects only the coefficients. Let Γ be some uncountable set of graphs embedded into some σ and let μ be a probability measure on Γ. Let P_{ψ_γ} be the projection onto the coherent state ψ_γ and consider the object $\rho := \int_\Gamma \mathrm{d}\mu(\gamma) P_{\psi_\gamma}$. Then it is not difficult to see [13] that in LQG this operator is the zero operator while in AQG

Table 1. Summary of the major differences in the mathematical structure of loop quantum gravity (LQG) and algebraic quantum gravity (AQG).

Object	LQG	AQG
Topology	Must be provided	Absent
Differentiable structure	Must be provided	Absent
Hilbert space	$\mathcal{H}_{\text{LQG}} := \mathcal{H}_{\text{AIL}}$	$\mathcal{H}_{\text{AQG}} := \mathcal{H}^{\otimes}$
Separability	Nonseparable	Nonseparable
Graphs	Embedded	Algebraic (combinatorial)
Number of graphs	Uncountably infinite	One
Structure of graphs	Finite	Countably infinite
Generating set of *-algebra $\mathfrak{A}$	Uncountably infinite	Countably infinite

this operator is trace class with unit trace. Even if we formally interchange the integral over Γ with taking the trace there are still qualitative differences, for generic operators A in LQG which admit an embedding independent lift to AQG such as the total volume of a compact manifold σ between the corresponding values of $\text{Tr}(A\rho)$.

1.7. Summary of differences between AQG and LQG

For the benefit of the reader we summarize the most important conceptual differences and similarities between AQG and LQG in table 1.

Note that the reason for the Hilbert spaces to be nonseparable is very different in the two cases. For LQG, it is due to the fact the set Γ of all finite embedded graphs is uncountable. For AQG, it is due to the fact that the ITP of a countable number of Hilbert spaces of which at least countably infinite many are at least two dimensional is not separable. Also the two Hilbert spaces of LQG and AQG are not directly related to each other. The only thing one can say is the following: given an algebraic graph, a manifold σ and an embedding X we can consider the set Γ_α^X of all finite subgraphs of $X(\alpha)$. Consider the closed linear span $\mathcal{H}_\alpha^X$ of spin network states over elements of Γ_α^X. Then $\mathcal{H}_\alpha^X \subset \mathcal{H}_{\text{LQG}}$ for all X. On the other hand, for all X the spaces $\mathcal{H}_\alpha^X$ are isomorphic to the sector of $\mathcal{H}_{\text{AQG}}$ which is the closed linear span of finite excitations of the vector $\otimes_e \mathbf{1}$, where $\mathbf{1}$ is the constant function equal to unity.

Note that in LQG one needs all graphs because the algebra of elementary operators contains the holonomies along all possible paths and those are obtained from a fixed given path through the natural action of the diffeomorphism group. In AQG the action of the infinitesimal diffeomorphisms preserves the algebraic graph and so there is no need to take all algebraic graphs into account. This is different from what was done in [14] where one worked in an embedding-dependent context and considered ITP Hilbert spaces over all possible countably infinite embedded graphs.

1.8. Organization of the paper

In section 2 we introduce the concept of an algebraic graph and define the abstract *-algebra labelled by it. For an arbitrary algebraic graph, we introduce the extended master constraint using the notion of a minimal loop.

In section 3 we review the framework of coherent states as developed in [9] as well as elements of the infinite tensor product construction of [14] and lift it to the algebraic level.

In section 4 we present the result of the calculation of our companion papers [22, 23] which establishes the correctness of the classical limit of the master constraint on cubic algebraic graphs.

In section 5 we forecast the tasks that can be addressed using the new AQG framework. In particular, we have in mind applications in quantum cosmology and the contact with the physics of the standard model. In order to do so one has to deal with the question in which sense one can perform trustable computations without solving the theory[7]. We present a possible scheme, elements of which were proposed in [25], which could be called *quantum gauge fixing*. In section 6 we sketch how ideas from AQG might help to solve two important problems for the spin foam approach to LQG, namely (1) to make contact with the canonical programme which as we prove in [22, 23] does have contact with the classical theory, and (2) to get rid of the triangulation dependence of spin foam models.

Finally, in section 7 we summarize and list some interesting open problems.

2. Algebraic quantum gravity

As appropriate for a top-to-bottom approach we introduce the basic ingredients of AQG axiomatically. In a second step we show how to extract physics from the mathematical notions and in particular reveal the connection with LQG. The latter is the subject of sections 3 and 4.

2.1. Algebraic graphs

Comprehensive monographs on algebraic graphs are listed in [26]. Here we just summarize what is needed at this point for our purposes. An (oriented) algebraic graph with N vertices can be defined in terms of its adjacency matrix. This is an $N \times N$ matrix α whose entries α_{IJ} take non-negative integer values n where n denotes the number of edges that start in vertex i and end in vertex j. Note that α_{IJ}, α_{JI} are not related to each other and that $e_{IJ} := \alpha_{IJ} + \alpha_{JI}$ is the total number of edges that connect vertices I, J. The valence of I is given by $v_I = \sum_J e_{IJ}$. We also use the symbols $V(\alpha), E(\alpha)$ to denote the set of vertices and edges respectively and $b(e), f(e)$ to denote the vertex at which e begins or finishes, respectively.

We are only interested in oriented algebraic graphs but for completeness we mention that for unoriented algebraic graphs the adjacency matrix is symmetric, its entries α_{IJ} being the total number of edges connecting vertices I, J and $v_I = \sum_J \alpha_{IJ}$ is the valence of vertex I. We will be interested in $N = \aleph$, i.e., graphs where the number of edges has countably infinite cardinality but where the valence of each vertex is bounded by a small number of order unity, typically by $2D$ for cubic algebraic graphs or $D + 1$ for simplicial algebraic graphs which we wish to embed into a D-dimensional manifold. This is necessary in order that the semiclassical limit of the theory is reached for arbitrary noncompact topologies σ.

There is no information contained in the adjacency matrix which tells us how the various edges are braided. Also no information is available whether the edges are smooth, or n-times differentiable, whether the tangents of two edges adjacent at a vertex intersect there at a non-vanishing angle etc. In particular, cubic algebraic graphs 'with defects', i.e. those obtained by deleting $D - 1$ edges adjacent at each vertex (the degrees of freedom on the deleted edges are then not excited) can be considered as algebraic simplicial graphs; hence the simplicial case is contained in the cubic one. This might be of some importance because it is easy to generate random, simplicial, embedded graphs by the Dirichlet–Voronoi procedure [24] which

[7] That is, the construction of (1) physical states annihilated by the master constraint, (2) operators commuting with it as well as (3) a definition of the quantum dynamics in terms of physical Hamiltonians.

improves the semiclassical properties of coherent states or density matrices constructed from them [13].

2.2. *Quantum kinematics*

2.2.1. Gauge field and gravitational sector. Given an algebraic graph α we associate with each of its (distinguishable) edges e an element $A(e)$ of a compact, connected, semisimple Lie group G and an element $E(e)$ of[8] its Lie algebra Lie(G).

These are subject to the algebraic relations

$$\begin{aligned}&[A(e), A(e')] = 0\\&[E_j(e), A(e')] = \mathrm{i}\hbar Q^2\delta_{e,e'}\tau_j/2A(e)\\&[E_j(e), E_k(e')] = -\mathrm{i}\hbar Q^2\delta_{e,e'}f_{jkl}E_l(e').\end{aligned} \tag{2.1}$$

Furthermore, the following *-relations hold:

$$A(e)^* = [A(e)^{-1}]^T, \qquad E_j(e)^* = E_j(e). \tag{2.2}$$

We will denote the resulting[9] *-algebra by $\mathfrak{A}$. Here Q^2 plays the role of the coupling constant of the gauge theory[10] in question and τ_j, $j = 1, \ldots, \dim(G)$ are generators of the Lie algebra of G which we take to be anti-Hermitian and trace-free for convenience since any compact, semisimple Lie group can be realized as a subgroup of some $SU(N)$. These satisfy $[\tau_j, \tau_k] = f_{jkl}\tau_l$ where the structure constants f_{jkl} are totally skew and we normalize according to $\mathrm{Tr}(\tau_j\tau_k) = -\frac{1}{2}\delta_{jk}$. Also $E_j(e) := -2\,\mathrm{Tr}(\tau_j E(e))$. Obviously, (2.1) takes the form of a direct sum of *-algebras, one for each edge e, each of which can be considered as the quantization of the cotangent bundle $T^*(G)$.

A natural representation of the algebra $\mathfrak{A}$ in (2.1) is the infinite tensor product (ITP) Hilbert space $\mathcal{H}^{\otimes} := \otimes_e\mathcal{H}_e$ where $\mathcal{H}_e \cong L_2(G, \mathrm{d}\mu_H)$ and μ_H is the Haar measure on G. Other representations are conceivable but this representation is natural if we want to match the uniqueness result [7, 8] of LQG valid for any (semianalytic) 3-manifold. For a review of the ITP and the associated von Neumann algebras connected with it, see, e.g., [14]. We just collect the necessary notions here.

The ITP Hilbert space is closure of the finite linear span of vectors of the form $\otimes_f := \otimes_e f_e$ where $f_e \in \mathcal{H}_e$. The inner product between these vectors is given by

$$\langle\otimes_f, \otimes_{f'}\rangle := \prod_e \langle f_e, f'_e\rangle_{\mathcal{H}_e}. \tag{2.3}$$

The infinite product $\prod_e z_e$ of complex numbers $z_e = |z_e|\,\mathrm{e}^{\mathrm{i}\phi_e}$ is defined by $\prod_e z_e := \left[\prod_e |z_e|\right]\mathrm{e}^{\mathrm{i}\sum_e\phi_e}, \phi_e \in [-\pi, \pi)$ provided that both of $\sum_e \|z_e| - 1|$ and $\sum_e |\phi_e|$ converge, in which case we also say that $\prod_e z_e$ is convergent. Otherwise we set $\prod_e z_e = 0$. One can show that for $z = \prod_e z_e \neq 0$ we can find for any $\delta > 0$ a finite subset $E_\delta(\alpha)$ of the set $E(\alpha)$ of edges of α such that $\left|z - \prod_{e\in E_\delta(\alpha)} z_e\right| < \delta$ for all $E_\delta(\alpha) \subset E \subset E(\alpha)$. Obviously we consider only elements such that $\|\otimes_f\| \neq 0$.

Two vectors $\otimes_f, \otimes_{f'}$ are said to be strongly equivalent if and only if $\sum_e |\langle f_e, f'_e\rangle_{\mathcal{H}_e} - 1|$ converges. We denote by $[f]$ the strong equivalence class of f. It follows that $\langle\otimes_f, \otimes_{f'}\rangle = 0$ if either $[f] \neq [f']$ or $[f] = [f']$ and $\langle f_e, f'_e\rangle = 0$ for at least one e.

We say that $\prod_e z_e$ is quasi-convergent if $\prod_e |z_e|$ converges. If we set $(z \cdot f)_e := z_e f_e$ then $\otimes_{z\cdot f} = \left(\prod_e z_e\right)\otimes_f$ fails to hold if $\prod_e z_e$ is not convergent. We say that f, f' are

[8] For LQG practitioners we stress that the notation $E(e)$ is no misprint: E is just labelled by the edges of the algebraic graph, surfaces will come in only when we consider semiclassical states.

[9] We have set the gravitational Immirzi parameter to unity, otherwise rescale Q appropriately.

[10] In particular $Q^2 = 8\pi G_{\mathrm{Newton}}$ for gravity. If needed we write Q_{GR} for gravity and Q_{YM} for the gauge field sector.

weakly equivalent provided that there exists z such that $[z \cdot f] = [f']$. This is equivalent to the convergence of $\sum_e ||\langle f_e, f'_e\rangle| - 1|$. We denote by (f) the weak equivalence class of f. Obviously, strong equivalence implies weak equivalence. One can show that the closure of the span of all vectors in the same strong equivalence class $[f]$, denoted by $\mathcal{H}^{\otimes}_{[f]}$, is separable, consisting of the completion of the finite linear span of the vectors of the form $\otimes_{f'}$ where $f'_e = f_e$ for all but finitely many e. The ITP Hilbert space $\mathcal{H}^{\otimes}$ is the direct sum of the $\mathcal{H}^{\otimes}_{[f]}$. Let also $\mathcal{H}^{\otimes}_{(f)}$ be the closure of the finite linear span of the $\otimes_{f'}$ with $(f') = (f)$. Then the strong equivalence subspaces of $\mathcal{H}^{\otimes}_{(f)}$ are unitarily equivalent, the corresponding unitary operators being of the form $U_z\otimes_f := \otimes_{z\cdot f}$ with $\prod_e z_e$ quasi-convergent.

Our basic operators act in the obvious way as

$$\begin{aligned} A(e)\otimes_f &:= [A(e)f_e] \otimes [\otimes_{e'\neq e} f_{e'}] \\ E_j(e)\otimes_f &:= [E_j(e)f_e] \otimes [\otimes_{e'\neq e} f_{e'}] \end{aligned} \tag{2.4}$$

where $[A(e)f_e](h) := hf_e(h)$ and $[E_j(e)f_e](h) := \mathrm{i}\hbar Q^2\big[\frac{d}{\mathrm{d}t}\big]_{|t=0} f_e(\mathrm{e}^{t\tau_j/2}h)$. It is not difficult to show that this makes $A(e)$ a unitary matrix valued (in particular bounded) multiplication operator and $E_j(e)$ an essentially self-adjoint derivation operator. Relations (2.4) define them densely on $\mathcal{H}^{\otimes}$. This concludes the definition of the quantum kinematics.

2.2.2. Fermionic sector. Given an algebraic graph α we associate with each vertex $v \in V(\alpha)$ Grassmann-valued variables $\theta_M(v), \bar\theta_M(v)$, where M is a compound index $M \equiv (m, I)$, where $m = \pm 1/2$ is a Weyl spinor index and $I = 1, \ldots, d$, where d is the dimension of the defining representation of the Yang–Mills group G. These are subject to the anti-commutation relations

$$[\theta_M(v), \theta_N(v')]_+ = [\bar\theta_M(v), \bar\theta_N(v')]_+ = 0, \qquad [\theta_M(v), \bar\theta_N(v')]_+ = Q_{\mathrm{F}}^2\hbar\delta_{MN}\delta_{v,v'} \tag{2.5}$$

as well as the *-relations

$$[\theta_M(v)]^* = \bar\theta_M(v). \tag{2.6}$$

Here $\hbar Q^2$ is dimensionfree if we take θ to be dimensionless[11]. We consider just one fermion species and only one helicity[12]. Again we will denote the resulting *-algebra by $\mathfrak{A}$. A natural representation thereof is again by an infinite tensor product: for each v we consider the 2^{2d}-dimensional Hilbert space of 'holomorphic' functions[13] $f_v(\theta) = \sum_{k=0}^{2d}\sum_{1\leqslant M_1<\cdots<M_{2d}} f_v^{M_1,\ldots,M_k}\theta_{M_1}(v)\cdots\theta_{M_k}(v)$ where the complex valued coefficients are totally skew. Set for one single Grassmann degree of freedom $\mathrm{d}\mu(\theta) = \mathrm{d}\theta\,\mathrm{d}\bar\theta\big(1+\bar\theta\theta\big/\big(\hbar Q_{\mathrm{F}}^2\big)\big)$ and define the usual Berezin 'integral' over superspace (better: linear functional) $\int \mathrm{d}\theta 1 = 0$, $\int \mathrm{d}\theta\,\theta = 1$. We now consider the infinite tensor product $\mathcal{H}^{\otimes} := \otimes_{v\in V(\alpha)}\mathcal{H}_v$ where $\mathcal{H}_v = L_2(\mathrm{d}\mu_v), \mathrm{d}\mu_v(\theta) = \prod_M \mathrm{d}\mu(\theta_M(v))$ which is a representation space of $\mathfrak{A}$ via $(\theta_M(v)f_v)(\theta) := \theta_M(v)f_v(\theta)$ and $(\bar\theta_M(v)f_v)(\theta) := \hbar Q_{\mathrm{F}}^2\partial\big/\partial\theta_M(v)f_v(\theta)$ (left derivative).

All remarks about the infinite tensor product from the last subsection apply, just that the label set has switched from edges to vertices.

2.2.3. Higgs sector. Given an algebraic graph α we associate with each vertex $v \in V(\alpha)$ Lie(G) valued (if the Higgs transforms in the adjoint representation) or vector valued (if

[11] The θ are related to the usual fermionic degrees of freedom of dimension cm$^{-3/2}$ by a canonical transformation which takes care of the dimensionalities (see below).

[12] By the canonical transformation (it preserves anti-Poisson brackets) $\theta_m \mapsto \bar\theta_m$ one can switch between left and right handed descriptions.

[13] Note that $\theta, \bar\theta$ are classically anticommuting Grassmann numbers but that in quantum theory classical identities such as the nilpotency $[\theta\bar\theta]^2 = 0$ no longer hold.

the Higgs transforms in the defining representation of G) fields $\phi_j(v), \pi_j(v)$ subject to the algebraic relations

$$[\phi_j(v), \phi_k(v')] = [\pi_j(v), \pi_k(v')] = 0, \qquad [\pi_j(v), \phi_k(v')] = \mathrm{i}\hbar Q_{\mathrm{H}}^2 \delta_{jk}\delta_{v,v'} \tag{2.7}$$

and the *-relations

$$\phi_j(v)^* = \phi_j(v), \qquad \pi_j(v)^* = \pi_j(v) \tag{2.8}$$

if the Higgs is Lie(G) valued. If it transforms in the defining representation so that it is complex valued then we split the Higgs into real and imaginary parts and impose (2.8) on those.

Again an infinite tensor product provides a representation of this *-algebra $\mathfrak{A}$. Consider the probability measure on $\mathbb{R}$ given by $\mathrm{d}\mu(x) = \mathrm{e}^{x^2/2}\,\mathrm{d}x/\sqrt{2\pi}$ and $\mathrm{d}\mu_v(\phi) := \prod_j \mathrm{d}\mu(\phi_j(v))$. Let $\mathcal{H}_v = L_2(\mathrm{d}\mu_v)$ and $\mathcal{H}^{\otimes} = \otimes_v \mathcal{H}_v$. We consider functions of the form $f_v(\phi) \equiv f_v(\{\phi_j(v)\}_j)$, which depend only on the $\phi_j(v)$. Then $[\phi_I(v) f_v](\phi) := \phi_I(v) f_v(\phi)$ and $[\pi_I(v) f_v](\phi) := \mathrm{i}\hbar[\partial/\partial\phi_I(v) - \phi_I(v)/2] f_v(\phi)$ provide a representation of $\mathfrak{A}$ on $\mathcal{H}^{\otimes}$.

2.3. *Quantum dynamics*

We turn now to the quantum dynamics. Pivotal for everything to come is the volume operator. Given a vertex v of the algebraic graph, we set

$$V_v := \ell_P^3 \sqrt{\left| \frac{1}{48} \sum_{e_1\cap e_2\cap e_3 = v} \epsilon_v(e_1, e_2, e_3)\epsilon^{ijk} E_i(e_1) E_j(e_2) E_k(e_3) \right|} \tag{2.9}$$

where the sum is over all triples of mutually distinct edges e_1, e_2, e_3 incident at v. The totally skew function $(e_1, e_2, e_3) \mapsto \epsilon_v(e_1, e_2, e_3)$ takes values ± 1, 0 and will be chosen according to the algebraic graph in question in such a way that it matches the embedding dependent volume operator of LQG [19] when embedding the algebraic graph in a generic[14] way. The functions $\epsilon_v(e_1, e_2, e_3)$ are then chosen once and for all, they are embedding independent. Note that the embedding independent operator [18] has been ruled out as inconsistent in a recent analysis [27, 28]. In formula (2.9) we have assumed that all edges are outgoing from v. If e is ingoing at v, then replace $E_j(e)$ by $-\mathrm{Ad}_{jk}(h_e) E_k(e)$ where $h\tau_j h^{-1} =: \mathrm{Ad}_{jk}(h)\tau_k$ defines the adjoint representation of G on Lie(G).

We will also need the total volume given by $V = \sum_{v\in V(\alpha)} V_v$. Finally we need the crucial operators

$$Q_v^{(r)} = \frac{1}{T_v} \sum_{e_1\cap e_2\cap e_3 = v} \epsilon_v(e_1, e_2, e_3)\,\mathrm{Tr}\Big(\big(A(e_1)\big[A(e_1)^{-1}, V_v^r\big]\big) \times \big(A(e_2)\big[A(e_2)^{-1}, V_v^r\big]\big)\big(A(e_3)\big[A(e_3)^{-1}, V_v^r\big]\big)\Big) \tag{2.10}$$

where T_v is the number of unordered triples of mutually distinct edges incident at v and r is any real number. They will be needed in order to ensure the correct density weight of the various expressions in the classical limit of the master constraint.

We now consider the following composite operators the classical limit of which are half densities.

[14] The possible embeddings of an algebraic graph fall into diffeomorphism equivalence classes. An embedding is called generic if a random embedding results with non-vanishing probability in an embedded graph of the same equivalence class. If there is more than one possibility then we must pick one. For our cubic graph to be considered later we consider half-generic embeddings in the sense that there is a neighbourhood of each vertex and a coordinate system in which the graph looks like the three coordinate axes in $\mathbb{R}^3$.

2.3.1. Gravitational sector.

A.1a *Gravitational Gauss constraint*. For any $v \in V(\alpha)$ we set

$$G_j^{GR}(v) := Q_v^{(1/2)} \left[\sum_{b(e)=v} E_j(v) - \sum_{f(e)=v} \mathrm{Ad}_{jk}(A(e)) E_k(v) \right] \tag{2.11}$$

where $j, k = 1, 2, 3$ for $G = SU(2)$.

B.1 *Spatial diffeomorphism constraint*. Given a vertex v of the algebraic graph α and two edges e, e' incident at and outgoing from v, a loop $\beta_{v,e,e'}$ within α starting at v along e and ending at v along $(e')^{-1}$ is said to be minimal [10] provided that there is no other loop within α satisfying the same restrictions with fewer edges traversed. We denote by $L(v, e, e')$ the set of minimal loops with the data indicated[15].

For any $v \in V(\alpha)$ we set

$$D_j^{GR}(v) := \frac{1}{T_v} \sum_{e_1 \cap e_2 \cap e_3 = v} \frac{\epsilon_v(e_1, e_2, e_3)}{|L(v, e_1, e_2)|} \times \sum_{\beta \in L(v, e_1, e_2)} \mathrm{Tr}(\tau_j [A(\beta) - A(\beta)^{-1}] A(e_3) [A(e_3)^{-1}, \sqrt{V_v}]) \tag{2.12}$$

where the sum is over unordered triples of mutually distinct edges adjacent to v and where again we assumed for convenience that all edges are outgoing from v. The quantity $T_v := |\{e_1 \cap e_2 \cap e_3 = v; |\epsilon_v(e_1, e_2, e_3)| = 1\}|$ is the number of contributing triples[16].

C.1a *Euclidean Hamiltonian constraint*. For any $v \in V(\alpha)$ and any $0 < r \in \mathbb{Q}$ we set

$$H_E^{(r)}(v) := \frac{1}{T_v} \sum_{e_1 \cap e_2 \cap e_3 = v} \frac{\epsilon_v(e_1, e_2, e_3)}{|L(v, e_1, e_2)|} \times \sum_{\beta \in L(v, e, e')} \mathrm{Tr}([A(\beta) - A(\beta)^{-1}] A(e_3) [A(e_3)^{-1}, (V_v)^r]) \tag{2.13}$$

where the conventions are the same as above. This constraint is just an auxiliary construction which we need in order to define various other quantities, it has no physical meaning in our manifestly Lorentzian theory.

C.1b *(Lorentzian) Hamiltonian constraint*. For any $v \in V(\alpha)$ we set

$$H^{GR}(v) - H_E^{(1/2)}(v) := \frac{1}{T_v} \sum_{e_1 \cap e_2 \cap e_3 = v} \epsilon_v(e_1, e_2, e_3) \,\mathrm{Tr}\big(\big(A(e_1) \big[A(e_1)^{-1}, \big[H_E^{(1)}, V \big] \big] \big) \times \big(A(e_2) \big[A(e_2)^{-1}, \big[H_E^{(1)}, V \big] \big] \big) \big(A(e_3) [A(e_3)^{-1}, \sqrt{V_v}] \big) \big) r \tag{2.14}$$

where the conventions are the same as above and $H_E^{(1)} := \sum_v H_E^{(1)}(v)$, $V := \sum_v V_v$.

2.3.2. Yang–Mills sector.

A.2b *Yang–Mills Gauss constraint*. For any $v \in V(\alpha)$ we set

$$\underline{G}_J^{\mathrm{YM}}(v) := Q_v^{(1/2)} \left[\sum_{b(e)=v} \underline{E}_J(v) - \sum_{f(e)=v} \mathrm{Ad}_{JK}(\underline{A}(e)) \underline{E}_K(v) \right] \tag{2.15}$$

[15] If we would work with the maximal algebraic graph, then the set $L(v, e, e')$ would need to be reduced to those minimal loops which are possible within the algebraic graph on which a given state depends, i.e., on which it is excited. This is equivalent to introducing suitable projection operators as described in [21].

[16] To T_v a comment similar to that for $L(v, e, e')$ applies when extending the framework to the maximal algebraic graph.

where $J, K = 1, \ldots, \dim(G)$ for G and we use underlined symbols to distinguish gravitational and Yang–Mills quantities.

B.2 *Spatial diffeomorphism constraint*. For any $v \in V(\alpha)$ we set

$$D_j^{\mathrm{YM}}(v) := Q_v^{(1/6)} \frac{1}{P_v} \sum_{e_1 \cap e_2 = v} \frac{1}{|L(v, e_1, e_2)|} \sum_{\beta \in L(v, e_1, e_2)} \mathrm{Tr}([\underline{A}(\beta) - A(\beta)^{-1}]\underline{E}(e_1)) E_j(e_2) \tag{2.16}$$

where the sum is over all pairs of distinct edges adjacent to v and P_v is their number.

C.2 *Hamiltonian constraint*. For any $v \in V(\alpha)$ we set

$$\begin{aligned} H^{\mathrm{YM}}(v) = \frac{1}{2Q^2} \frac{1}{P_v'} &\sum_{e_1 \cap e_2 = v} \left[\mathrm{Tr}\left(\tau_j A(e_1)\left[A(e_1)^{-1}, V_v^{1/4}\right]\right) \underline{E}_J(e_1)\right]^\dagger \\ &\times \left[\mathrm{Tr}\left(\tau_j A(e_2)\left[A(e_2)^{-1}, V_v^{1/4}\right]\right) \underline{E}_J(e_2)\right] \\ &+ \frac{1}{T_v} \sum_{e_1 \cap e_3 \cap e_4 = v} \frac{1}{T_v} \sum_{e_2 \cap e_5 \cap e_6 = v} \frac{\epsilon_v(e_1, e_3, e_4)}{|L(v, e_3, e_4)|} \frac{\epsilon_v(e_2, e_5, e_6)}{|L(v, e_5, e_6)|} \sum_{\beta \in L(v, e_3, e_4)} \\ &\times \sum_{\beta' \in L(v, e_5, e_6)} \left[\mathrm{Tr}\left(\tau_j A(e_1)\left[A(e_1)^{-1}, V_v^{1/4}\right]\right) \mathrm{Tr}(\underline{\tau}_J \underline{A}(\beta))\right] \\ &\times \left[\mathrm{Tr}\left(\tau_j A(e_2)\left[A(e_2)^{-1}, V_v^{1/4}\right]\right) \mathrm{Tr}(\underline{\tau}_J \underline{A}(\beta'))\right] \end{aligned} \tag{2.17}$$

Here in the electric term we sum over all pairs of edges incident at v and P_v' is their number.

2.3.3. Fermionic sector.

A.3a *Gravitational Gauss constraint*. For any $v \in V(\alpha)$ we define

$$G_j^F(v) = Q_v^{(1/2)} \sum_I \bar{\theta}_{mI}(v)(\tau_j)_{mn}\theta_{nI}(v). \tag{2.18}$$

A.3b *Yang–Mills Gauss constraint*. For any $v \in V(\alpha)$ we define

$$\underline{G}_J^F(v) = Q_v^{(1/2)} \sum_m \bar{\theta}_{mI}(v)(\underline{\tau}_J)_{IK}\theta_{mK}(v). \tag{2.19}$$

B.3 *Spatial diffeomorphism constraint*. For any $v \in V(\alpha)$ we define

$$D_j^F(v) := \frac{\mathrm{i}}{2} \sum_{b(e)=v} Q_v^{(1/6)} E_j(e)[\bar{\theta}_{mJ}(v)[A(e)]_{mn}[\underline{A}(e)]_{JK}\theta_{nK}(f(e)) - \mathrm{h.c.}] \tag{2.20}$$

where, as usual, h.c. denotes the adjoint, with respect to our chosen representation, of the expression in the bracket and the sum is over all edges adjacent to v which are outgoing from there[17].

C.3 *Hamiltonian constraint*. For any $v \in V(\alpha)$ we define

$$\begin{aligned} H^F(v) = \sum_{b(e)=v} Q_v^{(1/2)} E_j(e) &\left\{\left[Q_{f(e)}^{(1/2)}\right]^2 (A(e))_{jk} \bar{\theta}_{mJ}(f(e))(\tau_j)_{mn}\theta_{nJ}(f(e))\right. \\ &- \left[Q_v^{(1/2)}\right]^2 \bar{\theta}_{mJ}(v)(\tau_j)_{mn}\theta_{nJ}(v) \\ &+ i\left[Q_v^{(1/2)}\right]^2 [\bar{\theta}_{mJ}(v)[A(e)]_{pn}[\underline{A}(e)]_{JK}(\tau_j)_{mp}\theta_{nK}(f(e)) - \mathrm{h.c.}] \\ &\left. - \left[Q_v^{(1/2)}\right]^2 \mathrm{Tr}\left(\tau_j A(e)\left[A(e)^{-1}, \left[H_E^{(1)}(1), V\right]\right]\right) \bar{\theta}_{mJ}(v)\theta_{mJ}(v)\right\}. \end{aligned} \tag{2.21}$$

Here $(A(e))_{jk}$ denotes the matrix elements of the holonomy in the spin one representation.

[17] This corresponds to the forward lattice derivative. One can also add a term involving the incoming edges adjacent to v corresponding to the backward lattice derivative.

2.3.4. Higgs sector.

A.4b *Yang–Mills Gauss constraint*. For any $v \in V(\alpha)$ we define

$$\underline{G}_J^H(v) = Q_v^{(1/2)} \pi_K(v) (\underline{\tau}_J)_{KL} \phi_L(v). \tag{2.22}$$

B.4 *Spatial diffeomorphism constraint*. For any $v \in V(\alpha)$ we define

$$D_j^H = \left[Q_v^{(1/2)}\right]^3 \sum_{b(e)=v} E_j(e) \pi_J(v) [(\underline{A}(e))_{JK} \phi_K(f(e)) - \phi_J(v)]. \tag{2.23}$$

C.4 *Hamiltonian constraint*. For any $v \in V(\alpha)$ we define

$$\begin{aligned} H^H(v) = {} & \frac{1}{2}\left[Q_v^{(1/2)}\right]^3 \pi_J(v)\pi_J(v) + \frac{1}{2} V_v^{1/2} U(\phi(v)) + \frac{1}{2}\left[Q_v^{(1/2)}\right]^3 \\ & \times \left[\sum_{b(e)=b(e')=v} E_j(e) E_j(e') [(\underline{A}(e))_{JK} \phi_K(f(e)) - \phi_J(v)] \right. \\ & \times [(\underline{A}(e'))_{JL} \phi_L(f(e')) - \phi_J(v)] \end{aligned} \tag{2.24}$$

where U is a positive, gauge invariant function of the $\phi_I(v)$, called the potential term.

2.3.5. The (extended) master constraint. We now simply add all the various geometry and matter contributions[18]

$$\begin{aligned} G_j(v) &:= G_j^{GR}(v) + G_j^F(v) \\ \underline{G}_J(v) &:= \underline{G}_J^{\rm YM}(v) + \underline{G}_J^F(v) + \underline{G}_J^H(v) \\ D_j(v) &:= D_j^{GR}(v) + D_j^{\rm YM}(v) + D_j^F(v) + D_j^H(v) \\ H(v) &:= H^{GR}(v) + H^{\rm YM}(v) + H^F(v) + H^H(v) + \Lambda\sqrt{V_v} \end{aligned} \tag{2.25}$$

where we have added a possible cosmological term in the last line and can now simply define the master constraint as

$$\mathbf{M} := \sum_{v \in V(\alpha)} [G_j(v)^\dagger G_j(v)) + \underline{G}_J(v)^\dagger \underline{G}_J(v) + D_j(v)^\dagger D_j(v) + H(v)^\dagger H(v)]. \tag{2.26}$$

The master constraint is manifestly positive and we take as its self-adjoint extension the Friedrich's extension[19].

Several remarks are in order:

(I) *Difference with background dependent theories*. What is remarkable about all these formulae is that they are rather similar to the expressions familiar from (Hamiltonian) lattice gauge theory. For instance, on a regular cubic spatial lattice embedded in $\mathbb{R}^3$ with edge length ϵ with respect to the standard Euclidean metric the classical continuum Yang–Mills Hamiltonian

$$H_{\rm YM} = \frac{1}{2Q^2} \int_{\mathbb{R}^3} {\rm d}^3x \, \delta_{ab} \, {\rm Tr}[E^a E^b + B^a B^b], \tag{2.27}$$

[18] We suppress appropriate numerical coefficients which turn all the terms to be added dimensionless and such that in the semiclassical limit [22] these terms have the same coefficients as in the classical constraints. More precisely, for each commutator between a holonomy and a power r of the volume operator we should divide by $r\hbar Q_{\rm GR}^2$ and each contribution to either constraint comes with an additional factor of $1/Q_{\rm sector}^2$.

[19] Some care is required in order to do this. As we will show in section 5, the master constraint preserves all strong equivalence class Hilbert subspaces of the ITP. Since polynomials of elements of $\mathfrak{A}$ applied to the vector $\otimes_f$ lie dense in the sector defined by $\otimes_f$ it follows that the master constraint is densely defined on that sector if and only if $\|\mathbf{M} \otimes_f\|$ is finite. We will simply remove those sectors from the ITP on which $\mathbf{M}$ is not defined and take the Friedrich's extension on each of the remaining sectors. As we show in this series of papers, all sectors corresponding to classical background geometries lie in the domain of the master constraint.

where $B^a = \epsilon^{abc}F_{bc}/2$ is the magnetic field and F the curvature of the Yang–Mills connection, would be discretized in terms of our lattice variables as

$$H_{\mathrm{YM}} = \frac{1}{2Q^2\epsilon}\sum_v \delta_{ab}\,\mathrm{Tr}\big[E\big(e_v^a\big)E\big(e_v^b\big) + A\big(\beta_v^a\big)A\big(\beta_v^b\big)\big], \tag{2.28}$$

where the sum is over all vertices of the lattice, e_v^a is the edge in the ath direction beginning at v and β_v^a is the plaquette loop in the $x^a = \mathrm{const.}$ plane beginning at v.

This expression should be contrasted with the classical expression for the master constraint on a differential manifold σ (we just consider the contribution of the Euclidean Hamiltonian constraint for illustrative purposes),

$$\mathbf{M} = \int_\sigma \mathrm{d}^3x \frac{[\mathrm{Tr}(F_{ab}E^aE^b)]^2}{\sqrt{|\det(E)|}}, \tag{2.29}$$

which on a cubic algebraic graph could look like

$$\mathbf{M} = \sum_v \left[\sum_a \mathrm{Tr}\big(A\big(\beta_v^a\big)A\big(e_v^a\big)\big[A\big(e_v^a\big)^{-1}, \sqrt{V_v}\big]\big)\right]^2. \tag{2.30}$$

Expression (2.30), in contrast to (2.28), does not contain information about a background metric (there is none), a UV regulator ϵ or even the topology of σ. As long as the algebraic graph is infinite, it can be embedded arbitrarily densely into any manifold σ and therefore *no continuum limit has to be taken*. The theory is therefore UV finite.

(II) As we will see, the kernel of the master constraint defines the states which are invariant under internal gauge transformations and, when embedded, under spacetime diffeomorphisms[20] of GR. This is due to the simple fact that $\mathbf{M}$ vanishes if and only if the individual constraints hold[21].

(III) In order to see that the solutions of the master constraint are, in particular, what one intuitively expects of spatially diffeomorphism states that one can construct in the embedding dependent context [5], one must embed those solutions. At this point, the exact solutions of the master constraint in the new AQG context have not yet been constructed. However, one can perform tests that support our expectations. First of all, using coherent states one can show that the semiclassical limit of $\mathbf{M}$ is correct. Next, approximate solutions to the master constraint are coherent states which are peaked on the constraint hypersurface of the classical phase space and one can verify that the action of the diffeomorphism group derived in [5] leaves the state approximately invariant. Finally, one can try to improve the discretizations used in the above formulae which only use next neighbour terms to all neighbour terms in order to obtain a non-anomalous quantum algebra on the abstract graph. This could be done, for instance, by the method of perfect actions [29].

(IV) As already mentioned, it is tempting to drop the spatial diffeomorphism constraint from our analysis because at the abstract graph level no diffeomorphisms can be defined. However, that is inconsistent as it does not correctly reduce the degrees of freedom as required by the spatial diffeomorphism constraint, because the abstract theory and the embedded theory should be in one-to-one correspondence as far as the physical degrees

[20] The symmetries generated by the Hamiltonian and spatial diffeomorphism constraint have the interpretation of spacetime diffeomorphisms only when the equations of motion hold.

[21] The proof of this statement is trivial for the case that zero is only in the point spectrum of some set of (not necessarily self-adjoint) constraints C_I. Namely, $C_I\psi = 0$ for all I obviously implies $\mathbf{M}\psi = 0$ where $\mathbf{M} = \sum_I C_I^\dagger C_I$. Conversely, $\mathbf{M}\psi = 0$ implies $\langle\psi, \mathbf{M}\psi\rangle = \sum_I \|C_I\psi\|^2 = 0$, hence $C_I\psi = 0$ for all I. The general case is treated in complete detail in the first reference of [12].

of freedom are concerned, and when embedding the abstract graph, the diffeomorphism group acts nontrivially. See [15] for a more detailed discussion.

3. Semiclassical analysis

We review elements of [9, 17, 13] which can be consulted for more details.

We want to show that AQG is a canonical quantization of classical general relativity including matter. The canonical formulation of classical GR in the form we need is reviewed, for instance, in [1]. To begin with, the classical theory is formulated on manifolds diffeomorphic to $\mathbb{R} \times \sigma$, where σ is a three manifold of arbitrary topology. Thus, we must choose a differential manifold σ and embed the fundamental algebraic graph α into σ. Its image will be called $\gamma := X(\alpha)$. Note that any three manifold admits an infinite number of triangulations by tetrahedra or cubes and the graphs dual to such triangulations are simplicial (all vertices are four valent) or cubic (all vertices are six valent), respectively. Thus we focus on simplicial or cubic algebraic graphs. If there are topological obstructions to embed the total α into σ then we delete suitable parts of it until it can be embedded. We will then simply not excite the corresponding edges in the coherent state in what follows so that those edges drop out of all formulae (the coherent states are replaced by the function equal to 1). An example is when σ is compact so that embedding the infinite graph would lead to accumulation points.

3.1. Gravity and Yang–Mills sector

We will choose embeddings X such that γ is dual to a certain triangulation γ^*. Thus, for each $X(e)$ there is a face S_e in γ^* which intersects γ only in an interior point p_e of both S_e and $X(e)$. For each $x \in S_e$, we choose a path $\rho_e(x)$ which starts in $b(X(e))$ along $X(e)$ until p_e and then runs within S_e until x. Next, we choose a classical G-connection A_0 and a Lie(G) valued vector density E_0 of weight 1. With the help of these data we define the quantities

$$A_0(e) := A_0(X(e)) := \mathcal{P}\exp\left(\int_{X(e)} A_0\right) \tag{3.1}$$

$$E_0(e) := \int_{S_e} \epsilon_{abc}\, \mathrm{d}x^a \wedge \mathrm{d}x^b A_0(\rho_e(x))(E_0)^c(x) A_0(\rho_e(x))^{-1} \tag{3.2}$$

which we will refer to as holonomies and electric fluxes respectively.

As one can show [9], if the classical theory is equipped with the following Poisson brackets,

$$\begin{aligned} \left\{(A_0)_a^j(x), (A_0)_b^k(y)\right\} &= \left\{(E_0)_j^a(x), (E_0)_k^b(y)\right\} = 0, \\ \left\{(E_0)_j^a(x), (A_0)_b^k(y)\right\} &= Q^2\delta_b^a\delta_j^k\delta(x,y), \end{aligned} \tag{3.3}$$

where Q^2 is the coupling constant ($G = Q^2/(8\pi)$ is Newton's constant in GR), then the quantities (3.1) satisfy

$$\begin{aligned} &\{A_0(e), A_0(e')\} = 0 \\ &\{(E_0)_j(e), A_0(e')\} = \delta_{e,e'}\tau_j/2 A_0(e) \\ &\{(E_0)_j(e), (E_0)_k(e')\} = -Q^2\delta_{e,e'} f_{jkl}(E_0)_l(e) \end{aligned} \tag{3.4}$$

which precisely matches (2.1). Hence, our kinematical algebra $\mathfrak{A}$ can be regarded as the quantization of the reduction of the classical Poisson algebra to the quantities (3.1) and we have considered a specific representation of $\mathfrak{A}$.

We now consider coherent states. To that end, we construct elements of the complexification $G^{\mathbb{C}}$ of G by

$$g_{e;(A_0,E_0)} := \exp\left(\mathrm{i}E_0(e)\big/a_e^2\right)A_0(e) \tag{3.5}$$

where we have introduced a parameter a_e which may depend on e whose dimension is such that $E_0(e)\big/a_e^2$ is dimensionfree We now consider for $t > 0$ and $g \in G^{\mathbb{C}}$

$$\Psi_g^t(h) := \sum_\pi \dim(\pi)\,\mathrm{e}^{-t\lambda_\pi}\chi_\pi(gh^{-1}). \tag{3.6}$$

Here the sum extends over all equivalence classes of irreducible representations of G and $\dim(\pi)$, λ_π, χ_π respectively denote the dimension of π, eigenvalue of the Laplacian on G when restricted to the representation space of π and the character of π. For $G = SU(2)$ we have, for instance,

$$\Psi_g^t(h) := \sum_j (2j+1)\,\mathrm{e}^{-tj(j+1)/2}\chi_j(gh^{-1}) \tag{3.7}$$

where the sum extends over all non-negative half integers. The functions Ψ_g^t are elements of $L_2(G, \mathrm{d}\mu_H)$ and there is a measure ν^t on $G^{\mathbb{C}}$ such that the completeness relation holds:

$$\int_{G^{\mathbb{C}}} \mathrm{d}\nu(g)\frac{\overline{\Psi_g^t(h)}\Psi_g^t(h')}{\|\Psi_g\|\|\Psi_{g'}\|} = \delta_h(h') \tag{3.8}$$

where $\delta_h(h') = \Psi_h^0(h')$ is the δ-distribution on G with respect to the Haar measure.

We now set $t_e := \ell_P^2\big/a_e^2$ for gravity and

$$\Psi_{e;(A_0,E_0)}(A) := \Psi^{t_e}_{g_{e;(A_0,E_0)}}(A(e)) \tag{3.9}$$

and

$$\Psi_{(A_0,E_0)}(A) := \otimes_{e\in E(\alpha)}\Psi_{e;(A_0,E_0)}(A). \tag{3.10}$$

It is important to keep in mind that (3.9) is a state in the abstract graph Hilbert space; we just use all the data σ, X, γ^*, ρ_e, A_0, E_0, a_e in order to construct specific elements of the abstract ITP Hilbert space. These states are coherent for our kinematical abstract algebra $\mathfrak{A}$ in the following sense: consider the 'annihilation operators'

$$g_e := \exp\left(\mathrm{i}E(e)\big/a_e^2\right)A(e). \tag{3.11}$$

Then our states satisfy[22]

$$g_e\Psi_{(A_0,E_0)} = g_{e;(A_0,E_0)}\Psi_{(A_0,E_0)}, \tag{3.12}$$

that is, they are eigenstates of the annihilation operators. This is one of the defining properties of coherent states. These statements as well as other semiclassical properties such as peakedness properties are proved in [9]. Of most importance for our purposes is that for the normalized coherent states

$$\langle\Psi_{(A_0,E_0)}, A(e)\Psi_{(A_0,E_0)}\rangle = A_0(e), \qquad \langle\Psi_{(A_0,E_0)}, E(e)\Psi_{(A_0,E_0)}\rangle = E_0(e) \tag{3.13}$$

up to terms which vanish faster than any power of t_e as $t_e \to 0$. Also the fluctuations are small; see [9] for complete proofs.

This holds for the gravity sector for which E_j^a is dimensionfree while A_a^j has dimension cm^{-1}. This is why $\hbar Q^2 = \ell_P^2$ has dimension of area. For Yang–Mills theory $\underline{E}_J^a$ has dimension cm^{-2} and $\underline{A}_a^J$ has dimension cm^{-1} so that the Feinstrukturkonstante $\hbar Q^2$ is dimensionfree.

[22] Up to a multiplicative factor which depends only on t_e and tends to unity as $t_e \to 0$.

Thus, for Yang–Mills theory everything remains the same, the only difference being that the a_e are now dimensionfree.

For the mathematically inclined reader, we mention that these states follow from an application of the complexifier framework [13] which provides a constructive algorithm towards coherent states. We define the positive operator, the complexifier

$$C := -\frac{1}{2Q^2}\sum_{e\in E(\alpha)}\frac{1}{a_e^2}\,\mathrm{Tr}(E(e)^2) \tag{3.14}$$

and the δ-distribution on the ITP Hilbert space $\mathcal{H}^{\otimes}$

$$\delta_A(A') := \otimes\Psi^0_{A(e)}(A'(e)). \tag{3.15}$$

Then

$$\Psi_{(A_0,E_0)} = [\mathrm{e}^{-C/\hbar}\delta_A]_{A(e)\to g_{e,(A_0,E_0)}} \tag{3.16}$$

and

$$g_e = \mathrm{e}^{-C/\hbar}A(e)\,\mathrm{e}^{C/\hbar}. \tag{3.17}$$

That is, the coherent states are nothing else than heat kernel evolutions of the δ-distribution, analytically extended to complex group elements[23].

3.2. Fermionic sector

There is no such thing as a classical fermion. Only bilinear (commuting rather than anticommuting) expressions of the Grassmann fields ('current densities') have a classical interpretation. Hence, we are interested in semiclassical states which approximate the self-adjoint quantities

$$\begin{aligned} J^+_{MN}(v) &= [\bar\theta_M(v)\theta_N(v) + \bar\theta_N(v)\theta_M(v)]/2,\\ J^-_{MN}(v) &= [\bar\theta_M(v)\theta_N(v) - \bar\theta_N(v)\theta_M(v)]/(2\mathrm{i}). \end{aligned} \tag{3.18}$$

We will equivalently work with the non-self adjoint currents $J_{MN}(v) = \bar\theta_M(v)\theta_N(v)$. These satisfy the current algebra

$$[J_{MN}(v), J_{PQ}(v')] = \delta_{v,v'}[\delta_{NP}J_{MQ}(v) - \delta_{QM}J_{PN}(v)]. \tag{3.19}$$

We will construct semiclassical states for these currents; see [30] for other proposals made in the literature. It will be sufficient to do this for each v separately. For each v the Hilbert space is complex 2^N-dimensional while the number of currents is real N^2-dimensional where $N = 2\dim(G)$ due to the adjointness relation $J^*_{MN} = J_{NM}$. Since there are only N fermionic degrees of freedom[24] θ_M which count N complex degrees of freedom, we will not look for states which approximate all the currents but only the N currents $J_{MM} = \bar\theta_M\theta_M$ and the remaining freedom in the states will be used in order to approximate the phase of θ_M itself.

We note that the Hilbert space at fixed v is the span of states of the form

$$\Psi_{a,b} := (a_1 + b_1\theta_1)\cdots(a_N + b_N\theta_N) \tag{3.20}$$

for $a_k, b_k \in \mathbb{C}$ which are $2N$ complex degrees of freedom. In order to reduce those to N complex degrees of freedom we use the normalization $|a_k|^2/s + |b_k|^2 = 1$ for all k which leaves us with $3N$ real degrees of freedom. Here we have abbreviated $s = \hbar Q^2$. We compute

$$\langle\Psi_{a,b}, J_{MM}\Psi_{a,b}\rangle = |a_M|^2, \qquad \langle\Psi_{a,b}, \theta_M\Psi_{a,b}\rangle = (-1)^{M-1}\bar b_M a_M \tag{3.21}$$

[23] Also coherent states constructed for the harmonic oscillator or free field theories fit into that scheme.

[24] Note that $\bar\theta$ plays the role of the conjugate momentum of θ; hence one fermionic degree of freedom counts for one configuration and one momentum degree of freedom.

If we fix the expectation value of J_{MM} to j_M then $|a_M|^2 = s(1-|b_M|^2) = j_M$ which shows that $0 \leqslant j_M \leqslant s$, revealing that θ_M is a bounded operator[25]. Setting the expectation value of θ_M to be z_M we see that $|z_M| = \sqrt{j_M[1-j_M/s]}$ is already fixed, while the phase is free and we have $\arg(a_M) = (M-1)\pi + \arg(b_M) + \arg(z_M)$. The fluctuation of J_{MM} follows from the operator identities $\theta^2 = \bar{\theta}^2 = 0$ so that $(\bar{\theta}\theta)^2 = s\bar{\theta}\theta$, hence $\langle J_{MM}^2 \rangle - \langle J_{MM} \rangle^2 = j_M(s-j_M)$. The states (3.20) obey the resolution of identity

$$\mathbf{1} = \prod_{J=1}^{N} \int_0^1 r_J^{s-1}\, \mathrm{d}r_J \int_0^{2\pi} \frac{\mathrm{d}\phi_J}{2\pi} \int_0^{2\pi} \frac{\mathrm{d}\varphi_J}{2\pi} |\Psi_{a,b}\rangle\langle\Psi_{a,b}| \tag{3.22}$$

where $r_J = |a_J|^2/s$, $\phi_J = \arg(a_J) - \arg(b_J)$, $\varphi_J = \arg(a_J) + \arg(b_J)$.

In contrast to the semiclassical states defined for the gauge and gravitational sector, the states $\Psi_{a,b}$ defined for one vertex have large fluctuations. This is due to the fact that what we should consider are not current densities but rather currents, that is, expressions of the form $J_{MM}(B) = \sum_{v\in B} J_{MM}(v)$ where $B \subset V(\alpha)$. Then the relative fluctuation with respect to the states

$$\Psi = \otimes_{v\in V(\alpha)} \Psi_{a_v,b_v} \tag{3.23}$$

is given by

$$\frac{\langle \Psi, J_{MM}(B)^2\Psi\rangle - \langle \Psi, J_{MM}(B)\Psi\rangle^2}{\langle \Psi, J_{MM}(B)\Psi\rangle^2} = \frac{\sum_{v\in B} j_M^v\left(s-j_M^v\right)}{\left[\sum_{v\in B} j_M^v\right]^2} \propto \frac{1}{|B|} \tag{3.24}$$

if $j_v \approx j$ is not varying too much over B. We see that macroscopic currents have very small fluctuations.

Geometrically, the relation between the components of a Weyl spinor $\xi_M(x)$ (which transforms as a scalar under spatial diffeomorphisms) as it appears in the classical action and the $\theta_M(v)$ is given by the formula [6]

$$\sqrt[4]{\det(q)(x)}\xi_M(x) := \sum_{v\in V(\alpha)} \theta_M(v)\sqrt{\delta(X(v),x)} \tag{3.25}$$

where the three metric q_{ab} has appeared explicitly and X is the embedding again. The square root of the δ-distribution matches the density weight of the equation. Note that ξ vanishes away from the vertices of the embedded graph. Using $\sqrt{\delta(x,y)\delta(x,z)} := \delta_{x,y}\delta(x,z)$ it is easy to see that we have for the spatially diffeomorphism invariant quantity

$$\int_\sigma \mathrm{d}^3x \sqrt{\det(q)}(x)\bar{\xi}_M(x)\xi_M(x) = \sum_{v\in V(\alpha)} \bar{\theta}_M(v)\theta_M(v). \tag{3.26}$$

3.3. Higgs sector

For the Higgs sector we can construct coherent states of a more traditional type. Given a classical canonical pair $(\phi_0)_I, (\pi_0)_I$ equipped with the Poisson brackets

$$\begin{aligned} \{(\phi_0)_I(x), (\phi_0)_J(y)\} &= \{(\pi_0)_I(x), (\pi_0)_J(y)\} = 0, \\ \{(\pi_0)_I(x), (\phi_0)_J(y)\} &= Q^2\delta_{IJ}\delta(x,y) \end{aligned} \tag{3.27}$$

[25] This follows already from the anticommutation relations. Since both $\theta\bar{\theta}$, $\bar{\theta}\theta$ are positive operators while $\theta\bar{\theta}+\bar{\theta}\theta = s$, it follows that $\|\theta\|, \|\bar{\theta}\| \leqslant s$.

we consider for each vertex $X(v)$ of the embedded graph the variables $(\phi_0)_I(v) := (\phi_0)_I(X(v))$ and $(\pi_0)_I(v) := \int_{C_v} \mathrm{d}^3x(\pi_0)_I(x)$, where C_v is the cell of the dual cell complex γ^* which contains v. These variables induce the Poisson brackets

$$\{(\phi_0)_I(v), (\phi_0)_J(v')\} = \{(\pi_0)_I(v), (\pi_0)_J(v')\} = 0, \qquad \{(\pi_0)_I(v), (\phi_0)_J(v')\} = Q^2\delta_{IJ}\delta_{v,v'} \tag{3.28}$$

which is compatible with (2.7). If we take the Higgs field to be dimensionfree then $\hbar Q^2$ has dimension cm^2 and the $\phi_0(v)$ have dimension cm^2. Hence we introduce parameters L_v of dimension cm and from those annihilation operators

$$a_I(v) := \frac{1}{\sqrt{2}}\big[\phi_I(v) - \mathrm{i}\pi_I(v)\big/L_v^2\big] \tag{3.29}$$

and complex numbers

$$z_I(v) := \frac{1}{\sqrt{2}}\big[(\phi_0)I(v) - \mathrm{i}(\pi_0)I(v)\big/L_v^2\big]. \tag{3.30}$$

From these we construct

$$\Psi_z^t = \mathrm{e}^{-|z|^2/2}\,\mathrm{e}^{za}\Psi_0, \qquad \Psi_0(x) = \mathrm{e}^{-x^2/t_v}\Big/\sqrt{2\pi t_v} \tag{3.31}$$

and then

$$\Psi_{(\phi_0,\pi_0)} := \otimes_{v\in V(\alpha),I}\Psi_{z_I(v)}^{t_v} \tag{3.32}$$

where $t_v = \hbar Q^2\big/L_v^2$.

Remark. In contrast to the LQG representation which is necessarily discontinuous in the edge labels of the holonomy operators so that the connection operator (smeared over one dimensional paths) does not exist, in AQG we may indeed define such a representation. We simply define a new algebra by

$$[A_j(e), A_k(e')] = [E_j(e), E_k(e')] = 0, \qquad [E_j(e), A_k(e')] = \mathrm{i}\hbar Q^2\delta_{jk}\delta_{ee'} \tag{3.33}$$

where now both $E(e)$, $A(e)$ are Lie(G) valued. This *-algebra is represented on the infinite tensor product of Hilbert spaces, one for each edge, of the Hilbert space $L_2(\mathbb{R}^{\dim(G)}, d^{\dim(G)}x)$ on which $A_j(e)$ and $E_j(e)$ respectively act by multiplication and derivation by x_j. Such a representation is forbidden in LQG because one needs to relate the Hilbert spaces defined for different (infinite) graphs to each other in such a way as to respect the relations $A(e_1\circ e_2) = A(e_1)+A(e_2)$, $A(e^{-1}) = -A(e)$. One can easily see that there is no cylindrically consistent measure underlying such a Hilbert space because the divergence of the electric flux operator with respect to such a measure is not L_2 (see, e.g., [31]). ITP Hilbert spaces have no underlying measure; however, now the definition of the inner product between vectors belonging to two different ITP's based on different graphs becomes problematic (see [9]). In AQG there is only one graph and therefore the problem disappears.

One would then define $A_0(e) = \int_e A_0$ and then define harmonic oscillator type coherent states just as in (3.29)–(3.31). At least one could do that for the matter gauge fields such as the Maxwell field for which oscillator type coherent states were actually invented. For gravity one might want to stick with the algebra of section 2 in order to keep the discreteness of the spectrum of geometrical operators.

4. (One) semiclassical limit of the master constraint

In what follows we summarize the result of [22] where a semiclassical calculation for the extended algebraic master constraint operator based on a cubic algebraic graph is presented.

The calculation makes use of the following approximation. We substitute the gauge group $SU(2)$ by $U(1)^3$. This is of course incorrect; however, the results of [9, 23] together show that the results of the exact non-Abelian calculation match precisely the results of the Abelian approximate calculation, provided one substitutes in the result of the approximate expectation value calculation every Abelian holonomy and electric flux by the corresponding non-Abelian quantity. More precisely, the symplectic structure (3.3) does not know whether we are given a $SU(2)$ or $U(1)^3$ gauge theory, the phase space is the same, only if we add the constraints do we get this additional information. Hence, we may use a point (A_0, E_0) in the common phase space of both theories. In order to carry out the approximate calculation, the non-Abelian operators $(\mathrm{Tr}(\tau_j A(e)), \mathrm{Tr}(\tau_j E(e)))$ are replaced by the Abelian ones $(h^j(e), p_j(e))$ where $h^j(e)$ corresponds to the holonomy of the jth copy of $U(1)$ and likewise for the electric field. Note that on purpose we introduce new letters for the holonomy and the electric flux in order to distinguish more easily whether we are talking about $U(1)^3$ or $SU(2)$. Then, after the expectation value is calculated one replaces the classical $U(1)^3$ terms $((h_0)^j(e), (p_0)_j(e))$ by $(\mathrm{Tr}(\tau_j A_0(e)), \mathrm{Tr}(\tau_j E_0(e)))$. The result of that calculation turns out to be exactly the same as if directly doing the non-Abelian calculation, of course only to zeroth order in $\hbar$. The advantage of this indirect calculation is that it is much easier to perform.

In order to do this, all we have to do is to change the coherent states from those for $SU(2)$ to those of $U(1)^3$. This is rather easy: consider the state

$$\Psi^t_g(h) := \sum_{n\in\mathbb{Z}} \mathrm{e}^{-tn^2/2}(gh^{-1})^n \tag{4.1}$$

where $g \in \mathbb{C} - \{0\} = U(1)^{\mathbb{C}}$ and $h \in U(1)$. Function (4.1) is an element of $L_2(U(1), \mathrm{d}\mu_H)$. We set for $j = 1, 2, 3$

$$g^j_{e;(A_0,E_0)} := \mathrm{e}^{E^j(e)/a_e^2}\,\mathrm{e}^{\mathrm{i}\int_e A_0^j} \tag{4.2}$$

and with $t_e := \ell_p^2/a_e^2$

$$\Psi^{\{t_e\}}_{\alpha,(A_0,E_0)} := \otimes_{e\in E(\alpha)} \otimes_{j=1}^3 \Psi^{t_e}_{\alpha, g^j_{e;(A_0,B_0)}}. \tag{4.3}$$

For simplicity and since this will not affect the final result, we choose the same $t_e =: t$ for each edge. Moreover, we will introduce the shorthand $m := (A_0, E_0)$ for the phase space point. The coherent states are then denoted by

$$\Psi^t_{\gamma,m} = \otimes_{e\in E(\alpha)} \otimes_{j=1}^3 \Psi^t_{\alpha,g_e,m}. \tag{4.4}$$

Requiring the graph α to have cubic symmetry we know that each vertex is six-valent. We label these six edges by e^σ_J, whereby $\sigma \in \{+, -\}$ depending on the orientation with respect to the vertex v and $J \in \{1, 2, 3\}$. For more details, see [22]. Let us introduce the following notation for the $U(1)$-holonomies and electric fluxes:

$$\begin{aligned} h_{J\sigma jv} &:= h^j_{e^\sigma_J(v)} \\ p_{J\sigma jv} &:= p_j^{e^\sigma_J(v)} \\ h_{I_0\sigma_0 J_0\sigma_0' j_0 v} &:= h_{I_0\sigma_0 j_0 v} h_{J_0\sigma_0' j_0 v+\sigma_0\hat{I}_0} h^{-1}_{I_0\sigma_0 j_0 v+\sigma_0'\hat{J}_0} h^{-1}_{J_0\sigma_0' j_0 v} \end{aligned}. \tag{4.5}$$

For the considered algebraic graph of cubic symmetry, the algebraic master constraint operator denoted by $\widehat{\mathbf{M}}$ has the following form,

$$\begin{aligned}
\widehat{\mathbf{M}} &= \sum_{v\in V(\gamma)} \widehat{\mathbf{M}}_v \\
\widehat{\mathbf{M}}_v &= \sum_{\ell_0=0}^{3} \widehat{C}^{\dagger}_{\ell_0,v}\widehat{C}_{\ell_0,v} \\
\widehat{C}_{0,v} &= \sum_{I_0J_0K_0}\sum_{\sigma_0=+,-}\sum_{\sigma_0'=+,-}\sum_{\sigma_0''=+,-} \frac{4}{\kappa}\epsilon^{I_0J_0K_0}\widehat{h}_{\alpha_{I_0\sigma_0'J_0\sigma_0''\ell_0v}}\widehat{h}_{K_0\sigma_0\ell_0v}\frac{1}{\mathrm{i}\hbar}\left[\widehat{h}^{-1}_{K_0\sigma_0\ell_0v}, \widehat{V}^{\frac{1}{2}}_{\gamma,v}\right] \\
\widehat{C}_{\ell_0,v} &= \sum_{I_0J_0K_0}\sum_{\sigma_0=+,-}\sum_{\sigma_0'=+,-}\sum_{\sigma_0''=+,-} \frac{4}{\kappa}\epsilon^{I_0J_0K_0}\epsilon_{\ell_0m_0n_0}\widehat{h}_{\alpha_{I_0\sigma_0'J_0\sigma_0''m_0v}}\widehat{h}_{K_0\sigma_0n_0v}\frac{1}{\mathrm{i}\hbar}\left[\widehat{h}^{-1}_{K_0n_0\sigma_0v}, \widehat{V}^{\frac{1}{2}}_{\gamma,v}\right],
\end{aligned} \tag{4.6}$$

where the square root of the volume operator of the cubic graph denoted by $\widehat{V}^{\frac{1}{2}}_{\alpha,v}$ expressed in terms of right invariant vector fields $\widehat{X}^{e^{\sigma}_J}_j := \widehat{X}_{J\sigma jv} = \mathrm{i}\hat{h}_{J\sigma jv}\partial/\partial\hat{h}_{J\sigma jv}$ is given by

$$\widehat{V}^{\frac{1}{2}}_{\alpha,v} = \left(\ell_p^3\sqrt{\left|\epsilon^{jkl}\left[\frac{\widehat{X}_{1+jv}-\widehat{X}_{1-jv}}{2}\right]\left[\frac{\widehat{X}_{2+kv}-\widehat{X}_{k-jv}}{2}\right]\left[\frac{\widehat{X}_{3+lv}-\widehat{X}_{3-lv}}{2}\right]\right|}\right)^{\frac{1}{2}} \tag{4.7}$$

with its corresponding eigenvalue

$$\lambda^{\frac{1}{2}}(\{n_{J\sigma j\tilde{v}}\}) = \left(\ell_p^3\sqrt{\left|\epsilon^{jkl}\left[\frac{n_{1+jv}-n_{1-jv}}{2}\right]\left[\frac{n_{2+kv}-n_{k-jv}}{2}\right]\left[\frac{n_{3+lv}-n_{3-lv}}{2}\right]\right|}\right)^{\frac{1}{2}}. \tag{4.8}$$

Our task is now to show that the expectation value

$$\frac{\left\langle \Psi^t_{\gamma,m}, \widehat{\mathbf{M}}\Psi^t_{\gamma,m}\right\rangle}{\left\|\Psi^t_{\gamma,m}\right\|^2} \tag{4.9}$$

coincides with the classical $U(1)^3$ master constraint

$$\mathbf{M}[m] = \left\{\int_\sigma \mathrm{d}^3x\,\frac{\delta^{jk}C_jC_k + q^{ab}C_aC_b + C^2}{\sqrt{\det(q)}}(x)\right\}[m] \tag{4.10}$$

evaluated at the point $m = (A_0, E_0)$ in the classical phase space in the limit $\hbar \to 0$. Here the following functions were defined (we drop the subscript '0')

$$C_j = \partial_a E^a_j \qquad C_a = F^j_{ab}E^b_j \qquad C = \epsilon^{abc}\left[F^j_{ab} + \epsilon_{jkl}K^j_aK^k_b\right]e^l_c \tag{4.11}$$

where

$$\begin{aligned}
&F^j_{ab} = 2\partial_{[a}A_{b]} \qquad E^a_j = \left|\det\left(\left(e^k_b\right)\right)\right|e^a_j, \qquad e^a_je^j_b = \delta^a_b, \qquad e^a_je^k_a = \delta^k_j \\
&q_{ab} = e^j_ae^j_b \qquad K^j_a = A^j_a - \Gamma^j_a
\end{aligned}$$

and where Γ^j_a is the spin connection of the co-triad e^j_a, that is, $D_ae^j_b = \partial_ae^j_b - \Gamma^c_{ab}e^j_b + \epsilon_{jkl}\Gamma^k_ae^l_b = 0$, Γ^c_{ab} are the Christoffel symbols determined by the three metric q_{ab}.

These are the $U(1)^3$ quantities, the $SU(2)$ quantities are defined in exactly the same way, only the two following functions need to be changed to

$$\begin{aligned} C_j &\to \partial_a E^a_j + \epsilon_{jkl} A^k_a E^a_l \\ F^j_{ab} &\to \partial_{[a} A^j_{b]} + \epsilon_{jkl} A^k_a A^l_b. \end{aligned} \tag{4.12}$$

That the $U(1)^3$ calculation has anything to do with the result of the exact $SU(2)$ calculation relies on the fact, established in [23], that the $SU(2)$ volume operator can be semiclassically expanded in terms of polynomials of flux operators plus $\hbar$ corrections. However, as shown in [9], to zeroth order in $\hbar$, expectation values of polynomials of holonomy and flux operators agree in $U(1)^3$ and $SU(2)$ calculations and also extends to operators of type $Q^{(r)}$, as shown in [21]. As long as we arrive at the correct classical $U(1)^3$ master constraint in the leading order of the expectation value calculation we are also qualitatively done for $SU(2)$.

In [22] we prove that the expectation value of the algebraic master constraint operator associated with a graph of cubic topology yields in the leading order

$$\begin{aligned} \frac{\langle \Psi^t_{\{g,J,\sigma,j,L\}} | \widehat{\mathbf{M}} | \Psi^t_{\{g,J,\sigma,j,L\}} \rangle}{\| \Psi^t_{\{g,J,\sigma,j,L\}} \|^2} &= \sum_{v \in V(\alpha)} \frac{\langle \Psi^t_{\{g,J,\sigma,j,L\}} | \widehat{\mathbf{M}}_v | \Psi^t_{\{g,J,\sigma,j,L\}} \rangle}{\| \Psi^t_{\{g,J,\sigma,j,L\}} \|^2} \\ &= \sum_{v \in V(\alpha)} \sum_{I_0 J_0 K_0} \sum_{\widetilde{I}_0 \widetilde{J}_0 \widetilde{K}_0} \sum_{\sigma_0 = +,-} \sum_{\widetilde{\sigma}_0 = +,-} \epsilon^{I_0 J_0 K_0} \epsilon^{\widetilde{I}_0 \widetilde{J}_0 \widetilde{K}_0} \left(\delta_{m_0, n_0} \delta_{\widetilde{m}_0, \widetilde{n}_0} + \sum_{\ell_0 = 1}^{3} \epsilon_{\ell_0 m_0 n_0} \epsilon_{\ell_0 \widetilde{m}_0 \widetilde{n}_0} \right) \\ &\quad \times \left\{ \left(\frac{4 a^{\frac{3}{2}} |\det((p)^-)|^{\frac{1}{4}}}{\kappa \hbar} \right)^2 (sT)^2 \, \mathrm{e}^{+\mathrm{i} \sum_{\tilde{v} \in V} \sum_{(J,\sigma,j) \in L} \varphi_{J\sigma j \tilde{v}} \Delta(I_0, \widetilde{I}_0, J_0, \widetilde{J}_0, \sigma_0, \widetilde{\sigma}_0, m_0, \widetilde{m}_0, v, J, \sigma, j, \tilde{v})} \right. \\ &\quad \left. \times \left(f^{(1)}_{\frac{1}{8}}(1) \right)^2 (\mathrm{sgn}(\sigma_0)(q^{-1})^-_{K_0 n_0}) (\mathrm{sgn}(\widetilde{\sigma}_0)(q^{-1})^-_{\widetilde{K}_0 \widetilde{n}_0}) \right\} + O((sT/t)^2). \end{aligned} \tag{4.13}$$

The leading has to be understood as follows. The coherent are labelled with the so-called classicality parameter $t \propto \hbar$. Hence, the limit $\lim_{t \to 0}$ corresponds to the limit $\hbar \to 0$ and is the limit in which the expectation value should agree with the classical quantities to approximate. We show in detail in [22] that the result of the expectation value of $\mathbf{M}$ in the limit $t \to 0$ above can be identified with the classical discretized master constraint associated with a cubic lattice, denoted by $\mathbf{M}^{\mathrm{cubic}}$ from now on. Furthermore, we prove that $\mathbf{M}^{\mathrm{cubic}}$ agrees, indeed, with the classical continuum expression of the master constraint $\mathbf{M}$ in equation (4.10) when shrinking the parameter interval length ϵ to zero. This can be summarized in the following equation:

$$\boxed{\frac{\langle \Psi^t_{\alpha,m} | \widehat{\mathbf{M}} | \Psi^t_{\alpha,m} \rangle}{\| \Psi^t_{\alpha,m} \|^2} = \sum_{v \in V(\alpha)} \frac{\langle \Psi^t_{\{g,J,\sigma,j,L\}} | \widehat{\mathbf{M}}_v | \Psi^t_{\{g,J,\sigma,j,L\}} \rangle}{\| \Psi^t_{\{g,J,\sigma,j,L\}} \|^2} \underset{\lim_{t \to 0}}{=} \mathbf{M}^{\mathrm{cubic}}[m] \underset{\lim_{\epsilon \to 0}}{=} \mathbf{M}[m]}. \tag{4.14}$$

Consequently, with the calculation done in [22] we have shown that algebraic quantum gravity is a theory of quantum gravity which has the same infinitesimal generators as general relativity. Thus, the problem whether the semiclassical sector includes general relativity, which is still unsolved within the framework of loop quantum gravity, is significantly improved in the context of algebraic quantum gravity. Additionally, we discuss the next-to-leading order term of the expectation value which can be interpreted as fluctuations of $\widehat{\mathbf{M}}$. It turns out that these next-to-leading order contributions are finite. For a more detailed discussion, see [22].

Let us close this section with some remarks concerning the details of the analysis in [22]:

(1) In [22] we only considered the gravitational sector; however, the techniques used there carry over to all standard matter coupling.
(2) In [22] we also dropped the piece corresponding to the quantum Gauss constraint because it is just a linear combination of flux operators for which the correct classical limit has already been established in [9].
(3) In [22] we only considered the Euclidean part of the Hamiltonian constraint. The Lorentzian piece cannot be correctly produced using $U(1)^3$ because the classical identity $\left\{H_E^{(1)}, V\right\} = \int_\sigma \mathrm{d}^3x\, K_a^j E_j^a$ for $SU(2)$ on which (2.14) relies fails to hold. However, again the results of [22, 23] show that the correct $SU(2)$ calculation does reproduce the correct classical limit.

5. Computational AQG and quantum gauge fixing

The fact that the master constraint has the correct classical limit in AQG is a strong indicator that the theory has the correct classical limit because the master constraint determines both the physical Hilbert space and the quantum observables which are required to preserve the physical Hilbert space. Ideally, in order to establish this one needs to compute the physical Hilbert space, construct the gauge invariant quantum observables and define a dynamics among those[26].

As far as the first task is concerned, this can be done as follows. As is well known (see the first reference of [12] for all details), given a self-adjoint operator $\mathbf{M}$ on a separable Hilbert space $\mathcal{H}$, there is a unitarily equivalent representation of $\mathbf{M}$ on a direct integral Hilbert space,

$$\mathcal{H} \cong \mathcal{H}^{\oplus} = \int_{\mathrm{spec}(\mathbf{M})}^{\oplus} \mathrm{d}\mu(\lambda)\mathcal{H}_\lambda^{\oplus}, \tag{5.1}$$

where μ is a spectral measure for $\mathbf{M}$, spec($\mathbf{M}$) denotes the spectrum of $\mathbf{M}$ and the separable Hilbert spaces $\mathcal{H}_\lambda^{\oplus}$ are the generalized eigenspaces of $\mathbf{M}$ in the following sense. Given $\Psi \in \mathcal{H}$ we can represent it as a system of vectors $(\Psi_\lambda)_{\lambda\in\mathrm{spec}(\mathbf{M})}$ where $\Psi_\lambda \in \mathcal{H}_\lambda^{\oplus}$. Then $\mathbf{M}\Psi$ is represented as the system $(\lambda\Psi_\lambda)$. The inner product is given by

$$\langle \Psi, \Psi' \rangle = \int \mathrm{d}\mu(\lambda)\langle \Psi_\lambda, \Psi'_\lambda\rangle_{\mathcal{H}_\lambda} \tag{5.2}$$

This is really nothing else than a generalization of the Fourier transform to an arbitrary self-adjoint operator, the dimension of $\mathcal{H}_\lambda^{\oplus}$ has the interpretation of the multiplicity of λ.

The physical Hilbert space is the kernel of $\mathbf{M}$, that is $\mathcal{H}_{\mathrm{phys}} = \mathcal{H}_0^{\oplus}$. The construction of μ and $\mathcal{H}_\lambda^{\oplus}$ requires detailed knowledge of the spectrum of $\mathbf{M}$ but otherwise there is a clean algorithm for how to obtain these structures which are unique up to unitary equivalence[27]. While the assumption of separability does not apply to the ITP Hilbert space $\mathcal{H}^{\otimes}$, there is no problem because $\mathbf{M}$ preserves all the strong equivalence class Hilbert spaces. This follows from the fact that $\mathbf{M}$ is a countable sum of operators each of which changes only a finite number of entries in a vector of the form $\otimes_f$; hence we get a countable sum of vectors in the same equivalence class, which remains normalizable if $\otimes_f$ is in the domain of $\otimes_f$. Hence, we can apply the direct integral decomposition to each of these separable Hilbert spaces separately.

[26] Note that in background independent theories there is no natural Hamiltonian, the Hamiltonian constraint is constrained to vanish and observables need to commute with it. Hence the Hamiltonian constraint is unsuitable to define dynamics. Extra work is required in order to define evolution among observables (see below).

[27] There are some remaining ambiguities associated with the fact that equalities hold up to measure μ zero sets. For how to fix them, see [12].

The quantum observables are the self-adjoint operators on $\mathcal{H}_{\text{phys}}$. This is mathematically sufficient but we are interested in those observables with a classical interpretation, that is, those which can be defined on the kinematical Hilbert space $\mathcal{H}^{\otimes}$, which have a classical limit in the sense of our coherent states and which preserve the eigenspaces $\mathcal{H}^{\oplus}_{\lambda}$. As can be shown, a function F on the classical phase space is an observable provided that $\{F, \{F, \mathbf{M}\}\}_{\mathbf{M}=0} = 0$ (see [10]). A systematic way to construct such observables is via the partial observable ansatz due to Rovelli [32]; see [33–35] and references therein for recent improvements concerning the technical implementation. This is a classical framework which, given a set of constraints C_I, a set of phase space functions T_I subject to $\det((\{C_I, T_J\})) \neq 0$, a set of real numbers τ_I in the range of the T_I, and a phase space function f, constructs an observable $F^{\tau}_{f,T}$ as a power series in the variables $\tau_I - T_I$. Hence

$$F^{\tau}_{f,T} = f + \sum_I (\tau_I - T_I) f_I + \sum_{I,J} (\tau_I - T_I)(\tau_J - T_J) f_{IJ} + \cdots, \tag{5.3}$$

for certain phase space functions $f_I, f_{IJ}, \ldots$ which can be explicitly constructed. Physically, the T_I are gauge fixing functions and if we evaluate $F^{\tau}_{f,T}$ at a point in phase space for which $T_I = \tau_I$ then $F^{\tau}_{f,T} = f$. The meaning of the real parameters τ_I is that each of them defines a physical time evolution because $F^{\tau}_{f,T}$ is an observable for each value of the τ_I. One can also show that the evolution in τ_I is generated by a physical Hamiltonian $H_I(\tau)$ which in general, however, will be τ dependent. Of course, the Hamiltonian should be bounded from below and should reduce to the Hamiltonian of the standard model when the metric is close to the Minkowski metric plus small fluctuations.

Unfortunately, all of that framework is purely classical and difficult to quantize because the expression for $F^{\tau}_{f,T}$ faces, in general, severe operator ordering problems. In order to sidestep these problems it would be desirable to have a more direct procedure at one's disposal in order to generate a physical Hamiltonian. One way to do this is via the Brown–Kuchar mechanism based on a phantom field [36]. By choosing a suitable action for the phantom field, one can generate a physical Hamiltonian which reduces to that of the remaining matter and gravity when the phantom field distribution is homogeneous. That Hamiltonian is explicitly τ-independent and non-negative. Classical physical observables can be constructed as well which suffer from less severe ordering problems. This is due to the fact that the phantom field allows for an explicit deparametrization of the entire physical system. One might think that the drawback of this is that the phantom field is a scalar which has not been observed, but actually there is no problem because the phantom field is pure gauge anyway.

Thus we see that, apart from the technical problem of computing all of these quantities, there is a clear conceptual path for how to do physics with AQG. For instance, the physical Hamiltonian may be used in order to select the true vacuum of the universe, a quantity that is ambiguous in the framework of quantum field theory on curved spacetimes. Furthermore, it will be used in order to compute physical scattering amplitudes. However, in order to do so, we really need effective computational tools. The computation of the exact physical Hilbert space will be impossible due to the complexity of the theory so that we have to resort to approximations. In a background independent and therefore necessarily non-perturbative theory, only non-perturbative tools are allowed. These are precisely the coherent states defined in section 3. We will choose a point in the classical phase space which (1) lies on the constraint surface of the classical master constraint and (2) satisfies the gauge fixing conditions $T_I = \tau_I$ of our chosen functions T_I (in our case essentially the phantom field). This means that these states are approximately physical states because the norm of the master constraint (equivalently its fluctuation) is close to zero and expectation values of physical observables $F^{\tau}_{f,T}$ effectively reduce to the expectation value of f, of course only to lowest order in the fluctuations of the $T_I - \tau_I$.

One could call this approximation 'quantum gauge fixing' for the following reason. We are working at the level of the kinematical Hilbert space. We choose a state which is peaked on a point m of the constraint surface and within its orbit $[m]$ on that point which corresponds to the gauge cut $T = \tau$. However, there are still fluctuations of all degrees of freedom involved, not only physical ones, in particular in directions off the constraint surface and within the gauge orbit. This is in contrast to gauge fixing before quantizing. In a way we gauge fix after quantizing by choosing appropriate states which suppress the fluctuations into the unphysical directions. In the longer range, one has of course to answer the question how good this approximation is as compared to the exact calculation.

6. Algebraic quantum gravity and spin foams

The algebraic or embedding independent setting proposed for AQG also provides an interesting new perspective for the spin foam programme [37]. Spin foam models try to provide a path integral representation of LQG. Two of the most important tasks to be completed within the spin foam programme for 4D general relativity[28] are (1) to make contact with the canonical theory and (2) to remove the triangulation dependence of the models.

In more detail, the spin foam models currently discussed in the literature start from a path integral that involves a constrained BF theory action. Classically, if one solves those so-called simplicity constraints which impose that the B field is the exterior product of two vierbeine, then one obtains the Palatini action and a topological term. In order to define the path integral mathematically one regularizes it by choosing a triangulation and discretizes the constrained BF theory on this triangulation. However, to the best knowledge of the authors, none of the spin foam models currently discussed has properly implemented the quantum simplicity constraints nor has dealt with the fact that the Palatini theory leads to second class constraints in the canonical formulation which has a nontrivial effect on the path integral measure if the canonical and covariant theory are to compute the same thing. This is well known, see for instance [38], and has also been pointed out in [39] for the spin foam context.

The second issue has to do with the removal of the regulator, that is, triangulation dependence. A natural idea would be to sum over triangulations, the choice of the weights being motivated by the group field theory formulation of spin foams [37]. However, again to the present authors it is completely unclear how to make contact between the original path integral for the constrained BF theory which at least has a clear connection to the classical theory we want to quantize and the group field theory formulation. For instance, is it not more natural to study the infinite refinement limit of spin foam models and to look for critical points as in lattice gauge theory or dynamical triangulations [40]?

We will now show that AQG offers a clean solution to both problems. Indeed, as advertised in [10], the extended master constraint defines a new type of spin foam model which computes by means of the rigging map heuristically[29] given by

$$\eta : \mathcal{H} \to \mathcal{H}_{\mathrm{phys}};\ \psi \mapsto \int_{\mathbb{R}} \mathrm{d}t\, \exp(\mathrm{i}t\mathbf{M})\psi \tag{6.1}$$

the physical inner product

$$\langle \eta(\psi), \eta(\psi')\rangle_{\mathrm{phys}} := \int_{\mathbb{R}} \mathrm{d}t\,\langle \psi, \exp(\mathrm{i}t\mathbf{M})\psi'\rangle = \int_0^\infty \mathrm{d}t[\langle \psi, \exp(\mathrm{i}t\mathbf{M})\psi'\rangle + \langle \psi, \exp(-\mathrm{i}t\mathbf{M})\psi'\rangle] \tag{6.2}$$

[28] There are many promising results in 3D but this is hardly surprising since 2+1 gravity is a TQFT. Most of the results in 3D rely on the TQFT structure and therefore do not carry over to the 4D case.
[29] See the first reference of [12] for the rigorous definitions.

If the expression $\langle \psi, \exp(\pm it\mathbf{M})\psi' \rangle$ is analytic in t (for instance, if ψ or ψ' are analytic vectors for $\mathbf{M}$) then it can be considered as the analytic continuation $t \mapsto \mp it$ in t of the expression $\langle \psi, \exp(-t\mathbf{M})\psi' \rangle$. Since $\mathbf{M}$ is positive, the operators $\exp(-t\mathbf{M})$ are bounded for $t \geqslant 0$ (they form a contraction semi-group) and have improved convergence properties as compared to the unitarities $\exp(\pm it\mathbf{M})$.

Note that $\langle \psi, \exp(-t\mathbf{M})\psi' \rangle$ vanishes when ψ, ψ' do not belong to the same sector of the ITP. If we now write $\exp(-t\mathbf{M}) = [\exp(-t\mathbf{M}/N)]^N$ and insert $N - 1$ resolutions of unity $1_{\text{sector}} = \sum_s |s\rangle\langle s|$ where $|s\rangle$ denotes a countable orthonormal basis for the given sector then we arrive at a path integral formulation of the physical inner product. The orthogonality of the kinematical sectors carries over to their images under the rigging map.

Let us restrict ourselves, for the purpose of this paper, to the case that the semiclassical theories we want to quantize have compact σ. The appropriate sector of the ITP is then based on the vector $\otimes_{\mathbf{1}} = \otimes_e \mathbf{1}$, where $\mathbf{1}$ is the constant function equal to 1. An orthonormal basis for this sector is given by spin network functions defined over all finite subgraphs of the algebraic graph. Then (6.2) defines a concrete spin foam model of general relativity *for which the issue of triangulation dependence is absent*. Note that we may leave N large but finite, the formula one obtains is exact for any N. Depending on the 'boundary states' ψ, ψ' and the value of N, the non-vanishing contributions to the resulting sum will be over subgraphs of the algebraic graphs which reach a certain maximum size. This should be quite similar to the 3D model discussed in [41]. Details will follow in future publications.

7. Conclusions and outlook

Algebraic quantum gravity (AQG) offers a conceptually clear and technically simpler approach to quantum gravity than loop quantum gravity (LQG). The simplification occurs because in AQG one just has to deal with *one*, albeit countably infinite, algebraic graph while in LQG one deals with an uncountably infinite number of finite and embedded graphs. In LQG this has the effect that the Hamiltonian constraint always refines the graph on which it acts while in AQG the algebraic graph is the finest possible one. The search for semiclassical states for such refining or graph changing operators has so far been unsuccessful. However, as we have indicated here and as will be shown in [22, 23], the present semiclassical tools developed in [9] are already sufficient to establish the correct semiclassical limit of the master constraint. As a further bonus, AQG possibly can deal with topology change in the sense that it incorporates the semiclassical limits for all topologies while the corresponding states *belong to the same Hilbert space*.

A point worthy of note is that for convenience we used elementary variables whose classical limit coincides with those that are the starting point for LQG; hence our considerations are very much inspired by LQG. However, our purely combinatorial setup can be used in a much wider context, for instance it is conceivable that one can work with ADM variables rather than connection variables in the absence of fermionic matter. All one needs is to smear the ADM variables q_{ab}, P^{ab} over regions in σ whose smearing dimensions add up to 3. These smearing labels are then promoted to elements of an abstract countable labelling set of an algebra whose commutation relations mimic the Poisson brackets of the embedded objects.

Much has yet to be understood about AQG. For instance, what have the exact solutions of the master constraint of AQG, when embedded, to do with the exact solutions of at least the spatial diffeomorphism constraint of LQG? What we have established is that the semiclassical limit of the weighted square of the spatial diffeomorphism constraint agrees with the classical generator. Hence, semiclassical states peaked on the constraint surface of the spatial diffeomorphism constraint are approximate solutions of the spatial diffeomorphism constraint[30]. But are they approximately invariant under the finite diffeomorphisms of σ?

Another open question is the following. Basically the master constraint is the weighted sum of the Hamiltonian and spatial diffeomorphism constraints, which when embedded look similar to the discretizations used in [16]. While the master constraint itself is in any case *non-anomalous* we know that the constraints themselves do not close. Thus, the exact kernel of the master constraint could be empty or may contain too few solutions because the algebra of the constraints is anomalous. If this is the case then, as already mentioned, one must modify the master constraint. There are several proposals: either one subtracts from the master constraint the minimum of the spectrum, or one allows a whole interval of zero in the spectrum to define solutions [42] or one succeeds in defining non-anomalous constraints on the lattice, for instance by renormalization group techniques [29].

Finally, an interesting question is whether there is an algebraic version not only of the volume operator but also of area [18, 43] and length operators of LQG [44]. This requires a diffeomorphism invariant definition of the classical version of these operators in terms of matter whose analytical expression uses 3D rather than 2D or 1D integrals in order that there is an embedding independent lift; see [13] for an explanation. While the construction of these operators is not necessary because there are other functions on the classical, spatially diffeomorphism invariant phase space which separate the points, it would certainly be desirable to have those at one's disposal. We will leave this for future research.

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Appendix. Alternative quantization of the spatial diffeomorphism and Hamiltonian constraints

By using the operator Q_v defined in section (2.3) one can simplify the discretizations of the contributions of the spatial diffeomorphism constraint and Hamiltonian constraint and make the construction of the master constraint look more uniform. We just display the purely gravitational contributions; the general pattern should become clear.

C′ *Spatial diffeomorphism constraint*. For any $v \in V(\alpha)$ we set

$$\tilde{D}_j(v) := \frac{1}{P_v} \sum_{e_1 \cap e_2 = v} \frac{1}{|L(v, e_1, e_2)|} \sum_{\beta \in L(v,e_1,e_2)} \mathrm{Tr}(\tau_k[A(\beta) - A(\beta)^{-1}]) E_k(e_1) E_j(e_2) \tag{A.1}$$

where the sum is over unordered pairs of distinct edges adjacent to v and where again we assumed for convenience that all edges are outgoing from v. The quantity P_v is the number of contributing pairs.

D′ *Euclidean Hamiltonian constraint*. For any $v \in V(\alpha)$ we set

$$\tilde{H}_E(v) := \frac{1}{P_v} \sum_{e_1 \cap e_2 = v} \frac{1}{|L(v, e_1, e_2)|} \sum_{\beta \in L(v,e_1,e_2)} \mathrm{Tr}([A(\beta) - A(\beta)^{-1}] E(e_1) E(e_2)) \tag{A.2}$$

where we used the same notation as above.

[30] This does not contradict the fact that in LQG the infinitesimal generator of the diffeomorphism group does not exist due to the weight operator that is used in the definition of the square.

E′ *Lorentzian Hamiltonian constraint*. For any $v \in V(\alpha)$ we set

$$\tilde{H}(v) - \tilde{H}_E(v) := \frac{1}{P_v} \sum_{e_1 \cap e_2 = v} \mathrm{Tr}([(A(e_1)[A(e_1)^{-1}, [\tilde{H}'_E, V]]), (A(e_2)[A(e_2)^{-1}, [\tilde{H}'_E, V]])][E(e_1), E(e_2)]) \tag{A.3}$$

where we used the same notation as above and have set $\tilde{H}'_E := \sum_v \left[Q_v^{(1/2)}\right]^\dagger Q_v^{(1/2)} \tilde{H}_E(v)$.

F′ *(Extended) master constraint*. The extended master constraint is now simply given by

$$\mathbf{M}' := \sum_{v \in V(\alpha)} \Big[\big(Q_v^{(1/2)} G_j(v)\big)^\dagger \big(Q_v^{(1/2)} G_j(v)\big) + \big(\big[Q_v^{(1/6)}\big]^2 \tilde{D}_j(v)\big)^\dagger \big(\big[Q_v^{(1/6)}\big]^2 \tilde{D}_j(v)\big) + \big(\big[Q_v^{(1/6)}\big]^2 \tilde{H}(v)\big)^\dagger \big(\big[Q_v^{(1/6)}\big]^2 \tilde{H}(v)\big)\Big] \tag{A.4}$$

where appropriate coefficients are understood as in the main text in order to match dimensionalities and classical limit. Not only did the constraints simplify, also all terms involved in $\mathbf{M}'$ sandwich operators of the type $\big(\big(Q_v^{(r)}\big)^\dagger\big)^n \big(Q_v^{(r)}\big)^n$. This is because the operators $D_j(v)$, $H(v)$ transform as half-densities when embedded, $G_j(v)$ is a simple density and $\tilde{D}_j(v)$, $\tilde{H}(v)$ are double densities. The advantage is that the actual constraints (almost) remain polynomials in holonomies and electric fluxes, up to appearances of the operators $V(r)_v$.

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