

Algebraic quantum gravity (AQG): III. Semiclassical perturbation theory

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Abstract

In the two previous papers of this series we defined a new combinatorial approach to quantum gravity, algebraic quantum gravity (AQG). We showed that AQG reproduces the correct infinitesimal dynamics in the semiclassical limit, provided one incorrectly substitutes the non-Abelian group $SU(2)$ by the Abelian group $U(1)$ ³ in the calculations. The mere reason why that substitution was performed at all is that in the non-Abelian case the volume operator, pivotal for the definition of the dynamics, is not diagonalizable by analytical methods. This, in contrast to the Abelian case, so far prohibited semiclassical computations. In this paper, we show why this unjustified substitution nevertheless reproduces the correct physical result. Namely, we introduce for the first time semiclassical perturbation theory within AQG (and LQG) which allows us to compute expectation values of interesting operators such as the master constraint as a power series in \hbar with error control. That is, in particular, matrix elements of fractional powers of the volume operator can be computed with extremely high precision for sufficiently large power of \hbar in the \hbar expansion. With this new tool, the non-Abelian calculation, although technically more involved, is then exactly analogous to the Abelian calculation, thus justifying the Abelian analysis in retrospect. The results of this paper turn AQG into a calculational discipline.

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1. Introduction

In the two previous companion papers of this series [1, 2], we introduced a new combinatorial approach to quantum gravity, called algebraic quantum gravity (AQG) which uses ideas

from loop quantum gravity (LQG) [3, 4]. One of the advantages of AQG over LQG is that semiclassical tools for background independent quantum field theories already available in the literature [5–8] can also be applied to operators encoding the quantum dynamics while in LQG this has so far been possible only for kinematical operators. The difficulty, as explained in detail in [1], has to do with the fact that in LQG the Hamiltonian or master constraint operator [9–12] necessarily changes the number of degrees of freedom on which the semiclassical state, that it acts on, depends. The fluctuations of the degrees of freedom added by the operator are therefore not suppressed by the semiclassical state and their semiclassical behaviour is correspondingly bad. In AQG, on the other hand, the dynamics never changes the number of degrees of freedom and the just mentioned problem disappears. Furthermore, the annoying graph dependence of the normalizable coherent states of LQG disappears in AQG.

In the companion paper [2], we have displayed a non-trivial semiclassical calculation which shows that the Euclidean part of the master constraint for pure gravity in AQG reduces to its classical counterpart in the $\hbar \rightarrow 0$ limit. The same calculation demonstrates that its fluctuations and quantum corrections are small. It is trivially extendable to arbitrary matter coupling and the Lorentzian constraint. However, the calculation was done using an *a priori* physically unjustified, technical modification of the master constraint: namely we substituted the correct non-Abelian gauge group $SU(2)$ of the canonical formulation of general relativity by the incorrect Abelian gauge group $U(1)$ ³. The reason why this was done is that the $U(1)$ ³ analogue of the volume operator [13], without which none of the pieces of the master constraint (geometry and matter) can even be defined, can be diagonalized analytically. This is crucial in order that semiclassical calculations can be carried out.

For the non-Abelian case on the other hand, except in special cases, the volume operator cannot be diagonalized analytically³, which prohibited so far any explicit calculations involving the quantum dynamics. This is one of the many criticisms that have been spelled out recently in [17].

In the present paper we show that this criticism, as many others, is unjustified, thereby introducing *semiclassical perturbation theory* for AQG and LQG. Basically, what we are interested in are expectation values or matrix elements of (powers of) the master constraint operator in semiclassical states. We will show how to compute those as a power series in \hbar *with error control!* Note that in usual perturbative QFT there is no error control. In fact, there the perturbation series is known to be only asymptotic for all realistic theories.

As we will see, and as should be expected from [1, 2], the non-trivial part of the corresponding \hbar expansion consists in the evaluation of the matrix elements of fractional powers of the volume operator. The basic mathematical physics tools that we employ here are the spectral theorem for self-adjoint operators and the directed set structure of the cone of positive operators. More is not needed in order to define the series. The fact that makes the errors small on the other hand relies on the fantastically good semiclassical properties of the semiclassical states developed in [5]. For instance, if we are interested in cosmological questions, the power series is typically in terms of the classicality parameter $t = \Lambda \hbar G \approx 10^{-120}$ where Λ is the cosmological constant and G is Newton's constant. This should be contrasted with the Feinstrukturkonstante $\alpha \approx 1/137$ of QED. Here, the extreme weakness of the gravitational interaction comes to help. Note that our power expansion, for cosmological settings, is as fast converging as for the new spin foam model introduced in [15].

This paper is organized as follows. In section 2, we develop systematically semiclassical perturbation theory for AQG. For the benefit of the reader not interested in the detailed proofs,

³ The matrix elements of its fourth power are known in closed form [13]; however, the associated matrices on the invariant subspaces, while finite dimensional, are not diagonalizable by quadratures.

let us just state that if we are interested in the matrix elements $\langle \psi, V_v^{2q} \psi' \rangle$ of the volume operator V_v associated with a given vertex v which is not calculable analytically, then we can replace V_v^{2q} within the matrix element, up to \hbar^{k+1} corrections, by the calculable quantity

$$\langle \psi, Q_v \psi \rangle^{2q} \left[1 + \sum_{n=1}^{2k+1} (-1)^{n+1} \frac{q(1-q) \cdots (2k-q)}{n!} \left(\frac{Q_v^2}{\langle \psi, Q_v \psi \rangle^2} - 1 \right)^n \right] \tag{1.1}$$

where Q_v is related to the volume by $V_v = \sqrt[4]{Q_v^2}$. The operator Q_v is a polynomial in flux operators and its matrix elements are known in closed form.

In section 3, we show how the results of [5] can be used in order to perform concrete calculations (numbers!) using the tools of section 2. This will show semiclassical perturbation theory *at work!*

In section 4, we conclude and explain why the Abelian results of [2] carry over qualitatively and quantitatively to the non-Abelian case as far as the zeroth order in \hbar is concerned while there are finite differences in the first and higher order corrections. This justifies the calculation of [2] and demonstrates that AQG is a calculational discipline.

2. Semiclassical perturbation theory

2.1. The idea

In semiclassical applications of AQG we are interested, in particular, in expectation values of (finite powers of) the master constraint with respect to coherent states ψ . As displayed explicitly in [1], the operators of which we have to take expectation values are then linear combinations of expressions of the form (symbolically)

$$\langle \psi, p_1(h) F_1(V_{v_1}) p_2(h) F_2(V_{v_2}) \cdots F_N(V_{v_N}) p_{N+1}(h) \psi \rangle \tag{2.1}$$

where p_j are certain polynomials in the holonomies along edges or loops adjacent to the vertices v_1, \dots, v_N and F_I are certain functions of the volume operator V_{v_I} of the form $F_I(V_{v_I}) = (Q_{v_I}^2)^{q_I}$. Here $0 < q_I = m_I/n_I \leq 1/4$ is a rational number and m_I, n_I are relatively prime. The self-adjoint operator Q_v is explicitly given by⁴

$$Q_v = i t^3 \epsilon_{jkl} \frac{X_{e_{1+}(v)}^j - X_{e_{1-}(v)}^j}{2} \frac{X_{e_{2+}(v)}^k - X_{e_{2-}(v)}^k}{2} \frac{X_{e_{3+}(v)}^l - X_{e_{3-}(v)}^l}{2} \tag{2.2}$$

where we have used the notation from [1, 2]: by $e_{I\sigma}(v)$ we mean the edge of a cubic algebraic graph outgoing from vertex v into the positive ($\sigma = +$) or negative ($\sigma = -$) direction, respectively, and X_I^j denotes the corresponding right invariant vector field of $SU(2)$. Note that $e_{1+}(v) = e_1(v)$, $e_{1-}(v) = [e_1(v - \hat{I})]^{-1}$. The classicality parameter is $t = \ell_p^2/a^2$ where a is some length scale that enters the definition of the coherent states.

The naive idea to compute expectation values of (2.2) is as follows: consider

$$x_I := \frac{Q_{v_I}^2}{\langle \psi, Q_{v_I} \psi \rangle^2} - 1. \tag{2.3}$$

The operator x_I is bounded from below, $x_I \geq -1$ and the quantity $\langle \psi, Q_{v_I} \psi \rangle$ can be computed exactly by the methods of [5]. We have

$$F_I(V_{v_I}) = |\langle \psi, Q_{v_I} \psi \rangle|^{2q_I} f_I(x_I), \quad f_I(x_I) = (1 + x_I)^{q_I}. \tag{2.4}$$

⁴ For convenience we are working here with the dimensionless volume operator that results from the actual one by dividing it by a^3 , where a is the length scale parameter that enters the definition of the coherent states.

The idea is now to use the power expansion of the function $t \mapsto f(t) = (1+t)^q$, $-1 \leq t < \infty$, given by

$$f(t) := 1 + \sum_{n=1}^{\infty} \binom{q}{n} t^n, \quad \binom{q}{n} = (-1)^{n+1} \frac{q(1-q) \cdots (n-1+q)}{n!} \quad (2.5)$$

and to use the spectral theorem in order to get an expansion of $F_I(V_{v_I})$ in terms of x_I of which coherent state matrix elements are computable by the methods of [5], specifically

$$\begin{aligned} f_I(x_I) &= \int_{-1}^{\infty} f_I(t) dE_I(t) = \int_{-1}^{\infty} \left[1 + \sum_{n=1}^{\infty} \binom{q}{n} t^n \right] dE_I(t) \\ &= \left[1 + \sum_{n=1}^{\infty} \binom{q}{n} x_I^n \right] \end{aligned} \quad (2.6)$$

where E_I is the projection-valued measure associated with x_I . However, the second equality is wrong because the power expansion (2.6) does not converge outside the open interval $t \in (-1, 1)$. Hence, this naive idea is false and must be substituted by a rigorous argument.

Much of the effort that follows will be devoted to showing that the power expansion (2.5) can nevertheless be used in order to get reliable numerical results for the intended expectation value calculations including error estimates. More precisely, the power expansion up to order $2k+1$ gives reliable values for the matrix elements including \hbar^k corrections while the remainder can be estimated to be finite and of order \hbar^{k+1} . The proof of this fact is somewhat involved and we therefore split it into several subsections. The reader not interested in the proof can jump directly to section 3.

2.2. Some basic tools

In this subsection, we will state and prove some basic lemmas which will be employed over and over again in the core part of the proof.

Lemma 2.1. *For each $k \geq 0$ there exists $0 < \beta_k < \infty$ such that*

$$f_{2k+1}(t) - \beta_k t^{2k+2} \leq f(t) \leq f_{2k+1}(t) \quad (2.7)$$

where $f_k(t)$ denotes the partial Taylor series of $f(t) = (1+t)^q$, $0 < q \leq 1/4$, up to order t^k .

Proof. By Taylor's theorem we have

$$f(t) = f_k(t) + R_k(t), \quad f_k(t) = 1 + \sum_{n=1}^k \binom{q}{n} t^n, \quad R_k(t) = \int_0^t ds \frac{f^{(k+1)}(s)}{k!} (t-s)^k \quad (2.8)$$

for any $-1 < t < \infty$.

Since $f^{(2k+2)}(s) = -q(1-q) \cdots (2k+1-q)(1+s)^{q-2k-2} \leq 0$ for all $s > -1$ and $(t-s)^{2k+1}$ is positive (negative) for $t \geq s \geq 0$ ($-1 \leq t \leq s \leq 0$) it follows that $R_{2k+1}(t) \leq 0$ for all $t > -1$, hence $f_{2k+1}(t) \geq f(t)$ for all $t > -1$ and also for $t = -1$ by continuity.

On the other hand, $f^{(2k+3)}(s) = q(1-q) \cdots (2k+2-q)(1+s)^{q-2k-3} \geq 0$ for all $s > -1$. Since $(t-s)^{2k+2} \geq 0$ for all s we certainly have $f_{2k+2}(t) \leq f(t)$ for $t \geq 0$ and $f_{2k+2}(t) \geq f(t)$ for $-1 \leq t \leq 0$. Now $f_{2k+2}(t) = f_{k+1}(t) + \binom{q}{2k+2} t^{k+2}$. Then for any $\beta_k > -\binom{q}{2k+2}$ we still have $f_{2k+1}(t) - \beta_k t^{2k+2} \leq f(t)$ for $t \geq 0$. Moreover, since $0 \leq f_{2k+1}(t) - f(t)$ is bounded from above for $-1 < t \leq 0$ there exists a finite β_k such that also $f_{2k+1}(t) - \beta_k t^{2k+2} \leq f(t)$ for $-1 \leq t \leq 0$. \square

The optimal (lowest) value of β_k is not easy to obtain by analytical methods but certainly by numerical methods. As an example, let us determine it analytically for the case $k = 0$.

Lemma 2.2.

$$1 + qt - (1 - q)t^2 \leq (1 + t)^q \leq 1 + qt \tag{2.9}$$

for all $t \geq -1$ and all $0 < q < 1$.

Proof. Consider $f(t) = 1 + qt - (1 + t)^q$. We have $f'(t) = q(1 - (1 + t)^{q-1})$ so that $f' \geq 0$ for $t \geq 0$ and $f' \leq 0$ for $-1 \leq t \leq 0$. Thus, $f(0) = 0$ is the absolute minimum of f . Let now $g(t) = f(t) - \beta t^2$. In order that $g \leq 0$ we certainly need $g(-1) = 1 - q - \beta \leq 0$ so $\beta \geq 1 - q$. We have

$$g' = f' - 2\beta t = q \left[\frac{1 - \frac{1}{y^{1-q}}}{y - 1} - p \right] (y - 1) \tag{2.10}$$

where $y = 1 + t \geq 0$ and $p = 2\beta/q \geq 2(1 - q)/q > 1 - q$. Consider

$$h(y) = \frac{1 - \frac{1}{y^{1-q}}}{y - 1} \Rightarrow h'(y) = \frac{1}{y^{2-q}(y - 1)^2} k(y), \tag{2.11}$$

$$k(y) = (1 - q)(y - 1) - y^{2-q} + y.$$

It follows that $k' = (2 - q)(1 - y^{1-q})$ hence $y = 1$ is the maximum of k at which $k(1) = 0$. Hence $k \leq 0$ and thus $h' \leq 0$ is strictly monotonically decreasing. Since $h(1) = 1 - q$ and $p > 1 - q$ it follows that g' is strictly monotonically decreasing for $y > 1$. The equation $g' = 0$ or $h = p$ has one more solution $0 < y_{q,p} < 1$ because $h(0) = \infty$ and h is decreasing. It follows that $g' \leq 0$ for $0 \leq y \leq y_{q,p}$ and $y \geq 1$ and $g' \geq 0$ for $y_{q,p} \leq y \leq 1$. Hence $y_{q,p}$ is a local minimum of g and $y = 0, 1$ are local maxima at which $g(x = -1) = 1 - q - \beta$, $g(x = 0) = 0$. Thus, in order that $g \leq 0$ it is necessary and sufficient that $\beta \geq 1 - q$ where $\beta = 1 - q$ is the sharpest bound. \square

To see what lemma 2.1 is good for we state the following simple lemma.

Lemma 2.3. Let $B_- \leq B \leq B_+$ be self-adjoint operators and set $\bar{B} := [B_+ + B_-]/2$, $\Delta B := [B_+ - B_-]/4$. Then for any states ψ_1, ψ_2 in the common domain of all three operators we have

$$|\text{Re}(\langle \psi_1, [B - \bar{B}]\psi_2 \rangle)|, |\text{Im}(\langle \psi_1, [B - \bar{B}]\psi_2 \rangle)| \leq \langle \psi_1, [\Delta B]\psi_1 \rangle + \langle \psi_1, [\Delta B]\psi_1 \rangle \tag{2.12}$$

Proof. Consider the polarization identity for any self-adjoint operator B :

$$\langle \psi_1, B\psi_2 \rangle = \frac{1}{4} \sum_{i^4=1} \bar{i} \langle \psi_i, B\psi_i \rangle \tag{2.13}$$

where $\psi_i = \psi_1 + i\psi_2$. By assumption the operators $B_+ - B, B - B_-$ are positive. Thus,

$$\frac{\langle B_- \rangle_{+1} - \langle B_+ \rangle_{-1}}{4} \leq \text{Re}(\langle \psi_1, B\psi_2 \rangle) = \frac{\langle B \rangle_{11} - \langle B \rangle_{-1}}{4} \leq \frac{\langle B_+ \rangle_{11} - \langle B_- \rangle_{-1}}{4}$$

$$\frac{\langle B_- \rangle_{-i} - \langle B_+ \rangle_{+i}}{4} \leq \text{Im}(\langle \psi_1, B\psi_2 \rangle) = \frac{\langle B \rangle_{-i} - \langle B \rangle_{+i}}{4} \leq \frac{\langle B_+ \rangle_{-i} - \langle B_- \rangle_{+i}}{4} \tag{2.14}$$

where $\langle B \rangle_i := \langle \psi_i, B\psi_i \rangle$, etc. We conclude

$$|\text{Re}(\langle \psi_1, [B - \bar{B}]\psi_2 \rangle)| \leq \langle \psi_1, [\Delta B]\psi_1 \rangle + \langle \psi_2, [\Delta B]\psi_2 \rangle$$

$$|\text{Im}(\langle \psi_1, [B - \bar{B}]\psi_2 \rangle)| \leq \langle \psi_1, [\Delta B]\psi_1 \rangle + \langle \psi_2, [\Delta B]\psi_2 \rangle \tag{2.15}$$

as claimed. \square

The idea behind this lemma is that, even if we cannot compute the matrix elements of B , provided we can compute the matrix elements of \bar{B} and ΔB , formula (2.15) shows that the matrix element of \bar{B} is a good approximation to that of B provided that the respective expectation values of ΔB are small.

As an example, consider $B = (Q^2)^q$ with Q as in (2.2) polynomial in the elementary operators (flux operators for GR) so that matrix elements can be computed by the methods of [5]. We have for any $\lambda > 0$ that $B = \lambda^q [1 + (Q^2 - \lambda)/\lambda]^q$ and obviously $x = (X - \lambda)/\lambda \geq -1$. Then we may apply the above lemma and set $B_+ = \lambda^q [1 + x]$, $B_- = \lambda^q [1 + qx - (1 - q)x^2]$. Given a coherent state ψ_1 we could set $\lambda = \langle \psi_1, Q^2 \psi_1 \rangle$ so that $\langle \psi_1, B_+ \psi_1 \rangle = \lambda^q$ and $\langle \psi_1, B_- \psi_1 \rangle = \lambda^q [1 - (1 - q) \frac{\langle \psi_1, Q^4 \psi_1 \rangle}{\lambda^2} - 1]$. Hence, the expectation value of $B = (Q^2)^q$ is approximately given by $\langle Q^2 \rangle^q$ up to a correction which is controlled by the fluctuation of Q^2 as one would expect. Note that it has been established [5] that $\langle Q^2 \rangle$ agrees with the classical value to lowest order in \hbar .

2.3. The expansion

Turning to the general matrix element of interest (2.1) we expand

$$\begin{aligned} & \langle \psi, p_1(h) F_1(V_{v_1}) p_2(h) F_2(V_{v_2}) \cdots F_N(V_{v_N}) p_{N+1}(h) \psi \rangle \\ &= \prod_{j=1}^n F_j^0 \langle \psi, p_1 [1 + f_1] p_2 [1 + f_2] \cdots [1 + f_N] p_{N+1} \psi \rangle \\ &= \prod_{j=1}^n F_j^0 [\langle \psi, p_1 p_2 \cdots p_{N+1} \psi \rangle + R] \end{aligned} \tag{2.16}$$

where $F_j^0 = \langle \psi, Q_{v_j} \psi \rangle^{2q_j}$ and the remainder R is a linear combination of terms of the form

$$\langle \psi, p'_1 f'_1 \cdots f'_l p'_{l+1} \psi \rangle \tag{2.17}$$

with $l = 1, \dots, N$ and p'_j are products of p_j while each f'_j coincides with one of the f_k .

Hence we are left with computing (2.17). To do this, we use lemma 2.1 and find optimal $\beta_{j,k}$ such that

$$f_j^-(t) := f_{j,2k+1}(t) - \beta_{j,k} t^{2(k+2)} \leq f_j(t) \leq f_{j,2k+1}(t) =: f_j^+(t) \tag{2.18}$$

for all $t \geq -1$. We will now iterate lemma 2.3 frequently. In what follows we just consider the real part of (2.17), the imaginary part works analogously as displayed in (2.12).

We start with

$$\begin{aligned} \langle \psi, p_1 f_1 \cdots f_N p_{N+1} \psi \rangle &= \langle [\overline{p_{[(N+1)/2]} f_{[(N+1)/2]-1} \cdots f_1 p_1} \psi], \\ & f_{[(N+1)/2]} [p_{[(N+1)/2]-1} f_{[(N+1)/2]+1} \cdots f_N p_{N+1} \psi] \rangle \\ &= : \langle \psi_1, f_{[(N+1)/2]} \psi_2 \rangle. \end{aligned} \tag{2.19}$$

Here, $[t]$ denotes the Gauss bracket of the real number t . Application of (2.15) reveals that $|R - R_1| \leq R_2 + R_3$ where

$$\begin{aligned} R &= \text{Re}(\langle \psi, p_1 f_1 \cdots f_N p_{N+1} \psi \rangle) \\ R_1 &= \text{Re}(\langle \psi_1, \bar{f}_{[(N+1)/2]} \psi_2 \rangle) \\ R_2 &= \langle \psi_1, \Delta f_{[(N+1)/2]} \psi_1 \rangle \\ R_3 &= \langle \psi_2, \Delta f_{[(N+1)/2]} \psi_2 \rangle \end{aligned} \tag{2.20}$$

where $\bar{f}_j = (f_j^+ + f_j^-)/2$ and $\Delta f_j = (f_j^+ - f_j^-)/4$ are positive operators of which matrix elements can indeed be computed using the techniques of [5].

Note that in R_1 we have achieved a reduction from N operator insertions f_1, \dots, f_N of which matrix elements cannot be computed to $N - 1$ insertions $f_1, \dots, f_{[(N+1)/2]-1}, f_{[(N+1)/2]+1}, \dots, f_N$. However, in R_1 we still have $N - 2$ ($N - 1$) insertions if N is even (odd) while in R_2 we still have $N(N - 1)$ insertions if N is odd. It follows that by iterating (2.20) we can never achieve that all occurring expressions only contain operator insertions of which matrix elements are computable analytically. Note that judiciously we started the iteration with the ‘middle’ operator, otherwise we would create even more than N operator insertions of which matrix elements cannot be computed. By choosing the middle, the number of those insertions at least does not increase.

While the above procedure therefore never stops at a stage at which everything is exactly computable, we will show that after finitely many steps one reaches a stage at which the non-computable terms can be estimated from above by computable terms and those estimates are of higher order in \hbar than the order of the power expansion that we wanted to achieve. More precisely, we will show that after finitely many steps we obtain one expression which only involves \bar{f}_j and thus the associated expectation value can be computed by the methods of [5] and contains \hbar corrections up to order \hbar^k . In addition, there are a finite number of terms each of which still contains at most N operator insertions of which matrix elements cannot be computed plus at least $2l + 1$ insertions of operators of the form $\Delta f'_j$. These terms appear in the manifestly positive form, say for N even

$$\langle P'_1[\Delta F'_1]P'_2 \cdots P'_l[\Delta F'_l]P_1 F_1 P_2 \cdots F_{N/2} P_{N/2+1} \psi | \times [\Delta F'_{l+1}] | P'_1[\Delta F'_1]P'_2 \cdots P'_l[\Delta F'_l]P_1 F_1 P_2 \cdots F_{N/2} P_{N/2+1} \psi \rangle. \quad (2.21)$$

Here, F'_j, F_k are elements of the original set f_1, \dots, f_N . Likewise P'_j, P_k are elements of the original set p_1, \dots, p_{N+1} . Instead of $N/2$ insertions of type F_k we might also have some insertions of type \bar{F}_k but since the latter have computable matrix elements their estimate is of an even higher order in \hbar than (2.21) as we will see so that (2.21) provides the type of term whose estimate gives the lowest power of \hbar . Hence, we will show that for l sufficiently large (2.21) can be estimated by a computable expression of order at least \hbar^{k+1} .

2.4. Resolutions of unity

To estimate (2.21) we use the overcompleteness property of our coherent states [5] and insert resolutions of unity to cast (2.21) into the form

$$\begin{aligned} & \int dv_{1,1} \cdots \int dv_{1,N/2} \int dv_{2,1} \cdots \int dv_{2,N/2+1} \int dv'_{1,1} \cdots \int dv'_{1,l} \int dv'_{2,1} \cdots \int dv'_{2,l} \\ & \times \int dv_{3,1} \cdots \int dv_{3,N/2} \int dv_{4,1} \cdots \int dv_{4,N/2+1} \int dv'_{3,1} \cdots \int dv'_{3,l} \int dv'_{4,1} \cdots \\ & \times \frac{\prod_{j=1}^{l-1} \langle \psi'_{1,j}, [\Delta F'_j] \psi'_{2,j+1} \rangle \langle \psi'_{2,j}, P'_j \psi'_{1,j} \rangle \langle \psi'_{2,l}, P'_l \psi'_{1,l} \rangle \langle \psi'_{1,l}, [\Delta F'_l] \psi_{2,1} \rangle}{\prod_{k=1}^{N/2} \langle \psi_{1,k}, F_k \psi_{2,k+1} \rangle \langle \psi_{2,k}, P_k \psi_{1,k} \rangle \langle \psi_{2,N/2+1}, P_{N/2+1} \psi \rangle \langle \psi'_{2,1}, [\Delta F'_{l+1}] \psi'_{4,1} \rangle} \\ & \times \frac{\prod_{j=1}^{l-1} \langle \psi'_{3,j}, [\Delta F'_j] \psi'_{4,j+1} \rangle \langle \psi'_{4,j}, P'_j \psi'_{3,j} \rangle \langle \psi'_{4,l}, P'_l \psi'_{3,l} \rangle \langle \psi'_{3,l}, [\Delta F'_l] \psi_{4,1} \rangle}{\prod_{k=1}^{N/2} \langle \psi_{3,k}, F_k \psi_{4,k+1} \rangle \langle \psi_{4,k}, P_k \psi_{3,k} \rangle \langle \psi_{4,N/2+1}, P_{N/2+1} \psi \rangle}. \end{aligned} \quad (2.22)$$

Here the measures $\nu_{i,k}, \nu'_{i,j}$ are basically Liouville measures [5] over as many copies of the cotangent bundle $T^*(SU(2))$ as there are edges involved in the expectation value under consideration and each of the integrals is over the entire corresponding phase space. The states $\psi_{i,k}, \psi'_{i,j}$ are labelled by the points in that phase space, see [5] for details.

2.5. Sketch of the estimate

Before we give the rigorous argument, let us give its heuristic form in order to understand what we are driving at: in order to estimate (2.22) we make use of the following fact that has been proved in [5]:

$$\langle \psi_1, P\psi_2 \rangle = \langle \psi_1, \psi_2 \rangle [E_0(\psi_1, \psi_2) + \hbar E_1(\psi_1, \psi_2)] \tag{2.23}$$

where ψ_1, ψ_2 are arbitrary coherent states whose overlap function $\langle \psi_1, \psi_2 \rangle$ is sharply peaked (Gaussian) at the point in phase space by which they are labelled. The functions E_0, E_1 are absolutely integrable against that Gaussian over both resolution measures ν_1, ν_2 and are of zeroth order in \hbar .

Since (2.23) is sharply peaked at $\psi_1 = \psi_2$ we can basically set $E_0(\psi_1, \psi_2) = E_0(\psi_1, \psi_1) = \langle \psi_1, P\psi_1 \rangle$ in the essential support of the overlap function, the corrections being of higher order in \hbar , just like the E_1 contribution. This is what results after integrating over the measure corresponding to ψ_2 . Thus by integrating with respect to the measures $\nu_{2,k}, \nu'_{2,j}, \nu_{4,k}, \nu'_{4,j}$ we can simplify (2.22) up to \hbar corrections to

$$\begin{aligned} & \int d\nu_{1,1} \cdots \int d\nu_{1,N/2} \int d\nu'_{1,1} \cdots \int d\nu'_{1,l} \int d\nu_{3,1} \cdots \int d\nu_{3,N/2} \int d\nu'_{3,1} \cdots \int d\nu'_{3,l} \\ & \times \frac{\prod_{j=1}^{l-1} \langle \psi'_{1,j}, [\Delta F'_j] \psi'_{1,j+1} \rangle \langle \psi'_{1,j}, P'_j \psi'_{1,j} \rangle \langle \psi'_{1,l}, P'_l \psi'_{1,l} \rangle \langle \psi'_{1,l}, [\Delta F'_l] \psi_{1,1} \rangle}{\prod_{k=1}^{N/2-1} \langle \psi_{1,k}, F_k \psi_{1,k+1} \rangle \langle \psi_{1,k}, P_k \psi_{1,k} \rangle} \\ & \times \frac{\langle \psi_{1,N/2}, F_{N/2} \psi \rangle \langle \psi_{1,N/2}, P_{N/2} \psi_{1,N/2} \rangle \langle \psi, P_{N/2+1} \psi \rangle \langle \psi'_{1,1}, [\Delta F'_{l+1}] \psi'_{3,1} \rangle}{\prod_{j=1}^{l-1} \langle \psi'_{3,j}, [\Delta F'_j] \psi'_{3,j+1} \rangle \langle \psi'_{3,j}, P'_j \psi'_{3,j} \rangle \langle \psi'_{3,l}, P'_l \psi'_{3,l} \rangle \langle \psi'_{3,l}, [\Delta F'_l] \psi_{3,1} \rangle} \\ & \times \frac{\prod_{k=1}^{N/2-1} \langle \psi_{3,k}, F_k \psi_{3,k+1} \rangle \langle \psi_{3,k}, P_k \psi_{3,k} \rangle}{\langle \psi_{3,N/2}, F_{N/2} \psi \rangle \langle \psi_{3,N/2}, P_{N/2} \psi_{3,N/2} \rangle \langle \psi, P_{N/2+1} \psi \rangle}. \end{aligned}$$

In order to estimate (2.24) further we will prove that for arbitrary coherent states ψ_1, ψ_2 we have

$$\langle \psi_1, [\Delta F'] \psi_2 \rangle = \hbar^{k+1} \langle \psi_1, \psi_2 \rangle [G'_0(\psi_1, \psi_2) + \hbar G'_1(\psi_1, \psi_2)] \tag{2.24}$$

where G'_0, G'_1 are absolutely integrable with respect to both ν_1, ν_2 and of zeroth order in \hbar . Similarly, we will show that

$$|\langle \psi_1, F \psi_2 \rangle| \leq \hbar^{-3} \langle \psi_1, \psi_2 \rangle [G_0(\psi_1, \psi_2) + \hbar G_1(\psi_1, \psi_2)] \tag{2.25}$$

where G_0, G_1 are both of zeroth order in \hbar and both terms are separately absolutely integrable with respect to both ν_1, ν_2 . Note that while (2.23) and (2.24) are equalities, (2.25) is an inequality. It is due to this inequality, which provides the key part of the proof, that a negative

power of \hbar appears in (2.25) which is the price to pay for the estimate. The estimate is necessary to carry out because the left-hand side is not calculable analytically. We conjecture that the matrix element (2.25) admits a sharper bound which does not involve the \hbar^{-3} power; however, the estimate presented below is not able to deliver such a result.

Note also the prime at the overlap function $\langle \psi_1, \psi_2 \rangle$ in (2.25). This prime is in order to indicate that $\langle \psi_1, \psi_2 \rangle$ is a Gaussian only with respect to the momentum variables of the phase space but not with respect to the position variables of six copies of $SU(2)$ corresponding to the six edges adjacent to a vertex in the definition of the volume operator. This is unproblematic as far as integrability is concerned because the configuration space is a compact group and the Haar measure involved in ν is normalized. However, the missing damping factor leads to an additional negative \hbar power that comes from the measures ν as we will see momentarily.

Note that the polynomials P are bounded operators so that $|\langle \psi, P\psi \rangle| \leq \|P\|$ where $\|P\|$ is the sup norm on the Abelian C^* -algebra of functions of connections. Let us assume for simplicity that $\|P\| \leq 1$ which is typically the case because in our application P is a matrix element of a holonomy in some representation of $SU(2)$. Hence, we can estimate (2.24) by

$$\begin{aligned}
& \int d\nu_{1,1} \cdots \int d\nu_{1,N/2} \int d\nu'_{1,1} \cdots \int d\nu'_{1,l} \int d\nu_{3,1} \cdots \int d\nu_{3,N/2} \int d\nu'_{3,1} \cdots \int d\nu'_{3,l} \\
& \quad \times \prod_{j=1}^{l-1} |\langle \psi'_{1,j}, [\Delta F'_j] \psi'_{1,j+1} \rangle| |\langle \psi'_{1,l}, [\Delta F'_l] \psi_{1,1} \rangle| \\
& \quad \times \prod_{k=1}^{N/2-1} |\langle \psi_{1,k}, F_k \psi_{1,k+1} \rangle| |\langle \psi_{1,N/2}, F_{N/2} \psi \rangle| |\langle \psi'_{1,1}, [\Delta F'_{l+1}] \psi'_{3,1} \rangle| \\
& \quad \times \prod_{j=1}^{l-1} |\langle \psi'_{3,j}, [\Delta F'_j] \psi'_{3,j+1} \rangle| |\langle \psi'_{3,l}, [\Delta F'_l] \psi_{3,1} \rangle| \\
& \quad \times \left| \prod_{k=1}^{N/2-1} \langle \psi_{3,k}, F_k \psi_{3,k+1} \rangle \right| |\langle \psi_{3,N/2}, F_{N/2} \psi \rangle| \\
& \leq \hbar^{(2l+1)(k+1)-3N} \int d\nu_{1,1} \cdots \int d\nu_{1,N/2} \int d\nu'_{1,1} \cdots \int d\nu'_{1,l} \\
& \quad \times \int d\nu_{3,1} \cdots \int d\nu_{3,N/2} \int d\nu'_{3,1} \cdots \int d\nu'_{3,l} \\
& \quad \times \prod_{j=1}^{l-1} |\langle \psi'_{1,j}, \psi'_{1,j+1} \rangle G'_{0j}(\psi'_{1,j+1})| |\langle \psi'_{1,l}, \psi_{1,1} \rangle G'_{0l}(\psi_{1,1})| \\
& \quad \times \prod_{k=1}^{N/2-1} |\langle \psi_{1,k}, \psi_{1,k+1} \rangle' G_{0k}(\psi_{1,k+1})| |\langle \psi_{1,N/2}, \psi \rangle' G_{0N/2}(\psi)| \\
& \quad \times |\langle \psi'_{1,1}, \psi'_{3,1} \rangle G'_{0l+1}(\psi'_{3,1})| \\
& \quad \times \prod_{j=1}^{l-1} |\langle \psi'_{3,j}, \psi'_{3,j+1} \rangle G'_{0j}(\psi'_{3,j+1})| |\langle \psi'_{3,l}, \psi_{3,1} \rangle G'_{0l}(\psi_{3,1})| \\
& \quad \times \left| \prod_{k=1}^{N/2-1} \langle \psi_{3,k}, \psi_{3,k+1} \rangle' G_{0k}(\psi_{3,k+1}) \right| |\langle \psi_{3,N/2}, \psi \rangle' G_{0N/2}(\psi)| \tag{2.26}
\end{aligned}$$

where $G_{0k}(\psi) = G_{0k}(\psi, \psi)$, etc. Basically, the remaining integral has as many Gaussians in the momentum variables as there are measures. However, there are N factors of the form $\langle \psi_1, \psi_2 \rangle'$, hence there are $6N$ Gaussians in the configuration variables missing. Since the associated $6N$ resolution measures come with a factor of \hbar^{-3} for each of which only $\hbar^{-3/2}$ gets absorbed into the integration measure of the momentum Gaussian integral, the final integral in (2.26) is of order $\hbar^{-3/2(6N)} = \hbar^{-9N}$. Hence, the overall estimate is of order $\hbar^{(2l+1)(k+1)-12N}$ at most times a phase space point (defined by the state ψ) depending factor whose value is of the same order of magnitude as the leading order of \hbar coefficient of the calculable quantity:

$$\langle \psi, F_1^+ \cdots F_{N/2}^+ [\Delta F_1'] \cdots [\Delta F_{l+1}'] \psi \rangle. \tag{2.27}$$

From this heuristic reasoning we conclude that in order that the correction terms be of order \hbar^{k+1} we must have $l \geq l_{N,k} := 6N/(k+1)$. The actual value is slightly lower and is given by $l_{N,k} = (12N - 9)/(2(k+1))$. This holds for $N \geq 2$, for $N = 0$ there is nothing to do and the case $N = 1$ can be reduced to the case $N = 0$ by the expansion displayed in section 3. This means that the polarization identity must be invoked at least $l_{N,k}$ times. The number of correction terms that are produced in each iteration triples the number of terms, hence the number of correction terms is of the order of $3^{l_{N,k}}$ and they are all of the same order of magnitude as the lowest order in \hbar coefficient of (2.27).

What all of this means is of course that up to a controllable error of order \hbar^{k+1} the value of (2.19) is given by

$$\langle \psi, p_1 \bar{f}_1 \cdots \bar{f}_N p_{N+1} \psi \rangle = \langle \psi, p_1 f_1^+ \cdots f_N^+ p_{N+1} \psi \rangle + O(\hbar^{k+1}) \tag{2.28}$$

which is calculable and involves the fluctuations of Q up to order \hbar^k (we do not even have to calculate the constants β_k). In other words, we have shown that up to a controllable error of higher order in \hbar it is allowed to use the power expansion of $f(t) = (1+t)^q$ up to order $2k+1$ in order to get a reliable value of (2.19).

2.6. Rigorous estimate

2.6.1. *Step I: proof of (2.24).* We want to prove (2.24) and begin with the following observation⁵: since $(Q^2 - \langle Q \rangle^2)^{2(k+1)} = (Q - \langle Q \rangle)^{2(k+1)} (Q + \langle Q \rangle)^{2(k+1)}$ where $\langle Q \rangle = \langle \psi, Q \psi \rangle$ for our given coherent state ψ we insert resolutions of unity for the matrix element of $(Q - \langle Q \rangle)^{2(k+1)}$ between coherent states and can reduce our attention to the matrix element

$$\begin{aligned} \langle \psi_1, [Q - \langle Q \rangle] \psi_2 \rangle &= i \epsilon_{j_1 j_2 j_3} \left[\prod_{I=1}^3 \langle \psi_{1I}, Y_I^{j_I} \psi'_{2I} \rangle - \prod_{I=1}^3 \langle \psi_{1I}, \psi'_{2I} \rangle \langle \psi_I Y_I^{j_I} \psi_I \rangle \right] \\ &= i \epsilon_{j_1 j_2 j_3} \{ [\langle \psi_{11}, Y_1^{j_1} \psi'_{21} \rangle - \langle \psi_{11}, \psi'_{21} \rangle \langle \psi_1 Y_1^{j_1} \psi_1 \rangle] \langle \psi_{12}, Y_2^{j_2} \psi'_{22} \rangle \langle \psi_{13}, Y_3^{j_3} \psi'_{23} \rangle \\ &\quad + \langle \psi_{11}, \psi'_{21} \rangle \langle \psi_1 Y_1^{j_1} \psi_1 \rangle [\langle \psi_{12}, Y_2^{j_2} \psi'_{22} \rangle - \langle \psi_{12}, \psi'_{22} \rangle \langle \psi_2 Y_2^{j_2} \psi_2 \rangle] \langle \psi_{13}, Y_3^{j_3} \psi'_{23} \rangle \\ &\quad + \langle \psi_{11}, \psi'_{21} \rangle \langle \psi_1 Y_1^{j_1} \psi_1 \rangle \langle \psi_{12}, \psi'_{22} \rangle \langle \psi_2 Y_2^{j_2} \psi_2 \rangle \\ &\quad \times [\langle \psi_{13}, Y_3^{j_3} \psi'_{23} \rangle - \langle \psi_{13}, \psi'_{23} \rangle \langle \psi_3 Y_3^{j_3} \psi_3 \rangle] \}. \end{aligned} \tag{2.29}$$

Here ψ_1, ψ_2 are any coherent states and the operators Y_I^j for a given vertex v are defined by $Y_I^j(v) = t[X_{I+}^j(v) - X_{I-}^j(v)]/2$, see (2.2), where $t = \ell_p^2/a^2$ is the classicality parameter for some length scale a . Here, we have made use of the tensor product structure of the coherent

⁵ The factor $\frac{1}{\langle \psi, Q \psi \rangle}$ is of zeroth order in \hbar and thus is immaterial for the power counting. We will therefore understand that a suitable power of it has been factored out right from the beginning and neglect it in the following discussion.

states. It follows that

$$\begin{aligned} \langle \psi_{1I}, Y_{I'}^{j'} \psi'_{2I} \rangle - \langle \psi_{1I}, \psi'_{2I} \rangle \langle \psi_{1I} Y_{I'}^{j'} \psi_{1I} \rangle &= \frac{t}{2} \{ [\langle \psi_{1,I+}, X_{I+}^{j'} \psi_{2,I+} \rangle - \langle \psi_{I+}, X_{I+}^{j'} \psi_{I+} \rangle \langle \psi_{1,I+}, \psi_{2,I+} \rangle] \\ &\quad \times \langle \psi_{1,I-}, \psi_{2,I-} \rangle - \langle \psi_{1,I+}, \psi_{2,I+} \rangle [\langle \psi_{1,I-}, X_{I-}^{j'} \psi_{2,I-} \rangle \\ &\quad - \langle \psi_{I-}, X_{I-}^{j'} \psi_{I-} \rangle \langle \psi_{1,I-}, \psi_{2,I-} \rangle] \}. \end{aligned} \quad (2.30)$$

We now have reduced the computation to matrix elements of right invariant vector fields on a single copy of $SU(2)$ and can refer to [5] where we find

$$it \langle \psi_{g_1}, X^j \psi_{g_2} \rangle = t \langle \psi_{g_1}, \psi_{g_2} \rangle \left[\frac{2z_{12}}{t} + \frac{1}{z_{12}} - \coth(z_{12}) \right] \frac{\text{Tr}(g_1^\dagger g_2 \tau_j)}{\text{sh}(z_{12})} + O(t^\infty) \quad (2.31)$$

where z_{12} is defined by $\text{ch}(z_{12}) := \text{Tr}(g_1^\dagger g_2)/2$ for $g_1, g_2 \in SL(2, \mathbb{C})$. Thus,

$$\begin{aligned} it [\langle \psi_{g_1}, X^j \psi_{g_2} \rangle - \langle \psi_{g_1}, \psi_{g_2} \rangle \langle \psi_g, X^j \psi_g \rangle] \\ = \langle \psi_{g_1}, \psi_{g_2} \rangle \left[\frac{z_{12}}{\text{sh}(z_{12})} \text{Tr}(g_1^\dagger g_2 \tau_j) - \frac{z}{\text{sh}(z)} \text{Tr}(g^\dagger g \tau_j) + O(t) \right] \end{aligned} \quad (2.32)$$

where $\text{ch}(z) := \text{Tr}(g^\dagger g)/2$.

Thus, using resolutions of unity we have

$$\begin{aligned} \langle \psi, (Q^2 - \langle Q \rangle)^{2(k+1)} \psi' \rangle &= \int dv_1 \cdots dv_{2(k+1)} \langle \psi, [Q - \langle Q \rangle] \psi_1 \rangle \langle \psi_1, [Q - \langle Q \rangle] \psi_2 \rangle \cdots \\ &\quad \times \langle \psi_{2k+1}, [Q - \langle Q \rangle] \psi_{2(k+1)} \rangle \langle \psi_{2(k+1)}, [Q + \langle Q \rangle]^{2(k+1)} \psi' \rangle \end{aligned} \quad (2.33)$$

Now consider for $a = 1, \dots, 2k+1$ with $\psi_0 := \psi$

$$\begin{aligned} \int dv_a \langle \psi_{a-1}, [Q - \langle Q \rangle] \psi_a \rangle \langle \psi_a, [Q - \langle Q \rangle] \psi_{a+1} \rangle &= \frac{1}{t^{18}} \prod_{I\sigma} \int_{\mathbb{R}^3} d^3 p_{a,I\sigma} \int_{SU(2)} d\mu_H(U_{I\sigma}) \\ &\quad \times \exp(-[\|p_{a,I\sigma} - p_{a-1,I\sigma}\|^2 + \|p_{a,I\sigma} - p_{a+1,I\sigma}\|^2 + \|\theta_{a,I\sigma} - \theta_{a-1,I\sigma}\|^2 \\ &\quad + \|\theta_{a,I\sigma} - \theta_{a+1,I\sigma}\|^2]/t) \text{Pol}_4(z_{a,a-1,I\sigma}, z_{a,a+1,I\sigma}) \text{Pol}_2(z_{a,a-1,I\sigma}) \\ &\quad - 2p_{I\sigma}, z_{a,a+1,I\sigma} - 2p_{I\sigma}). \end{aligned} \quad (2.34)$$

Here, we have explicitly written out the resolution measures $dv = c d^3 p d\mu_H(U)[1 + O(\hbar^\infty)]/t^3$ where $g = \exp(p_j \sigma_j)U$, $U = \exp(-i\sigma_j \theta^j)$, c is a positive constant of order unity (essentially π^{-3}) and have used the form of the overlap function $\langle \psi_g, \psi_{g'} \rangle = \exp(-[\|p - p'\|^2 + \|\theta - \theta'\|^2]/t)[1 + O(\hbar^\infty)]$ close to $g' = g$ where it is sharply peaked. Moreover, close to $g = g'$ we have $z = p + p' + i(\theta' - \theta)$. The notation Pol_n denotes a homogeneous polynomial of degree n . In order to evaluate (2.34) we use translation invariance of the Haar measure $d\mu_H(U_{a,I\sigma}) = d\mu_H(U_{a,I\sigma} U_{a-1,I\sigma})^{-1}$ and note that close to $\theta = 0$ the Haar measure $d\mu_H = \sin^2(\theta) \sin(\theta) d\theta d\phi d\varphi$ with $\theta^2 = \theta_j^2$, $\vec{\theta}/\theta = (\sin(\phi) \cos(\varphi), \sin(\phi) \sin(\varphi), \cos(\phi))$ approaches the Lebesgue measure $d^3\theta$. Let us introduce $x_{a,I\sigma}^j := (p_{a,I\sigma}^j - p^j)/s$ and $y_{a,I\sigma}^j = (\theta_{a,I\sigma}^j - \theta_{a-1,I\sigma}^j)/s$ where $s = \sqrt{t}$. Note that $\theta/s \leq \pi/s$ but we can estimate the integral from above by extending the integral over θ to infinity and in any case this just gives an error of higher order in \hbar . Then we can perform the Gaussian integrals and find to leading order in $t \propto \hbar$

$$\begin{aligned} \int dv_a \langle \psi_{a-1}, [Q - \langle Q \rangle] \psi_a \rangle \langle \psi_a, [Q - \langle Q \rangle] \psi_{a+1} \rangle \\ = t \prod_{I\sigma} \exp(-3[\|p_{a+1,I\sigma} - p_{a-1,I\sigma}\|^2 + \|\theta_{a+1,I\sigma} - \theta_{a-1,I\sigma}\|^2]/(4t)) \\ \times \text{Pol}'_4(p, s x_{a-1,I\sigma}, s x_{a+1,I\sigma}, s y_{a-1,I\sigma}, s y_{a+1,I\sigma}) \\ \times \text{Pol}'_2(x_{a-1,I\sigma}, x_{a+1,I\sigma}, y_{a-1,I\sigma}, y_{a+1,I\sigma}) \end{aligned} \quad (2.35)$$

where Pol'_n is not necessarily a homogeneous polynomial of degree n . Hence, after carrying out all $2(k + 1)$ integrals we arrive to leading order in \hbar at an expression of the form

$$\langle \psi_1, (Q^2 - \langle Q \rangle^2)^{2(k+1)} \psi_2 \rangle = t^{k+1} \langle \psi_1, \psi_2 \rangle \text{Pol}'_{10(k+1)}(p) \text{Pol}'_{2(k+1)}(p_1, p_2, \theta_1, \theta_2) \tag{2.36}$$

as claimed.

2.6.2. *Step II: proof of (2.25).* In order to prove (2.25) we must estimate $|\langle \psi_1, f \psi_2 \rangle|$. The method of estimation displayed below is not contained in [5] so we must be more detailed here. In fact, the estimate (2.25) provides the *key part of the proof*, all the manipulations with Gaussian integrals that follow later are fairly standard.

Hence, we need the explicit form of our coherent states [5] which for a single copy of $SU(2)$ are given by

$$\psi_g(h) = \frac{\tilde{\psi}_g(h)}{\|\tilde{\psi}_g(h)\|}, \quad \tilde{\psi}_g(h) = \sum_j d_j e^{-tj(j+1)/2} \chi_j(gh^{-1}) \tag{2.37}$$

where the sum is over all spin quantum numbers $j = 0, 1/2, 1, \dots$, while $d_j = 2j + 1$ and χ_j is the character of the j th irreducible representation π_j of $SU(2)$. Relevant for a given function f of the volume operator located at a vertex v are only the states $\psi_{g_{I\sigma}}$ with $I = 1, 2, 3$ and $\sigma = \pm$ associated with the six outgoing edges adjacent to v . Here, we use the notation of [1, 2], that is, the edges of the algebraic graphs are labelled as $e_I(v) =: e_{I+}(v)$ denoting an edge outgoing from v into the I th direction. The other three edges are ingoing but we can define the outgoing edges $e_{I-}(v) = [e_I(v - \hat{I})]^{-1}$. This has the consequence that

$$\begin{aligned} \tilde{\psi}_{e_{I-}(v)}(A) &:= \tilde{\psi}_{e_I(v-I)}(A) = \sum_j d_j e^{-tj(j+1)} [\pi_j(g_I(v - I))]_{mn} [\pi_j(A(e_I(v - I))^{-1})]_{nm} \\ &= \sum_j \sqrt{d_j} e^{-tj(j+1)} [(\pi_j(g_I(v - I)))^T]_{mn} T_{jmn}(A(e_{I-}(v))) \\ \tilde{\psi}_{e_{I+}(v)}(A) &:= \tilde{\psi}_{e_I(v)}(A) = \sum_j d_j e^{-tj(j+1)} [\pi_j(g_I(v))]_{mn} [\pi_j(A(e_I(v))^{-1})]_{nm} \\ &= \sum_j \sqrt{d_j} e^{-tj(j+1)} [\pi_j(g_I(v))]_{mn} \overline{T_{jmn}(A(e_{I+}(v)))} \end{aligned} \tag{2.38}$$

where $h \mapsto T_{jmn}(h) = \sqrt{2j + 1} \pi_{jmn}(h)$ denotes a spin network function and we have used unitarity in the second step. Let $g_{I+} := g_I(v)$, $g_{I-} := [g_I(v - I)]^T$ then we obtain

$$\begin{aligned} \langle \psi_1, f \psi_2 \rangle \prod_{I,\sigma} \|\tilde{\psi}_{1,I\sigma}\| \|\tilde{\psi}_{2,I\sigma}\| &= \sum_{\{j_{I\sigma}\}} \prod_{I\sigma} [2j_{I\sigma} + 1] e^{-tj_{I\sigma}(j_{I\sigma}+1)} \sum_{|m_{I\sigma}|, |m'_{I\sigma}|, |n_{I\sigma}|, |n'_{I\sigma}| \leq j_{I\sigma}} \\ &\times \prod_{I\sigma} \overline{\pi_{j_{I\sigma} m_{I\sigma} n_{I\sigma}}(g_{1,I\sigma})} \pi_{j_{I\sigma} m'_{I\sigma} n'_{I\sigma}}(g_{2,I\sigma}) \\ &\times \langle \otimes_I [\overline{T_{j_{I+}, m_{I+}, n_{I+}}} \otimes T_{j_{I-}, m_{I-}, n_{I-}}], f \otimes_I [\overline{T_{j_{I+}, m'_{I+}, n'_{I+}}} \otimes T_{j_{I-}, m'_{I-}, n'_{I-}}] \rangle \\ &= \sum_{\{j_{I\sigma}\}} \prod_{I\sigma} [2j_{I\sigma} + 1] e^{-tj_{I\sigma}(j_{I\sigma}+1)} \sum_{|m_{I\sigma}|, |m'_{I\sigma}| \leq j_{I\sigma}} \prod_{I\sigma} \pi_{j_{I\sigma} m'_{I\sigma} m_{I\sigma}}(g_{2,I\sigma} g_{1,I\sigma}^\dagger) \\ &\times \langle \otimes_I [\overline{|j_{I+}, m_{I+}\rangle} \otimes |j_{I-}, m_{I-}\rangle], f \otimes_I [|\overline{j_{I+}, m'_{I+}\rangle} \otimes |j_{I-}, m'_{I-}\rangle] \rangle \end{aligned} \tag{2.39}$$

where in the first step it was crucially exploited that the volume operator preserves the mutually orthogonal subspaces of the Hilbert space labelled by fixed spin quantum numbers. In the second step we exploited that the spin network matrix element in the third line is non-vanishing only when $n_{I\sigma} = n'_{I\sigma}$ and in that case it is actually independent of $n_{I\sigma}$. The calculation then

equals that in an abstract spin system so that we dropped the dependence on $n_{I\sigma}$ of the matrix element, thereby replacing T_{jmn} by $|jm\rangle$. The latter fact follows for instance from the detailed analysis in [16].

Denote the factor in the fifth line of (2.39) by $\overline{A_{\{m,m'\}}}$ and the factor in the sixth line by $B_{\{m,m'\}}$ then (2.39) can be estimated by using the Schwarz inequality

$$\begin{aligned}
 |\langle \psi_1, f \psi_2 \rangle| & \prod_{I,\sigma} \|\tilde{\psi}_{1,I\sigma}\| \|\tilde{\psi}_{2,I\sigma}\| \\
 & \leq \sum_{\{j_{I\sigma}\}} \prod_{I\sigma} [2j_{I\sigma} + 1] e^{-tj_{I\sigma}(j_{I\sigma}+1)} \sqrt{\sum_{\{m,m'\}} |A_{m,m'}|^2} \sqrt{\sum_{\{m,m'\}} |B_{m,m'}|^2} \\
 & = \sum_{\{j_{I\sigma}\}} \prod_{I\sigma} [2j_{I\sigma} + 1] e^{-tj_{I\sigma}(j_{I\sigma}+1)} \sqrt{\prod_{I\sigma} \chi_{j_{I\sigma}}(g_{1,I\sigma}^\dagger g_{2,I\sigma} g_{2,I\sigma}^\dagger g_{1,I\sigma}^\dagger)} \\
 & \quad \times \sqrt{\sum_{\{m\}} \langle \otimes_I [\overline{|j_{I+}, m_{I+}\rangle} \otimes |j_{I-}, m_{I-}\rangle], f \otimes_I [\overline{|j_{I+}, m_{I+}\rangle} \otimes |j_{I-}, m_{I-}\rangle] \rangle} \quad (2.40)
 \end{aligned}$$

where in the second step we used the completeness relation of the states $|jm\rangle$ on the fixed j subspace of the abstract spin system.

We will now estimate the second square root in (2.40). Since certainly $f(t) = (1+t)^q - 1 \leq 1 + (1+t) - 1 = 1+t$ for all $t \geq -1$ we have $f \leq Q^2/\langle Q \rangle^2$ and therefore have

$$\begin{aligned}
 & \sum_{\{m\}} \langle \otimes_I [\overline{|j_{I+}, m_{I+}\rangle} \otimes |j_{I-}, m_{I-}\rangle], f \otimes_I [\overline{|j_{I+}, m_{I+}\rangle} \otimes |j_{I-}, m_{I-}\rangle] \rangle \\
 & \leq \sum_{\{m\}} \frac{1}{\langle Q \rangle^2} \|Q \otimes_{I\sigma} |j_{I\sigma}, m_{I\sigma}\rangle\|^2 \quad (2.41)
 \end{aligned}$$

where we exploited that in the angular momentum representation $\overline{|j, m\rangle} = |j, -m\rangle$ and that we sum over all m . Now we just have to evaluate the norm squared in (2.41) by using elementary angular momentum calculus, we do not need to work in the recoupling basis [13].

Since Y_I^j, Y_J^k mutually commute for $I \neq J$ we have

$$\begin{aligned}
 Q^2 & = -t^6 \epsilon^{IJK} \epsilon_{MNP} Y_I^1 Y_J^2 Y_K^3 Y_M^1 Y_N^2 Y_P^3 \\
 & = -3! t^6 Y_I^1 Y_J^2 Y_K^3 Y_I^1 Y_J^2 Y_K^3 \\
 & = \sum_{I \neq J, K: J \neq K} \{ [Y_I^1 Y_I^1] [Y_J^2 Y_J^2] [Y_K^3 Y_K^3] - [Y_I^1 Y_I^1] [Y_J^2 Y_J^2] [Y_K^3 Y_K^2] + [Y_I^1 Y_I^3] [Y_J^2 Y_J^1] [Y_K^3 Y_K^2] \\
 & \quad - [Y_I^1 Y_I^2] [Y_J^2 Y_J^1] [Y_K^3 Y_K^3] + [Y_I^1 Y_I^2] [Y_J^2 Y_J^3] [Y_K^3 Y_K^1] - [Y_I^1 Y_I^3] [Y_J^2 Y_J^2] [Y_K^3 Y_K^1] \}. \quad (2.42)
 \end{aligned}$$

When we take the expectation value of (2.42) with respect to $\otimes_{I\sigma} |j_{I\sigma}, m_{I\sigma}\rangle$, only the first and fourth terms survive because all other terms contain factors of the form $Y_I^1 Y_I^3, Y_I^3 Y_I^1, Y_I^2 Y_I^3, Y_I^3 Y_I^2$ of which Y_I^3 is diagonal but Y_I^1, Y_I^2 is a linear combination of raising and lowering operators.

Since $iY_I^j = i(X_{I+}^j - X_{I-}^j)/2$ is represented on the abstract spin system as $J_{I+}^j - J_{I-}^j$ where $J_{I\sigma}^j$ are the usual angular momentum operators we can now evaluate (2.42) further

$$\begin{aligned}
 \|Q \otimes_{I\sigma} |j_{I\sigma}, m_{I\sigma}\rangle\|^2 & = \frac{t^6}{4} \sum_{I \neq J, K: J \neq K} \langle m_{I+}, m_{I-} | [J_{I+}^+]^2 + [J_{I-}^-]^2 + J_{I+}^+ J_{I-}^- + J_{I-}^- J_{I+}^+ | m_{I+}, m_{I-} \rangle \\
 & \quad \times \langle m_{J+}, m_{J-} | -[J_{J+}^+]^2 - [J_{J-}^-]^2 + J_{J+}^+ J_{J-}^- + J_{J-}^- J_{J+}^+ | m_{J+}, m_{J-} \rangle [m_{K+} - m_{K-}]^2
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{t^6}{4} \sum_{I \neq J, K; J \neq K} \langle m_{I+}, m_{I-} | [J_I^+]^2 - [J_I^-]^2 + [J_I^-, J_I^+] | m_{I+}, m_{I-} \rangle \\
 & \times \langle m_{J+}, m_{J-} | [J_J^+]^2 - [J_J^-]^2 + [J_J^+, J_J^-] | m_{J+}, m_{J-} \rangle [m_{K+} - m_{K-}]^2 \\
 & = \frac{t^6}{16} \sum_{I \neq J, K; J \neq K} (m_{K+} - m_{K-})^2 \{ (j_{I+}(j_{I+} + 1) + j_{I-}(j_{I-} + 1) - m_{I+}^2 - m_{I-}^2) \\
 & \times (j_{J+}(j_{J+} + 1) + j_{J-}(j_{J-} + 1) - m_{J+}^2 - m_{J-}^2) - (m_{I+} + m_{I-})(m_{J+} + m_{J-}) \}
 \end{aligned} \tag{2.43}$$

where $J_I^\pm = J_I^1 \pm iJ_I^2 = J_{I+}^\pm - J_{I-}^\pm$ is a linear combination of the usual ladder operators and we have used the algebra $[J_{I\sigma}^j, J_{J\sigma'}^k] = i\delta_{IJ}\delta_{\sigma\sigma'}\epsilon_{jkl}J_{I\sigma}^l$.

The second term in (2.43) vanishes when summing over $\{m\}$ while the first can be evaluated using the relation

$$\sum_{|m| \leq j} m^2 = \frac{j}{3}(j + 1)(2j + 1) \tag{2.44}$$

resulting in the compact formula

$$\sum_{\{m\}} \|Q \otimes_{I\sigma} |j_{I\sigma}, m_{I\sigma}\rangle\|^2 = \left(\frac{2}{3}\right)^5 t^6 \left[\prod_{I\sigma} (2j_{I\sigma} + 1) \right] \left[\prod_I \left[\sum_{\sigma} j_{I\sigma}(j_{I\sigma} + 1) \right] \right]. \tag{2.45}$$

As we wish to apply the Poisson resummation formula in order to evaluate the estimate (2.40), we should estimate (2.45) in terms of the integers $n_{I\sigma} = 2j_{I\sigma} + 1$. We have

$$\begin{aligned}
 & \left(\frac{2}{3}\right)^5 t^6 \left[\prod_{I\sigma} (2j_{I\sigma} + 1) \right] \left[\prod_I \left[\sum_{\sigma} j_{I\sigma}(j_{I\sigma} + 1) \right] \right] \\
 & = \left(\frac{1}{3}\right)^5 \frac{t^6}{2} \left[\prod_{I\sigma} n_{I\sigma} \right] \left[\prod_I \left[\sum_{\sigma} n_{I\sigma}^2 - 1 \right] \right] \\
 & \leq \left(\frac{1}{3}\right)^5 \frac{t^6}{2} \prod_I [n_{I+}^3 n_{I-} + n_{I-}^3 n_{I+}] \\
 & \leq \left(\frac{1}{3}\right)^5 \frac{t^6}{16} \prod_I [n_{I+} + n_{I-}]^4
 \end{aligned} \tag{2.46}$$

where we used $\sqrt{mm'} \leq (m + m')/2$ and $m^2 + (m')^2 \leq (m + m')^2$ for non-negative integers m, m' .

Next we evaluate the characters

$$\chi_{j_{I\sigma}}(g_{1,I\sigma} g_{2,I\sigma}^\dagger g_{2,I\sigma} g_{1,I\sigma}) = \frac{\text{sh}(n_{I\sigma} z_{I\sigma})}{\text{sh}(z_{I\sigma})} \tag{2.47}$$

where

$$\text{ch}(z_{I\sigma}) := \frac{1}{2} \text{Tr}(g_{1,I\sigma} g_{2,I\sigma}^\dagger g_{2,I\sigma} g_{1,I\sigma}). \tag{2.48}$$

Since the argument of the trace in (2.48) is a positive definite, Hermitian, unimodular matrix, it is clear that $z_{I\sigma}$ is positive. Since (2.47) appears under a square root which would make the

application of the Poisson resummation formula cumbersome, we use the elementary estimate

$$\sqrt{2\text{sh}(z)} \leq 2\text{ch}(z/2) \quad (2.49)$$

which holds for all $z \geq 0$. Putting everything together we find

$$\begin{aligned} |(\psi_1, f\psi_2)| \prod_{I,\sigma} \|\tilde{\psi}_{1,I\sigma}\| \|\tilde{\psi}_{2,I\sigma}\| &\leq \frac{t^3}{36\sqrt{3}} \sum_{\{n_{I\sigma}=1\}} \left[\prod_{I\sigma} e^{-t(n_{I\sigma}^2-1)/4} \frac{\text{ch}(n_{I\sigma} z_{I\sigma}/2)}{\sqrt{\text{sh}(z_{I\sigma})}} \right] \\ &\times \left[\prod_I n_{I+} n_{I-} (n_{I+} + n_{I-})^2 \right] \leq \frac{2t^3}{9\sqrt{3}} \sum_{\{n_{I\sigma}=1\}} \left[\prod_{I\sigma} e^{-t(n_{I\sigma}^2-1)/4} \frac{\text{ch}(n_{I\sigma} z_{I\sigma}/2)}{\sqrt{\text{sh}(z_{I\sigma})}} \right] \\ &\times \left[\prod_I (n_{I+}^4 + n_{I-}^4) \right] = \frac{2t^3 e^{3t/2}}{9\sqrt{3}} \frac{1}{\sqrt{\prod_{I\sigma} \text{sh}(z_{I\sigma})}} \sum_{\sigma_1, \sigma_2, \sigma_3} \\ &\times \prod_I \left\{ \left[\sum_{n_{I,\sigma_1}=1}^{\infty} n_{I,\sigma_1}^4 e^{-tn_{I,\sigma_1}^2/4} \text{ch}(n_{I,\sigma_1} z_{I,\sigma_1}/2) \right] \right. \\ &\times \left. \left[\sum_{n_{I,-\sigma_1}=1}^{\infty} e^{-tn_{I,-\sigma_1}^2/4} \text{ch}(n_{I,-\sigma_1} z_{I,-\sigma_1}/2) \right] \right\} = \frac{t^3 e^{3t/2}}{2^5 9\sqrt{3}} \frac{1}{\sqrt{\prod_{I\sigma} \text{sh}(z_{I\sigma})}} \sum_{\sigma_1, \sigma_2, \sigma_3} \\ &\times \prod_I \left\{ \left[\sum_{n_{I,\sigma_1}=-\infty}^{\infty} n_{I,\sigma_1}^4 e^{-tn_{I,\sigma_1}^2/4} e^{n_{I,\sigma_1} z_{I,\sigma_1}/2} \right] \right. \\ &\times \left. \left[-1 + \sum_{n_{I,-\sigma_1}=-\infty}^{\infty} e^{-tn_{I,-\sigma_1}^2/4} e^{n_{I,-\sigma_1} z_{I,-\sigma_1}/2} \right] \right\} \leq \frac{t^3 e^{3t/2}}{2^5 9\sqrt{3}} \frac{1}{\sqrt{\prod_{I\sigma} \text{sh}(z_{I\sigma})}} \sum_{\sigma_1, \sigma_2, \sigma_3} \\ &\times \prod_I \left\{ \left[\sum_{n_{I,\sigma_1}=-\infty}^{\infty} n_{I,\sigma_1}^4 e^{-tn_{I,\sigma_1}^2/4} e^{n_{I,\sigma_1} z_{I,\sigma_1}/2} \right] \right. \\ &\times \left. \left[\sum_{n_{I,-\sigma_1}=-\infty}^{\infty} e^{-tn_{I,-\sigma_1}^2/4} e^{n_{I,-\sigma_1} z_{I,-\sigma_1}/2} \right] \right\} \end{aligned} \quad (2.50)$$

where we have used $2mn \leq m^2 + n^2$ in the second step.

Note that

$$\|\psi_{j,I\sigma}\|^2 = e^{t/4} \sum_{n=1}^{\infty} n e^{-tn^2/4} \frac{\text{sh}(ny_{j,I\sigma})}{\text{sh}(y_{j,I\sigma})} = \frac{e^{t/4}}{2\text{sh}(y_{j,I\sigma})} \sum_{n=-\infty}^{\infty} n e^{-tn^2/4} e^{ny_{j,I\sigma}} \quad (2.51)$$

where $\text{ch}(y_{j,I\sigma}) := \text{Tr}(g_{j,I\sigma}^\dagger g_{j,I\sigma})/2$, $j = 1, 2$. Using the Poisson resummation formula [5] we can now evaluate

$$\sum_n n e^{-n^2 t/4} e^{yn} = \frac{2\sqrt{\pi}}{s^3} e^{y^2/t} y \quad (2.52)$$

$$\sum_n e^{-n^2 t/4} e^{zn/2} = \frac{2\sqrt{\pi}}{s} e^{z^2/(4t)}$$

$$\sum_n n^4 e^{-n^2 t/4} e^{zn/2} = \frac{2\sqrt{\pi}}{s^9} e^{z^2/(4t)} \left(\frac{3}{2} s^4 + \frac{3}{4} z^2 s^2 + z^4/16 \right) \quad (2.53)$$

where $s = \sqrt{t}$ and we have neglected $O(t^\infty)$ terms. We obtain the final formula

$$|\langle \psi_1, f \psi_2 \rangle| \leq \frac{1}{2^{59} \sqrt{3}} t^{-3} \sqrt{\prod_{I\sigma} \frac{\text{sh}(y_{1,I\sigma}) \text{sh}(y_{2,I\sigma})}{y_{1,I\sigma} y_{2,I\sigma} \text{sh}(z_{I\sigma})}} \prod_{I\sigma} e^{[z_{I\sigma}^2 - 2y_{1,I\sigma}^2 - 2y_{2,I\sigma}^2]/(4t)} \times \sum_{\sigma_1, \sigma_2, \sigma_3} \prod_I \left[\frac{3}{2} s^4 + \frac{3}{4} z_{I\sigma_I}^2 s^2 + z_{I\sigma_I}^4 / 16 \right]. \tag{2.54}$$

We want to show that (2.54) is sharply peaked at $H_{1,I\sigma} = H_{2,I\sigma}$ where $g_{j,I\sigma} = U_{j,I\sigma} H_{j,I\sigma}$ is a polar decomposition into unitary and positive definite Hermitian matrices of the $SL(2, \mathbb{C})$ matrices. Parametrizing $H = \exp(p_j \sigma_j) = \text{ch}(p) + \sigma_j p_j \text{sh}(p)/p$ where $p^2 = p_j p_j$ and σ_j are the Pauli matrices we easily find $y_{j,I\sigma} = 2p_{j,I\sigma}$ and

$$\text{ch}(z_{I\sigma}) = \text{ch}(y_{1,I\sigma}) \text{ch}(y_{2,I\sigma}) + c_{I\sigma} \text{sh}(y_{1,I\sigma}) \text{sh}(y_{2,I\sigma}), \quad c_{I\sigma} = \frac{p_{1,I\sigma}^j p_{2,I\sigma}^j}{p_{1,I\sigma} p_{2,I\sigma}}. \tag{2.55}$$

Since $c_{I\sigma} \in [-1, 1]$ we find from basic hyperbolic identities

$$|y_{1,I\sigma} - y_{2,I\sigma}| \leq z_{I\sigma} \leq (y_{1,I\sigma} + y_{2,I\sigma}). \tag{2.56}$$

Hence the argument of the exponent in (2.54) satisfies the following relation:

$$z_{I\sigma}^2 - 2y_{1,I\sigma}^2 - 2y_{2,I\sigma}^2 \leq -(y_{1,I\sigma} - y_{2,I\sigma})^2 \leq 0 \tag{2.57}$$

where equality is reached if and only if $c_{I\sigma} = 1$ and $y_{1,I\sigma} = y_{2,I\sigma}$, that is, $H_{1,I\sigma} = H_{2,I\sigma}$ as claimed. This property of (2.54) will be crucial for what follows, because otherwise in the resolutions of unity that will follow below, if we did not have the Gaussian decay just established, the corresponding integrals would blow up.

2.6.3. *Step III: proof of (2.23).* Next we prove (2.23). From [5] we recall that the measure ν involved in the resolution of unity is given, up to an $O(t^\infty)$ correction, by

$$d\nu(g) = \frac{c}{t^3} d^3 p d\mu_H(U), \quad g = \exp(p^j \sigma_j) U \tag{2.58}$$

where c is a numerical constant of order unity. Hence, the resolution measure is up to the important factor $1/t^3$ essentially given by the Liouville measure on $T^*(SU(2))$. Also recall [5] that the overlap function is essentially (i.e. up to $O(t^\infty)$ corrections) given by

$$|\langle \psi_g, \psi_{g'} \rangle|^2 = \frac{|\tilde{z}| \text{sh}(p) \text{sh}(p')}{|\text{sh}(\tilde{z})| p p'} \exp\left(-\frac{4\Delta^2 + 2\tilde{\theta}^2 + 2\delta^2}{t}\right) \tag{2.59}$$

$$\Delta^2 = p^2 + (p')^2 - \tilde{p}^2/2$$

$$\text{ch}(2\tilde{p}) = (1+c)\text{ch}^2(p+p') + (1-c)\text{ch}^2(p-p') - 1, \quad c = \frac{p_j p'_j}{p p'}$$

$$\text{ch}(\tilde{z}) = \text{ch}(\tilde{p}) \cos(\tilde{\theta}) + i \text{sh}(\tilde{p}) \sin(\tilde{\theta}) \cos(\tilde{\alpha}), \quad \cos(\tilde{\alpha}) = \frac{\tilde{p}_j \tilde{\theta}_j}{\tilde{p} \tilde{\theta}}$$

$$\tilde{\delta}^2 = \tilde{p}^2 - \tilde{s}^2 + \tilde{\phi}^2 - \tilde{\theta}^2, \quad \tilde{z} = \tilde{s} + i\tilde{\phi}$$

where the notation is as follows: we set $g = \exp(p_j \sigma_j) U$, $g' = \exp(p'_j \sigma_j) U'$ and $\exp(p_j \sigma_j) \exp(p'_j \sigma_j) = \exp(\tilde{p}_j \sigma_j) U''$ and finally $\tilde{U} = U^{-1} U' (U'')^{-1} = \exp(-i\tilde{\theta}_j \sigma_j)$. Here, $\tilde{\theta}^2 = \tilde{\theta}_j \tilde{\theta}_j$ and all angles $\beta, \tilde{\alpha}, \tilde{\phi}$ lie in $[0, \pi]$. All quantities are uniquely determined by the above formulae. Moreover one can show

- (1) $\Delta^2 \geq 0$ where equality is reached if and only if $p_j = p'_j$;
- (2) $\delta^2 \geq 0$ where equality is reached if and only if either (a) $\tilde{\phi} = \tilde{\theta}$, $|s| = \tilde{p}$, $|\cos(\tilde{\alpha})| = 1$ or (b) $s = \tilde{p} = 0$ or (c) $\tilde{\phi} = \tilde{\theta} = 0, \pi$.

Since $\delta^2 \geq 0$, in order that the exponent in the first line of (2.59) vanishes we must have at least $\Delta^2 = \tilde{\theta}^2 = 0$. Since $\Delta^2 = 0$ is equivalent to $p_j = p'_j$ this means that $U'' = 1$ and therefore $\tilde{\theta} = 0$ means $U = U'$, that is, $g = g'$ altogether. Clearly, then $z = \tilde{p} = 2p$, hence $\tilde{\phi} = \tilde{\theta} = 0$ so we are in case (c) and thus $\delta^2 = 0$.

What all of this means is that the overlap function essentially takes the form of a Gaussian peaked at $g = g'$. The more complicated structure is due to the fact that we are dealing with a non-Abelian group rather than a vector space. Since the overlap function vanishes rapidly when $g \neq g'$, let us expand (2.59) in terms of $\Delta p_j = p'_j - p_j$ and $\Delta \theta_j = \theta'_j - \theta_j$ where $U = \exp(-i\theta_j \sigma_j)$, $U' = \exp(-i\theta'_j \sigma_j)$. Then it is not difficult to see that

$$\begin{aligned}
 |\langle \psi_g, \psi_{g'} \rangle|^2 &= \exp\left(-\frac{[\Delta p]^2 + [\Delta \theta]^2}{t}\right) \\
 |\langle \psi_1, f \psi_2 \rangle| &\leq \frac{a^{627}}{9\sqrt{3}} t^{-3} \exp\left(-\sum_{I,\sigma} \frac{[\Delta p_{I\sigma}]^2}{t}\right) \left[\prod_I p_{I+}^4 + p_{I-}^4 \right]
 \end{aligned}
 \tag{2.60}$$

where $\Delta p_{I\sigma}^j = p_{1,I\sigma}^j - p_{2,I\sigma}^j$.

A further result that we need from [5] is the following. The polynomials P are typically just (products of) holonomy operators in the spin 1/2 representation for which holds

$$\begin{aligned}
 \langle \psi_g, h_{AB} \psi_{g'} \rangle &= \langle \psi_g, \psi_{g'} \rangle \left\{ \left[g'_{AB} \text{ch}(z/2) + (g' \tau_j)_{AB} \frac{\text{Tr}(g^\dagger g' \tau_j)}{2\text{sh}(z)} \text{sh}(z/2) \right] \right. \\
 &\quad \left. + t \left[\frac{\text{sh}(z/2)}{2z} + (\text{ch}(z)g' - g^\dagger (g')^2)_{AB} \frac{\text{coth}(z) \text{sh}(z/2) - \frac{1}{2} \text{ch}(z/2)}{z \text{sh}(z)} \right] + [O(t^\infty)] \right\}.
 \end{aligned}
 \tag{2.61}$$

Here $g^\dagger g' = \text{ch}(z) - i\tau_j z_j / z \text{sh}(z)$. Due to the Gaussian prefactor $\langle \psi_g, \psi_{g'} \rangle$ the first term of order $O(t^0)$ contributes only at $g = g'$ and there it is given by U_{AB} where $g = HU$, $g' = H'U'$ are the polar decompositions of g, g' with $H = \exp(ip_j \tau_j)$, $U = \exp(\theta_j \tau_j)$. At $g \neq g'$ this term as well as the one of order t^1 and t^∞ is exponentially bounded in p, p' which is suppressed by the Gaussian. More precisely, in the vicinity of $g' = g$ where the main contribution to the integral that we are interested in comes from, the term of zeroth order (in t) in (2.23) becomes approximately

$$\langle \psi_g, h_{AB} \psi_{g'} \rangle = \exp\left(-\frac{1}{t} [(p'_j - p_j)^2 + (\theta'_j - \theta_j)^2]\right) \exp(i(p'_j - p_j) \tau_j / 2) \exp((\theta'_j + \theta_j) \tau_j / 2)
 \tag{2.62}$$

as claimed.

2.6.4. *Step IV: performance of the resolution integrals.* The way to extract the leading \hbar power of the correction terms is now as follows: recalling (2.22), the final integrals to be computed are of the form

$$\begin{aligned}
 &\int d\nu_1 \cdots \int d\nu_{N/2} \int d\nu'_1 \cdots \int d\nu'_{N/2} \int d\tilde{\nu}_1 \cdots \int d\tilde{\nu}_l \int d\tilde{\nu}'_1 \cdots \int d\tilde{\nu}'_l \int d\nu_{00} \cdots \int d\nu_{0N/2} \\
 &\quad \times \int d\nu'_{00} \cdots \int d\nu'_{0N/2} \int d\tilde{\nu}_{01} \cdots \int d\tilde{\nu}_{0l} \int d\tilde{\nu}'_{01} \cdots \int d\tilde{\nu}'_{0l} \\
 &\quad \times |\langle \psi, P_1 \psi_{00} \rangle| |\langle \psi_{00}, f_1 \psi_1 \rangle| |\langle \psi_1, P_2 \psi_{01} \rangle| |\langle \psi_{01}, f_2 \psi_2 \rangle| \cdots \\
 &\quad \times |\langle \psi_{0N/2-1}, f_{N/2} \psi_{N/2} \rangle| |\langle \psi_{N/2}, P_{N/2+1} \psi_{0N/2} \rangle|
 \end{aligned}$$

$$\begin{aligned}
 & \times |\langle \psi, P_1 \tilde{\psi}_{00} \rangle| |\langle \tilde{\psi}_{00}, f_1 \tilde{\psi}_1 \rangle| |\langle \tilde{\psi}_1, P_2 \tilde{\psi}_{01} \rangle| |\langle \tilde{\psi}_{01}, f_2 \tilde{\psi}_2 \rangle| \cdots \\
 & \times |\langle \tilde{\psi}_{N/2-1}, f_{N/2} \tilde{\psi}_{N/2} \rangle| |\langle \tilde{\psi}_{N/2}, P_{N/2+1} \tilde{\psi}_{0N/2} \rangle| \\
 & \times |\langle \psi_{0N/2}, [\Delta f'_1] \psi'_1 \rangle| |\langle \psi'_1, P'_1 \psi'_{01} \rangle| |\langle \psi'_{01}, [\Delta f'_2] \psi'_2 \rangle| |\langle \psi'_2, P'_2 \psi'_{02} \rangle| \cdots \\
 & \times |\langle \psi'_{0l-1}, [\Delta f'_l] \psi'_l \rangle| |\langle \tilde{\psi}_{0N/2}, [\Delta f'_1] \tilde{\psi}'_1 \rangle| \\
 & \times |\langle \tilde{\psi}'_1, P'_1 \tilde{\psi}'_{01} \rangle| |\langle \tilde{\psi}'_{01}, [\Delta f'_2] \tilde{\psi}'_2 \rangle| |\langle \tilde{\psi}'_2, P'_2 \tilde{\psi}'_{02} \rangle| \cdots \\
 & \times |\langle \tilde{\psi}'_{0l-1}, [\Delta f'_l] \tilde{\psi}'_l \rangle| |\langle \psi'_l, P'_l \psi'_{0l} \rangle| |\langle \tilde{\psi}'_l, P'_l \tilde{\psi}'_{0l} \rangle| |\langle \psi'_{0l}, [\Delta f'_{l+1}] \tilde{\psi}'_{0l} \rangle|. \tag{2.63}
 \end{aligned}$$

As we have seen, factors of the form $|\langle \psi_1, f \psi_2 \rangle|$ are to leading order in \hbar essentially estimated by products of Gaussians in $p_{2,I\sigma}^j - p_{1,I\sigma}^j$ times a homogeneous polynomial in $p_{1,I\sigma}^j, p_{2,I\sigma}^j$ of order twelve. On the other hand, factors of the form $\langle \psi_1, [\Delta F] \psi_2 \rangle$ could be written to leading order in \hbar as Gaussians in $p_{2,I\sigma}^j - p_{1,I\sigma}^j$ and $\theta_{2,I\sigma}^j - \theta_{1,I\sigma}^j$ times a homogeneous polynomial of order $5(2k+2)$ in $z_{I\sigma}^j$ times a homogeneous polynomial of order $2k+2$ in $z_{I\sigma}^j - p_{I\sigma}^j$ where for $g_{1,I\sigma} \approx g_{2,I\sigma}$ we have

$$z_{I\sigma}^j \approx [p_{2,I\sigma} + p_{1,I\sigma}] + i[\theta_{2,I\sigma} - \theta_{1,I\sigma}] \tag{2.64}$$

as follows from the explicit formulae proved in [5]. Here $p_{I\sigma}^j = i\langle \psi, X_{I\sigma}^j \psi \rangle$ is the expectation value of the right invariant vector fields with respect to the state ψ under consideration. Finally, factors of the form $\langle \psi_1, P \psi_2 \rangle$ could be written, to leading order in \hbar , as Gaussians in $p_{2,I\sigma}^j - p_{1,I\sigma}^j$ and $\theta_{2,I\sigma}^j - \theta_{1,I\sigma}^j$ times sums of products of matrix elements of the $SL(2, \mathbb{C})$ elements $\exp(-i(p_{2,I\sigma}^j - p_{1,I\sigma}^j)\tau_j/2), \exp(\theta_{2,I\sigma}^j + \theta_{1,I\sigma}^j)\tau_j/2$.

We now carry out the various integrals to leading order in \hbar . We will drop the indices I, σ, j in the following power counting argument, hence for instance $d\nu_a$ means an integral over all $p_{a,I\sigma}, \theta_{a,I\sigma}^j$ and by p_a we mean the 18 variables $p_{a,I\sigma}$, etc and correspondingly $\|p_a\|^2 := \sum_{I\sigma j} (p_{a,I\sigma}^j)^2$. Further $dp_a := \prod_{I\sigma j} dp_{a,I\sigma}^j, d\mu_H(U_a) := \prod_{I\sigma} d\mu_H(U_{a,I\sigma})$. Also, for simplicity we assume that, as happens in our application, all volume operators are with respect to same vertex v , otherwise we just have to introduce more notation which however does not change the argument. Also for simplicity we assume that we just have to consider holonomies along the edges $e_{I\sigma}(v)$, for loops we would otherwise just have to use more resolution integrals but they do not change the \hbar power.

The first type of integral we consider is (for $a = 0, \dots, N/2 - 1$ with $\psi_0 := \psi$)

$$\begin{aligned}
 \int d\nu_{0a} |\langle \psi_a, P_{a+1} \psi_{0a} \rangle| |\langle \psi_{0a}, F_{a+1} \psi_{a+1} \rangle| & \leq \frac{1}{t^3} \int \frac{dp_{0a} d\mu_H(U_{0a})}{t^{18}} \\
 & \times \exp(-[\|p_{0a} - p_a\|^2 + \|p_{0a} - p_{a+1}\|^2 + \|\theta_{0a} - \theta_a\|^2]/t) \\
 & \times |\text{Pol}_{12}(p_{0a}, p_{a+1}) \text{Pol}_1(\exp([p_{0a} - p_a]/2), \exp(i[\theta_{0a} + \theta_a]/2))| \tag{2.65}
 \end{aligned}$$

where Pol_n denotes a homogeneous polynomial of degree n of the variables indicated and we have only kept the leading order in \hbar . Here $\exp([p_{0a} - p_a]/2)$ stands for the collection of group elements $\exp(\sigma_j(p_{0a,I\sigma}^j - p_{a,I\sigma}^j)/2)$, etc.

Using translation invariance of the Haar measure, we get $d\mu_H(U_{0a}) = d\mu_H(U_{0a}U_a^{-1})$ and since the Gaussian in (2.64) receives its essential support at $U_{0a} = U_a$ the Haar measure can be replaced by the Lebesgue measure, that is, for small θ^j the Haar measure $d\mu_H(U) = \sin^2(\theta) \sin(\phi) d\theta d\phi d\varphi$ with $\vec{\theta}/\theta = (\sin(\phi) \cos(\varphi), \sin(\phi) \sin(\varphi), \cos(\phi))$ approaches $d^3\theta$ with integration domain of a ball of radius π . The polynomial involving $\exp((\theta_{0a} + \theta_a)/2)$ can be estimated from above by a polynomial in $\exp((p_{0a} - p_a)/2)$ alone because matrix elements of $SU(2)$ group elements are bounded by 1.

Let us introduce $x_{0a} := (p_{0a} - p)/s$, $x_a := (p_a - p)/s$, $y_{0a} := (\theta_{0a} - \theta_a)/s$ where $s = \sqrt{t}$ then (2.64) can be further estimated by

$$\begin{aligned}
 & \int dv_{0a} |\langle \psi_a, P_{a+1} \psi_{0a} \rangle| |\langle \psi_{0a}, f_{a+1} \psi_{a+1} \rangle| \\
 & \leq \frac{1}{t^{21}} \int_{\mathbb{R}^{18}} dx_{0a} \int_{\|y_{0a}\| \leq \pi} dy_{0a} \exp(-[\|x_{0a} - x_a\|^2 + \|x_{0a} - x_{a+1}\|^2 + \|y_{0a}\|^2]) \\
 & \quad \times |\text{Pol}_{12}(p + sx_{0a}, p + sx_{a+1}) \text{Pol}_1(\exp(s[x_{0a} - x_a]), \exp(\theta_{0a} + \theta_a))| \\
 & \leq \frac{1}{t^{21}} \int_{\mathbb{R}^{18}} dx_{0a} \int_{\|y_{0a}\| \leq \pi/s} dy_{0a} \exp(-[\|x_{0a} - x_a\|^2 + \|x_{0a} - x_{a+1}\|^2 + \|y_{0a}\|^2]) \\
 & \quad \times |\text{Pol}_{12}(p + sx_{0a}, p + sx_{a+1}) \text{Pol}_1(\exp(s[x_{0a} - x_a]))| \\
 & = \frac{\pi^{18}}{t^3} \text{Pol}_{12}(p) \exp(-3(\|x_a - x_{a+1}\| + s)^2/4) \tag{2.66}
 \end{aligned}$$

where we have used basic properties of Gaussian integrals, used $\int_0^{\pi/s} d\theta I \leq \int_0^\infty d\theta I$ for positive integrand I and dropped subleading orders of s . We will also drop the factor $\exp(-3s\|x_a - x_{a+1}\|)$ in (2.66) in what follows as it just leads to higher order corrections.

We can now perform the integrals corresponding to the measures $\nu_1, \dots, \nu_{N/2-1}$. Since the integrand no longer depends on $U_1, \dots, U_{N/2-1}$ the Haar measure part of those measures just integrates to one and we are left with an integral of the form

$$\begin{aligned}
 & \frac{c}{t^{3N/2}} \frac{1}{s^{18(N/2-1)}} \text{Pol}_{6N}(p) \\
 & \quad \times \int dx_1 \cdots \int dx_{N/2-1} \exp(-3\|x - x_1\|^2/4) \cdots \exp(-3\|x_{N/2-1} - x_{N/2}\|^2) \\
 & = \frac{c'}{t^{3N/2}} \frac{1}{s^{18(N/2-1)}} \text{Pol}_{6N}(p) \exp(-3\|x - x_{N/2}\|^2/(2N)) \tag{2.67}
 \end{aligned}$$

where c, c' are numerical constants and we have made use of the Markov property $\int dy K_s(x, y) K_t(y, z) = c K_{s+t}(x, z)$ with $K_t(x, y) = \exp(-(x - y)^2/t)$. The negative power of $s = \sqrt{t}$ comes from the fact that $dv = dp d\mu_H(U)/t^{18}$ but only $1/s^{18}$ was absorbed by the Gaussian integral in p .

Next we perform the integral corresponding to $\nu_{0N/2}$ which is of the form

$$\begin{aligned}
 & \int dv_{0N/2} |\langle \psi_{N/2}, P_{N/2+1} \psi_{0N/2} \rangle| |\langle \psi_{0N/2}, [\Delta F'_1] \psi'_1 \rangle| \\
 & \leq c \int \frac{dp_{0N/2} d\mu_H(U_{0N/2})}{t^{18}} \exp(-[\|p_{0N/2} - p_{N/2}\|^2 + \|p_{0N/2} - p'_1\|^2 \\
 & \quad + [\|\theta_{0N/2} - \theta_{N/2}\|^2 + \|\theta_{0N/2} - \theta'_1\|^2]/t]) \\
 & \quad \times |\text{Pol}_1(\exp((p_{0N/2} - p_{N/2})/2), \exp(i(\theta_{0N/2} - \theta_{N/2})/2))| \\
 & \quad \times \text{Pol}_{10(k+1)}(p_{0N/2} + p'_1 + i(\theta'_1 - \theta_{0N/2})) \\
 & \quad \times \text{Pol}_{2(k+1)}(p_{0N/2} + p'_1 - 2p + i(\theta'_1 - \theta_{0N/2}))|. \tag{2.68}
 \end{aligned}$$

We introduce new integration variables $x_{0N/2} = (p_{0N/2} - p)/s$, $x'_1 = (p'_1 - p)/s$, $y_{0N/2} = (\theta_{0N/2} - \theta'_1)/s$, replace the Haar measure by the Lebesgue measure after an appropriate shift in $SU(2)$ as above, extend the integration domain from a ball to all of \mathbb{R}^3 , estimate entries of $SU(2)$ group elements from above by 1 and can estimate (2.68) further by

$$\begin{aligned}
 & \int dv_{0N/2} |\langle \psi_{N/2}, P_{N/2+1} \psi_{0N/2} \rangle| |\langle \psi_{0N/2}, [\Delta F'_1] \psi'_1 \rangle| \\
 & \leq \int dx_{0N/2} dy_{0N/2} \exp(-[\|x_{0N/2} - x_{N/2}\|^2 + \|x_{0N/2} - x'_1\|^2] \\
 & \quad + \|y_{0N/2}\|^2)
 \end{aligned}$$

$$\begin{aligned}
 & + [\|y_{0N/2} + (\theta'_1 - \theta_{N/2})/s\|^2 + \|y_{0N/2}\|^2]) \\
 & \times |\text{Pol}_1(\exp(s(x_{0N/2} - x_{N/2})/2) \text{Pol}_{10(k+1)}(2p + s[x_{0N/2} + x'_1 - iy_{0N/2}]) \\
 & \times \text{Pol}_{2(k+1)}(s[x_{0N/2} + x'_1 - iy_{0N/2}]))| \\
 & = ct^{k+1} \text{Pol}_{10(k+1)}(p) \text{Pol}'_{2(k+1)}(x'_1) \exp([-3\|x'_1 - x_{N/2}\|^2 + \|\theta'_1 - \theta_{N/2}\|^2/t]/4)
 \end{aligned} \tag{2.69}$$

where we again used basic properties of Gaussian integrals and dropped subleading orders in \hbar . We used the notation Pol'_n for a generic inhomogeneous polynomial of degree n and performed estimates of the form $|\text{Pol}_{2(k+1)}(x + iy)| = P_{k+1}(x^2 + y^2)$.

Combining (2.67) and (2.69) we can perform the integral over $\nu_{N/2}$ using the same manipulations and end up with an error estimate so far given by

$$= ct^{k+1-3N/2} \frac{1}{s^{18(N/2-1)}} \text{Pol}_{6N+10(k+1)}(p) \text{Pol}'_{2(k+1)}(x'_1) \exp(-3\|x - x'_1\|^2/(2N + 4)) \tag{2.70}$$

where $x = p/s$ and p corresponds to the external state ψ . Next consider the integral for $(a = 1, \dots, l - 1)$

$$\int dv'_{0a} |\langle \psi'_a, P'_a \psi'_{0a} \rangle| |\langle \psi'_{0a}, [\Delta F'_{a+1} \psi'_{a+1}] \rangle| \tag{2.71}$$

which can be estimated just like (2.68) with the obvious changes in the integration variables, resulting in

$$\begin{aligned}
 \int dv'_{0a} |\langle \psi'_a, P'_a \psi'_{0a} \rangle| |\langle \psi'_{0a}, [\Delta F'_{a+1} \psi'_{a+1}] \rangle| & \leq ct^{k+1} \text{Pol}_{10(k+1)}(p) \text{Pol}'_{2(k+1)}(x'_{a+1}) \\
 & \times \exp([-3\|x'_{a+1} - x'_a\|^2 + \|\theta'_{a+1} - \theta'_a\|^2/t]/4).
 \end{aligned} \tag{2.72}$$

We can now combine (2.70) and (2.72) and perform the integrals corresponding to the measures $\nu'_1, \dots, \nu'_{l-1}$. We use translation invariance of $d\mu_H(U'_a) = d\mu_H(U'_a(U'_{a+1})^{-1})$ and introduce $y'_a = (\theta'_a - \theta'_{a+1})/s$ and arrive at the estimate so far

$$ct^{l(k+1)-3N/2} \frac{1}{s^{18(N/2-1)}} \text{Pol}_{6N+10l(k+1)}(p) \text{Pol}'_{2(k+1)}(x'_l) \exp(-3\|x - x'_l\|^2/(2N + 4l)) \tag{2.73}$$

where we absorbed the constants corresponding to Gaussian integrals over polynomials into c .

All we did so far can also be done to the corresponding integrals involving the measures with a tilde resulting in the estimate

$$ct^{l(k+1)-3N/2} \frac{1}{s^{18(N/2-1)}} \text{Pol}_{6N+10l(k+1)}(p) \text{Pol}'_{2(k+1)}(\tilde{x}'_l) \exp(-3\|x - \tilde{x}'_l\|^2/(2N + 4l)). \tag{2.74}$$

It remains to perform the integral

$$\begin{aligned}
 ct^{2l(k+1)-3N} \frac{1}{t^{18(N/2-1)}} \text{Pol}_{12N+20l(k+1)}(p) & \int dv'_l \int d\tilde{v}'_l \int dv'_{0l} \int d\tilde{v}'_{0l} \text{Pol}'_{2(k+1)}(x'_l) \\
 & \times \exp(-3\|x - x'_l\|^2/(2N + 4l)) \text{Pol}'_{2(k+1)}(\tilde{x}'_l) \exp(-3\|x - \tilde{x}'_l\|^2/(2N + 4l)) \\
 & \times |\langle \psi'_l, P'_l \psi'_{0l} \rangle| |\langle \tilde{\psi}'_l, P'_l \tilde{\psi}'_{0l} \rangle| |\langle \psi'_{0l}, [\Delta F'_{l+1} \tilde{\psi}'_{0l}] \rangle|.
 \end{aligned} \tag{2.75}$$

We perform first the integral corresponding to ψ'_{0l} using the same manipulations as in (2.69) leading to an estimate of (2.75) given by

$$\begin{aligned}
 ct^{(2l+1)(k+1)-3N} \frac{1}{t^{18(N/2-1)}} \text{Pol}_{12N+10(2l+1)(k+1)}(p) & \int dv'_l \int d\tilde{v}'_l \int d\tilde{v}'_{0l} \text{Pol}'_{2(k+1)}(x'_l) \\
 & \times \exp(-3\|x - x'_l\|^2/(2N + 4l)) \text{Pol}'_{2(k+1)}(\tilde{x}'_l) \exp(-3\|x - \tilde{x}'_l\|^2/(2N + 4l)) \\
 & \times \text{Pol}'_{2(k+1)}(x'_{l+1}) \exp([-3\|x'_l - \tilde{x}'_{0l}\|^2 + \|\theta'_l - \tilde{\theta}'_{0l}\|^2/t]/4) |\langle \tilde{\psi}'_l, P'_l \tilde{\psi}'_{0l} \rangle|
 \end{aligned}$$

$$\begin{aligned}
&\leq c t^{(2l+1)(k+1)-3N} \frac{1}{t^{18(N/2-1)}} \text{Pol}_{12N+10(2l+1)(k+1)}(p) \int d\nu'_l \int d\tilde{\nu}'_l \int d\tilde{\nu}'_{0l} \\
&\quad \times \text{Pol}'_{2(k+1)}(x'_l) \exp(-3\|x - x'_l\|^2/(2N + 4l)) \text{Pol}'_{2(k+1)}(\tilde{x}'_l) \\
&\quad \times \exp(-3\|x - \tilde{x}'_l\|^2/(2N + 4l)) \text{Pol}'_{2(k+1)}(x'_{l+1}) \exp([-3[\|x'_l - \tilde{x}'_{0l}\|^2 \\
&\quad + \|\theta'_l - \tilde{\theta}'_{0l}\|^2/t]/4) \exp([-[\|\tilde{x}'_l - \tilde{x}'_{0l}\|^2 + \|\tilde{\theta}'_l - \tilde{\theta}'_{0l}\|^2/t]) \\
&\quad \times \text{Pol}_1(\exp((\tilde{p}'_l - \tilde{p}'_{0l})/2), \exp(i[\tilde{\theta}'_l + \tilde{\theta}'_{0l}]/2)). \tag{2.76}
\end{aligned}$$

Using translation invariance $d\mu_H(U'_l) = d\mu_H(U'_l(\tilde{U}'_{0l})^{-1})$ and $d\mu_H(\tilde{U}'_l) = d\mu_H(\tilde{U}'_l(\tilde{U}'_{0l})^{-1})$ we can perform the Gaussians in $\theta'_l - \tilde{\theta}'_{0l}, \tilde{\theta}'_l - \tilde{\theta}'_{0l}$. After this, nothing depends on $\tilde{\theta}'_{0l}$ any more and the corresponding Haar measure integrates to unity. Finally, the remaining three momentum Gaussian integrals can also be performed and give a constant of order unity times s^{-18} from one missing Gaussian factor in the angle variables. Thus, the final estimate is given by

$$c t^{(2l+1)(k+1)-3N-18(N/2-1/2)} \text{Pol}_{12N+10(2l+1)(k+1)}(p). \tag{2.77}$$

The constant c is of order unity because the resolution measures contain the right power of π , see [5]. The bound depends on the point in phase space as expected. In order that the power of t equals at least $k + 1$ we should have $l \geq (12N - 9)/(2(k + 1))$. Thus we see that we need, for given N , fewer terms in order to arrive at corrections of order \hbar^{k+1} the higher is the power of $Q^2 - \langle Q \rangle^2$ in terms of which we expand the operator.

Note that the power counting argument outlined here can be supplemented by an analytical calculation in order to obtain the actual value of the integral because after the estimate of $\langle \psi_1, f \psi_2 \rangle$ everything is computable [14]. However, the leading power in t of the integral will be at least the one we stated. Such an analytical calculation would also reveal all the finite, higher order (in \hbar) corrections that we have dropped in the power counting argument.

3. Explicit example: the case $N = 2$

Note that for standard matter for the terms appearing in the master constraint we have $2 \leq N \leq 12$. We will exemplify the procedure for $N = 2$ and consider arbitrary k . This means that in order to obtain a computable error bound of order $k + 1$ we must have $l \geq 15/(2(k + 1))$.

We will work out only the real part, the imaginary part works completely analogously. We have with $R = \langle \psi, p_1 f_1 p_2 f_2 p_3 \psi \rangle$ and $\psi_1 := \bar{p}_1, \psi_2 := p_2 f_2 p_3 \psi$

$$|\text{Re}(R) - R_1| \leq R_2 + R_3 \tag{3.1}$$

where

$$\begin{aligned}
R_1 &= \text{Re}(\langle \psi, p_1 \bar{f}_1 p_2 f_2 p_3 \psi \rangle) \\
R_2 &= \langle \psi, p_1 [\Delta f]_1 \bar{p}_1 \psi \rangle \\
R_3 &= \langle \psi, \bar{p}_3 f_2 \bar{p}_2 [\Delta f_1] p_2 f_2 p_3 \psi \rangle.
\end{aligned} \tag{3.2}$$

The term R_2 is already computable by the methods of [5]. The first term involves the non-polynomial expression f_2 only linearly. Thus we can get rid of it by iterating the steps (2.18) and (2.19). Hence we set $\psi_1 := \bar{p}_2 \bar{f}_1 \bar{p}_1 \psi, \psi_2 := p_3 \psi$ and find

$$|R_1 - R_4| \leq R_5 + R_6 \tag{3.3}$$

where

$$\begin{aligned}
R_4 &= \text{Re}(\langle \psi, p_1 \bar{f}_1 p_2 \bar{f}_2 p_3 \psi \rangle) \\
R_5 &= \langle \psi, p_1 \bar{f}_1 p_2 [\Delta f_2] \bar{p}_2 \bar{f}_1 \bar{p}_1 \psi \rangle \\
R_6 &:= \langle \psi, \bar{p}_3 [\Delta f_2] p_3 \psi \rangle.
\end{aligned} \tag{3.4}$$

All terms R_4, R_5, R_6 are computable by the methods of [5].

The term R_3 contains f_2 quadratically and iterating the steps (2.18) and (2.19) will not change that. However, we will iterate to generate terms of higher order. We set $\psi_1 = p_3\psi$, $\psi_2 := \bar{p}_2[\Delta f_1]p_2f_2p_3\psi$ and get

$$|R_3 - R_7| \leq R_8 + R_9 \tag{3.5}$$

where

$$\begin{aligned} R_7 &= \text{Re}(\langle \psi, \bar{p}_3\bar{f}_2\bar{p}_2[\Delta f_1]p_2f_2p_3\psi \rangle) \\ R_8 &= \langle \psi, \bar{p}_3[\Delta f_2]p_3\psi \rangle \\ R_9 &:= \langle \psi, \bar{p}_3f_2\bar{p}_2[\Delta f_1]p_2[\Delta f_2]\bar{p}_2[\Delta f_1]p_2f_2p_3\psi \rangle. \end{aligned} \tag{3.6}$$

The term R_8 is computable by the methods of [5] while R_7 involves f_2 only linearly and thus can be gotten rid of by performing once more steps (2.20) and (2.21). Hence we set $\psi_1 = \bar{p}_2[\Delta f_1]p_2\bar{f}_2p_3$ and $\psi_2 = p_3\psi$ and find

$$|R_7 - R_{10}| \leq R_{11} + R_{12} \tag{3.7}$$

where

$$\begin{aligned} R_{10} &= \text{Re}(\langle \psi, \bar{p}_3\bar{f}_2\bar{p}_2[\Delta f_1]p_2\bar{f}_2p_3\psi \rangle) \\ R_{11} &= \langle \psi, \bar{p}_3[\Delta f_2]p_3\psi \rangle \\ R_{12} &:= \langle \psi, \bar{p}_3\bar{f}_2\bar{p}_2[\Delta f_1]p_2[\Delta f_2]\bar{p}_2[\Delta f_1]p_2\bar{f}_2p_3\psi \rangle. \end{aligned} \tag{3.8}$$

All terms R_{10}, R_{11}, R_{12} are computable by the methods of [5].

Note that we have shown so far that

$$|R - R_4| \leq R_2 + R_5 + R_6 + R_8 + R_{10} + R_{11} + R_{12} + R_9 \tag{3.9}$$

where only the last term R_9 is not computable. Since it contains three factors of the form Δf meaning $l = 1$, given that it already leads to order $k + 1$ corrections we must take $k = 7$. For lower values of k the procedure must be iterated again in order to generate more factors of the form Δf in the non-computable expressions. We will not do that as the general pattern should be clear by now. Note that $R_2, R_5, R_6, R_8, R_{10}, R_{11}$ are of order \hbar^{k+1} while R_{12} is of order $\hbar^{3(k+1)}$ and thus is subleading.

We finish this section by noting that in the special case $N = 2$ we actually do not need to use the iteration process in order to arrive at a calculable bound because we can use the Cauchy–Schwarz inequality to obtain

$$\begin{aligned} |\langle \psi, p_1f_1p_2f_2p_3\psi \rangle| &\leq \|f_1\bar{p}_1\psi\| \|p_2f_2p_3\psi\| \\ &\leq \|f_1\bar{p}_1\psi\| \|f_2p_3\psi\| \\ &\leq \|f_1^+\bar{p}_1\psi\| \|f_2^+p_3\psi\| \end{aligned} \tag{3.10}$$

where in the second step we used that p_2 is a bounded operator with bound $\|p_2\| \leq 1$ and in the third we used $\|f\psi\|^2 = \langle \psi, f^2\psi \rangle \leq \langle \psi, (f^+)^2\psi \rangle \leq \|f^+\psi\|^2$. Both factors in the last line of (3.10) are computable by the methods of [5]. This will suffice to get a bound of order \hbar but unfortunately not better than that.

4. Conclusion

In this paper, we established that semiclassical calculations within AQG and LQG can be done analytically as a perturbation expansion in \hbar . The errors can be controlled, i.e. they are finite and can be estimated from above and are of higher order in \hbar than the order that one is interested in. This is despite the fact that the spectrum of the volume operator of AQG is not known analytically, hence disproving the negative claims made in [17] ‘that nothing is

calculable in LQG'. All one has to do in order to compute expectation values, correct to the order \hbar^k , with respect to coherent states which involve the volume operator for a vertex v in the form $(Q_v^2)^q$, is to replace this operator within the matrix element by its power expansion up to order $2k + 1$ that is

$$\langle \psi, Q_v \psi \rangle^{2q} \left[1 + \sum_{n=1}^{2k+1} (-1)^{n+1} \frac{q(1-q) \cdots (2k-q)}{n!} \left(\frac{Q_v^2}{\langle \psi, Q_v \psi \rangle^2} - 1 \right)^n \right]. \quad (4.1)$$

The exact matrix elements of the operator Q_v are known in closed form. The error term can be estimated from above and is of order \hbar^{k+1} . Hence we are in a situation similar to ordinary QFT, in fact the situation is even better because we have error control.

Formula (4.1) is what one would naively do anyway in order to do semiclassical calculations; however, since the operator Q_v is unbounded, one cannot use the spectral theorem to show that (4.1) really approximates the actual operator. In this paper, we showed that the naive guess is nevertheless correct by using methods from functional analysis and properties of our coherent states.

Note that while we have done perturbation theory within AQG for cubic algebraic graphs, everything goes through also in LQG for arbitrary graphs. Hence, the present paper turns AQG and LQG into a calculational discipline.

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References

- [1] Giesel K and Thiemann T 2007 Algebraic quantum gravity (AQG): I. Conceptual setup *Class. Quantum Grav.* **24** 2465–97 (Preprint [gr-qc/0607099](#))
- [2] Giesel K and Thiemann T 2007 Algebraic quantum gravity (AQG): II. Semiclassical analysis *Class. Quantum Grav.* **24** 2499–564 (Preprint [gr-qc/0607100](#))
- [3] Rovelli C 2004 *Quantum Gravity* (Cambridge: Cambridge University Press)
Thiemann T 2007 *Modern Canonical Quantum General Relativity* (Cambridge: Cambridge University Press) (Draft available as Preprint [gr-qc/0110034](#))
- [4] Rovelli C 1998 Loop quantum gravity *Living Rev. Rel.* **1** 1 (Preprint [gr-qc/9710008](#))
Thiemann T 2003 Lectures on loop quantum gravity *Lect. Notes Phys.* **631** 41–135 (Preprint [gr-qc/0210094](#))
Ashtekar A and Lewandowski J 2004 Background independent quantum gravity: a status report *Class. Quantum Grav.* **21** R53 (Preprint [gr-qc/0404018](#))
Smolin L 2004 An invitation to loop quantum gravity Preprint [hep-th/0408048](#)
- [5] Thiemann T 2001 Quantum spin dynamics (QSD): VII. Symplectic structures and continuum lattice formulations of gauge field theories *Class. Quantum Grav.* **18** 3293–338 (Preprint [hep-th/0005232](#))
Thiemann T 2001 Gauge field theory coherent states (GCS): I. General properties *Class. Quantum Grav.* **18** 2025–64 (Preprint [hep-th/0005233](#))
Thiemann T and Winkler O 2001 Gauge field theory coherent states (GCS): II. Peakedness properties *Class. Quantum Grav.* **18** 2561–636 (Preprint [hep-th/0005237](#))
Thiemann T and Winkler O 2001 Gauge field theory coherent states (GCS): III. Ehrenfest theorems *Class. Quantum Grav.* **18** 4629–81 (Preprint [hep-th/0005234](#))
- [6] Sahlmann H, Thiemann T and Winkler O 2001 Coherent states for canonical quantum general relativity and the infinite tensor product extension *Nucl. Phys. B* **606** 401–40 (Preprint [gr-qc/0102038](#))
- [7] Thiemann T and Winkler O 2001 Gauge field theory coherent states (GCS): IV. Infinite tensor product and thermodynamic limit *Class. Quantum Grav.* **18** 4997–5033 (Preprint [hep-th/0005235](#))
- [8] Thiemann T 2006 Complexifier coherent states for canonical quantum general relativity *Class. Quantum Grav.* **23** 2063–118 (Preprint [gr-qc/0206037](#))

- [9] QSD papers Thiemann T 1996 Anomaly-free formulation of non-perturbative, four-dimensional Lorentzian quantum gravity *Phys. Lett. B* **380** 257–64 (Preprint [gr-qc/9606088](#))
 Thiemann T 1998 Quantum spin dynamics (QSD) *Class. Quantum Grav.* **15** 839–73 (Preprint [gr-qc/9606089](#))
 Thiemann T 1998 Quantum spin dynamics (QSD): II. The kernel of the Wheeler–deWitt constraint operator *Class. Quantum Grav.* **15** 875–905 (Preprint [gr-qc/9606090](#))
 Thiemann T 1998 Quantum spin dynamics (QSD): III. Quantum constraint algebra and physical scalar product in quantum general relativity *Class. Quantum Grav.* **15** 1207–47 (Preprint [gr-qc/9705017](#))
 Thiemann T 1998 Quantum spin dynamics (QSD): IV. 2+1 Euclidean quantum gravity as a model to test 3+1 Lorentzian quantum gravity *Class. Quantum Grav.* **15** 1249–80 (Preprint [gr-qc/9705018](#))
 Thiemann T 1998 Quantum spin dynamics (QSD): V. Quantum gravity as the natural regulator of the Hamiltonian constraint of matter quantum field theories *Class. Quantum Grav.* **15** 1281–314 (Preprint [gr-qc/9705019](#))
 Thiemann T 1998 Quantum spin dynamics (QSD): VI. Quantum Poincaré algebra and a quantum positivity of energy theorem for canonical quantum gravity *Class. Quantum Grav.* **15** 1463–85 (Preprint [gr-qc/9705020](#))
 Thiemann T 1998 Kinematical Hilbert spaces for fermionic and Higgs quantum field theories *Class. Quantum Grav.* **15** 1487–512 (Preprint [gr-qc/9705021](#))
- [10] Thiemann T 2006 The Phoenix project: master constraint programme for loop quantum gravity *Class. Quantum Grav.* **23** 2211–48 (Preprint [gr-qc/0305080](#))
- [11] Thiemann T 2006 Quantum spin dynamics: VIII. The master constraint *Class. Quantum Grav.* **23** 2249–66 (Preprint [gr-qc/0510011](#))
- [12] Dittrich B and Thiemann T 2006 Testing the master constraint programme for loop quantum gravity: I. General framework *Class. Quantum Grav.* **23** 1025–66 (Preprint [gr-qc/0411138](#))
 Dittrich B and Thiemann T 2006 Testing the master constraint programme for loop quantum gravity: II. Finite-dimensional systems *Class. Quantum Grav.* **23** 1067–88 (Preprint [gr-qc/0411139](#))
 Dittrich B and Thiemann T 2006 Testing the master constraint programme for loop quantum gravity: III. $SL(2R)$ models *Class. Quantum Grav.* **23** 1089–120 (Preprint [gr-qc/0411140](#))
 Dittrich B and Thiemann T 2006 Testing the master constraint programme for loop quantum gravity: IV. Free field theories *Class. Quantum Grav.* **23** 1121–42 (Preprint [gr-qc/0411141](#))
 Dittrich B and Thiemann T 2006 Testing the master constraint programme for loop quantum gravity: V. Interacting field theories *Class. Quantum Grav.* **23** 1143–62 (Preprint [gr-qc/0411142](#))
- [13] Rovelli C and Smolin L 1995 Discreteness of volume and area in quantum gravity *Nucl. Phys. B* **442** 593
 Rovelli C and Smolin L 1995 *Nucl. Phys. B* **456** 734 (erratum)
 Ashtekar A and Lewandowski J 1997 Quantum theory of geometry: II. Volume operators *Adv. Theor. Math. Phys.* **1** 388–429
 Thiemann T 1998 Closed formula for the matrix elements of the volume operator in canonical quantum gravity *J. Math. Phys.* **39** 3347–71 (Preprint [gr-qc/9606091](#))
 Brunnemann J and Thiemann T 2006 Simplification of the spectral analysis of the volume operator in loop quantum gravity *Class. Quantum Grav.* **23** 1289–346 (Preprint [gr-qc/0405060](#))
- [14] Brunneman J in preparation
- [15] Smolin L and Starodubtsev A 2003 General relativity with a topological phase: an action principle Preprint [hep-th/0311163](#)
 Freidel L and Starodubtsev A 2005 Quantum gravity in terms of topological observables Preprint [hep-th/0501191](#)
- [16] Giesel K and Thiemann T 2006 Consistency check on volume and triad operator quantisation in loop quantum gravity: I *Class. Quantum Grav.* **23** 5667–92 (Preprint [gr-qc/0507036](#))
 Giesel K and Thiemann T 2006 Consistency check on volume and triad operator quantisation in loop quantum gravity: II *Class. Quantum Grav.* **23** 5693–772 (Preprint [gr-qc/0507037](#))
- [17] Nicolai H, Peeters K and Zamaklar M 2005 Loop quantum gravity: an outside view *Class. Quantum Grav.* **22** R193 (Preprint [hep-th/0501114](#))